LECTURE NOTES

ON

HEAT TRANSFER

B. Tech V semester

IARE – R16

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UNIT-I
INTRODUCTORY CONCEPTS AND BASIC LAWS OF HEAT TRANSFER

Introduction:- We recall from our knowledge of thermodynamics that heat is a form of energy transfer that takes place from a region of higher temperature to a region of lower temperature solely due to the temperature difference between the two regions. With the knowledge of thermodynamics we can determine the amount of heat transfer for any system undergoing any process from one equilibrium state to another. Thus the thermodynamics knowledge will tell us only how much heat must be transferred to achieve a specified change of state of the system. But in practice we are more interested in knowing the rate of heat transfer (i.e. heat transfer per unit time) rather than the amount. This knowledge of rate of heat transfer is necessary for a design engineer to design all types of heat transfer equipments like boilers, condensers, furnaces, cooling towers, dryers etc.The subject of heat transfer deals with the determination of the rate of heat transfer to or from a heat exchange equipment and also the temperature at any location in the device at any instant of time.

The basic requirement for heat transfer is the presence of a "temperature difference". The temperature difference is the driving force for heat transfer, just as the voltage difference for electric current flow and pressure difference for fluid flow. One of the parameters ,on which the rate of heat transfer in a certain direction depends, is the magnitude of the temperature gradient in that direction. The larger the gradient higher will be the rate of heat transfer.

1.2. Heat Transfer Mechanisms:- There are three mechanisms by which heat transfer can take place. All the three modes require the existence of temperature difference. The three mechanisms are: (i) conduction, (ii) convection and (iii) radiation

1.2.1Conduction:- It is the energy transfer that takes place at molecular levels. Conduction is the transfer of energy from the more energetic molecules of a substance to the adjacent less energetic molecules as a result of interaction between the molecules. In the case of liquids and gases conduction is due to collisions and diffusion of the molecules during their random motion. In solids, it is due to the vibrations of the molecules in a lattice and motion of free electrons.

Fourier's Law of Heat Conduction:- The empirical law of conduction based on experimental results is named after the French Physicist Joseph Fourier. The law states that the rate of heat flow by conduction in any medium in any direction **is proportional to the area normal to the direction of heat flow and also proportional to the temperature gradient in that direction**. For example the rate of heat transfer in x-direction can be written according to Fourier"s law as

$$Q_x \ \alpha - A\,(dT / dx) \qquad \text{……………………….(1.1)}$$

or

$$Q_x = - k\,A\,(dT / dx) \ \ W \text{…………………….. ..(1.2)}$$

In equation (1.2), Q_X is the rate of heat transfer in positive x-direction through area A of the medium normal to x-direction, (dT/dx) is the temperature gradient and k is the constant of proportionality and is a material property called *"thermal conductivity"*. Since heat transfer has to take place in the direction of decreasing temperature, (dT/dx) has to be negative in the direction of heat transfer. Therefore negative sign has to be introduced in equation (1.2) to make Q_X positive in the direction of decreasing temperature, thereby satisfying the second law of thermodynamics. If equation (1.2) is divided throughout by A we have

$$q_X = (Q_X / A) = - k\,(dT / dx) \quad W/m^2 \text{………..(1.3)}$$

q_X is called the *heat flux*.

Thermal Conductivity: - The constant of proportionality in the equation of Fourier"s law of conduction is a material property called the thermal conductivity.The units of thermal conductivity can be obtained from equation (1.2) as follows:

Solving for k from Eq. (1.2) we have $k = - q_X / (dT/dx)$

Therefore units of k = (W/m^2) (m/ K) = W / (m – K) or W / (m – ^{0} C). Thermal conductivity is a measure of a material"s ability to conduct heat. The thermal conductivities of materials vary over a wide range as shown in Fig. 1.1.

It can be seen from this figure that the thermal conductivities of gases such as air vary by a factor of 10^4 from those of pure metals such as copper. The kinetic theory of gases predicts and experiments confirm that the thermal conductivity of gases is proportional to the *square root of the absolute temperature,* and inversely proportional to the *square root of the molar mass M.* Hence, the thermal conductivity of gases increases with increase in temperature and decrease with increase in molar mass. It is for these reasons that the thermal conductivity of helium (M=4) is much higher than those of air (M=29) and argon (M=40).For wide range of pressures encountered in practice the thermal conductivity of gases is *independent of pressure*.

The mechanism of heat conduction in liquids is more complicated due to the fact that the molecules are more closely spaced, and they exert a stronger inter-molecular force field. The values of k for liquids usually lie between those for solids and gases. Unlike gases, the thermal conductivity for most liquids decreases with increase in temperature except for water. Like gases the thermal conductivity of liquids decreases with increase in molar mass.

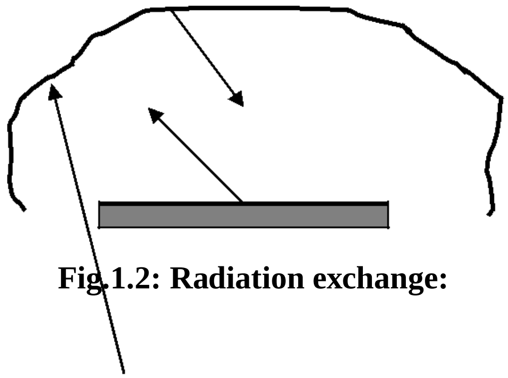


Fig.1.2: Radiation exchange:

is emitted by the surface originates from the thermal energy of matter bounded by the surface, and the rate at which this energy is released per unit area is called as the surface *emissive power E.*An ideal surface is one which emits maximum emissive power and is called *an ideal radiator or a black body.*Stefan-Boltzman‟s law of radiation states that the emissive power of a black body is proportional to the fourth power of the absolute temperature of the body. Therefore if E_b is the emissive power of a black body at temperature T ^{0}K, then

$$E_b \; \alpha \; T^4$$

(or)

$$E_b = \zeta T^4 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1.7)$$

ζ is the *Stefan-Boltzman constant* (σ = 5.67 x 10^{-8} W / (m^2 – K^4)). For a non black surface the emissive power is given by

$$E = \varepsilon \zeta T^4 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1.8)$$

where ε is called the emissivity of the surface ($0 \le \varepsilon \le 1$).The emissivity provides a measure of how efficiently a surface emits radiation relative to a black body. The emissivity strongly depends on the surface material and finish.

Radiation may also *incident* on a surface from its surroundings. The rate at which the radiation is incident on a surface per unit area of the surface is calle the *"irradiation"* of the surface and is denoted by G. The fraction of this energy absorbed by the surface is called *"absorptivity"* of the surface and is denoted by the symbol α. The fraction of the incident energy is reflected and is called the *"reflectivity"* of the surface denoted by ρ and the remaining fraction of the incident energy is transmitted through the surface and is called the *"transmissivity"* of the surface denoted by η. It follows from the definitions of α, ρ, and η that

$$\alpha + \rho + \eta = 1 \quad \text{..(1.9)}$$

Therefore the energy absorbed by a surface due to any radiation falling on it is given by

$$G_{abs} = \alpha G \quad \text{......................................(1.10)}$$

The absorptivity α of a body is generally different from its emissivity. However in many practical applications, to simplify the analysis α is assumed to be equal to its emissivity ε.

Radiation Exchange:- When two bodies at different temperatures "see" each other, heat is exchanged between them by radiation. If the intervening medium is filled with a substance like air which is transparent to radiation, the radiation emitted from one body travels through the intervening medium without any attenuation and reaches the other body, and vice versa. Then the hot body experiences a net heat loss, and the cold body a net heat gain due to radiation heat exchange between the two. The analysis of radiation heat exchange among surfaces is quite complex which will be discussed in chapter 10. Here we shall consider two simple examples to illustrate the method of calculating the radiation heat exchange between surfaces.

As the first example" let us consider a small opaque plate (for an opaque surface η = 0) of area A, emissivity ε and maintained at a uniform temperature T_s. Let this plate is exposed to a large surroundings of area A_{su} (A_{su} >> A) whish is at a uniform temperature T_{sur} as shown in Fig. 1.2b.The space between them contains air which is transparent to thermal radiation.

The radiation energy emitted by the plate is given by

$$Q_{em} = A\, \varepsilon\, \zeta\, T_s^4$$

The large surroundings can be approximated as a black body in relation to the small plate. Then the radiation flux emitted by the surroundings is $\zeta\, T_{sur}^4$ which is also the radiaton flux incident on the plate. Therefore the radiation energy absorbed by the plate due to emission from the surroundings is given by

$$Q_{ab} = A\, \alpha\, \zeta\, T_{sur}^4.$$

The net radiation loss from the plate to the surroundings is therefore given by

$$Q_{rad} = A\, \varepsilon\, \zeta\, T_s^4 - A\, \alpha\, \zeta\, T_{sur}^4.$$

Assuming $\alpha = \varepsilon$ for the plate the above expression for Q_{net} reduces to

$$\mathbf{Q_{rad} = A\, \varepsilon\, \zeta\, [T_s^4 - T_{sur}^4]} \quad \textbf{……………….(1.11)}$$

The above expression can be used to calculate the net radiation heat exchange between a small area and a large surroundings.

As the second example, consider two finite surfaces A_1 and A_2 as shown in Fig. 1.3.

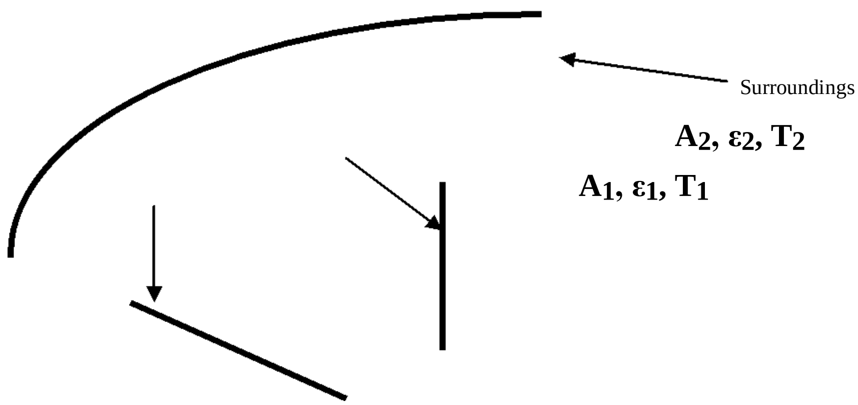


Fig.1.3: Radiation exchange between surfaces A_1 and A_2

The surfaces are maintained at absolute temperatures T_1 and T_2 respectively, and have emissivities ε_1 and ε_2. Part of the radiation leaving A_1 reaches A_2, while the remaining energy is lost to the surroundings. Similar considerations apply for the radiation leaving A_2.If it is assumed that the radiation from the surroundings is negligible when compared to the radiation from the surfaces A_1 and A_2 then we can write the expression for the radiation emitted by A_1 and reaching A_2 as

$$Q_{1\rightarrow 2} = F_{1-2}\, A_1 \varepsilon_1 \zeta\, T_1^4 \text{……………………………(1.12)}$$

where F_{1-2} is defined as the fraction of radiation energy emitted by A_1 and reaching A_2. Similarly the radiation energy emitted by A_2 and reaching A_1 is given by

$$Q_{2\rightarrow 1} = F_{2-1}\, A_2\, \varepsilon_2\, \zeta\, T_2^4 \text{ …………………………..(1.13)}$$

where F_{2-1} is the fraction of radiation energy leaving A_2 and reaching A_1. Hence the net radiation energy transfer from A_1 to A_2 is given by

$$Q_{1-2} = Q_{1\to2} - Q_{2\to1}$$

$$= [F_{1-2}\, A_1 \varepsilon_1 \zeta\, T_1^4] - [F_{2-1}\, A_2\, \varepsilon_2\, \zeta\, T_2^4]$$

$F_{1\text{-}2}$ is called the view factor (or geometric shape factor or configuration factor) of A_2 with respect to A_1 and F_{2-1} is the view factor of A_1 with respect to A_2.It will be shown in chapter 10 that the view factor is purely a geometric property which depends on the relative orientations of A_1 and A_2 satisfying the reciprocity relation, $A_1 F_{1-2} = A_2 F_{2-1}$.

Therefore $$\mathbf{Q_{1-2} = A_1F_{1-2}\, \zeta\, [\varepsilon_1\, T_1^4 - \varepsilon_2\, T_2^4] \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.13)}$$

Radiation Heat Transfer Coefficient:- Under certain restrictive conditions it is possible to simplify the radiation heat transfer calculations by defining a radiation heat transfer coefficient h_r analogous to convective heat transfer coefficient as

$$Q_r = h_r A\, \Delta T$$

For the example of radiation exchange between a surface and the surroundings [Eq. (1. 11)] using the concept of radiation heat transfer coefficient we can write

$$Q_r = h_r A[T_s - T_{sur}] = A\, \varepsilon\, \zeta\, [T_s^4 - T_{sur}^4]$$

Or $$h_r = \frac{\varepsilon\, \zeta\, [T_s^4 - T_{sur}^4]}{[T_s - T_{sur}]} = \frac{\varepsilon\, \zeta\, [T_s^2 + T_{sur}^2][T_s + T_{sur}][T_s - T_{sur}]}{[T_s - T_{sur}]}$$

Or $$\mathbf{h_r = \varepsilon\, \zeta\, [T_s^2 + T_{sur}^2][T_s + T_{sur}] \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.14)}$$

1.3.<u>First Law of Thermodynamics (Law of conservation of energy) as applied to Heat Transfer Problems</u> :-

The first law of thermodynamics is an essential tool for solving many heat transfer problems. Hence it is necessary to know the general formulation of the first law of thermodynamics.

First law equation for a control volume:- A *control volume* is a region in space bounded by a *control surface* through which energy and matter may pass.There are two options of formulating the first law for a control volume. One option is formulating the law on a *rate basis*. That is, at any instant, there must be a balance between all *energy rates*. Alternatively, the first law must also be satisfied over any *time interval* Δt. For such an interval, there must be a balance between the *amounts* of all energy changes.

First Law on rate basis: - The rate at which thermal and mechanical energy enters a control volume, plus the rate at which thermal energy is generated within the control volume, minus the rate at which thermal and mechanical energy leaves the control volume must be equal to the rate of increase of stored energy within the control volume. Consider a control volume shown in Fig. 1.4 which shows that thermal and mechanical energy are entering the control volume at a rate denoted by $\dot{E}_{in}$, thermal and

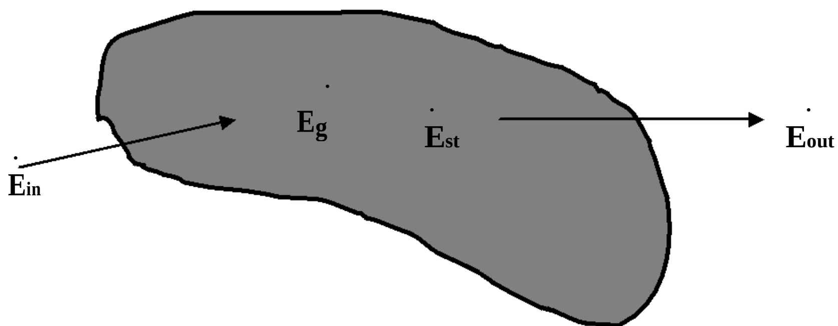


Fig. 1.4: Conservation of energy for a control volume on rate basis

mechanical energy are leaving the control volume at a rate denoted by $\dot{E}_{out}$. The rate at which energy is generated within the control volume is denoted by $\dot{E}_g$ and the rate at which energy is stored within the control volume is denoted by $\dot{E}_{st}$. The general form of the energy balance equation for the control volume can be written as follows:

$$\dot{E}_{in} + \dot{E}_g - \dot{E}_{out} = \dot{E}_{st} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.15)$$

$\dot{E}_{st}$ is nothing but the rate of increase of energy within the control volume and hence can be written as equal to dE_{st} / dt.

First Law over a Time Interval Δt:- Over a time interval Δt, the amount of thermal and mechanical energy that enters a control volume, plus the amount of thermal energy generated within the control volume minus the amount of thermal energy that leaves the control volume is equal to the increase in the amount of energy stored within the control volume.

The above statement can be written symbolically as

$$E_{in} + E_g - E_{out} = \Delta E_{st} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.16)$$

The inflow and outflow energy terms are *surface phenomena.* That is they are associated exclusively with the processes occurring at the boundary surface and are proportional to the surface area.

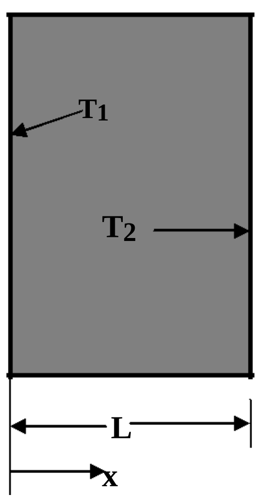
The energy generation term is associated with conversion from some other form (chemical, electrical, electromagnetic, or nuclear) to thermal energy. It is a *volumetric phenomenon.*That is, it occurs within the control volume and is proportional to the magnitude of this volume. For example, exothermic chemical reaction may be taking place within the control volume. This reaction converts chemical energy to thermal energy and we say that energy is generated within the control volume. Conversion of electrical energy to thermal energy due to resistance heating when electric current is passed through an electrical conductor is another example of thermal energy generation

Energy storage is also a volumetric phenomenon and energy change within the control volume is due to the changes in kinetic, potential and internal energy of matter within the control volume.

1.4. Illustrative Examples: A. Conduction

Example 1.1:- *Heat flux through a wood slab 50 mm thick, whose inner and outer surface temperatures are 40 ^{0} C and 20 ^{0} C respectively, has been determined to be 40 W/m^2. What is the thermal conductivity of the wood slab?*

Solution:



Given:- $T_1 = 40\ ^0$ C; $T_2 = 20\ ^0$ C; L = 0.05

m q = Q/A = 40 W / m^2.

To find: *k*

.

Assuming steady state conduction across the thickness of the slab and noting that the slab is not generating any thermal energy, the first law equation for the slab can be written as

Rate at which thermal energy (conduction) is entering the slab at the surface x = 0 is equal to the rate at which thermal energy is leaving the slab at the surface x = L That is

$$Q_x|_{x=0} = Q_x|_{x=L} = Q_x = \text{constant}$$

By Fourier‟s law we have $Q_x = -\, kA\,(dT / dx)$.

Separating the variables and integrating both sides w.r.t. „x‟ we have

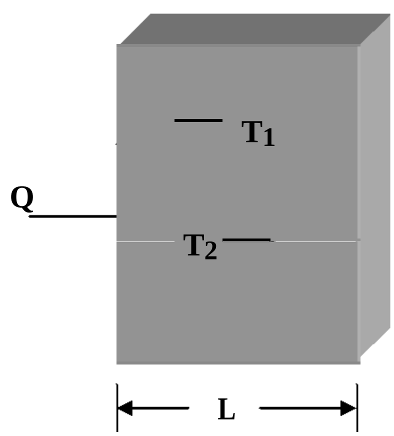
$$Q_x \int_0^L dx = -\, kA \int_{T1}^{T2} dT \text{ . Or } Q_x = kA\,(T_1 - T_2) / L$$

Heat flux = $q = Q_x / A = k(T_1 - T_2) / L$

Hence $k = q\,L / (T_1 - T_2) = 40 \text{ x } 0.05 / (40 - 20) = 0.1 \text{ W} / (\text{m} - \text{K})$

Example 1.2:- *A concrete wall, which has a surface area of 20 m^2 and thickness 30 cm, separates conditioned room air from ambient air.The temperature of the inner surface of the wall is 25 ^{0} C and the thermal conductivity of the wall is 1.5 W / (m-K).Determine the heat loss through the wall for ambient temperature varying from − 15 ^{0} C to 38 ^{0} C which correspond to winter and summer conditions and display your results graphically.*

Solution:



Data:- $T_1 = 25\ ^0$ C ; $A = 20$ m^2; $L = 0.3$ m

K = 1.5 W /(m-K) ;

By Fourier‟s law,

$$Q = kA(T_1 - T_2) / L$$

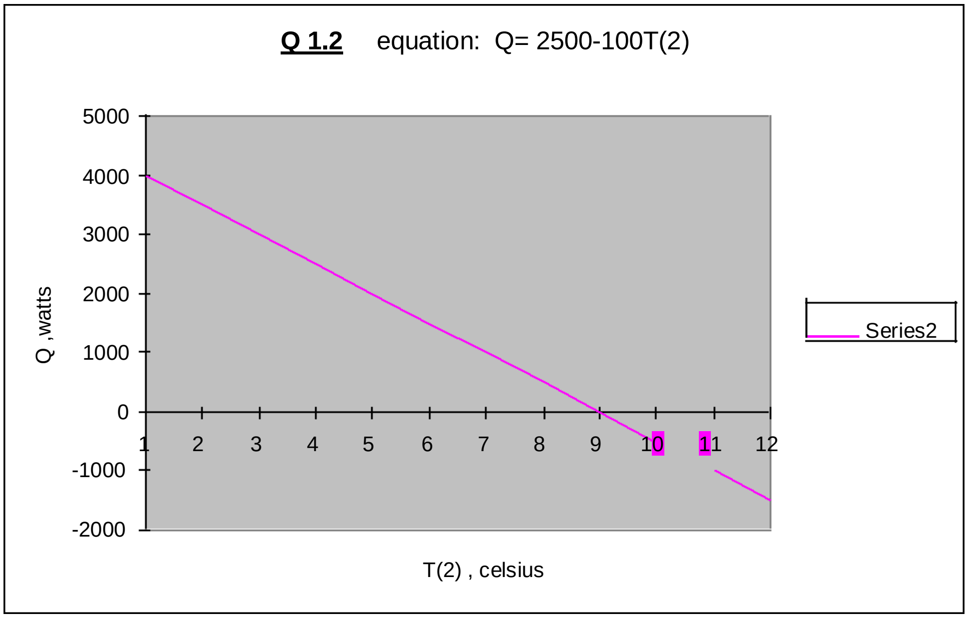
$$= \frac{1.5 \text{ x } 20 \text{ x } (25 - T_2)}{0.30}$$

Or $Q = 2500 - 100\,T_2$ ………..(1)

Heat loss Q for different values of T_2 ranging from – 15 ^{0} C to + 38 ^{0} C are obtained from Eq. (1) and the results are plotted as shown

Scale x-axis : 1cm= 5 C

y-axis : 1cm =1000 W



***Example 1.3:-**What is the thickness required of a masonry wall having a thermal conductivity of 0.75 W/(m-K), if the heat transfer rate is to be 80 % of the rate through another wall having thermal conductivity of 0.25 W/(m-K) and a thickness of 100 mm? Both walls are subjected to the same temperature difference.*

Solution:- Let subscript 1 refers to masonry wall and subscript 2 refers to the other wall.

By Fourier‟s law, $Q_1 = k_1 A(T_1 - T_2) / L_1$

And $Q_2 = k_2 A(T_1 - T_2) / L_2$

Therefore

$$\frac{Q_1}{Q_2} = \frac{k_1 L_2}{k_2 L_1}$$

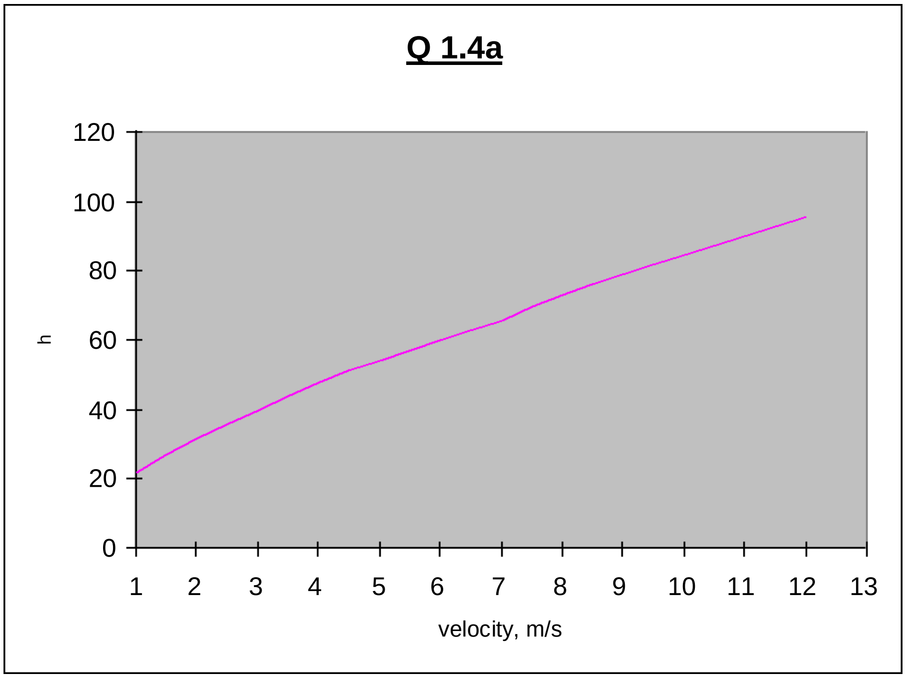
$$L_1 = \frac{Q_2 k_1}{Q_1 k_2} L_2$$

$$= (1 / 0.80) \times (0.75/0.25) \times 100 = 375 \text{ mm}$$

Air Velocity, V (m/s) :	1	2	4	8	12
Power,P (W/m)	: 450	658	983	1507	1963
h, (W / (m^2 – K))	: 22.04	32.22	48.14	73.8	96.13

(a) A graph of h versus V can now be plotted as shown in Fig. P 1.4 (a). Scale: X axis 1cm= 1m/s

Y axis 1cm= 10 W/m^2k

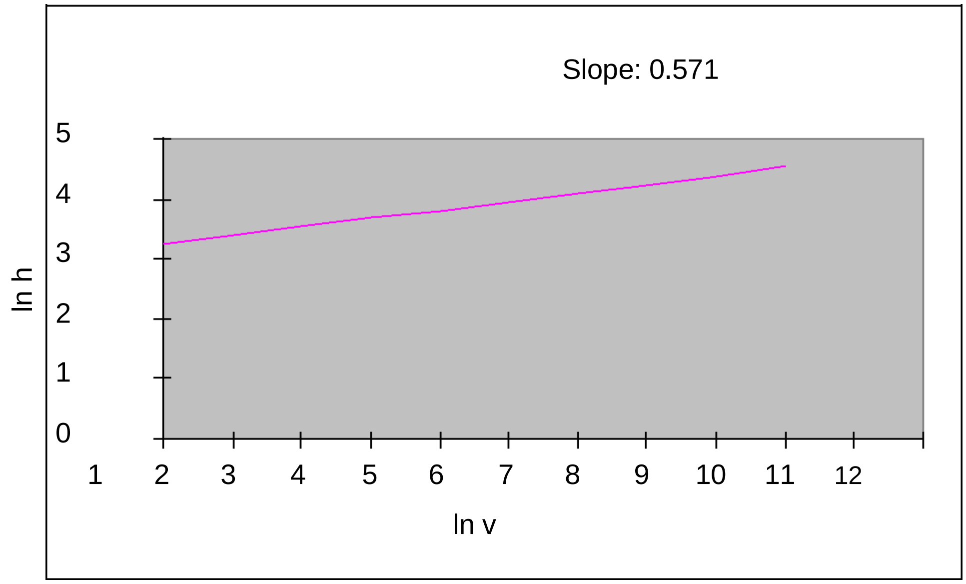


(b) $$h = CV^n$$

Therefore $$\ln h = \ln C + n \ln V \qquad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

If ln h is plotted against ln V it will be straight line and the slope of which will give the value of n. Also the intercept of this line w.r.t the axis on which ln V is plotted will give the value of ln C from which C can be determined. The log –log plot is as shown in Fig. P 1.4(b). Scale X axis 1cm=0.25

Y axis 1cm=0.5



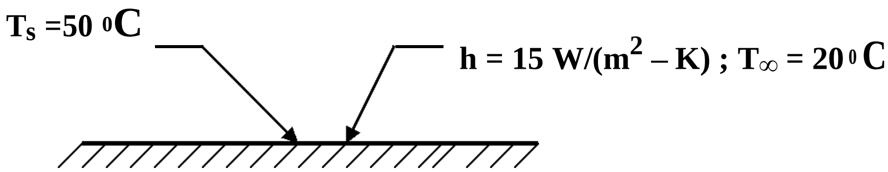
$$\ln C = 3.1 \text{ or } C = 22$$

and
$$n = \frac{(\ln h - \ln C)}{\ln V} = \frac{(4.55 - 3.10)}{2.5}$$

$$= 0.571$$

Therefore $h = 22.2\, V^{0.571}$ is the empirical relation between h and V.

Example 1.5:- *A large surface at 50 ^{0} C is exposed to air at 20 ^{0} C. If the heat transfer coefficient between the surface and the air is 15 W/(m^2-K), determine the heat transferred from 5 m^2 of the surface area in 7 hours.*



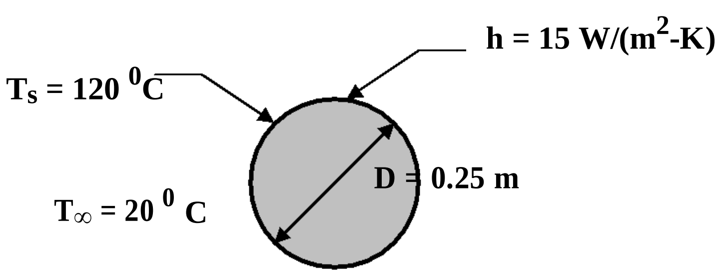
A = 5 m^2 : time = t = 7 h ;

$Q_{total} = Q\,t = hA(T_S - T_\infty)\,t = 15 \times 5 \times (50 – 20) \times 7 \times 3600$ J

$= 56.7 \times 10^{6}$ J = 56.7 MJ

Example 1.6:- *A 25 cm diameter sphere at 120 ^{0} C is suspended in air at 20 ^{0} C. If the convective heat transfer coefficient between the surface and air is 15 W/(m^2-K), determine the heat loss from the sphere.*

Solution:-



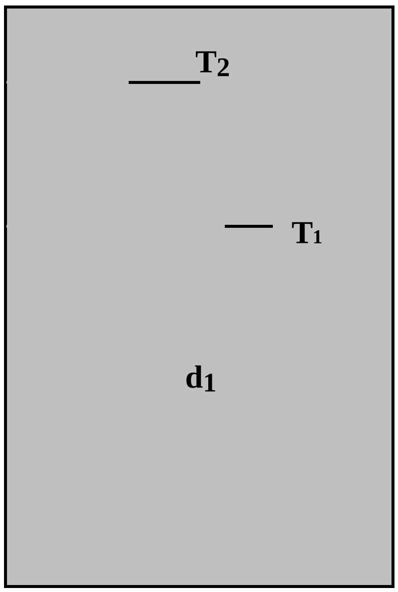
$Q = hA_S(T_S - T_\infty) = h\,4\pi R^2\,(T_S - T_\infty) = 15 \times 4\pi \times (0.25/2)^2 \times (120 –$

$20) = 294.52$ W

C. Radiation:

Example 1.7:- *A sphere 10 cm in diameter is suspended inside a large evacuated chamber whose walls are kept at 300 K. If the surface of the sphere is black and maintained at 500 K what would be the radiation heat loss from the sphere to the walls of the chamber?. What would be the heat loss if the surface of the sphere has an emissivity of 0.8?*

Solution:



$T_1 = 500$ K ; $T_2 = 300$ K ; $d_1 = 0.10$ m

Surface area of the sphere $= A_s = 4\pi R_1^2$

$= 4\pi x\ (0.1/2)^2$

$= 0.0314\ m^2$

If the surface of the sphere is black then

$Q_{black} = \zeta\ A_s\ (T_1^4 - T_2^4)$

$= 5.67 \text{ x } 10^{-8} \text{ x } 0.0314 \text{ x } (500^4 - 300^4)$

$= 96.85$ W

If the surface is having an emissivity of 0.8 then

$Q = 0.8\ Q_{black} = 0.8 \text{ x } 96.85 = 77.48$ W.

Example 1.8:- *A vacuum system as used in sputtering conducting thin films on micro circuits, consists of a base plate maintained at a temperature of 300 K by an electric heater and a shroud within the enclosure maintained at 77 K by circulating liquid nitrogen. The base plate insulated on the lower side is 0.3 m in diameter and has an emissivity of 0.25.*

(a) How much electrical power must be provided to the base plate heater?

(b) At what rate must liquid nitrogen be supplied to the shroud if its latent heat of vaporization is 125 kJ/kg?

Solution:- $T_1 = 300$ K ; $T_2 = 77$ K ; d = 0.3 m ; $\varepsilon_1 = 0.25$

Surface area of the top surface of the base plate $= A_s = (\pi / 4)d_1^2 = (\pi / 4) \text{ x } 0.3^2$

$= 0.0707\ m^2$

(a) $Q_r = \varepsilon_1 \zeta A_s (T_1^4 - T_2^4)$

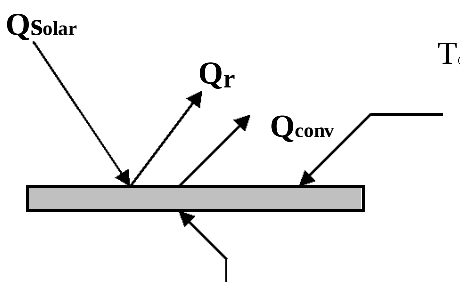
$= 0.25 \times 5.67 \times 10^{-8} \times 0.0707 \times (300^4 - 77^4) = 8.08$ W

(b) If $\dot{m}_{N2}$ = mass flow rate of nitrogen that is vapourised then

$$\dot{m}_{N2} = Q_r / h_{fg} = \frac{8.08}{125 \times 1000} = 6.464 \times 10^{-5} \text{ kg/s or } 0.233 \text{ kg/s}$$

Example 1.9:- *A flat plate has one surface insulated and the other surface exposed to the sun. The exposed surface absorbs the solar radiation at a rate of 800 W/m² and dissipates heat by both convection and radiation into the ambient at 300 K. If the emissivity of the surface is 0.9 and the surface heat transfer coefficient is 12 W/(m²-K), determine the surface temperature of the plate.*

Solution:-



$T_\infty = 300$ K ; $q_{solar} = 800$ W / m^2

T_s ; $\varepsilon = 0.9$; $h = 12$ W / (m^2 – K)

Energy balance equation for the top surface of the plate is given by

$$Q_{solar} = Q_r + Q_{conv}$$

$$q_{solar} A_s = \varepsilon \zeta A_s (T_s^4 - T_\infty^4) + h A_s (T_s - T_\infty)$$

Therefore $\quad 800 = 0.9 \times 5.67 \times 10^{-8} \times (T_s^4 - 300^4) + 12 \times (T_s - 300)$

On simplifying the above equation we get

$$(T_s / 100)^4 + 2.35\, T_s = 943 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Equation (1) has to be solved by trial and error.

Trial 1:- Assume T_S = 350 K. Then LHS of Eq. (1) = 972.6 which is more than RHS of Eq.(1). Hence T_S < 350 K.

Trial 2 :- Assume T_S = 340 K. Then LHS of Eq. (1) = 932.6 which is slightly less than RHS. Therefore T_S should lie between 340 K and 350 K but closer to 340 K. ***Trial 3:-*** Assume T_S = 342.5 K. Then LHS of Eq.(1) = 942.5 = RHS of Eq. (1). Therefore Ts = 342.5 K

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UNIT-II

GOVERNING EQUATIONS OF CONDUCTION

Introduction: In this chapter, the governing basic equations for conduction in Cartesian coordinate system is derived. The corresponding equations in cylindrical and spherical coordinate systems are also mentioned. Mathematical representations of different types of boundary conditions and the initial condition required to solve conduction problems are also discussed. After studying this chapter, the student will be able to write down the governing equation and the required boundary conditions and initial condition if required for any conduction problem.

One – Dimensional Conduction Equation : In order to derive the one-dimensional conduction equation, let us consider a volume element of the solid of thickness Δx along x – direction at a distance „x" from the origin as shown in Fig. 2.1.Q_X represents the rate

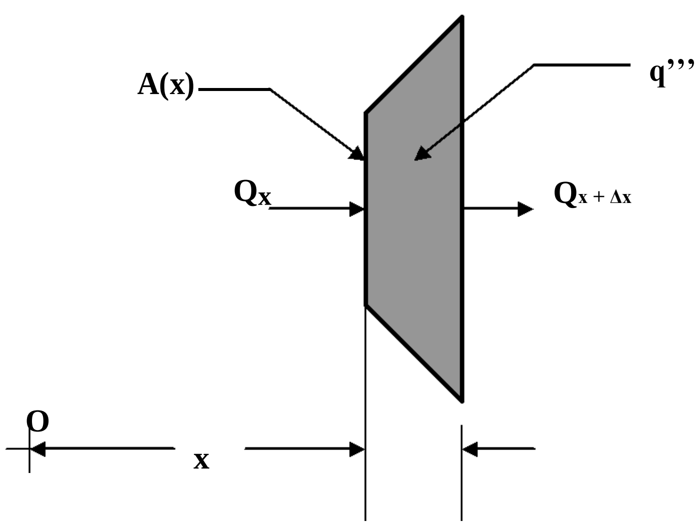

Fig. 2.1: Nomenclature for one dimensional conduction equation

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of heat transfer in x – direction entering into the volume element at x, A(x) area of heat flow at the section x ,q""" is the thermal energy generation within the element per unit volume and $Q_{X+\Delta X}$ is the rate of conduction out of the element at the section x + Δx. The energy balance equation per unit time for the element can be written as follows:

[Rate of heat conduction into the element at x + Rate of thermal energy generation within the element − Rate of heat conduction out of the element at x + Δx]

= Rate of increase of internal energy of the element.

i.e., $Q_x + Q_g - Q_{x+\Delta x} = \partial E / \partial t$

or $Q_x + q''' A(x)\,\Delta x - \{Q_x + (\partial Q_x / \partial x)\Delta x + (\partial^2 Q_x / \partial x^2)(\Delta x)^2 / 2! + \ldots\ldots\}$

$$= \partial / \partial t\,(\rho A(x)\Delta x C_p T)$$

Neglecting higher order terms and noting that ρ and Cp are constants the above equation simplifies to

$$Q_x + q''' A(x)\,\Delta x - \{Q_x + (\partial Q_x / \partial x)\Delta x = \rho A(x)\Delta x C_p\,(\partial T / \partial t)$$

Or $$-(\partial Q_x / \partial x) + q''' A(x) = \rho A(x)\,C_p\,(\partial T / \partial t)$$

Using Fourier"s law of conduction , $Q_x = -k\,A(x)\,(\partial T / \partial x)$, the above equation simplifies to

$$-\partial / \partial x\,\{-k\,A(x)\,(\partial T / \partial x)\} + q''' A(x) = \rho A(x)\,C_p\,(\partial T / \partial t)$$

Or $$\mathbf{\{1/A(x)\}\,\partial / \partial x\,\{k\,A(x)\,(\partial T / \partial x)\} + q''' = \rho\,C_p\,(\partial T / \partial t)\ \ldots\ldots\ldots\ldots\ldots(2.1)}$$

Eq. (2.1) is the most general form of conduction equation for one-dimensional unsteady state conduction.

2.2.1.Equation for one-dimensional conduction in plane walls :- For plane walls, the area of heat flow A(x) is a constant. Hence Eq. (2.1) reduces to the form

$$\mathbf{\partial / \partial x\,\{k\,(\partial T / \partial x)\} + q''' = \rho\,C_p\,(\partial T / \partial t)\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2.2)}$$

(i) If the thermal conductivity of the solid is constant then the above equation reduces to

$$\mathbf{(\partial^2 T / \partial x^2) + (q''' / k) = (1/\alpha)(\partial T / \partial t)\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2.3)}$$

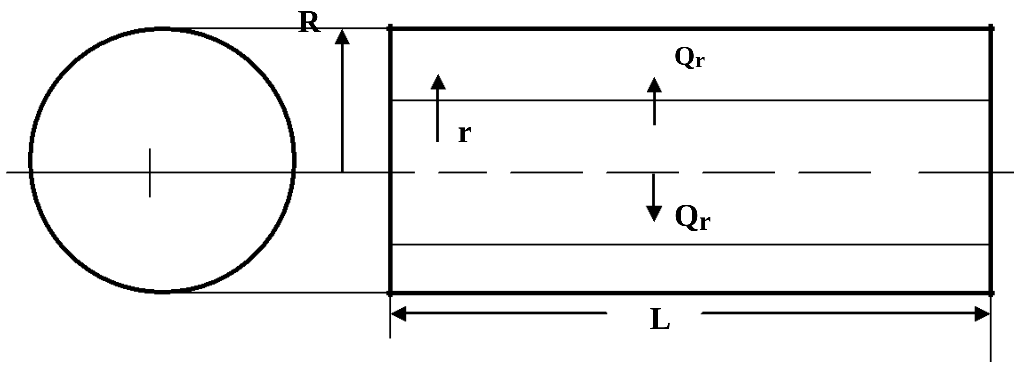
(ii) For steady state conduction problems in solids of constant thermal conductivity temperature within the solid will be independent of time (i.e.$(\partial T / \partial t) = 0$) and hence Eq. (2.3) reduces to

$$\mathbf{(d^2 T / dx^2) + (q''' / k) = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2.4)}$$

(iii) For a solid of constant thermal conductivity for which there is no thermal energy generation within the solid $q''''''=0$ and the governing for steady state conduction is obtained by putting $q''''''=0$ in Eq. (2.4) as

$$\left(d^2T / dx^2\right) = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.4)$$

2.2.2.Equation for one-dimensional radial conduction in cylinders:-



For radial conduction in cylinders, by convention the radial coordinate is denoted by „r‟ instead of „x‟ and the area of heat flow through the cylinder of length L,at any radius r is given by $A(x) = A(r) = 2\pi rL$. Hence substituting this expression for A(x) and replacing x by r in Eq. (2.1) we have

$$\{1/(2\pi rL)\partial/\partial r\,\{k\,2\pi rL\,(\partial T/\partial r)\} + q'''''' = \rho\,C_p\,(\partial T/\partial t)$$

Or
$$\mathbf{(1/r)\,\partial/\partial r\,\{k\,r\,(\partial T/\partial r)\} + q''' = \rho\,C_p\,(\partial T/\partial t)\ldots\ldots\ldots\ldots\ldots(2.5)}$$

(i) For cylinders of constant thermal conductivity the above equation reduces to

$$\mathbf{(1/r)\,\partial/\partial r\,\{\,r\,(\partial T/\partial r)\} + q'''/k = (1/\alpha)\,(\partial T/\partial t)\ldots\ldots\ldots\ldots\ldots(2.6)}$$

(ii) For steady state radial conduction (i.e. $(\partial T/ \partial t) = 0$) in cylinders of constant k, the above equation

reduces to $$(1/r)\, d/ dr \{ r (dT / \partial r)\} + q''' / k = 0 \text{ ……………………….(2.7)}$$

(iii) For steady state radial conduction in cylinders of constant k and having no thermal energy generation (i.e. q'''''' = 0) the above equation reduces to

$$d/ dr \{ r (dT / \partial r)\} = 0 \text{ ………………………………(2.8)}$$

2.2.3.Equation for one-dimensional radial conduction in spheres:- For one-imensional radial conduction in spheres, the area of heat flow at any radius r is given by $A(r) = 4\pi r^2$. Hence Eq.(2.1) for a sphere reduces to

$$\{1/(4\pi r^2)\}\partial/ \partial r \{k\, 4\pi r^2 (\partial T / \partial r)\} + q'''''' = \rho C_p (\partial T/ \partial t)$$

Or $$1/r^2\, \partial/ \partial r \{k\, r^2 (\partial T / \partial r)\} + q''' = \rho\, C_p (\partial T/ \partial t) \text{ …………………(2.9)}$$

(i) For spheres of constant thermal conductivity the above equation reduce to

$$1/r^2\, \partial/ \partial r \{ r^2 (\partial T / \partial r)\} + q''' / k = (1 / \alpha) (\partial T/ \partial t) \text{ ……………..(2.10)}$$

(ii) For steady state conduction in spheres of constant k the above equation further reduce to

$$1/r^2\, \partial/ \partial r \{ r^2 (\partial T / \partial r)\} + q''' / k = 0 \text{ ……………………………(2.11)}$$

(iii) For steady state conduction in spheres of constant k and without any thermal energy generation the above equation further reduces to

$$1/r^2\, d/ dr \{ r^2 (dT / dr)\} = 0 \text{ ……………………………………(2.12)}$$

Equation in compact form:- The general form of one – dimensional conduction equations for plane walls, cylinders and spheres {equations (2..2), (2.5) and (2.9)} can be written in a compact form as follows:

$$1/r^n\, \partial/ \partial r \{k\, r^n (\partial T / \partial r)\} + q''' = \rho\, C_p (\partial T/ \partial t) \text{ ………….(2.13)}$$

Where n = 0 for plane walls,
n = 1 for radial conduction in cylinders
n = 2 for radial conduction in spheres,

and for plane walls it is customary to replace the „r‟ variable by „x‟ variable.

2.3.Three dimensional conduction equations: While deriving the one – dimensional conduction equation, we assumed that conduction heat transfer is taking place only along one direction. By allowing conduction along the remaining two directions and following the same procedure we obtain the governing equation for conduction in three dimenions.

2.3.1. Three dimensional conduction equation in Cartesian coordinate system: Let us consider a volume element of dimensions Δx, Δy and Δz in x y and z directions respectively. The conduction heat transfer across the six surfaces of the element is shown in Fig. 2.3.

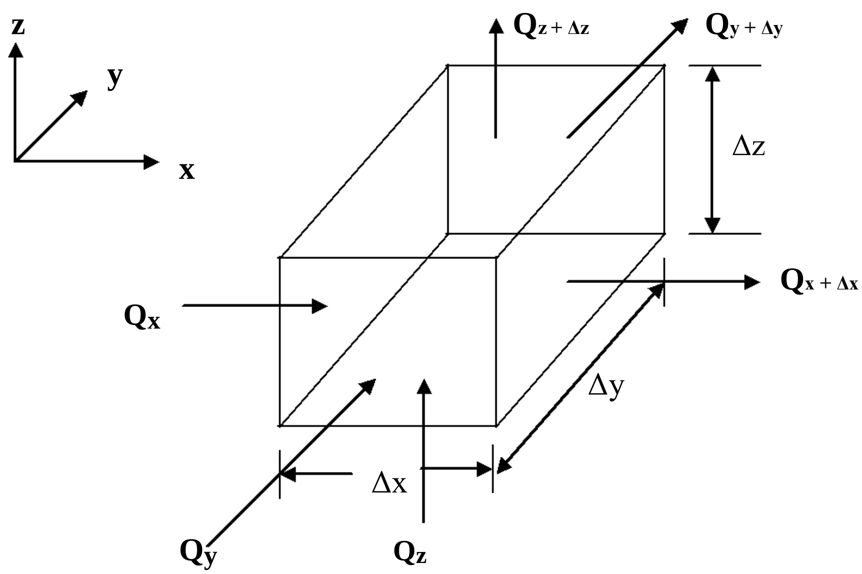


Fig. 2.3: Conduction heat transfer across the six faces of a volume element

Net Rate of conduction into the element in x-direction = $Q_x - Q_{x+\Delta x}$

$$= Q_x - [Q_x + (\partial Q_x/\partial x)\,\Delta x + (\partial^2 Q_x/\partial x^2)(\Delta x)^2 / 2! + \ldots.]$$

$$= -\,(\partial Q_x/\partial x)\,\Delta x \quad \text{by neglecting higher order terms.}$$

$$= -\,\partial / \partial x\,[-\,k_x\,\Delta y\,\Delta z(\partial T / \partial x)]\,\Delta x$$

$$= \partial / \partial x[k_x\,(\partial T / \partial x)]\,\Delta x\,\Delta y\,\Delta z$$

Similarly the net rate of conduction into the element in y – direction $= \partial / \partial y[k_y\,(\partial T / \partial y)]\,\Delta x\,\Delta y\,\Delta z$

and in z – direction $= \partial / \partial z[k_z\,(\partial T / \partial z)]\,\Delta x\,\Delta y\,\Delta z$.

Hence the net rate of conduction into the element from all the three directions

$Q_{in} = \{\partial / \partial x[k_x (\partial T / \partial x)] + \partial / \partial y[k_y (\partial T / \partial y)] + \partial / \partial z[k_z (\partial T / \partial z)] \} \Delta x \Delta y \Delta z$

Rate of heat thermal energy generation in the element = $Q_g = q''''''\Delta x \Delta y \Delta z$

Rate of increase of internal energy within the element = $\partial E / \partial t = \rho \Delta x \Delta y \Delta z C_p (\partial T / \partial t)$ Applying I law of thermodynamics for the volume element we have

$$Q_{in} + Q_g = \partial E / \partial t$$

Substituting the expressions for Q_{in}, Q_g and $\partial E / \partial t$ and simplifying we get

$$\{\partial / \partial x[k_x (\partial T / \partial x)] + \partial / \partial y[k_y (\partial T / \partial y)] + \partial / \partial z[k_z (\partial T / \partial z)] \} + q''' = \rho C_p (\partial T / \partial t) \quad \ldots\ldots\ldots\ldots\ldots\ldots(2.14)$$

Equation (2.14) is the most general form of conduction equation in Cartesian coordinate system. This equation reduces to much simpler form for many special cases as indicated below.

Special cases:- (i) For isotropic solids, thermal conductivity is independent of direction; i.e., $k_x = k_y = k_z = k$. Hence Eq. (2.14) reduces to

$$\{\partial / \partial x[k (\partial T / \partial x)] + \partial / \partial y[k (\partial T / \partial y)] + \partial / \partial z[k (\partial T / \partial z)] \} + q''' = \rho C_p (\partial T / \partial t) \quad \ldots\ldots\ldots\ldots\ldots\ldots(2.15)$$

(ii) For isotropic solids with constant thermal conductivity the above equation further reduces to

$$\partial^2 T / \partial x^2 + \partial^2 T / \partial y^2 + \partial^2 T / \partial z^2 + q''' / k = (1 / \alpha) (\partial T / \partial t) \ldots\ldots\ldots\ldots\ldots\ldots(2.16)$$

Eq.(2.16) is called as the *"Fourier – Biot equation"* and it reduces to the following forms under specified conditions as mentioned below:

(iii) Steady state conduction [i.e., $(\partial T / \partial t) = 0$]

$$\partial^2 T / \partial x^2 + \partial^2 T / \partial y^2 + \partial^2 T / \partial z^2 + q''' / k = 0 \ldots\ldots\ldots\ldots\ldots\ldots(2.17)$$

Eq. (2.17) is called the *"Poisson equation"*.

(iv) No thermal energy generation [i.e. $q'''''' = 0$]:

$$\partial^2 T / \partial x^2 + \partial^2 T / \partial y^2 + \partial^2 T / \partial z^2 = (1 / \alpha) (\partial T / \partial t) \ldots\ldots\ldots\ldots\ldots\ldots(2.18)$$

Eq. (2.18) is called the *"diffusion equation"*.

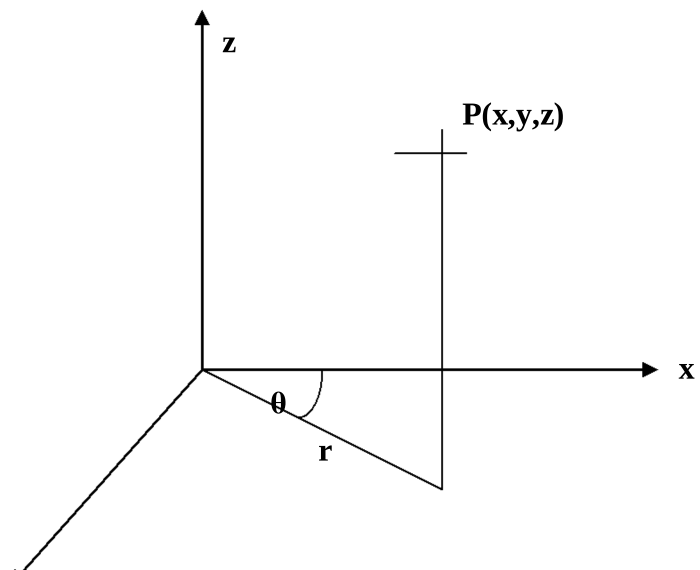
(v) Steady state conduction without heat generation [i.e., $(\partial T / \partial t) = 0$ and $q''' = 0$]:

$$\partial^2 T / \partial x^2 + \partial^2 T / \partial y^2 + \partial^2 T / \partial z^2 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2.19)$$

Eq. (2.19) is called the *"Laplace equation"*.

<u>2.3.2. Three dimensional conduction equation in cylindrical coordinate system:</u>

It is convenient to express the governing conduction equation in cylindrical coordinate system when we want to analyse conduction in cylinders. Any point P in space can be located by using the cylindrical coordinate system r, θ and z and its relation to the Cartesian coordinate system (See Fig. 2.4) can be written as follows:



y

Fig.2.4: Cylindrical coordinate system

$x = r \cos\theta$; $y = r \sin\theta$; $z = z$. Using these transformations and after laborious simplifications Eq. (2.15) simplifies to

$$\frac{1}{r}\frac{\partial}{\partial r}\left[k\, r \frac{\partial T}{\partial r}\right] + \frac{1}{r^2}\frac{\partial}{\partial \theta}\left[k \frac{\partial T}{\partial \theta}\right] + \frac{\partial}{\partial z}\left[k \frac{\partial T}{\partial z}\right] + q''' = \rho\, C_p \frac{\partial T}{\partial t} \quad \ldots\ldots\ldots\ldots(2.20)$$

The above equation is valid for only for isotropic solids.

2.3.2. Three dimensional conduction equation in Spherical coordinate system:

For spherical solids,it is convenient to express the governing conduction equation in spherical coordinate system. Any point P on the surface of a sphere of radius r can be located by using the spherical coordinate system r, θ and θ and its relation to the Cartesian coordinate system (See Fig. 2.5) can be written as follows:

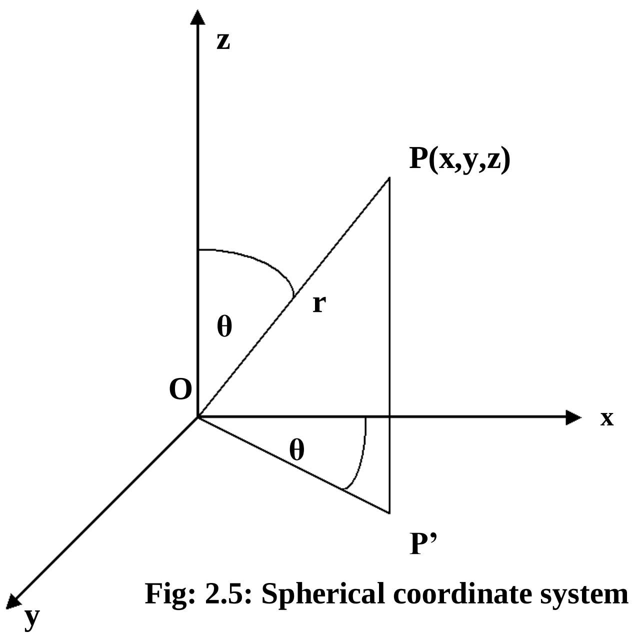


Fig: 2.5: Spherical coordinate system

OP'' = r sin θ.Hence

$x = r \sin\theta \cos\theta$;

$y = r \sin\theta \sin\theta$;

$z = r \cos\theta$

Using the relation between x, y ,z and r, θ and θ, the conduction equation (2.15) can be transformed into the equation in terms of r, θ and θ as follows.

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left[kr^2\frac{\partial T}{\partial r}\right]+\frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial\theta}\left[k\frac{\partial T}{\partial\theta}\right]+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left[k\sin\theta\frac{\partial T}{\partial\theta}\right]+q'''=\rho C_p\frac{\partial T}{\partial t}\quad\ldots\ldots\ldots\ldots\ldots(2.21).$$

2.4.Boundary and Initial Conditions:

The temperature distribution within any solid is obtained by integrating the above conduction equation with respect to the space variable and with respect to time.The solution thus obtained is called the *"general solution"* involving arbitrary constants of integration. The solution to a particular conduction problem is arrived by obtaining these constants which depends on the conditions at the bounding surfaces of the solid as well as

the initial condition. The thermal conditions at the boundary surfaces are called the *"boundary conditions"* **.** Boundary conditions normally encountered in practice are:
*(i) Specified temperature (*also called as *boundary condition of the first kind),*
(ii) Specified heat flux (also known as *boundary condition of the second kind),*
(iii) Convective boundary condition (also known as *boundary condition of the third kind)* and *(iv) radiation boundary condition.* The mathematical representations of these boundary conditions are illustrated by means of a few examples below.

2.4.1. Specified Temperatures at the Boundary:- Consider a plane wall of thickness L whose outer surfaces are maintained at temperatures T_0 and T_L as shown in Fig.2.6. For one-dimensional unsteady state conduction the boundary conditions can be written as

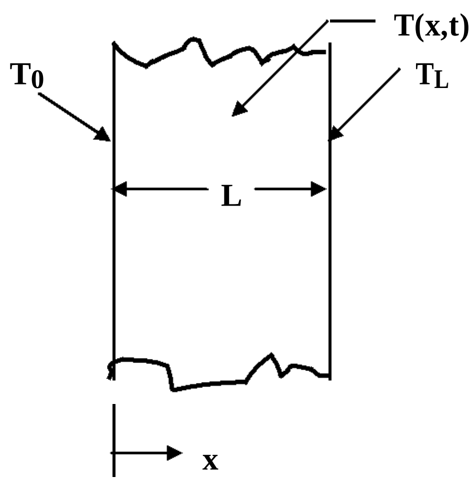


Fig. 2.6: Boundary condition of first kind for a plane wall

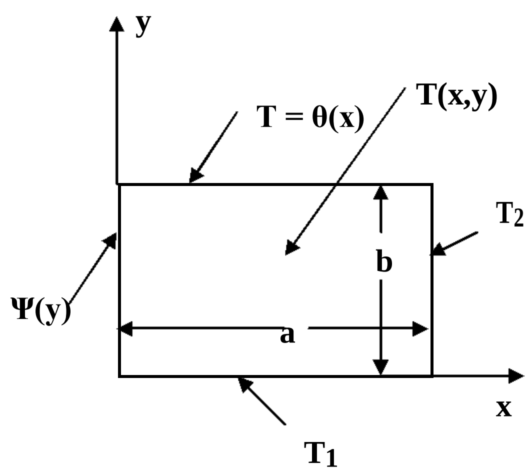


Fig.2.7: Boundary conditions of first kind for a rectangular plate

(i) at x = 0, $T(0,t) = T_0$; (ii) at x = L, $T(L,t) = T_L$.

Consider another example of a rectangular plate as shown in Fig. 2.7. The boundary conditions for the four surfaces to determine two-dimensional steady state temperature distribution T(x,y) can be written as follows.

(i) at x = 0, $T(0,y) = \Psi(y)$; (ii) at y = 0, $T(x,0) = T_1$ for all values of y

(iii) at x = a, $T(a,y) = T_2$ for all values of y; (iv) at y = b, $T(x,b) = \theta(x)$

2.4.2. Specified heat flux at the boundary:- Consider a rectangular plate as shown in Fig. 2.8 and whose boundaries are subjected to the prescribed heat flux conditions as shown in the figure. Then the boundary conditions can be mathematically expressed as follows.

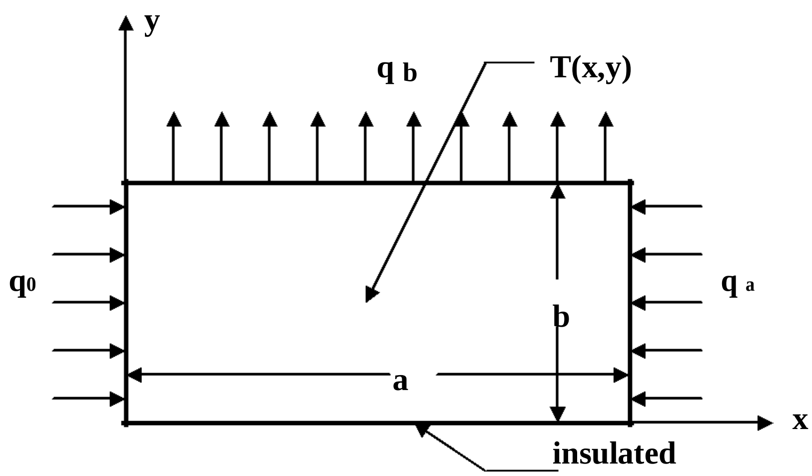


Fig.2.8: Prescribed heat flux boundary conditions

(i) at $x = 0, -k(\partial T/\partial x)|_{x=0} = q_0$ for $0 \le y \le b$;

(ii) at $y = 0$, $(\partial T/\partial y)|_{y=0} = 0$ for $0 \le x \le a$;

(iii) at $x = a$, $k(\partial T/\partial x)|_{x=a} = q_a$ for $0 \le y \le b$;

(iv) at $y = b, -k(\partial T/\partial y)|_{y=b} = 0$ for $0 \le x \le a$;

Boundary surface subjected to convective heat transfer:- Fig. 2.9 shows a plane wall whose outer surfaces are subjected to convective boundary conditions.The surface at $x = 0$ is in contact with a fluid which is at a uniform temperature T_i and the surface heat transfer coefficient is h_i. Similarly the other surface at $x = L$ is in contact with another fluid at a uniform temperature T_0 with a surface heat transfer coefficient h_0. This type of boundary condition is encountered in heat exchanger wherein heat is transferred from hot fluid to the cold fluid with a metallic wall separating the two fluids. This type of boundary condition is normally referred to as the boundary condition of third kind. The mathematical representation of the boundary conditions for the two surfaces of the plane wall can be written as follows.

(i) at $x = 0$, $q_{convection} = q_{conduction}$; i.e., $h_i[T_i - T|_{x=0}] = -k(dT/dx)|_{x=0}$

(ii) at $x = L$, $-k(dT/dx)|_{x=L} = h_0[T|_{x=L} - T_0]$

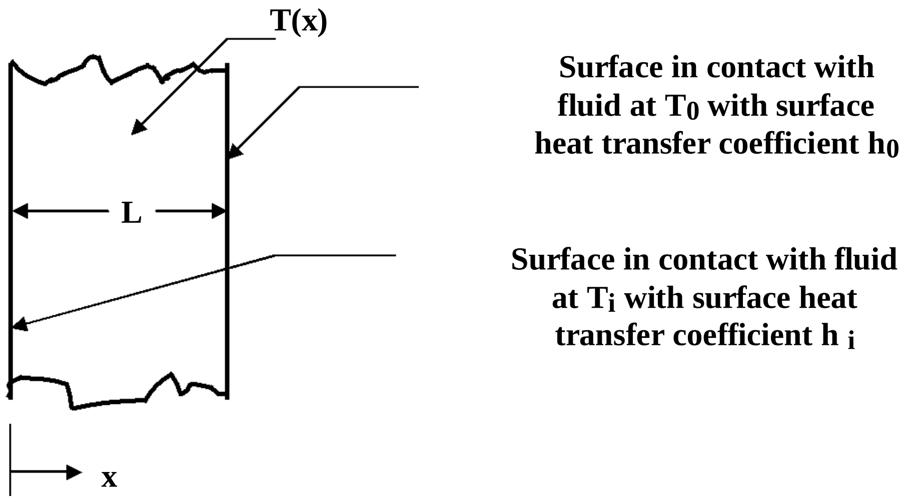


Fig. 2.9: Boundaries subjected to convective heat transfer for a plane wall

Radiation Boundary Condition: Fig. 2.10 shows a plane wall whose surface at x =L is having an emissivity „ε‟ and is radiating heat to the surroundings at a uniform temperature T_S. The mathematical expression for the boundary condition at x = L can be written as follows:

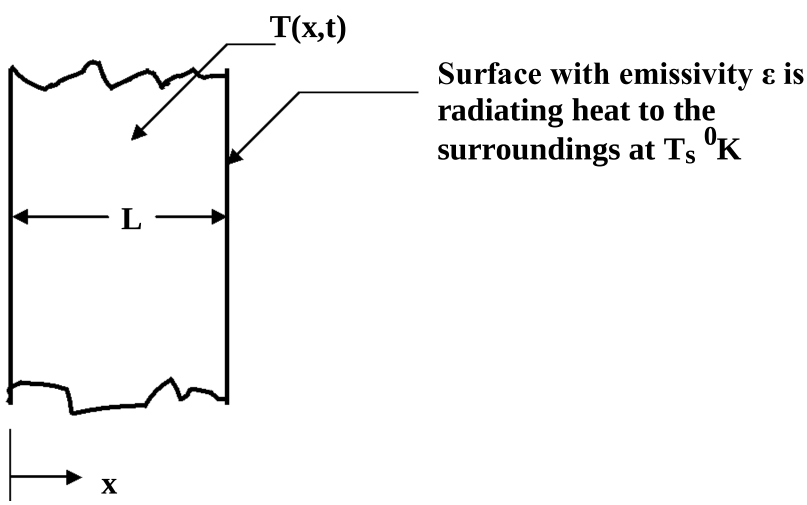


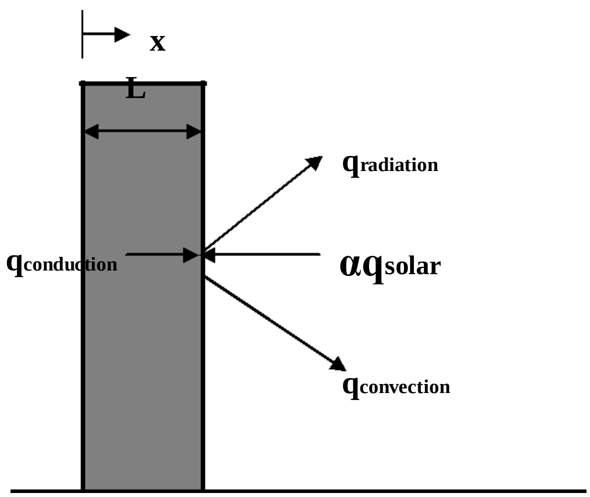
Fig. 2.10: Boundary surface at x = L subjected to radiation heat transfer

(i) at x = L, $q_{conduction} = q_{radiation}$; i.e., $-k\,(dT/dx)|_{x=L} = \zeta\,\varepsilon\,[(T|_{x=L})^4 - T_S{}^4]$

In the above equation both $T|_{x=L}$ and T_S should be expressed in degrees Kelvin.

General form of boundary condition (combined conduction, convection and radiation boundary condition):

There are situations where the boundary surface is subjected to combined conduction, convection and radiation conditions as illustrated in Fig. 2.11.It is a south wall of a house and the outer surface of the wall is exposed to solar radiation. The interior of the room is at a uniform temperature $T_{i.}$ The outer air is at uniform temperature T_0 . The sky, the ground and the surfaces of the surrounding structures at this location is modeled as a surface at an effective temperature of $T_{sky.}$



Schematic for general form of boundary condition

Energy balance for the outer surface is given by the equation

$$q_{conduction} + \alpha\, q_{solar} = q_{radiation} + q_{convection}$$

$$-\,k\,(dT / dx)|_{x = L} + \alpha q_{solar} = \varepsilon\, \zeta\, [(T|_{x = L})^4 - T_{sky}^4] + h_0[T|_{x = L} - T_0]$$

B. Mathematical Formulation of Boundary conditions:

- A plane wall of thickness L is subjected to a heat supply at a rate of q_0 W/m^2 at one boundary surface and dissipates heat from the surface by convection to the ambient which is at a uniform temperature of $T\infty$ with a surface heat transfer coefficient of h_∞. Write the mathematical formulation of the boundary conditions for the plane wall.

- Consider a solid cylinder of radius R and height Z. The outer curved surface of the cylinder is subjected to a uniform heating electrically at a rate of q_0 W / m^2.Both the circular surfaces of the cylinder are exposed to an environment at a uniform temperature T_∞ with a surface heat transfer coefficient h. Write the mathematical formulation of the boundary conditions for the solid cylinder.

- A hollow cylinder of inner radius r_i, outer radius r_0 and height H is subjected to the following boundary conditions.
  (a) The inner curved surface is heated uniformly with an electric heater at a constant rate of q_0 W/m^2,
  (b) the outer curved surface dissipates heat by convection into an ambient at a uniform temperature, T_∞ with a convective heat transfer coefficient, h
  (c) the lower flat surface of the cylinder is insulated, and
  (d) the upper flat surface of the cylinder dissipates heat by convection into the ambient at T_∞ with surface heat transfer coefficient h. Write the mathematical formulation of the boundary conditions for the hollow cylinder.

C. Formulation of Heat Conduction Problems:

- A plane wall of thickness L and with constant thermal properties is initially at a uniform temperature T_i. Suddenly one of the surfaces of the wall is subjected to heating by the flow of a hot gas at temperature T_∞ and the other surface is kept insulated. The heat transfer coefficient between the hot gas and the surface exposed to it is h. There is no heat generation in the wall. Write the mathematical formulation of the problem to determine the one-dimensional unsteady state temperature within the wall.

- A copper bar of radius R is initially at a uniform temperature T_i. Suddenly the heating of the rod begins at time t=0 by the passage of electric current, which generates heat at a uniform rate of q''' W/m^2. The outer surface of the dissipates heat into an ambient at a uniform temperature T_∞ with a convective heat transfer coefficient h. Assuming that thermal conductivity of the bar to be constant, write the mathematical formulation of the heat conduction problem to determine the one-dimensional radial unsteady state temperature distribution in the rod.

- Consider a solid cylinder of radius R and height H. Heat is generated in the solid at a uniform rate of q''' W/m^3. One of the circular faces of the cylinder is insulated and the other circular face dissipates heat by convection into a medium at a uniform temperature of T_∞ with a surface heat transfer coefficient of h. The outer curved surface of the cylinder is maintained at a uniform temperature of T_0. Write the mathematical formulation to determine the two-dimensional steady state temperature distribution T(r, z) in the cylinder.

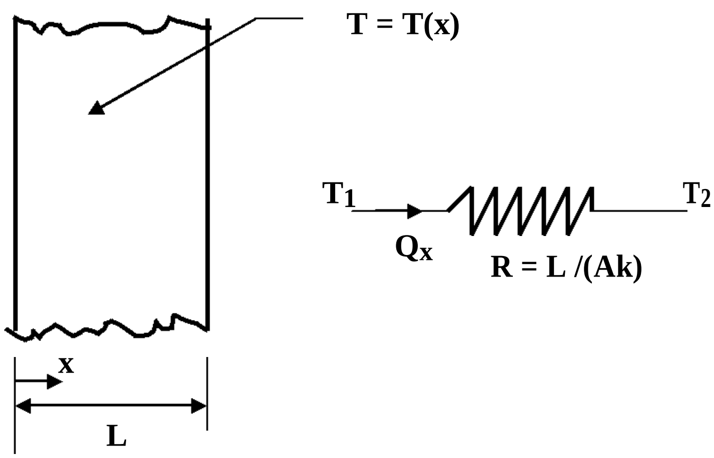
- Consider a rectangular plate as shown in Fig. P2.10.The plate is generating heat at a uniform rate of q''' W/m^3. Write the mathematical formulation to determine two-dimensional steady state temperature distribution in the plate.

Consider the north wall of a house of thickness L. The outer surface of the wall exchanges heat by both convection and radiation. The interior of the house is maintained at a uniform temperature of T_i, while the exterior of the house is at a uniform temperature T_0. The sky, the ground, and the surfaces of the surrounding structures at this location can be modeled as a surface at an effective temperature of T_{sky} for radiation heat exchange on the outer surface. The radiation heat exchange between the inner surface of the wall and the surfaces of the other walls, floor and ceiling are negligible. The convective heat transfer coefficient for the inner and outer surfaces of the wall under consideration are h_i and h_0 respectively. The thermal conductivity of the wall material is K and the emissivity of the outer surface of the wall is 'ε_0'. Assuming the heat transfer through the wall is steady and one dimensional, express the mathematical formulation (differential equation and boundary conditions) of the heat conduction problem

ONE DIMENSIONAL STEADY STATE CONDUCTION

Conduction Without Heat Generation

The Plane Wall (The Slab):- The statement of the problem is to determine the temperature distribution and rate of heat transfer for one dimensional steady state conduction in a plane wall without heat generation subjected to specified boundary conditions.



One dimensional steady state conduction in a slab

The governing equation for one − dimensional steady state conduction without heat generation is given by

$$\frac{d^2T}{dx^2} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3.1)$$

Integrating Eq. (3.1) twice with respect to x we get

$$T = C_1x + C_2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3.2)$$

where C_1 and C_2 are constants which can be evaluated by knowing the boundary conditions.

Plane wall with specified boundary surface temperatures:- If the surface at x = 0 is maintained at a uniform temperature T_1 and the surface at x = L is maintained at another uniform temperature T_2, then the boundary conditions can be written as follows:
(i) at x = 0, $T(x) = T_1$; (ii) at x = L, $T(x) = T_2$.

Condition (i) in Eq.(3.2) gives $\quad T_1 = C_2$.

Condition (ii) in Eq. (3.2) gives $\quad T_2 = C_1L + T_1$

Or $$C_1 = \frac{T_2 - T_1}{L}.$$

Substituting for C_1 and C_2 in Eq. (3.2), we get the temperature distribution in the plane wall as

$$T(x) = (T_2 - T_1)\frac{x}{L} + T_1$$

Or
$$\frac{T(x) - T_1}{(T_2 - T_1)} = \frac{x}{L} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3.3)$$

Expression for Rate of Heat Transfer:

The rate of heat transfer at any section x is given by Fourier‟s law as

$$Q_x = -\,k\,A(x)\,(dT / dx)$$ ʋ K / W.

For a plane wall A(x) = constant = A. From Eq. (3.3), $dT/dx = (T_2 - T_1) / L$.

Hence $$Q_x = -\,k\,A\,(T_2 - T_1) / L.$$

Or
$$Q_x = \frac{kA(T_1 - T_2)}{L} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.4)$$

Concept of thermal resistance for heat flow:

It can be seen from the above equation that Q_x is independent of x and is a constant. Eq. (3.4) can be written as

$$Q_x = \frac{(T_1 - T_2)}{\{L/(kA)\}} = \frac{(T_1 - T_2)}{R} \quad \ldots\ldots\ldots\ldots\ldots\ldots(3.5)$$

Where R = L / (A k).

Eq. (3.5) is analogous to Ohm‟s law for flow of electric current. In this equation $(T_1 - T_2)$ can be thought of as *"thermal potential",* R can be thought of as *"thermal resistance",* so that the plane wall can be represented by an equivalent *"thermal circuit"* as shown in Fig.3.1.The units of thermal resistance R are

Plane wall whose boundary surfaces subjected to convective boundary conditions:

The expression for rate of heat transfer Q_x can be written as follows:

$$Q_x = h_i A [T_i - T_1]$$

or
$$Q_x = \frac{(T_i - T_1)}{1/(h_i A)} = \frac{(T_i - T_1)}{R_{ci}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3.6a)$$

$R_{ci} = 1 / (h_i A)$ is called thermal resistance for convection at the surface at x = 0

Similarly
$$Q_x = \frac{(T_1 - T_2)}{R} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3.6b)$$

where R = L /(Ak) is the thermal resistance offered by the wall for conduction and

$$Q_x = \frac{(T_2 - T_o)}{R_{co}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(3.6c)$$

Where $R_{co} = 1 / (h_o A)$ is the thermal resistance offered by the fluid at the surface at x = L for convection. It follows from Equations (3.6a), (3.6b) and (3.6c) that

$$Q_x = \frac{(T_i - T_1)}{R_{ci}} = \frac{(T_1 - T_2)}{R} = \frac{(T_2 - T_0)}{R_{co}}$$

Or
$$\mathbf{Q_x = \frac{(T_i - T_o)}{[R_{ci} + R + R_{co}]}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\mathbf{(3.7)}$$

Radial Conduction in a Hollow Cylinder:

The governing differential equation for one-dimensional steady state radial conduction in a hollow cylinder of constant thermal conductivity and without thermal energy generation is given by Eq.(2.10b) with n = 1: i.e.,

$$\frac{d}{dr}[r (dT / dr)] = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3.8)$$

Integrating the above equation once with respect to „r" we get

$$r (dT / dr) = C_1$$

or
$$(dT / dr) = C_1 / r$$

Integrating once again with respect to „r" we get

$$T(r) = C_1 \ln r + C_2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3.9)$$

where C_1 and C_2 are constants of integration which can be determined by knowing the boundary conditions of the problem.

Hollow cylinder with prescribed surface temperatures: Let the inner surface at $r = r_1$ be maintained at a uniform temperature T_1 and the outer surface at $r = r_2$ be maintained at another uniform temperature T_2 as shown in Fig. 3.3.

Substituting the condition at r_1 in Eq.(3.9) we get

$$T_1 = C_1 \ln r_1 + C_2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3.10a)$$

and the condition at r_2 in Eq. (3.9) we get

$$T_2 = C_1 \ln r_2 + C_2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3.10b)$$

Solving for C_1 and C_2 from the above two equations we get

$$C_1 = \frac{(T_1 - T_2)}{[\ln r_1 - \ln r_2]} = \frac{(T_1 - T_2)}{\ln (r_1 / r_2)}$$

and

$$C_2 = T_1 - \frac{(T_1 - T_2)}{\ln (r_1 / r_2)} \ln r_1$$

Substituting these expressions for C_1 and C_2 in Eq. (3.9) we have

$$T(r) = \frac{(T_1 - T_2)}{\ln (r_1 / r_2)} \ln r + T_1 - \frac{(T_1 - T_2)}{\ln (r_1 / r_2)} \ln r_1$$

or

$$\boldsymbol{\frac{[T(r) - T_1]}{[T_2 - T_1]} = \frac{\ln (r / r_1)}{\ln (r_2 / r_1)}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\mathbf{(3.11)}$$

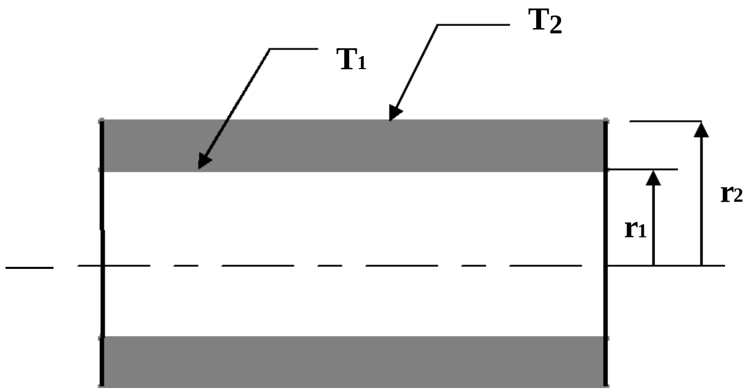


Fig.3.3: Hollow cylinder with prescribed surface temperatures

Eq. (3.11) gives the temperature distribution with respect to the radial direction in a hollow cylinder. The plot of Eq. (3.11) is shown in Fig. 3.4.

Expression for rate of heat transfer:- For radial steady state heat conduction in a hollow cylinder without heat generation energy balance equation gives

$$Q_r = Q_r|_{r = r1} = Q_r|_{r = r2}$$

Hence $$Q_r = -\, k\, [A(r)\, (dT / dr)]\, |_{r = r1} \quad \text{.................................(3.12)}$$

Now $A(r)\, |_{r = r1} = 2\, \pi\, r_1\, L$ .From Eq. (3.11) we have

$$(dT / dr) = \{[\, T_2 - T_1] / \ln\, (r_2 / r_1)\, \}(1/r)$$

Hence $$(dT / dr)|_{r = r1} = \{[\, T_2 - T_1] / \ln\, (r_2 / r_1)\, \}(1/ r_1).$$

Substituting the expressions for $A(r)|_{r = r1}$ and $(dT / dr)|_{r = r1}$ in Eq. (3.12) we get the expression for rate of heat transfer as

$$Q_r = \frac{2\, \pi\, L\, k\, (T_1 - T_2)}{\ln\, (r_2 / r_1)} \quad \text{.....................................(3.13)}$$

Thermal resistance for a hollow cylinder: Eq. 3.13 can be written as:

$$Q_r = (T_1 - T_2) / R \quad \text{...(3.14a)}$$

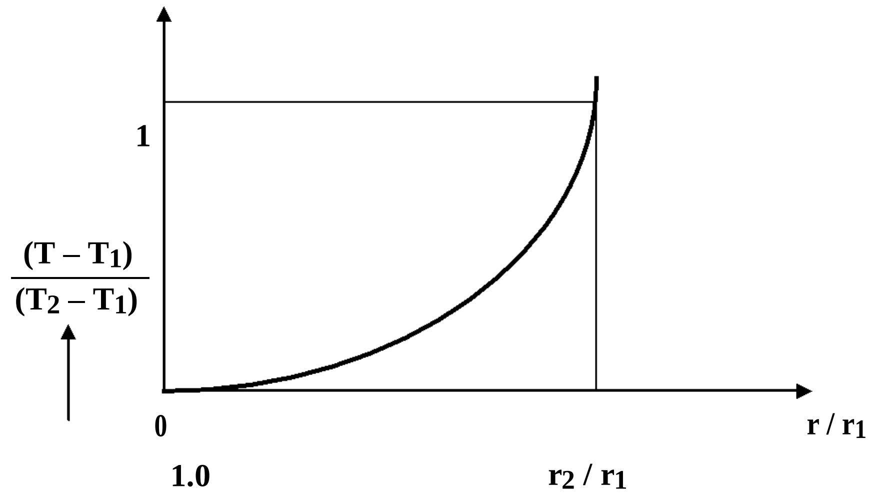


Fig. 3.4: Radial temperature distribution for a hollow cylinder

Where $A_m = (A_2 - A_1) / \ln (A_2 / A_1)$, when $A_2 = 2\pi\, r_2\, L$ = Area of the outer surface of the

cylinder and $A_1 = 2\pi\, r_1\, L$ = Area of the inner surface of the cylinder, and A_m is logarithmic mean area.

Hollow cylinder with convective boundary conditions at the surfaces:- Let for the hollow cylinder, the surface at $r = r_1$ is in contact with a fluid at temperature T_i with a surface heat transfer coefficient h_i and the surface at $r = r_2$ is in contact with another fluid at a temperature T_o as shown in Fig.3.5.By drawing the thermal circuit for this problem and using the concept of thermal resistance it is easy and straight forward to write down the expression for the rate of heat transfer as shown.

Now $Q_r = h_i A_i (T_i - T_1) = 2\pi\, r_1 L\, h_i\, (T_i - T_1) = \dfrac{(T_i - T_o)}{R_{ci}}$ ……………..(3.15a)

where $R_{ci} = 1 / (2\pi\, r_1 L h_i)$……………………………………………..(3.15b)

Also $Q_r = \dfrac{(T_1 - T_2)}{R}$ …………………………………………………..(3.15c) where $R = \ln (r_2 / r_1) / (2\pi Lk)$……………………………….(3.15d)

$R_{ci} + R + R_{co}$

where R_{ci}, R and R_{co} are given by Eqs.(3.15b), (3.15d) and (3.15f) respectively.

$$\frac{d}{dr}\left[r^2 (dT / dr)\right] = 0 \quad \text{……………………………(3.17)}$$

Integrating the above equation once with respect to „r" we get

$$r^2 (dT / dr) = C_1$$

or $$(dT / dr) = C_1/ r^2$$

Integrating once again with respect to „r" we get

$$T(r) = - C_1 / r + C_2 \quad \text{………………………...(3.18)}$$

where C_1 and C_2 are constants of integration which can be determined by knowing the boundary conditions of the problem.

Hollow sphere with prescribed surface temperatures:

(i) Expression for temperature distribution:-Let the inner surface at $r = r_1$ be maintained at a uniform temperature T_1 and the outer surface at $r = r_2$ be maintained at another uniform temperature T_2 as shown in Fig. 3.6.

The boundary conditions for this problem can be written as follows:

(i) at $r = r_1$, $T(r) = T_1$ and (ii) at $r = r_2$, $T(r) = T_2$.

Condition (i) in Eq. (3.18) gives $T_1 = - C_1 / r_1 + C_2$ ………………………….(3.19a)

Condition (ii) in Eq. (3.18) gives $T_2 = - C_1 / r_2 + C_2$ ………………………….(3.19b)

Solving for C_1 and C_2 from Eqs. (3.19a) and (3.19b) we have

$$C_1 = \frac{(T_1 - T_2)}{[1 / r_2 - 1 / r_1]} \quad \text{and} \quad C_2 = T_1 + \frac{(T_1 - T_2)}{r_1[1 / r_2 - 1 / r_1]}$$

Substituting these expressions for C_1 and C_2 in Eq. (3.18) we get

$$T(r) = - \frac{(T_1 - T_2) / r}{[1 / r_2 - 1 / r_1]} + T_1 + \frac{(T_1 - T_2) / r_1}{[1 / r_2 - 1 / r_1]}$$

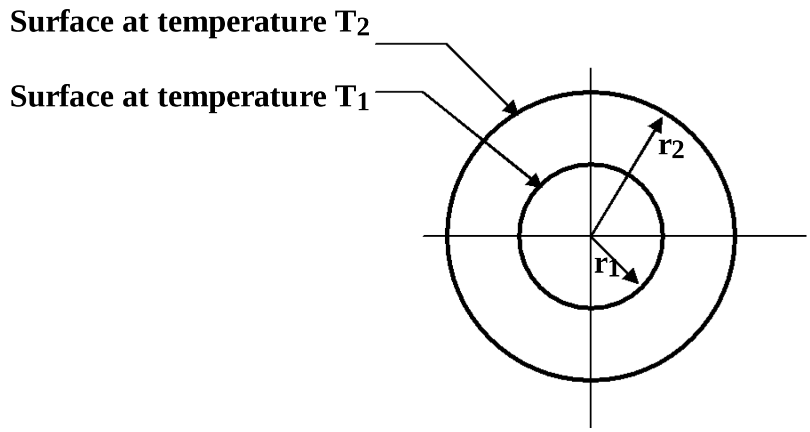


Fig. 3.6: Radial conduction in a hollow sphere with prescribed surface temperatures

Or

$$\frac{T(r) - T_1}{[T_1 - T_2]} = \frac{[1/r_2 - 1/r]}{[1/r_2 - 1/r_1]} \quad \text{......(3.20)}$$

(ii) <u>Expression for Rate of Heat Transfer:-</u> The rate of heat transfer for the hollow sphere is given by

$$Q_r = -k\, A(r)(dT/dr) \quad \text{......(3.21)}$$

Now at any radius for a sphere $A(r) = 4\pi\, r^2$ and from Eq. (3.20)

$$dT/dr = [T_1 - T_2] \frac{1}{[1/r_2 - 1/r_1]} (1/r^2)$$

Substituting these expressions in Eq. (3.21) and simplifying we get

$$\mathbf{Q_r = \frac{4\pi k r_1 r_2 [T_1 - T_2]}{[r_2 - r_1]}} \quad \text{......(3.22)}$$

Eq.(3.22) can be written as $\mathbf{Q_r = [T_1 - T_2] / R}$ **......(3.23a)**

Where R is the thermal resistance for the hollow sphere and is given by

$$\mathbf{R = (r_2 - r_1) / \{4\pi k r_1 r_2\}} \quad \text{......(3.23b)}$$

Hollow sphere with convective conditions at the surfaces: - Fig. 3.7 shows a hollow sphere whose boundary surfaces at radii r_1 and r_2 are in contact with fluids at temperatures T_i and T_0 with surface heat transfer coefficients h_i and h_0 respectively.

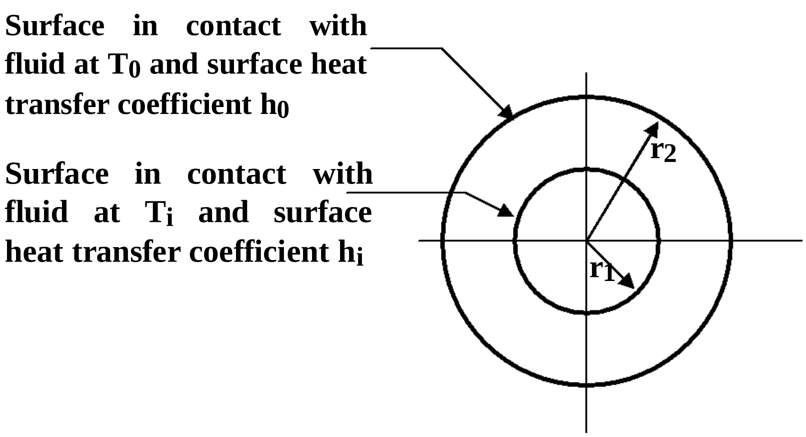


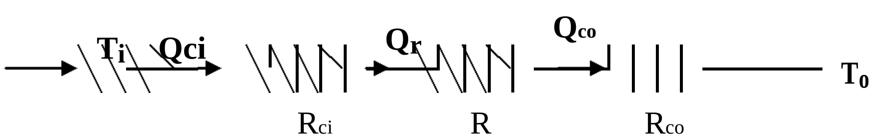
Fig. 3.7: Radial conduction in a hollow sphere with convective conditions at the two boundary surfaces

The thermal resistance network for the above problem is shown in Fig.3.8

$$Q_{ci} = Q_r = Q_{co} \quad \ldots\ldots\ldots\ldots\ldots\ldots(3.24)$$

Where Q_{ci} = heat transfer by convection from the fluid at T_i to the inner surface of the hollow sphere and is given by

$$Q_{ci} = h_i A_i [T_i - T_1] = \frac{[T_i - T_1]}{R_{ci}} \quad \ldots..(3.25)$$



Thermal circuit for a hollow sphere with convective boundary conditions 3.12

When T_1 = the inside surface temperature of the sphere and

$R_{ci} = 1 / (h_i A_i)$ = the thermal resistance for convection for the inside surface

Or $R_{ci} = 1 / (4 \pi r_1^2 h_i)$ ……………………………………………………….(3.25b)

Q_r = Rate of heat transfer by conduction through the hollow sphere

$= [T_1 - T_2] / R$ with $R = (r_2 - r_1) / \{4 \pi k r_1 r_2\}$

And Q_{co} = Rate of heat transfer by convection from the outer surface of the sphere to the outer fluid and is given by

$$Q_{co} = h_o A_o [T_2 - T_o] = \frac{[T_2 - T_0]}{R_{co}} \quad \text{……………(3.26a)}$$

Where T_2 = outside surface temperature of the sphere and

A_o = outside surface area of the sphere = $4 \pi r_2^2$ so that

$$R_{co} = 1 / \{4 \pi r_2^2 h_o\} \quad \text{……………………………(3.26b)}$$

Now Eq.(3.24) can be written as

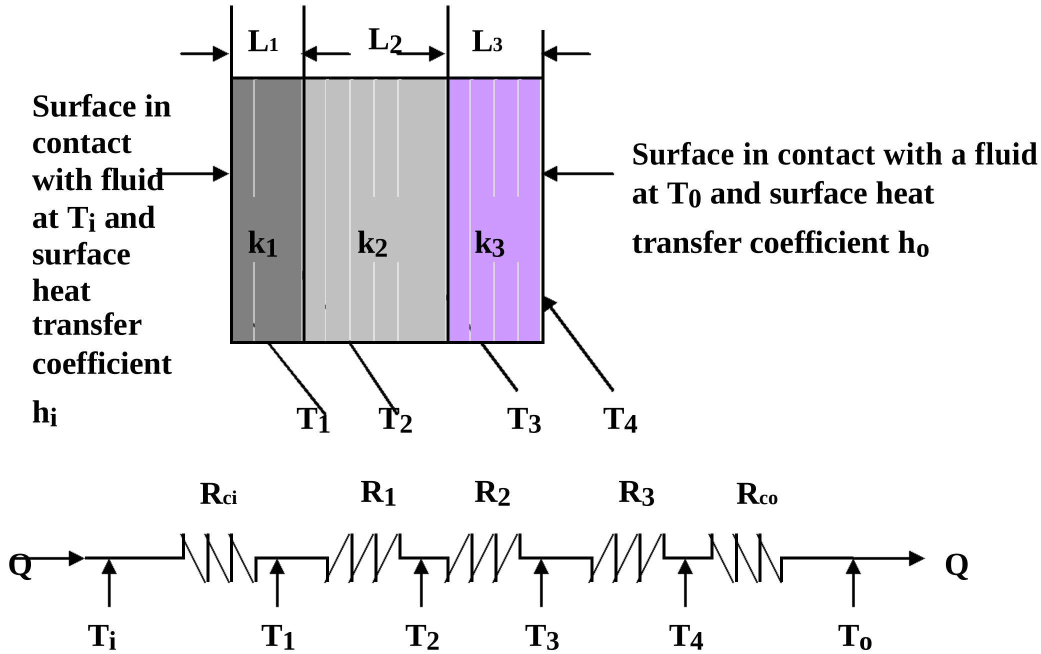
$$Q_r = h_i A_i [T_i - T_1] = \frac{[T_i - T_1]}{R_{ci}} = \frac{[T_1 - T_2]}{R} = \frac{[T_2 - T_1]}{R_{co}}$$

$$\mathbf{Q_r = \frac{[T_i - T_o]}{[R_{ci} + R + R_{co}]}} \quad \text{…………………………………………(3.27)}$$

<u>Steady State conduction in composite medium:</u>

There are many engineering applications in which heat transfer takes place through a medium composed of several different layers, each having different thermal conductivity. These layers may be arranged in series or in parallel or they may be arranged with combined series-parallel arrangements. Such problems can be conveniently solved using electrical analogy as illustrated in the following sections.

Composite Plane wall:- **(i) Layers in series:** Consider a plane wall consisting of three layers in series with perfect thermal contact as shown in Fig. 3.10.The equivalent thermal resistance network is also shown. If Q is the rate of heat transfer through an area A of the composite wall then we can write the expression for Q as follows:



A composite plane wall with three layers in series and the equivalent thermal resistance network

$$Q = \frac{(T_2 - T_3)}{R_{co}} = \frac{(T_1 - T_2)}{R_1} = \frac{(T_1 - T_2)}{R_2} = \frac{(T_2 - T_3)}{R_3} = \frac{(T_3 - T_{co})}{R_{co}}$$

$$\text{Or } Q = \frac{(T_i - T_0)}{R_{ci} + R_1 + R_2 + R_3 + R_{co}} = \frac{T_i - T_0)}{R_{total}} \quad \ldots\ldots(3.28)$$

***Overall heat transfer coefficient for a composite wall:* -** It is sometimes convenient to express the rate of heat transfer through a medium in a manner which is analogous to the Newton"s law of cooling as follows:

If U is the overall heat transfer coefficient for the composite wall shown in Fig. (3.10) then

$$Q = U\,A\,(T_i - T_o) \quad \ldots\ldots(3.29)$$

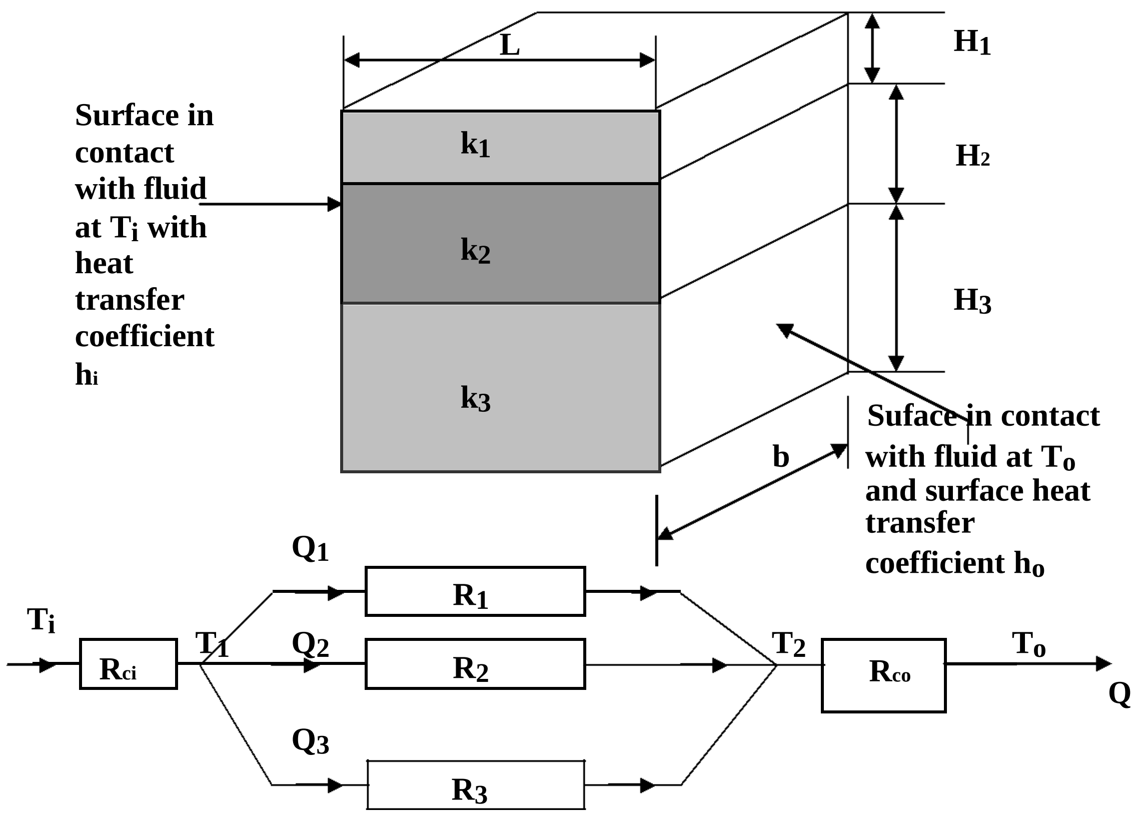
Comparing Eq. (3.28) with Eq. (3.29) we have the expression for U as

$$U = \frac{1}{A\,R_{total}} \quad \ldots\ldots(3.30)$$

$$\text{Or } U = \frac{1}{A[R_{ci} + R_1 + R_2 + R_3]} = \frac{1}{A[1/(h_iA) + L_1/(Ak_1) + L_2/(Ak_2) + L_3/(Ak_3)]}$$

$$\text{Or } \mathbf{U} = \frac{1}{[1/h_i + L_1/k_1 + L_2/k_2 + L_3/k_3]} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3.31)$$

(ii) Layers in Parallel:- Fig.3.11 shows a composite plane wall in which three layers are



Schematic and equivalent thermal circuit for a composite wall with layers in parallel

arranged in parallel. Let „b‟ be the dimension of these layers measured normal to the plane of the paper. Let one surface of the composite wall be in contact with a fluid at temperature T_i and surface heat transfer coefficient h_i and the other surface of the wall be in contact with another fluid at temperature T_o with surface heat transfer coefficient h_o. The equivalent thermal circuit for the composite wall is also shown in Fig. 3.11. The rate of heat transfer through the composite wall is given by

$$Q = Q_1 + Q_2 + Q_3 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(3.32)$$

where Q_1 = Rate of heat transfer through layer 1,

Q_2 = Rate of heat transfer through layer 2, and

Q_3 = Rate of heat transfer through layer 3.

$$\text{Now } Q_1 = \frac{(T_1 - T_2)}{R_1} \quad \ldots\ldots(3.33a)$$

Where $R_1 = \{L / (H_1 b k_1)\}$

$$\text{Similarly } Q_2 = \frac{(T_1 - T_2)}{R_2} \quad \ldots\ldots(3.33b)$$

Where $R_2 = \{L / (H_2 b k_2)\}$

$$\text{and } Q_3 = \frac{(T_1 - T_2)}{R_3} = \quad \ldots\ldots(3.33c)$$

Where $R_3 = \{L / (H_3 b k_3)\}$

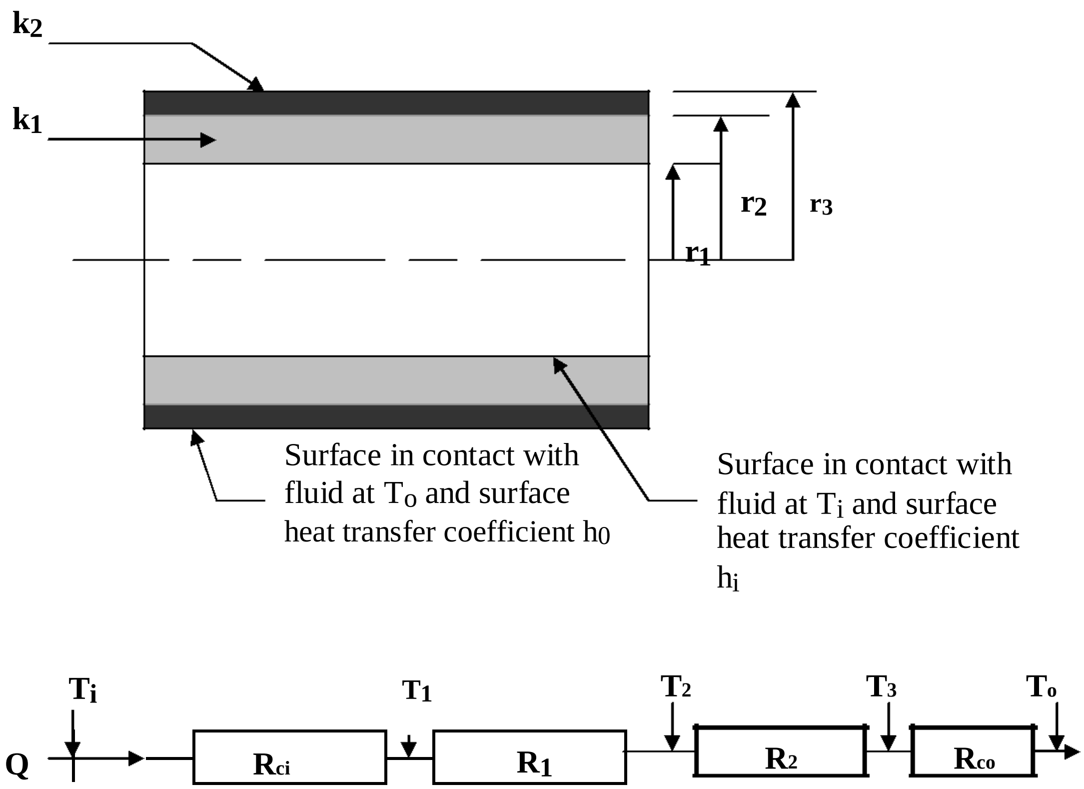
Substituting these expressions in Eq. (3.32) and simplifying we get

$$Q = \frac{(T_1 - T_2)}{R_1} + \frac{(T_1 - T_2)}{R_2} + \frac{(T_1 - T_2)}{R_3} = \frac{(T_1 - T_2)}{R_e} \quad \ldots\ldots(3.34)$$

Where $1 / R_e = 1/R_1 + 1/R_2 + 1/R_3$

$$\text{Hence } Q = \frac{(T_i - T_1)}{R_{ci}} = \frac{(T_1 - T_2)}{R_e} = \frac{(T_2 - T_o)}{R_{co}} = \frac{(T_i - T_o)}{[R_{ci} + R_e + R_{co}]} \quad \ldots\ldots(3.35)$$

Composite Coaxial Cylinders:- Fig. 3.12. shows a composite cylinder having two layers in series. The equivalent thermal circuit is also shown in the figure. The rate of heat transfer through the composite layer is given by



: Schematic and thermal circuit diagrams for a composite cylinder

$$\text{Now } Q = \frac{(T_i - T_1)}{R_{ci}} = \frac{(T_1 - T_2)}{R_1} = \frac{(T_2 - T_3)}{R_2} = \frac{(T_3 - T_o)}{R_{co}} = \frac{(T_i - T_0)}{[R_{ci} + R_1 + R_2 + R_{co}]} \qquad \text{……………..(3.36)}$$

$$\text{Where } R_{ci} = 1 / [h_i A_i] = \frac{1}{2\,\pi\, r_1 L\, h_i}\,; \qquad R_1 = \frac{1}{2\,\pi\, L\, k_1} \ln (r_2 / r_1)$$

$$R_{co} = 1 / [h_o A_o] = \frac{1}{2\,\pi\, r_3 L\, h_o}\,; \qquad R_2 = \frac{1}{2\,\pi\, L\, k_2} \ln (r_3 / r_2)$$

The above expression for Q can be extended to any number of layers.

Overall Heat Transfer Coefficient for a Composite Cylinder:- For a cylinder the area of heat flow in radial direction depends on the radius r we can define the overall heat transfer coefficient either based on inside surface area or based on outside surface area of the composite cylinder. Thus if U_i is the overall heat transfer coefficient based on inside surface area A_i and U_o is the overall heat transfer coefficient based on outside surface area A_o then

$$Q = U_i A_i (T_i - T_o) \quad \ldots\ldots(3.37)$$

From equations (3.36) and (3.37) we have

$$\text{Now } U_i A_i (T_i - T_o) = \frac{(T_i - T_0)}{[R_{ci} + R_1 + R_2 + R_{co}]}$$

Substituting the expressions for A_i, R_{ci},R_1,R_2 and R_{co} in the above equation we have

$$2\pi r_1 L U_i = \frac{1}{[1/(2\pi r_1 L h_i) + \{1/(2\pi L k_1)\}\ln(r_2/r_1) + \{1/(2\pi L k_2)\}\ln(r_3/r_2) + 1/(2\pi r_3 L h_o)]}$$

$$\textbf{Or} \quad U_i = \frac{1}{[1/h_i + (r_1/k_1)\ln(r_2/r_1) + (r_1/k_2)\ln(r_3/r_2) + (r_1/r_3)(1/h_o)]} \quad \ldots\ldots(3.38)$$

Similarly it can be shown that

$$U_o = \frac{1}{[(r_3/r_2)(1/h_i) + (r_3/k_1)\ln(r_2/r_1) + (r_3/k_2)\ln(r_3/r_2) + (1/h_o)]} \quad \ldots\ldots(3.39)$$

Composite Concentric Spheres:- Fig.3.13 shows a composite sphere having two layers with the inner surface of the composite sphere in contact with fluid at a uniform temperature T_i and surface heat transfer coefficient h_i and the outer surface in contact with another fluid at a uniform temperature T_o and surface heat transfer coefficient h_o. The corresponding thermal circuit diagram is also shown in the figure.

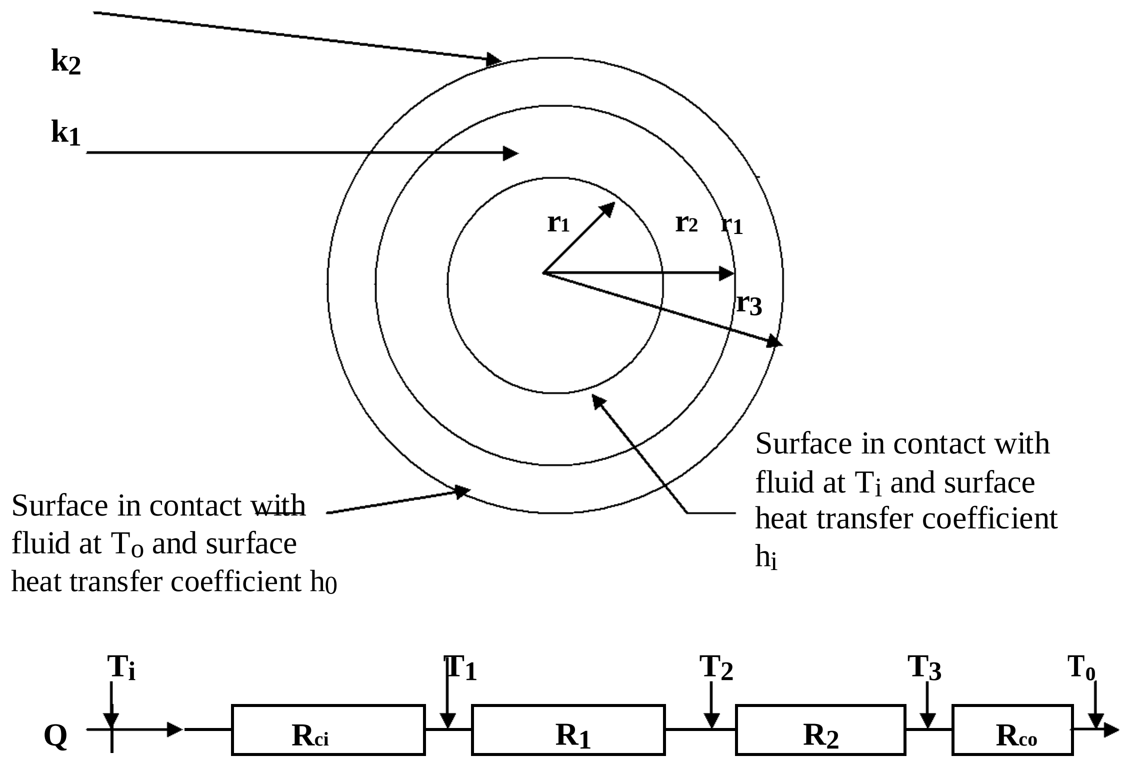


Fig. 3.13: Schematic and thermal circuit diagrams for a composite sphere

Eq. (3.36) is also applicable for the composite sphere of Fig. 3.13 except that the expression for individual resistance will be different. Thus

$$Q = \frac{(T_i - T_o)}{[R_{ci} + R_1 + R_2 + R_{co}]} \qquad \text{.......................................}(3.40)$$

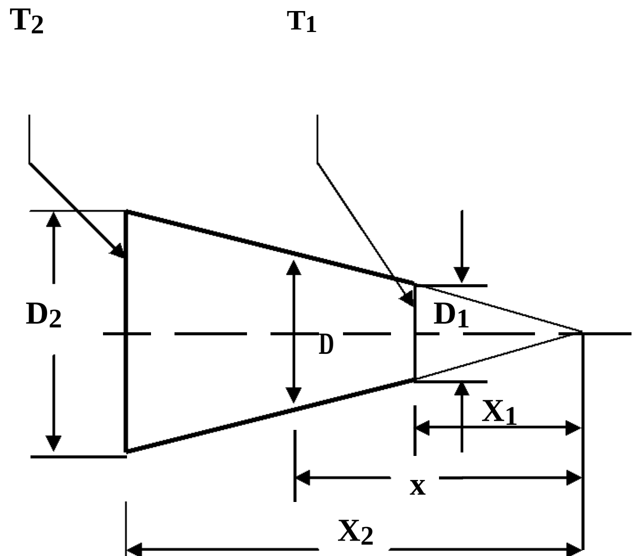
where

$$R_{ci} = \frac{1}{h_i A_i} = \frac{1}{4\pi r_1^2 h_i}\ ; \quad R_1 = \frac{(r_2 - r_1)}{4\pi k_1 r_1 r_2}\ ;$$

$$R_{co} = \frac{1}{h_o A_o} = \frac{1}{4\pi r_3^2 h_o}\ ; \quad R_2 = \frac{(r_3 - r_2)}{4\pi k_2 r_2 r_3}\ ;$$

Example 3.2:-Fig. P3.2 shows a frustum of a cone (k = 3.46 W/m-K). It is of circular cross section with the diameter at any x is given by D = ax, where a = 0.25. The smaller cross section is at x_1 = 50 mm and the larger cross section is at x_2 = 250 mm. The corresponding surface temperatures are T_1 = 400 K and T_2 = 600 K. The lateral surface of the cone is completely insulated so that conduction can be assumed to take place in x-direction only.

(i) Derive an expression for steady state temperature distribution, T(x) in the solid and
(ii) calculate the rate of heat transfer through the solid.(T(x) = 400 + 12.5{20 – 1/x} ; Q_x = - 2.124 W)



By Fourier‟s law, the rate of heat transfer in x-direction across any plane at a distance x from the origin „o‟ is given by

$$Q_X = - k\, A_X\, (dT/dx).$$

For steady state conduction without heat generation Q_X will be a constant. Also at any x, D = ax.

Therefore, $Q_X = - k\, (\pi D^2/4)\, (dT/dx) = - k\, [\pi (ax)^2/4]\, (dT/dx)$.

Separating the variables we get, $dT = - (4/\pi a^2 k)\, Q_X\, (dx/x^2)$

Integrating the above equation we have

$$\int_{T_1}^{T} dT = -(4Q_x / \pi\, a_2\, k) \int_{X_1}^{x} (dx / x_2)$$

Or $$T - T_1 = -(4Q_x / \pi\, a_2\, k)\, [(1 / x) - (1 / X_1)]$$

Or $$T = T_1 - \frac{(4\, Q_x)}{(\pi\, a_2\, k)}\, ((1 / x) - (1 / X_1)) \;\ldots\ldots\ldots\ldots(1)$$

At x = X_2, T = T_2. Substituting this condition in Eq.(1) and solving for Q_x we get

$$Q_x = \frac{(\pi\, a_2\, k)\, (T_2 - T_1)}{4\, (1/X_2 - 1/X_1)} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Substituting this expression for Q_X in Eq. (1) we get the temperature distribution in the cone as follows:

$$T(x) = T_1 + \frac{(T_2 - T_1)(1/x - 1/X_1)}{(1/X_2 - 1/X_1)} \quad \ldots\ldots\ldots\ldots\ldots\ldots..(3)$$

Substituting the given numerical values for X_1, X_2, T_1 and T_2 in Eq.(3) we get the temperature distribution as follows:

$$T(x) = 400 + \frac{(600 - 400)\,[\,1/x - 1/0.05]}{[\,1/0.25 - 1/0.05\}}$$

Or $\qquad T(x) = 400 + 12.5\,[20 - 1/x] \rightarrow$ Temperature distribution

And $$Q_x = \frac{\pi \times (0.25)^2 \times 3.46 \times [600 - 400]}{4 \times [\,1/0.25 - 1/0.05\,]} = -\,2.123\ W$$

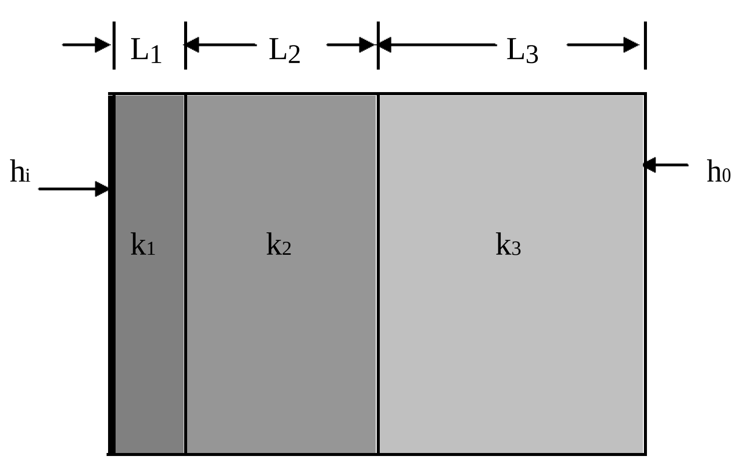
Example 3.3: -A plane composite wall consists of three different layers in perfect thermal contact. The first layer is 5 cm thick with k = 20 W/(m-K), the second layer is 10 cm thick with k = 50 W/(m-K) and the third layer is 15 cm thick with k = 100 W/(m-K). The outer surface of the first layer is in contact with a fluid at 400 ^{0}C with a surface heat transfer coefficient of 25 W/ (m ^{2} – K), while the outer surface of the third layer is exposed to an ambient at 30 ^{0}C with a surface heat transfer coefficient of 15 W/(m ^{2}- K).Draw the equivalent thermal circuit indicating the numerical values of all the thermal resistances and calculate the heat flux through the composite wall. Also calculate the overall heat transfer coefficient for the composite wall.

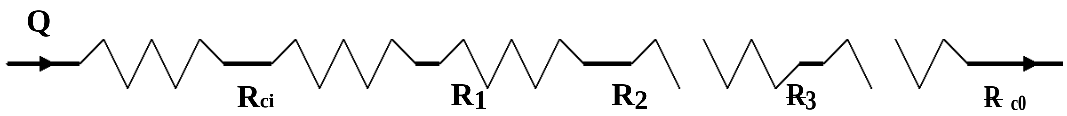
Solution: Data :- L_1 = 0.05 m ; L_2 = 0.10 m ; L_3 = 0.15 m ; k_1 = 20 W /(m-K) ;

k_2 = 50 W /(m-K) ; k_3 = 100 W/(m-K) ; h_i = 25 W /(m^2 – K) ; h_0 = 15 W/(m^2 – K)

; T_i = 400 ^{0} C ; T_0 = 30 ^{0} C.

$$R_{ci} = 1 / (h_i A_1) = \frac{1}{25 \times 1} = 0.04\ m^2 - K / W \quad (A_1 = A_2 = A_3 = A_4 = 1\ m^2)$$





$$R_1 = L_1/(k_1A_1) = \frac{0.05}{20 \times 1} = 0.0025\ m^2 - K/W.$$

$$R_2 = L_2/(k_2A_2) = \frac{0.10}{50 \times 1} = 0.002\ m^2 - K/W.$$

$$R_3 = L_3/(k_3A_3) = \frac{0.15}{100 \times 1} = 0.0015\ m^2 - K/W.$$

$$R_{co} = 1/(h_0A_4) = \frac{1}{15 \times 1} = 0.067\ m^2 - K/W.$$

$$\sum R = R_{ci} + R_1 + R_2 + R_3 + R_{co} = 0.04 + 0.0025 + 0.002 + 0.0015 + 0.067$$

Or $$\sum R = 0.113\ m^2\text{-}K/W.$$

Heat Flux through the composite slab = $$q = \frac{(T_i - T_0)}{\sum R} = \frac{(400 - 30)}{0.113}$$

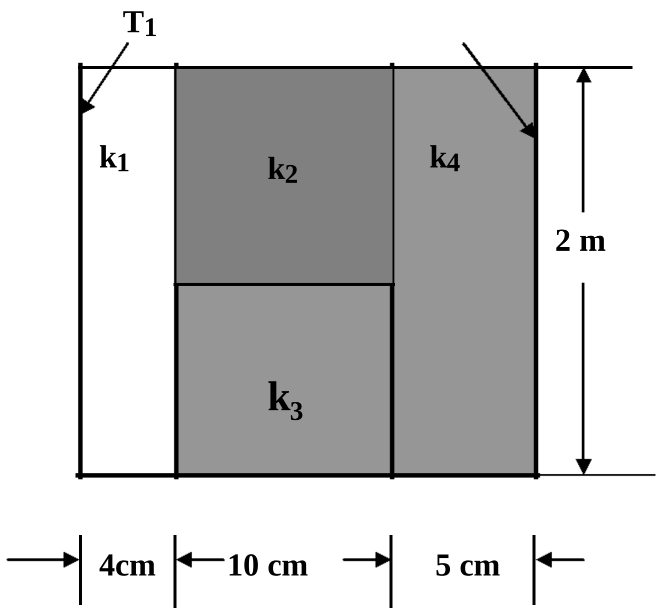
$$= 3274.34\ W/m^2.$$

If „U" is the overall heat transfer coefficient for the given system then

$$U = \frac{Q}{(T_i - T_0)} = \frac{1}{\sum R} = \frac{1}{0.113}$$

$$= 8.85\ W/(m^2 - K).$$

Example 3.4:-A composite wall consisting of four different materials is shown in Fig P3.10. Using the thermal resistance concept determine the heat transfer rate per m^2 of the exposed surface for a temperature difference of 300 ^{0} C between the two outer surfaces. Also draw the thermal circuit for the composite wall.



k_1 = 100 W/(m-K) ; L_1 =0.04 m;

k_2 = 0.04 W/(m-K) ; L_2 = 0.1 m;

k_3 = 20 W/(m-K) ; L_3 = 0.1 m ;

k_4 = 70 W/(m-K) ; L_4 = 0.05 m;

W = Width of the wall perpendicular to the plane of paper = 1 m (assumed).

$T_1 - T_4 = 300\ ^0$ C.
$A_1 = A_4 = 1 \times 2 = 2\ m^2$.
$A_2 = A_3 = 1 \times 1 = 1\ m^2$.

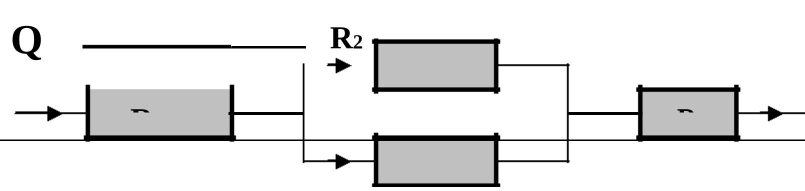
Solution:

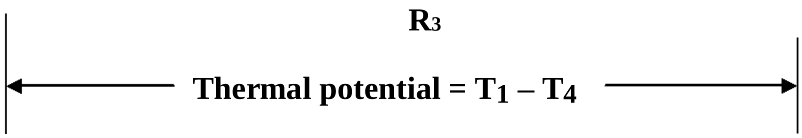
$$R_1 = L_1 / (A_1 k_1) = \frac{0.04}{2 \times 100} = 0.0002\ ^0\text{C / W.}$$

$$R_2 = L_2 / (A_2 k_2) = \frac{0.10}{1 \times 70} = 0.00143\ ^0\text{C / W.}$$

$$R_3 = L_3 / (A_3 k_3) = \frac{0.10}{1 \times 20} = 0.005\ ^0\text{C / W.}$$

$$R_4 = L_4 / (A_4 k_4) = \frac{0.05}{2 \times 70} = 0.00036\ ^0\text{C / W.}$$





R_2 and R_3 are resistances in parallel and they can be replaced by a single equivalent resistance R_e, where

$$1 / R_e = 1 / R_2 + 1 / R_3 \text{ or } R_e = \frac{R_2 R_3}{(R_2 + R_3)} = \frac{0.00143 \times 0.005}{(0.00143 + 0.005)} = 0.0011\ ^0C/W$$

Now R_1, R_e and R_4 are resistances in series so that

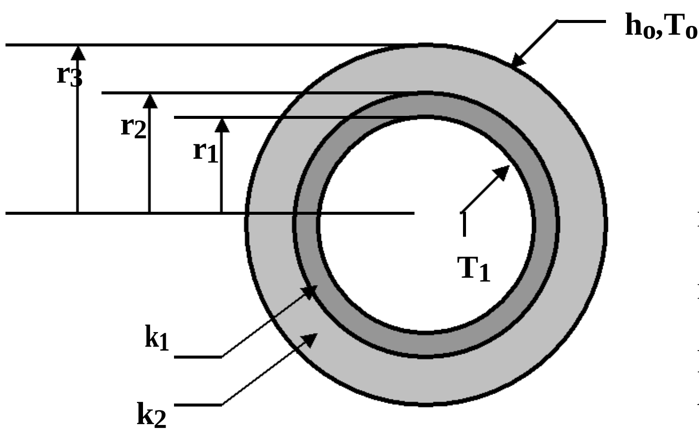
$$Q = \frac{(T_1 - T_4)}{(R_1 + R_e + R_4)} = \frac{300}{[0.002 + 0.0011 + 0.00036]} = 86.705 \times 10^3 \text{ W}$$

Heat transfer per unit area of the exposed surface is given by

$$q = Q / A_1 = 86.705 / 2.0 = 43.35 \text{ kW.}$$

Example 3.8:- A hollow aluminum sphere with an electrical heater in the centre is used to determine the thermal conductivity of insulating materials. The inner and outer radii of the sphere are 15 cm and 18 cm respectively and testing is done under steady state conditions with the inner surface of the aluminum maintained at 250^0 C. In a particular test, a spherical shell of insulation is cast on the outer surface of the aluminum sphere to a thikness of 12 cm. The system is in a room where the air temperature is 20 ^{0} C and the convection coefficient is 30 W/(m^2 – K). If 80 W are dissipated by the heater under steady state conditions, what is the thermal conductivity of the insulating material?

Solution:



$r_1 = 0.15$ m ; $r_2 = 0.18$ m ;

$r_3 = 0.18 + 0.12 = 0.3$ m ;

$k_1 = 204$ W/(m-K) from tables; $k_2 = 0.30$ W/(m-K)

$h_o = 30$ W/(m^2-K);Q = 60 W

$T_1 = 250\ ^0$ C ; $T_o = 20\ ^0$ C.

$$R_1 = \frac{(r_2 - r_1)}{4\pi k_1 r_1 r_2} = \frac{(0.18 - 0.15)}{4 \pi \times 204 \times 0.18 \times 0.15} = 4.335 \times 10^{-4} \ {}^0C / W.$$

$$R_2 = \frac{(r_3 - r_2)}{4\pi k_2 r_2 r_3} = \frac{(0.30 - 0.18)}{4 \pi \times k_2 \times 0.30 \times 0.18} = 0.177 / k_2 \ {}^0C / W.$$

$$R_{co} = 1 / (h_0 A_0) = \frac{1}{4\pi r_3^2 h_o} = \frac{1}{4\pi \times (0.3)^2 \times 30} = 0.0295 \ {}^0C / W.$$

$$Q = \frac{(T_1 - T_o)}{R_1 + R_2 + R_{co}} \quad \text{or} \quad R_2 = (T_1 - T_o) / Q - (R_1 + R_{co})$$

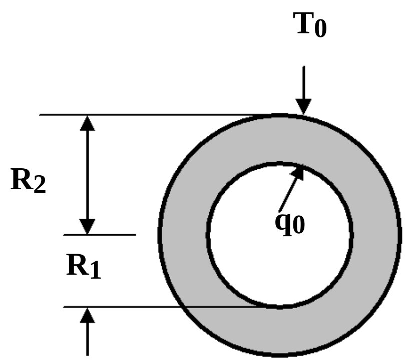
$$\text{Or} \quad R_2 = (250 - 20) / 80 - (4.335 \times 10^{-4} + 0.0295) = 2.874$$

Therefore $\quad 0.177 / k_2 = 2.845$

Or $\quad k_2 = 0.177 / 2.845 = 0.062 \ W / (m\text{-}K)$

Example 3.8:- In a hollow sphere of inner radius 10 cm and outer radius 20, the inner surface is subjected to a uniform heat flux of 1.6×10^5 W/m^2 and the outer surface is maintained at a uniform temperature of 0 ^{0}C.The thermal conductivity of the material of the sphere is 40 W /(m – K).Assuming one-dimensional radial steady state conduction determine the temperature of the inner surface of the hollow sphere.

Solution:-



The governing equation for one-dimensional steady-state radial conduction in a sphere without heat generation is given by

$$d/dr \, (r^2 \, dT / dr) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

The boundary conditions are : (i) at $r = R_1$, $-k\,(dT/dr)|_{r=R1} = q_0$

(ii) at $r = R_2$ $\quad T(r) = 0$.

Integrating Eq. (1) w.r.t. r once, we get

$$r^2\,(dT/dr) = C_1$$

$$\text{or } dT / dr = C_1 / r^2 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Integrating once again w.r.t. r we get

$$T(r) = -C_1 / r + C_2 \;\ldots\ldots\ldots\ldots\ldots\ldots.. \qquad (3)$$

From (2) $\qquad (dT/dr)_{r\,=\,R1} = C_1 / R_1^2$

Hence condition (i) gives

$$-kC_1 / R_1^2 = q_0$$

Or $\qquad C_1 = -q_0\,R_1^2 / k$

Condition (ii) in Eq.(2) gives $\quad 0 = -C_1 / R_2 + C_2$

Or $\qquad C_2 = C_1 / R_2 = -(q_0 R_1^2) / (kR_2)$

Substituting the expressions for C_1 and C_2 in Eq. (2) we have

$$T(r) = \frac{q_0\,R_1^2}{k\,r} - \frac{q_0\,R_1^2}{k\,R_2}$$

Substituting the numerical values for q_0, k, R_1 and R_2 we have

$$T(r) = \frac{1.6 \times 10^5 \times 0.1^2}{40} / r - \frac{1.6 \times 10^5 \times 0.1^2}{40 \times 0.2}$$

Or $\qquad T(r) = (40 / r) - 200$

Therefore $\qquad T(r)\,|_{r\,=\,R1} = (40 / 0.1) - 200 = 200\ {}^0\,C.$

3.2.5. Thermal Contact Resistance: In the analysis of heat transfer problems for composite medium it was assumed that there is "perfect thermal contact" at the interface of two layers. This assumption is valid only the two surfaces are smooth and they produce a perfect contact at each point.But in reality, even flat surfaces that appear smooth to the naked eye would be rather rough when examined under a microscope as shown in Fig. 3.14 with numerous peaks and valleys.

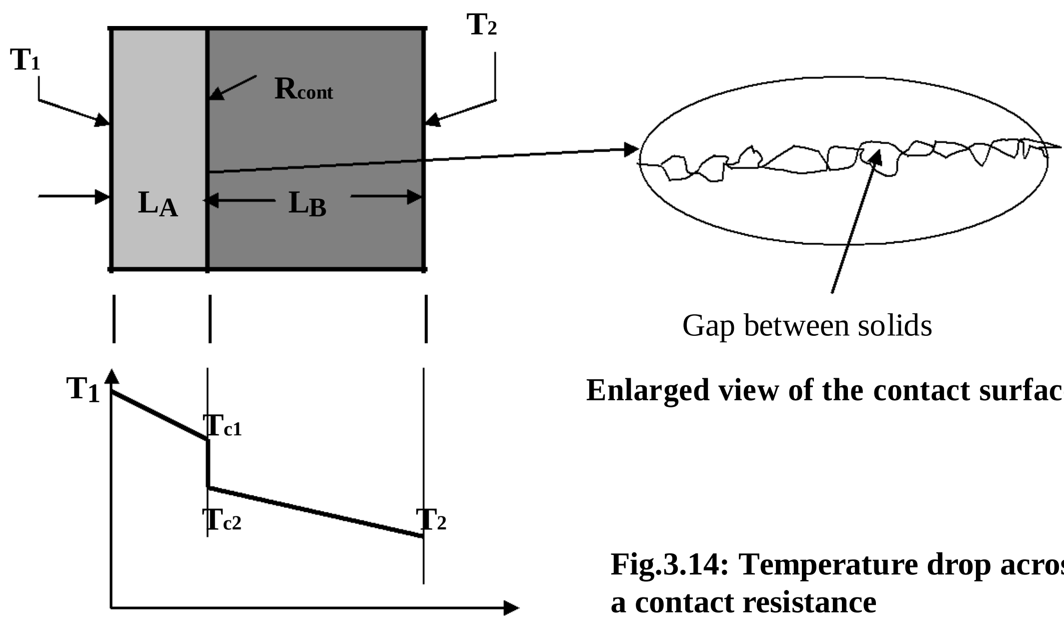
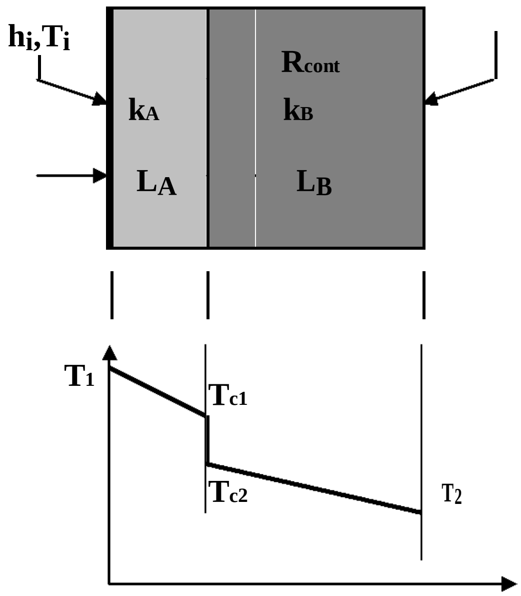


Fig.3.14: Temperature drop across a contact resistance

The physical significance of thermal contact resistance is that the peaks will form good thermal contact, but the valleys will form voids filled with air.As a result the air gaps act as insulation because of poor thermal conductivity of air.Thus the interface offers some resistance to heat conduction and this resistance is called the *"thermal contact resistance,R_{cont}"*. The value of R_{cont} is determined experimentally and is taken into account while analyzing the heat conduction problems involving multi-layer medium.The procedure is illustrated by means of a few examples below.

Example 3.4:- A composite wall consists of two different materials A [k = 0.1 W/(m-k)] of thickness 2 cm and B[k = 0.05 W/(m-K)] of the thickness 4 cm. The outer surface of layer A is in contact with a fluid at 200^0C with a surface heat transfer coefficient of 15 W/(m^2-K) and the outer surface of layer B is in contact with another fluid at 50 ^{0} C with a surface heat transfer coefficient of 25 W/(m^2-K). The contact resistance between layer A and layer B is 0.33 (m^2-K) /W. Determine the heat transfer rate through the composite wall per unit area of the surface. Also calculate the interfacial temperatures and the inner and outer surface temperatures.

Solution:



$T_i = 200\,^0$ C ; $T_0 = 50\,^0$ C ;

$h_i = 15$ W/(m^2 – K) ; $h_0 = 25$W/(m^2– K)

$k_A = 0.1$ W/(m-K) ; $k_B = 0.05$ W/(m-K)

$R_{cont} = 0.33$ (m^2 – K) /W.

The equivalent thermal circuit is also shown in the figure.

$$R_{ci} = 1/(h_i A_A) = \frac{1}{(15 \times 1)} = 0.067 \text{ m}^2\text{-K/W}$$

$R_1 = L_A/(k_A A_A) = 0.02 / (0.1 \times 1)$
$= 0.2$ m^2-K / W.

$R_2 = L_B / (k_B A_B) = 0.04 / (0.05 \times 1) = 0.8$ m^2 – K / W.

$R_{co} = 1 / (h_o A_B) = 1 / (25 \times 1) = 0.04$ m^2 – K / W.

$\sum R = R_{ci} + R_1 + R_{cont} + R_2 + R_{co} = 0.067 + 0.2 + 0.33 + 0.8 + 0.04 = 1.437$ m^2 – K / W.

$$\text{Heat flux} = q = (T_i - T_o) / \sum R = \frac{(200 - 50)}{1.437} = 104.4 \text{ W/m}^2$$

Now $q = (T_i - T_A) / R_{ci}$ or $T_A = T_i - q\,R_{ci} = 200 - (104.4 \times 0.067) = 193\,^0$ C.

Similarly $T_{c1} = T_A - q\,R_1 = 193 - (104.4 \times 0.2) = 172.12\,^0$ C.

$T_{c2} = T_{c1} - q\,R_{cont} = 172.12 - (104.4 \times 0.33) = 137.67\,^0$ C.

$T_B = T_{c2} - q\,R_2 = 137.67 - (104.4 \times 0.8) = 54.15\,^0$ C.

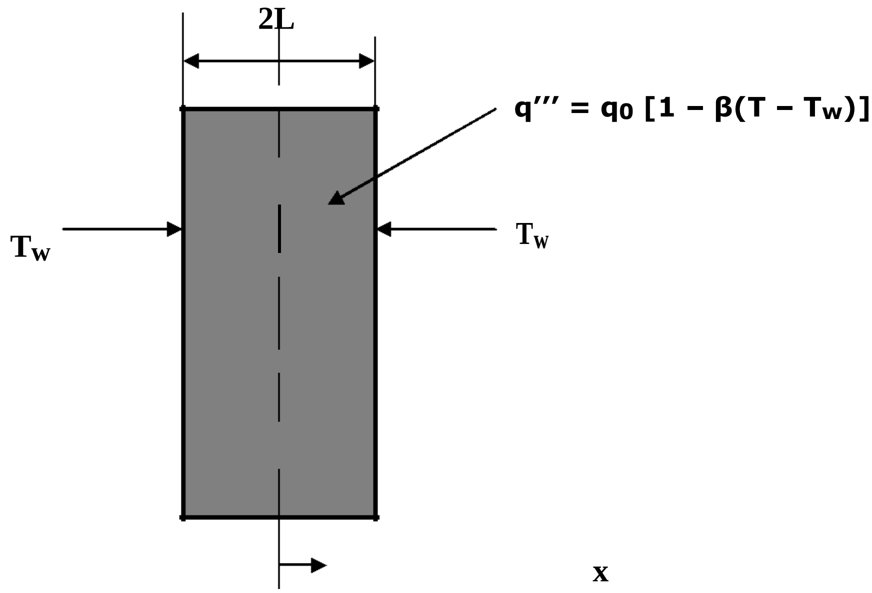
Check : $T_o = T_B - q\,R_{co} = 54.15 - (104.4 \times 0.04) = 49.97\,^0$ C

Example 3.11:- A plane wall of thickness 2L is generating heat according to the law

$q''' = q_0 [1 - \beta(T - T_w)]$

where q_o, β, and T_w are constants and T is the temperature at any section x from the mid-plane of the wall. The two outer surfaces of the wall are maintained at a uniform temperature T_w. Determine the one-dimensional steady state temperature distribution, T(x) for the wall.

Solution:



Governing differential equation for one-dimensional steady state conduction in a plane wall which generating heat is given by

$$d^2T / dx^2 + q''' / k = 0.$$

Substituting for q''' we have

$$d^2T / dx^2 + q_o [1 – \beta(T – T_w)] /k = 0$$

Defining a new variable $\theta = T – T_w$, the above equation can be written as

$$d^2\theta / dx^2 + q_o [1 – \beta\theta] /k = 0$$

or $$d^2\theta / dx^2 - q_o \beta\theta /k = q_o / k$$

or $$d^2\theta / dx^2 - m^2\theta = q_o / k \quad \text{……………………………..(1a)}$$

where $$m^2 = q_0 \beta /k \quad \text{……………………………...(1b)}$$

Eq.(1a) is a second order linear non-homogeneous differential equation whose solution is given by

$$\theta (x) = \theta_h(x) + \theta_p(x) \quad \text{………………………………….(2)}$$

where $\theta_h(x)$ satisfies the differential equation

$$d^2\theta_h / dx^2 - m^2\theta_h = 0 \quad \text{……………………………….(3)}$$

Solution to Eq.(3) is given by

$$\theta_h(x) = A_1 e^{mx} + A_2 e^{-mx} \quad \text{…………………………….(4)}$$

$\theta_p(x)$ satisfies the differential equation

$$d^2\theta_p / dx^2 - m^2\theta_p = q_o / k \quad \text{…………………………..(5)}$$

The term q_o/k makes the governing differential equation non-homogeneous. Since this is a constant $\theta_p(x)$ is also assumed to be constant. Thus let $\theta_p(x) = C$, where C is a constant. Substituting this solution in Eq. (5) we get

$$- m^2C = q_o / k$$

Or $$C = - q_o /(km^2)$$

Substituting for m^2 we get $$C = - 1 / \beta.$$

Hence $$\theta_p(x) = - 1 / \beta \quad \text{…………………………….(6)}$$

The complete solution $\theta(x)$ is therefore given by

$$\theta(x) = A_1 e^{mx} + A_2 e^{-mx} - 1 / \beta \quad \text{…………………..(7)}$$

The constants A_1 and A_2 in Eq.(7) can be determined by using the two boundary conditions, which are:

(i)at x = 0, dT / dx = 0 (axis of symmetry) i.e., $d\theta / dx = 0$

(ii) at x = L, $T = T_W$; i.e., $\theta = 0$

From Eq.(7), $\quad d\theta / dx = m[A_1 e^{mx} - A_2 e^{-mx}]$

Substituting condition (i) we get $m[A_1 - A_2] = 0$

Or $\quad A_1 = A_2.$

Substituting condition (ii) in Eq.(7) we get $A_1[e^{mL} + e^{-mL}] = 1 / \beta$

Or
$$A_1 = \frac{(1/\beta)}{[e^{mL} + e^{-mL}]}$$

Substituting the expressions for A_1 and A_2 in Eq. (7) we get the temperature distribution in the plane wall as

$$\theta(x) = T(x) - T_W = \frac{(1/\beta)}{[e^{mL} + e^{-mL}]} [e^{mx} + e^{-mx}] - 1/\beta$$

Or
$$T(x) - T_W = \frac{1}{\beta}\left[\frac{e^{mx} + e^{-mx}}{e^{mL} + e^{-mL}} - 1\right]$$

or
$$T(x) - T_W = (1/\beta)\frac{[e^{m(L-x)} + e^{-m(L-x)}]}{[e^{mL} + e^{-mL}]} = (1/\beta)\frac{\cosh m(L-x)}{\cosh mL}$$

<u>3.4. Critical Radius of Insulation</u>:- For a plane wall adding more insulation will result in a decrease in heat transfer as the area of heat flow remains constant .But adding insulation to a cylindrical pipe or a conducting wire or a spherical shell will result in an increase in thermal resistance for conduction at the same will result in a decrease in the convection resistance of the outer surface because of increase in surface area for convection. Therefore the heat transfer may either increase or decrease depending on the relative magnitude of these two resistances.

Critical Radius of Insulation for Cylinder:- Let us consider a cylindrical pipe of outer radius r_s maintained at a constant temperature of T_s. Let the pipe now be insulated with a material of thermal conductivity k and outer radius r. Let the outer surface of the insulation be in contact with a fluid at a uniform temperature T_∞ with a surface heat transfer coefficient h. Then the thermal circuit for this arrangement will be as shown in Fig.3.15.

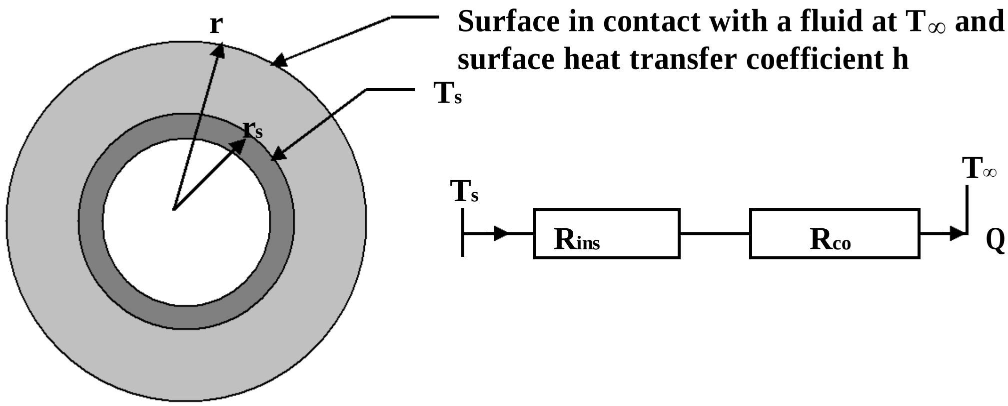


Fig.3.15: Schematic of a cylindrical pipe covered with an insulation and exposed to an ambient and the corresponding thermal circuit

The rate of heat transfer from the pipe to the ambient is given by

$$Q = \frac{(T_s - T_\infty)}{[R_{ins} + R_{co}]} = \frac{(T_s - T_o)}{\dfrac{\ln (r / r_s)}{2 \pi L k} + \dfrac{1}{2 \pi r L h}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3.44)$$

It can be seen from Eq. (3.44) that if T_S and h are assumed not to vary with „r" then Q depends only on r and the nature of variation of Q with r will be as shown in Fig.3.16. The value of r at which Q reaches a maximum can be determined as follows.

Eq. (3.44) can be written as $\quad Q = \dfrac{(T_S - T_O)}{F(r)}$

where $\quad F(r) = \dfrac{\ln (r / r_S)}{2 \pi L k} + \dfrac{1}{2 \pi r L h}$

Hence for Q to be maximum, F(r) has to be minimum: i.e., dF(r) / dr = 0

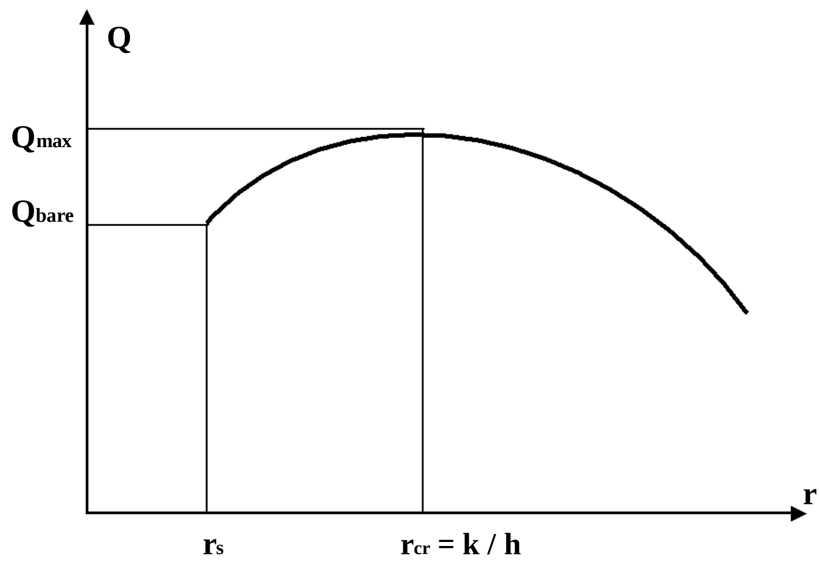


Fig.3.16: Variation of Q with outer radius of insulation

Now dF / dr = $(1 / 2\pi Lk)(1/r) - (1/2\pi Lh)(1/r^2) = 0$

Or $r = k / h$.

This value of r is called *"critical radius of insulation, r_{cr}".*

Therefore **$r_{cr} = k / h$ ………………………………………...(3.45)**

It can be seen from Fig.(3.16) that if the outer radius of the bare tube or bare wire is greater than the critical radius then, any addition of insulation on the tube surface decreases the heat loss to the ambient. But if the outer radius of the tube is less than the critical radius , the heat loss will increase continuously with the addition of insulation until the outer radius of insulation equals the critical radius. The heat loss becomes maximum at the critical radius and begins to decrease with addition of insulation beyond the critical radius.

The value of critical radius r_{cr} will be the largest when k is large and h is small. The lowest value of h encountered in practice is about 5 W/(m^2 – K) for free convection in a gaseous medium and the thermal conductivity of common insulating materials is about 0.05 W/(m – K). Hence the largest value of r_{cr} that we may likely to encounter is given by

$$r_{cr} = \frac{0.05}{5} = 0.01 \text{ m} = 1 \text{ cm}$$

The critical radius would be much less in forced convection (it may be as low as 1mm) because of large values of h associated with forced convection. Hence we can insulate hot water or steam pipes freely without worrying about the possibility of increasing the heat loss to the surroundings by insulating the pipes.

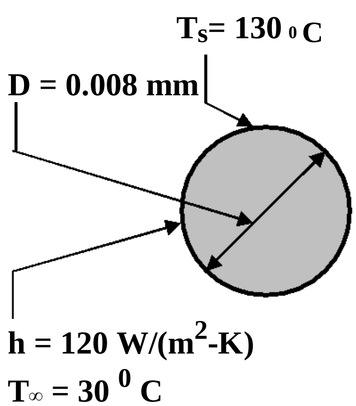
The radius of electric wires may be smaller than the critical radius. Therefore, the plastic electrical insulation may enhance the heat transfer from electric wires, there by keeping their steady operating temperatures at lower and safer levels.

Critical Radius Insulation for a Sphere:- The analysis described above for cylindrical pipes can be repeated for a sphere and it can be shown that for a sphere the critical radius of insulation is given by

$$\mathbf{r_{cr} = \frac{2k}{h}} \quad \text{...(3.46)}$$

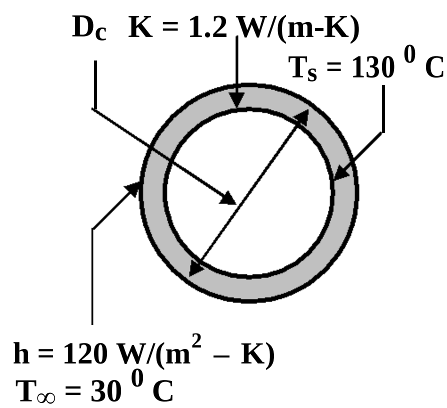
Example 3.12:-A conductor with 8 mm diameter carrying an electric current passes through an ambient at 30 ^{0} C with a convection coefficient of 120 W/(m^2 – K). The temperature of the conductor is to be maintained at 130 ^{0} C. Calculate the rate of heat loss per metre length of the conductor when (a) the conductor is bare and (b) conductor is covered with bakelite insulation [k = 1.2 W/(m-K)] with radius corresponding to the critical radius of insulation.

Solution:



(a) Conductor without

Insulation.



(b) Conductor with critical thickness of insulation

(a) When the conductor is bare the rate of heat loss to the ambient is given by

$$Q = h\,\pi D\,L\,(T_S - T_\infty) = 120 \times \pi \times 0.008 \times 1 \times (130 - 30) = 301.6 \text{ W/m}.$$

(b) When the conductor is covered with critical thickness of insulation,

$D_c = 2\,r_c = 2\,(k/h) = 2 \times (1.2 / 120) = 0.02$ m.

$$R_{insulation} = \frac{1}{2\pi L k} \ln (D_c / D) = \frac{1}{2\,\pi \times 1.0 \times 1.2} \ln (0.02 / 0.008)$$

$$= 0.1215\ (m - {}^0C) / W.$$

$$R_{co} = 1 / (h\,A_c) = \frac{1}{\pi D_c L h} = \frac{1}{\pi \times 0.02 \times 1 \times 120} = 0.133\ (m - {}^0C)/W.$$

$$\sum R = R_{insulation} + R_{co} = 0.1215 + 0.133 = 0.2545\ (m - {}^0C) / W.$$

$$Q_{insulation} = \frac{(T_s - T_\infty)}{\sum R} = \frac{(130 - 30)}{0.145} = 392.93\ W / m.$$

Example 3.13: -An electrical current of 700 A flows through a stainless steel cable having a diameter of 5 mm and an electrical resistance of 6×10^{-4} ohms per metre length of the cable. The cable is in an environment at a uniform temperature of 30 ^{0}C and the surface heat transfer coefficient of 25 W/(m^2 – ^{0}C).

(a) What is the surface temperature of the cable when it is bare?

(b)What thickness of insulation of k = 0.5 W/(m – K) will yield the lowest value of the maximum insulation temperature? What is this temperature when the thickness is used?

Solution: <u>(a) When the cable is Bare: -</u> Electrical Resistance = $R_e = 6 \times 10^{-4}\ \Omega$ / m

Current through the cable = I = 700 A; D = 0.005 m ; h = 25 W/(m^2-K) ; $T_\infty = 30\ ^0$

C. Power dissipated = $Q = I^2 R_e = (700)^2 \times 6 \times 10^{-4} = 294$ W / m.

But $Q = hA(T_s - T_\infty)$ or $T_s = T_\infty + Q / [(\pi D L) \times h]$

Or $T_s = 30 + 294 / [(\pi \times 0.005 \times 1) \times 25] = 779\ ^0C$.

(b) <u>When the cable is covered with insulation:</u>

k = 0.5 W/(m-K) ; Hence critical radius = $r_c = k / h = 0.5 / 25 = 0.02$ (m-K) /

W. Thickness of insulation = $r_c - D/2 = 0.02 - 0.005 / 2 = 0.0175$ m

$$R_{insulation} = \frac{1}{(2\pi L k)} \ln (r_c / r_o) = \frac{1}{(2\pi \times 1 \times 0.5)} \ln (0.02 / 0.0025) = 0.662 \text{ (m-K)/W}$$

$$R_{co} = 1 / (hA_o) = \frac{1}{(2\pi r_c L h)} = \frac{1}{(2 \times \pi \times 0.02 \times 1 \times 25)} = 0.318 \text{ (m-K) / W.}$$

$$Q = \frac{(T_s - T_\infty)}{R_{insulation} + R_{co}} \quad \text{or} \quad T_s = T_\infty + Q (R_{insulation} + R_{co})$$

Or $$T_s = 30 + 294 \times (0.662 + 0.318)$$

$$= 318.12\ {}_0\text{C.}$$

Example 3.14:- A 2 mm-diameter and 10 m-long electric wire is tightly wrapped With a 1 mm-thick plastic cover whose thermal conductivity is 0.15 W / (m-K). Electrical measurements indicate that a current of 10 A passes through the wire and there is a voltage drop of 8 V along the wire. If the insulated wire is exposed to a medium at 30 ^{0}C with a heat transfer coefficient of 24 W / (m^2 – K), determine the temperature at the interface of the wire and the plastic cover in steady operation. Also determine if doubling the thickness of the plastic cover will increase or decrease this interface temperature.

Given: Outer radius of the bare wire = r_S = 1 mm = 0.001 m ; Length of the wire = L = 10 m ; outer radius of plastic insulation = r = 1 + 1 = 2 mm = 0.002 m ;
Current through the wire = I = 10 A ; Voltage drop in the wire = V = 8 V ; Ambient temperature = T_∞ = 30 ^{0}C ; Thermal conductivity of the plastic cover = k = 0.15 W /(m– K) ; Surface heat transfer coefficient = h = 24 W /(m^2 – K).

To find: (i) Interface temperature = T_S ; (ii) Whether T_S increases or decreases when the thickness of insulation is doubled.

Solution: (i) Q = VI = 8 x 10 = 80 W.

The thermal circuit for the problem is shown in Fig. P3.14.

$$R_{ins} = \frac{\ln (r / r_s)}{2 \pi L k} = \frac{\ln (0.002 / 0.001)}{2 \times \pi \times 10 \times 0.15} = 0.0735 \text{ K / W}$$

$$R_{co} = \frac{1}{h A_o} = \frac{1}{2 \pi L r h} = \frac{1}{2 \times \pi \times 10 \times 0.002 \times 24} = 0.3316 \text{ K / W}$$

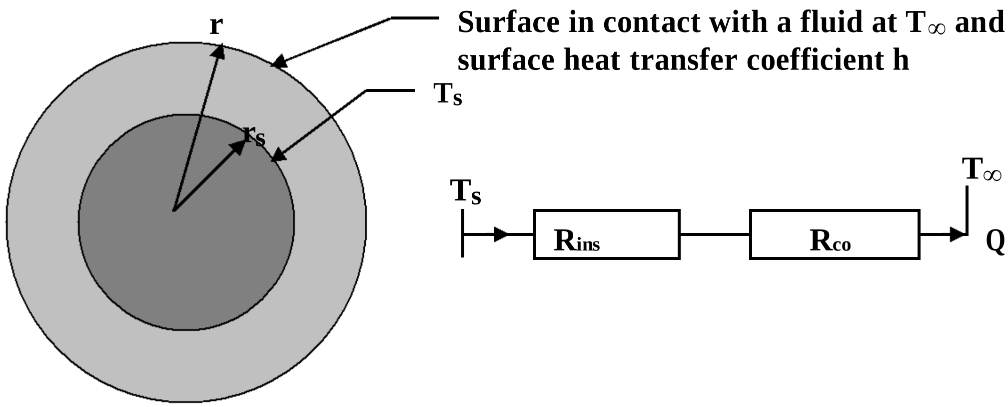


Fig.P3.14: Schematic of an electric wire covered with an insulation and exposed to an ambient and the corresponding thermal circuit

Hence $R_{total} = R_{ins} + R_{co} = 0.0735 + 0.3316 = 0.405$ K / W.

Now $Q = (T_S - T_\infty) / R_{total}$.

Hence $T_S = T_\infty + Q\,R_{total} = 30 + 80 \times 0.405 = 62.4\ ^0C$

(ii) Critical radius of insulation = $r_{cr} = k / h = 0.15 / 24 = 0.00625$.

Since $r_{cr} > r$, increasing the thickness of plastic insulation will increase the heat transfer rate if Ts is held constant or for a given heat transfer rate the interface temperature Ts will decrease till the critical radius is reached. Now when the thickness is doubled then r = 3 mm = 0.003 m . Therefore

$$R_{ins} = \frac{\ln(0.003 / 0.001)}{2 \times \pi \times 10 \times} = 0.1166 \text{ K / W}$$

$$R_{co} = \frac{1}{2 \times \pi \times 10 \times 0.003 \times 24} = 0.221 \text{ K / W}$$

Therefore $R_{total} = 0.1166 + 0.221 = 0.3376$ K / W.

and $T_S = 30 + 80 \times 0.3376 = 57\ ^0C$

3.5. Extended Surfaces (Fins):-

Solution to tutorial problems:

Example 3.15:- A steel rod of diameter 2 cm and thermal conductivity 50 W/(m – K) is exposed to ambient air at 20 ^{0}C with a heat transfer coefficient 64 W/(m^2 – K). One end of the rod is maintained at a uniform temperature of 120 ^{0}C. Determine the rate of heat from the rod to the ambient and the temperature of the tip of the rod exposed to ambient if (i) the rod is very long, (ii) rod is of length 10 cm with negligible heat loss from its tip, (ii) rod is of length 25 cm with heat loss from its tip.

Solution: (i) Given:- D = 0.02 m ; k = 50 W/(m-K); T_∞ = 20 ^{0}C; T_o = 120 ^{0}C; h = 64 W/(m^2-K); Very long fin ($x \rightarrow \infty$)

$$m = \sqrt{[(hP) / (kA)]} = \sqrt{[(h\pi D / (\pi D^2/4)]} = \sqrt{[(4h) / (kD)]} = \sqrt{\left[\frac{4 \times 64}{50 \times 0.02}\right]} = 16$$

For a very long fin the rate of heat transfer is given by

$$Q = kmA(T_0 - T_\infty) = 50 \times 16 \times (\pi / 4) \times 0.02^2 \times [120 - 20] = 25.13 \text{ W}$$

(ii) L = 0.10 m. Hence mL = 16 x 0.1 = 1.6

$$Q = kmA(T_0 - T_\infty) \tanh mL = 50 \times 16 \times (\pi / 4) \times 0.02^2 \times (120 - 20) \times \tanh 1.6 = 23.16 \text{ W}$$

(iii) When the heat loss from the rod tip is not negligible, then we can use the same formula as in case (ii) with modified length L_e given by

$$L_e = L + A /P = L + (\pi D^2/4) /(\pi D) = L + D / 4 = 0.1 + 0.02/4 = 0.105$$

Hence mL_e = 16 x 0.105 = 1.68 and tanh mL_e = tanh 1.68 = 0.9329 Hence Q = 25.13 x 0.933 = 23.44 W

Example 3.16:-A thin rod of uniform cross section A, length L and thermal conductivity k is thermally attached from its ends to two walls which are maintained at temperatures T_1 and T_2. The rod is dissipating heat from its lateral surface to an ambient at temperature T_∞ with a surface heat transfer coefficient h.

(a) Obtain an expression for the temperature distribution along the length of the rod
(b) Also obtain an expression for the heat dissipation from the rod to the ambient

Solution: The general solution for the one-dimensional steady-state temperature distribution along the length of a rod dissipating heat by convection from its lateral surface is given by

$$\theta(x) = C_1 \cosh mx + C_2 \sinh mx \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(1)$$

where $\theta(x) = T(x) - T_\infty$; $m = \sqrt{(hP) / (kA)}$:
P = perimeter of the rod = πD and A = Area of cross section of the rod = $\pi D^2 / 4$.

The boundary conditions are: (i) at $x = 0$, $T = T_1$ or $\theta = T_1 - T_\infty = \theta_o$ (say).

(ii) at $x = L$, $T = T_2$ or $\theta = T_2 - T_\infty = \theta_L$ (say).

Condition(i) in Eq. (1) gives $\theta_o = C_1$.

Condition (ii) in Eq. (1) gives $\theta_L = \theta_0 \cosh mL + C_2 \sinh mL$

$$C_2 = \frac{(\theta_L - \theta_o \cosh mL)}{\sinh mL} .$$

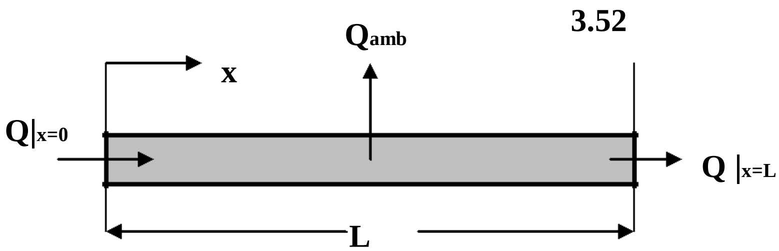
Substituting for C_1 and C_2 in Eq. (1) we have

$$\theta(x) = \theta_o \cosh mx + \frac{(\theta_L - \theta_o \cosh mL)}{\sinh mL} \sinh mx$$

$$\text{0r } \theta(x) = \frac{\theta_o \cosh mx \sinh mL + \theta_L \sinh mx - \theta_o \cosh mL \sinh mx}{\sinh mL}$$

$$\text{0r } \theta(x) = \frac{\theta_L \sinh mx + \theta_o \sinh m(L - x)}{\sinh mL} \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(2)$$

Expression for the rate of heat dissipation from the rod:



Energy balance for the rod is given by

$$Q_{amb} = Q|_{x=0} - Q|_{x=L}$$

$$= -kA\,(d\theta / dx)|_{x=0} + kA\,(d\theta / dx)|_{x=L} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

From Eq. (2) we have $$(d\theta / dx) = \frac{-m\,[\theta_L \cosh mx + \theta_0 \cosh m(L-x)]}{\sinh mL}$$

Therefore $$(d\theta / dx)|_{x=0} = \frac{-m\,[\theta_L + \theta_0 \cosh mL]}{\sinh mL}$$

and $$(d\theta / dx)|_{x=L} = \frac{-m\,[\theta_L \cosh mL + \theta_0]}{\sinh mL}$$

Hence $$Q_{amb} = \frac{kmA\,[\theta_L + \theta_0 \cosh mL - \theta_L \cosh mL - \theta_0\,]}{\sinh mL}$$

$$= \frac{kmA\,[(\theta_L - \theta_0) - (\theta_L - \theta_0) \cosh mL\,]}{\sinh mL}$$

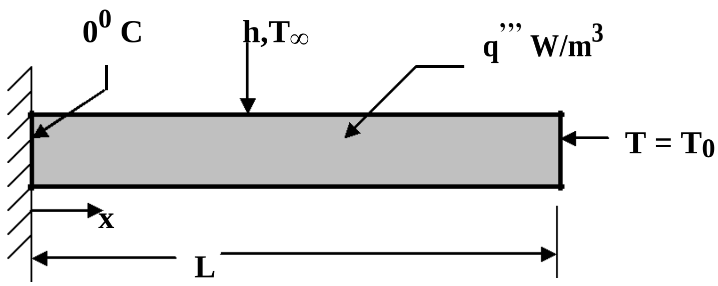
or $$Q_{amb} = \frac{kmA(\theta_L - \theta_0)\,(1 - \cosh mL)}{\sinh mL}$$

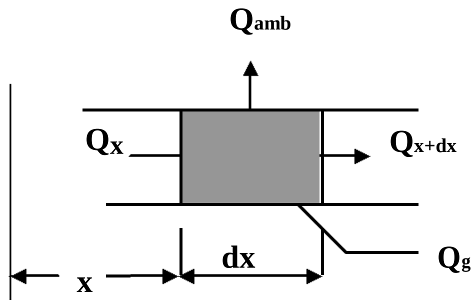
Example 3.17:-Heat is generated at a constant rate of q''' W/m^3 in a thin circular rod of length L and diameter D by the passage of electric current. The two ends of the rod are maintained at uniform temperatures with one end at temperature T_0 and the other end at 0 ^{0}C, while heat is being dissipated from the lateral surface of the rod to an ambient at 0 ^{0}C with a surface heat transfer coefficient h.

(a) Derive the one-dimensional steady state energy equation to determine the temperature distribution along the length of the rod

(b) Solve the above equation and obtain the temperature distribution.

Solution: Since the rod is generating heat and dissipating heat to the ambient, the governing differential equation to determine the one-dimensional steady state temperature distribution has to be obtained from first principles as illustrated below.





Consider an elemental length „dx"of the rod as shown in the figure above. The various energies crossing the boundaries of the rod as well as the energy generated are also shown in the figure. For steady state condition the energy balance equation for the rod element can be written as

$$Q_x + Q_g = Q_{x+dx} + Q_{amb}$$

Or $$Q_x + Q_g = Q_x + (dQ_x/dx)\, dx + Q_{amb}$$

Or $$(dQ_x/dx)\, dx + Q_{amb} = Q_g$$

$$d/dx(-kA\, dT/dx)\, dx + hPdx\,(T - T_\infty) = Adx\, q'''$$

Or $$(d^2T / dx^2) - (hP / kA)(T - T_\infty) = -(q''' / k)$$

Let $T - T_\infty = \theta$ and $(hP / kA) = m^2$. then the above equation reduces to

$$(d^2 \theta / d x^2) - m^2 \theta = -(q''' / k) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Eq.(1) is a non-homogeneous linear second order ordinary differential equation whose solution can be written as

$$\theta(x) = \theta_h(x) + \theta_p(x) \text{ ------------------------------------ } (2)$$

where $\theta_h(x)$ satisfies the homogeneous part of the differential equation namely

$$(d^2 \theta_h / d x^2) - m^2 \theta_h = 0 \text{ ------------------------------------ } (3)$$

and $\theta_p(x)$ is the particular integral which satisfies Eq. (1). Solution to Eq.(3) is given by

$$\theta(x) = C_1 e^{mx} + C_2 e^{-mx} \text{ ---------------------------- } (4)$$

To find $\theta_p(x)$:- Since the RHS of Eq.(1) is a constant let us assume $\theta_p(x) = B$, where B is a constant. Substituting this solution in Eq.(1) we have

$$0 - m^2 B = -(q''' / k)$$

Or $$B = (q''' / km^2)$$

Therefore the complete solution for Eq. (1) can be written as

$$\theta(x) = C_1 e^{mx} + C_2 e^{-mx} + (q''' / km^2)$$

Or $T(x) = T_\infty + C_1 e^{mx} + C_2 e$ Boundary $^{-mx} + (q''' / km^2)$………………..(5)

conditions are: (i) at $x = 0$, $T = 0$

(ii) at $x = L$, $T = T_0$

Condition (i) in Eq. (5) gives

$$0 = T_\infty + C_1 + C_2 + (q''' / km^2)$$

Or $$C_1 + C_2 = -T_\infty - (q'''/km^2) \text{ ------------------------------- } (a)$$

Condition(ii) in Eq.(5) gives $T_0 = T_\infty + C_1 e^{mL} + C_2 e^{-mL} + (q'''/km^2)$

Or $$C_1 e^{mL} + C_2 e^{-mL} = T_0 - T_\infty - (q'''/km^2) \text{ ----------------------------- } (b)$$

From Eq.(a) $C_2 = -C_1 - T_\infty - (q'''/km^2)$. Substituting this expression in Eq.(b) we

have $C_1 e^{mL} - [C_1 + T_\infty + (q'''/km^2)] e^{-mL} = T_0 - T_\infty - (q'''/km^2)$

Solving for C_1 we get $C_1 = \dfrac{T_0 - \{T_\infty + (q'''/km^2)\}\{1 - e^{-mL}\}}{\{e^{mL} - e^{-mL}\}}$

$$C_2 = -\{T_\infty + (q'''/km^2)\} - \frac{T_0 - \{T_\infty + (q'''/km^2)\}\{1 - e^{-mL}\}}{\{e^{mL} - e^{-mL}\}}$$

$$C_2 = \frac{-\{T_\infty + (q'''/km^2)\}\{e^{mL} - e^{-mL}\} - T_0 + \{T_\infty + (q'''/km^2)\}\{1 - e^{-mL}\}}{\{e^{mL} - e^{-mL}\}}$$

$$C_2 = \frac{\{T_\infty + (q'''/km^2)\}[-e^{mL} + e^{-mL} + 1 - e^{-mL}] - T_0}{\{e^{mL} - e^{-mL}\}}$$

$$C_2 = \frac{\{T_\infty + (q'''/km^2)\}[1 - e^{mL}] - T_0}{\{e^{mL} - e^{-mL}\}}$$

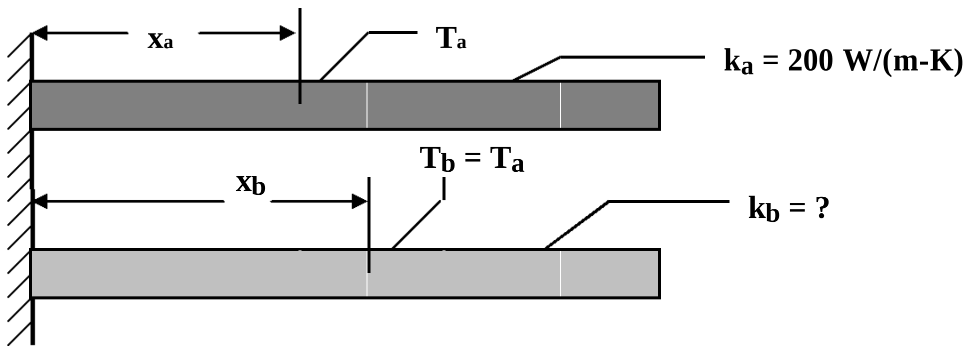
Substituting the expressions for C_1 and C_2 in Eq. (5) and simplifying we get

$$T(x) = T_\infty + (q'''/km^2) + \frac{[T_0 - \{T_\infty + (q'''/km^2)\}\{1 - e^{-mL}\}]\, e^{mx}}{\{e^{mL} - e^{-mL}\}} + \frac{[\{T_\infty + (q'''/km^2)\}[1 - e^{mL}] - T_0]\, e^{-mx}}{\{e^{mL} - e^{-mL}\}}$$

$$T(x) = T_\infty + (q'''/km^2) + \frac{T_0(e^{mx} - e^{-mx})}{\{e^{mL} - e^{-mL}\}} + \frac{[-\{T_\infty + (q'''/km^2)\}\{1 - e^{-mL}\}]\, e^{mx} + \{T_\infty + (q'''/km^2)\}[1 - e^{mL}]\, e^{-mx}]}{\{e^{mL} - e^{-mL}\}}$$

Example 3.18:- Two very long slender rods of the same diameter are given. One rod is of aluminum (k = 200 W/(m-K)). The thermal conductivity of the other rod is not known. To determine this, one end of each rod is thermally attached to a metal surface maintained at a uniform temperature T_0. Both rods are losing heat to the ambient air at T_∞ by convection with a surface heat transfer coefficient h. The surface temperature of each rod is measured at various distances from hot base surface. The temperature of the aluminum rod at 40 cm from the base is same as that of the rod of unknown thermal conductivity at 20 cm from the base. Determine the unknown thermal conductivity.

Solution:



For very long slender rods the steady-state one-dimensional temperature distribution along the length of the rod is given by

$$\theta(x) = \theta_0 e^{-mx}$$

……………………………..(1) where $\theta(x) = T(x) - T_\infty$ and $\theta_0 = T_0 - T_\infty$.

For rod A Eq.(1) can be written as $\theta_a(x) = \theta_0 e^{-m_a x_a}$ …………………………...(2)

And for rod B it can be written as $\theta_b(x) = \theta_0 e^{-m_b x_b}$ ……………………………(3)

It is given that when $x_a = 0.4$ m and $x_b = 0.2$ m, $\theta_a(x_a) = \theta_b(x_b)$

Therefore we have $\theta_0 e^{-0.4 m_a} = \theta_0 e^{-0.2 m_b}$

Or $m_b = 2 m_a$

Or $\sqrt{[(hP_b)/(k_b A_b)]} = 2\sqrt{[(hP_a)/(k_a A_a)]}$

Since $P_a = P_b$ and $A_a = A_b$, we have $\sqrt{k_a} = 2\sqrt{k_b}$ or $k_a = 4 k_b$

Therefore $k_b = 200/4 = 50$ W/(m-K).

Example 3.19:- Show that for a finned surface the total heat transfer rate is given by

$$Q_{total} = [\eta\ \beta + (1 - \beta)]\ a\ h\ \theta_0 = \acute{\eta}\ a\ h\ \theta_0$$

Where η = fin efficiency ; $\beta = a_f / a$: a_f = surface area of the fin, a = total heat transfer area (i.e. finned surface + unfinned surface) ; $\theta_0 = T_0 - T_\infty$, with T_0 = fin base temperature and T_∞ = ambient temperature, and $\acute{\eta}$ = area – weighted fin efficiency.

Solution:

$$Q_{total} = Q_{fin} + Q_{bare}$$

Where Q_{total} = Total heat transfer rate, Q_{fin} = Heat transfer rate from the finned surface and Q_{bare} = Heat transfer rate from the bare surface.

Therefore $Q_{total} = \eta\ h\ a_f\ \theta_0 + h(a - a_f)\ \theta_0$

$= h\ a\ \theta_0\ [(\eta\ a_f) / a + (1 - a_f/a)]$

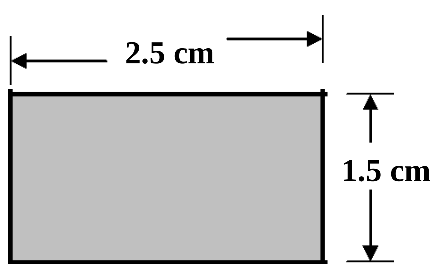
here, $\beta = a_f/a$

$= ha\ \theta_0\ [\eta\beta + (1 - \beta)]$

$= \acute{\eta}\ ha\ \theta_0$, where $\acute{\eta} = [\eta\beta + (1 - \beta)]$

Example 3.20:- The handle of a ladle used for pouring molten lead at 327 ^{0} C is 30 cm long and is made of 2.5 cm x 1.5 cm mild steel bar stock (k = 43 W/(m-K)). In order to reduce the grip temperature, it is proposed to make a hollow handle of mild steel plate 1.5 mm thick to the same rectangular shape. If the surface heat transfer coefficient is 14.5 W/(m^2-K) and the ambient temperature is 27 ^{0}C, estimate the reduction in the temperature of the grip. Neglect the heat transfer from the inner surface of the hollow shape.

Solution: (a) When the handle is made of solid steel bar:



Cross section of the handle

h = 14.5 W/(m^2-K) ;

k = 43 W/(m-K)

$\theta_0 = 327 - 27 = 300\ ^0$ C

Area of cross section of the bar = A = 2.5 x 1.5 x 10^{-4} m^2 = 3.75 x 10^{-4} m^2 Perimeter of the bar = P = 2 [2.5 + 1.5] x 10^{-2} m = 8 x 10^{-2} m

$$\text{Therefore } m = \frac{(hP)^{(1/2)}}{(kA)^{(1/2)}} = \frac{\sqrt{[14.5 \times 8 \times 10^{-2}]}}{\sqrt{[43 \times 3.75 \times 10^{-4}]}} = 8.48 \text{ (1/m)}$$

Therefore $mL = 8.48 \times 0.3 = 2.54$.

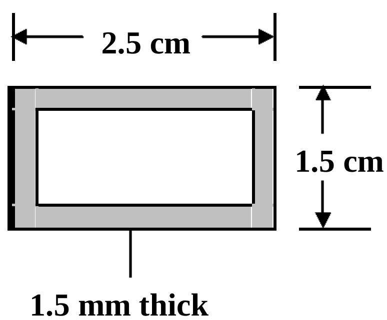
When the heat loss from the tip of the handle is neglected the temperature at any point along the length of the handle is given by

$$\theta(x) = \theta_0 \frac{\cosh m(L - x)}{\cosh mL}$$

Therefore $\theta(x)|_{x=L} = \theta_0 / \cosh mL = 300 / \cosh 2.54 = 47\ {}^0\text{C}$.

Or $T(x)|_{x=L} = 47 + 27 = 74\ {}^0\text{C}$.

(b) When the handle is hollow made out of a sheet:



Area of the cross section of the fin is

$A = [(2.5 \times 1.5) – (2.5 – 0.3) \times (1.5 – 0.3)]$

$= 1.11 \text{ cm}^2 = 1.11 \times 10^{-4} \text{ m}^2$

$P = 2 \times [\ 2.5 + 1.5\ ] = 8 \text{ cm} = 8 \times 10^{-2} \text{ m}$

$$m = \sqrt{(hP) / (kA)} = \frac{\sqrt{(14.5 \times 8 \times 10^{-2})}}{\sqrt{(43 \times 1.11 \times 10^{-4})}}$$

Or $m = 15.59$ 1/m. Therefore $mL = 15.59 \times 0.3 = 4.68$

$$\theta(x)|_{x=L} = \frac{\theta_0}{\cosh mL} = \frac{(327 – 27)}{\cosh 4.68} = 5.57\ {}^0\text{C}.$$

Therefore $T(x)|_{x=L} = 5.57 + 27 = 32.57\ {}^0\text{C}$.

Reduction in grip temperature $= 74 – 32.57 = 41.43\ {}^0\text{C}$.

Example 3.21:- Derive an expression for the overall heat transfer coefficient across a plane wall of thickness 'b' and thermal conductivity 'k' having rectangular fins on both sides. Given that over an overall area A of the wall, the bare area on both sides, not covered by the fins are A_{u1} and A_{u2}, the fin efficiencies are η_1 and η_2, and the heat transfer coefficients h_1 and h_2.

Solution:

Let T_i be the temperature of the fluid in contact with the surface 1, T_0 be the temperature of the fluid in contact with surface 2, T_1 be the temperature of surface 1 and T_2 be the temperature of surface 2.Let $T_i > T_0$. Then the rate of heat transfer from T_i to T_0 is given by

$$Q = Q_{bare} + Q_{fin}$$
$$= h_i A_{u1} (T_i - T_1) + h_i \eta_1 A_{f1}(T_i - T_1)$$

Or
$$Q = \frac{(T_i - T_1)}{(1/h_1 A_{u1})} - \frac{(T_i - T_1)}{(1/h_1 \eta_1 A_{f1})}$$

$$Q = \frac{(T_i - T_1)}{[(1/h_1 A_{u1}) + (1/h_1 \eta_1 A_{f1})]} \quad \ldots\ldots(1)$$

$$Q = \frac{(T_2 - T_0)}{[(1/h_2 A_{u2}) + (1/h_2 \eta_2 A_{f2})]} \quad \ldots\ldots(2)$$

Rate of heat transfer is also given by

$$Q = \frac{(T_1 - T_2)}{(b/Ak)} \quad \ldots\ldots(3)$$

Therefore as A/B = C/D = E/F = (A+C+E)/(B+D+F) →

$$Q = \frac{(T_i - T_1) + (T_1 - T_2) + (T_2 - T_0)}{[(1/h_1 A_{u1}) + (1/h_1 \eta_1 A_{f1}) + (1/h_1 A_{u1}) + (1/h_1 \eta_1 A_{f1}) + (b/Ak)]}$$

$$Q = \frac{(T_i - T_0)}{[(1/h_1 A_{u1}) + (1/h_1 \eta_1 A_{f1}) + (1/h_1 A_{u1}) + (1/h_1 \eta_1 A_{f1}) + (b/Ak)]} \quad \ldots\ldots(4)$$

If U = overall heat transfer coefficient for the plane wall then

$$Q = UA(T_i - T_0)$$

$$= \frac{(T_i - T_0)}{} \quad \ldots\ldots(5)$$

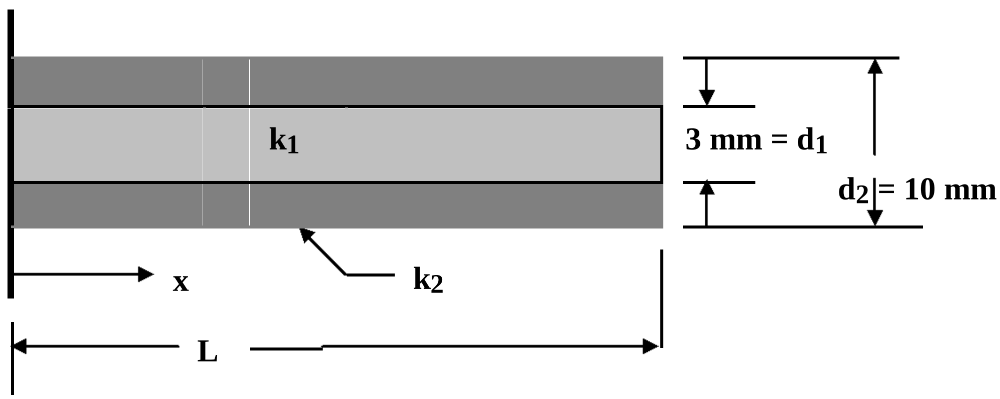
$$(1/UA)$$

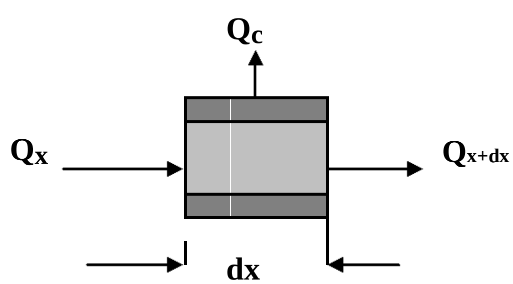
From Eqs. (4) and (5) we have

$$U = \frac{1}{A\,[(1/h_1A_{u1}) + (1/h_1\eta_1A_{f1}) + (1/h_1A_{u1}) + (1/h_1\eta_1A_{f1}) + (b/Ak)]}$$

Example 3.22:- Calculate the effectiveness of the composite pin fin shown in Fig.P3.22. Assume k_1 = 15 W/(m-K), k_2 = 50 W/(m-K) and h = 12 W/(m^2 – K).

Solution:





K_1=15 W/m-k , K_2=50W/m-k, K_3=12 W/m-k.

(b) Energy transfer across the surfaces of the fin element

Energy balance equation for the fin element is given by

$$Q_x = Q_{x+dx} + Q_c$$

$$= Q_x + (dQ_x/dx)\,dx + Q_c$$

Or $\quad dQ_x / dx + Q_c = 0$ ……………………………………….(1)

Q_x consists of two components namely the heat transfer Q_{x1} through the material of thermal conductivity k_1 and the rate of heat transfer Q_{x2} through the material of conductivity k_2.

Therefore $\quad Q_x = Q_{x1} + Q_{x2} = -\,k_1A_1\,(dT / dx) - k_2A_2\,(dT / dx)$

$$= -\,(k_1A_1 + k_2A_2)\,(dT / dx)$$

And $\quad Q_c = (hP_2\,dx)\,(T - T_\infty).$

Substituting these expressions for Q_X and Q_C in equation (1) we get

$$(d^2T / dx^2) - \frac{hP_2}{(k_1A_1 + k_2A_2)} (T - T_\infty) = 0$$

Or $$(d^2\theta / dx^2) - m^2 \theta = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Where $\theta = T - T\infty$ and $m = \sqrt{[hP_2 / (k_1A_1 + k_2A_2)]}$.

When the heat loss from the fin tip is negligible , the solution to equation (2) is given by

$$\theta(x) = \theta_0 \frac{\cosh [m(L - x)]}{\cosh mL} \quad \ldots\ldots\ldots..(3)$$

The rate of heat transfer from the fin base is given by

$$Q_x|_{x=0} = - (k_1A_1 + k_2A_2) (d\theta / dx)|_{x=0}$$

$$= \frac{- (k_1A_1 + k_2A_2) \sinh [m(L - x)]_{x=0} (- m) \theta_0}{\cosh mL}$$

$$= m\theta_0 (k_1A_1 + k_2A_2) \tanh mL$$

Now $$\eta = Q_x|_{x=0} / Q_{max}$$

$$= \frac{m\theta_0 (k_1A_1 + k_2A_2) \tanh mL}{hP_2L \theta_0}$$

Noting that $hP_2 / (k_1A_1 + k_2A_2) = m^2$, the above expression for η simplifiers to

$$\eta = \frac{\tanh mL}{mL} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

In the given problem $A_1 = (\pi / 4) \times (0.003)^2 = 7.1 \times 10^{-6}\ m^2$.

$$A_2 = (\pi / 4) \times [(0.01)^2 - (0.003)^2] = 7.15 \times 10^{-5}$$

$$P_2 = \pi \times 0.01\ 0.0314\ m.$$

$$m = \frac{\sqrt{[12 \times 0.0314]}}{\sqrt{[(15 \times 7.1 \times 10^{-6}) + (50 \times 7.15 \times 10^{-5})]}} = 10.12$$

Therefore $mL = 10.12 \times 0.1 = 1.012$

$$\eta = \frac{\tanh(1.012)}{1.012} = 0.757$$

Example 3.23:- Why is it necessary to derive a fresh differential equation for determining the one-dimensional steady state temperature distribution along the length of a fin?

Solution:- While deriving the conduction equation in differential form we will have considered a differential volume element within the solid so that the heat transfer across the boundary surfaces of the element is purely by conduction. But in the case of a fin the lateral surface is exposed to an ambient so that the heat transfer across the lateral surfaces is by convection. Therefore we have to derive the differential equation afresh taking into account the heat transfer by convection across the lateral surfaces of the fin.

Solutions to Problems on Conduction in solids with variable thermal conductivity

Example 3.24:- A plane wall 4 cm thick has one of its surfaces in contact with a fluid at 130 ^{0}C with a surface heat transfer coefficient of 250 W/(m^2 – K) and the other surface is in contact with another fluid at 30 ^{0}C with a surface heat transfer coefficient of 500 W/(m^2-K). The thermal conductivity of the wall varies with temperature according to the law

$$k = 20\,[\,1 + 0.001\,T]$$

where T is the temperature. Determine the rate of heat transfer through the wall and the surface temperatures of the wall.

Given:- L = 0.04 m; T_i = 130 ^{0}C; h_i = 250 W/(m^2-k); T_o = 30 ^{0}C; h_o = 500 W/(m^2-K); k = 20 [1 + 0.001 T].

To find:- (i) Q_x (ii) T_1 and T_2

Solution:

R_{ci} = Thermal resistance for convection at the surface at T_i = 1/(h_iA) = 1 / (250 x 1) = 0.004 m^2 – K /W

R_{co} = Thermal resistance for convection at the surface at T_o = 1/(h_oA) = 1/(500 x 1)

Or $R_{co} = 0.002\ m^2$-K/W

Now $Q = (T_i - T_1) / R_{ci}$, where T_1 = Surface temperature in contact with fluid at T_i.

Hence $T_1 = T_i - QR_{ci} = 130 - 0.004\ Q$ ……………………………………………(1)

Similarly $Q = (T_2 - T_o) / R_{co}$

Or $T_2 = T_o + QR_{co} = 30 + 0.002Q$ ……………………………………………(2)

From equations (1) and (2) we have

$T_1 - T_2 = 100 - 0.006Q$ …………………………………………………………(3)

And $T_m = (T_1 + T_2) / 2 = 80 - 0.001Q$ ...………………………………………(4)

Hence $k_m = k_o [1 + \beta T_m] = 20 \times [1 + 0.001 \times \{80 - 0.001Q\}]$

$$= 21.6 - 2 \times 10^{-5}Q$$

Hence thermal resistance offered by the wall = $R = L/(Ak_m)$

Or $$R = \frac{0.04}{[21.6 - 2 \times 10^{-5}Q]}$$

$$Q = \frac{[T_1 - T_2]}{R} = \frac{[100 - 0.006Q]}{0.04} \times \frac{[21.6 - 2 \times 10^{-5}Q]}{}$$

Cross multiplying we have

$$0.04Q = 2160 - 0.1316Q + 1.2 \times 10^{-7} Q^2$$

Or $Q^2 - 1.41 \times 10^6 Q + 1.8 \times 10^{10} = 0$.Hence $Q = (1.41 \times 10^6 \pm 1.39 \times 10^6) / 2$

For physically meaningful solution T_1 should lie between T_i and T_o. This is possible only If

$$Q = (1.41 \times 10^6 - 1.39 \times 10^6) / 2 = 10000\ \text{W}.$$

Now $T_1 = T_i - QR_{ci} = 130 - 10000 \times 0.004 = 90\ ^0C$

and $T_2 = T_0 + Q\ R_{co} = 30 + 10000 \times 0.002 = 50\ ^0C$.

Example 3.25:- The thermal conductivity of a plane wall varies with temperature according to the equation

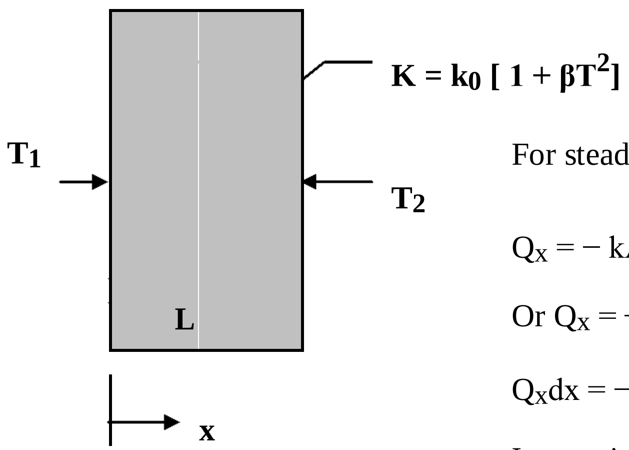
$$\mathbf{k(T) = k_0\,[\,1 + \beta\, T^2\,]}$$

where k_0 and β are constants.

(a) Develop an expression for the heat transfer through the wall per unit area of the wall if the two surfaces are maintained at temperatures T_1 and T_2 and the thickness of the wall is L.

(b) Develop a relation for the thermal resistance of the wall if the heat transfer area is A.

Solution:



For steady state conduction we have

$$Q_x = -\,kA(dT / dx) = \text{constant.}$$

Or $Q_x = -\,k_0[1 + \beta T^2]A(dT/dx)$

$$Q_x dx = -\,k_0[1 + \beta T^2]A\, dT$$

Integrating the above equation between x = 0 and x = L we have

$$\int_0 Q_x dx = -\,k_0 A \int_{T_1}^{T_2} [1 + \beta T^2]\, dT$$

Or

$$Q_x L = -\,k_0 A\,[(T_2 - T_1) + (\beta/3)(T_2^3 - T_1^3)]$$

Or

$$Q_x = (k_0 A / L)(T_1 - T_2)\,[1 + (\beta/3)(T_1^2 + T_1 T_2 + T_2^2)]$$

$$Q_x = \frac{(T_1 - T_2)}{\dfrac{1}{(k_0 A/L)\,[1 + (\beta/3)(T_1^2 + T_1 T_2 + T_2^2)]}}$$

Therefore thermal resistance of the wall is given by

$$R = \frac{1}{(k_0 A/L)\,[1 + (\beta/3)(T_1^2 + T_1 T_2 + T_2^2)]}$$

Transient Conduction

<u>4.1.Introduction</u>:- In general, the temperature of a body varies with time as well as position. In chapter 3 we have discussed conduction in solids under steady state conditions for which the temperature at any location in the body do not vary with time. But there are many practical situations where in the surface temperature of the body is suddenly altered or the surface may be subjected to a prescribed heat flux all of a sudden. Under such circumstances the temperature at any location within the body varies with time until steady state conditions are reached. In this chapter, we take into account the variation of temperature with time as well as with position. However there are many practical applications where in the temperature variation with respect to the location in the body at any instant of time is negligible. The analysis of such heat transfer problems is called the *"lumped system analysis"*. Therefore in lumped system analysis we assume that the temperature of the body is a function of time only.

<u>4.2. Lumped system analysis</u>:- Consider a solid of volume V, surface area A, density ρ, Specific heat C_p and thermal conductivity k be initially at a uniform temperature T_i.Suddenly let the body be immersed in a fluid which is maintained at a uniform temperature T_∞, which is different from T_i.The problem is illustrated in Fig.4.1.Now if

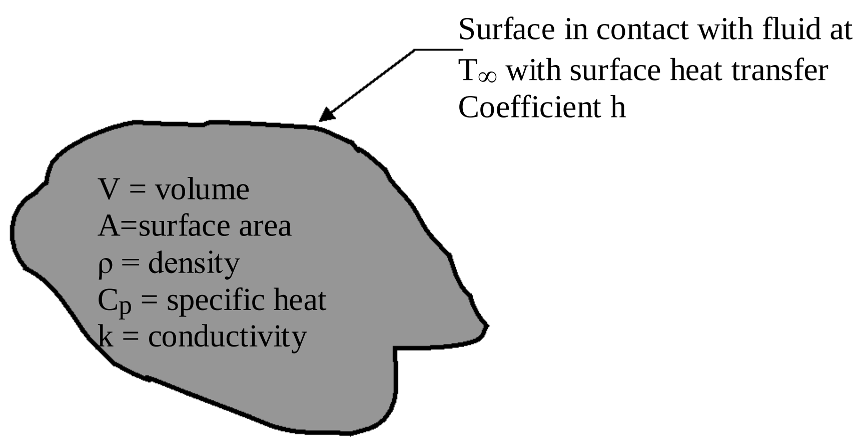


Fig.4.1: Nomenclature for lumped system analysis of transient Conduction heat transfer

T(t) is the temperature of the solid at any time t, then the energy balance equation for the solid at time t can be written as

Rate of increase of energy of the solid = Rate of heat transfer from the fluid to the solid

i.e., $$\rho V C_p \,(dT / dt) = hA[T_\infty - T(t)]$$

Or $$dT / dt = \frac{h\,A}{\rho\, V\, C_p} [T_\infty - T(t)]$$

For convenience, a new temperature $\theta(t) = T(t) - T_\infty$ is defined and denoting $m = (hA)/(\rho V C_p)$ the above equation can be written as

$$(d\theta / dt) = -m\,\theta \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4.1)$$

Eq.(4.1) is a first order linear differential equation and can be solved by separating the variables. Thus

$$d\theta / \theta = -m\,dt$$

Integrating we get $\ln \theta = -mt + \ln C$, where $\ln C$ is a constant.

Or $$\theta = C\,e^{-mt} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4.2)$$

At time $t = 0$, $T(t) = T_i$ or $\theta = T_i - T_\infty = \theta_i$ (say). Substituting this condition in Eq. (4.2) we get

$$C = \theta_i.$$

Substituting this value of C in eq. (4.2) we get the temperature $\theta(t)$ as follows.

$$\theta(t) = \theta_i\,e^{-mt}$$

or

$$\frac{\theta(t)}{\theta_i} = e^{-mt} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4.3)$$

Since LHS of Eq.(4.3) is dimensionless, it follows that 1/m has the dimension of time and is called the time constant.Fig. 4.2 shows the plot of Eq.(4.3) for different values of m. Two observations can be made from this figure and Eq. (4.3).

1. Eq. (4.3) can be used to determine the temperature T(t) of the solid at any time t or to determine the time required by the solid to reach a specified temperature.

2. The plot shows that as the value of m increases the solid approaches the surroundings temperature in a shorter time. That is any increase in m will cause the solid to respond more quickly to approach the surroundings temperature.

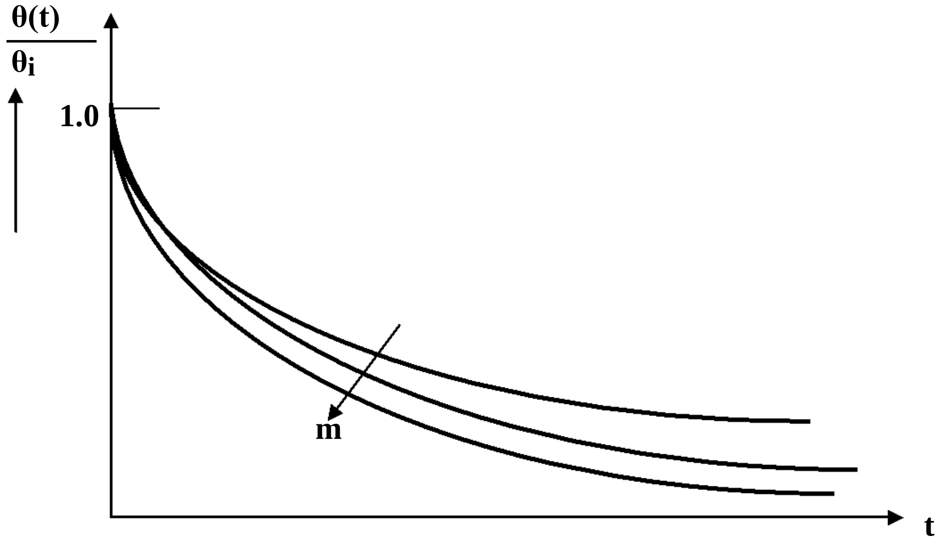


Fig.4.2: Dimensionless temperature as a function of time for a solid with negligible internal temperature gradients

The definition of m reveals that increasing the surface area for a given volume and the heat transfer coefficient will increase m. Increasing the density, specific heat or volume decreases m.

4.2.2. Criteria for Lumped System Analysis:- To establish a criterion to neglect internal temperature gradient of the solid so that lumped system analysis becomes applicable, a *Characteristic length* L_S is defined as

$$L_S = V / A \quad \ldots\ldots(4.4)$$

and the Biot. number Bi as

$$Bi = \frac{h\, L_S}{k} \quad \ldots\ldots(4.5)$$

For solids like slabs, infinite cylinder, and sphere, it has been found that the error by neglecting internal temperature gradients is less than 5 %, if

$$Bi < 0.1 \quad \ldots\ldots(4.6)$$

The physical significance of Biot number can be understood better by writing the expression for Biot number as follows

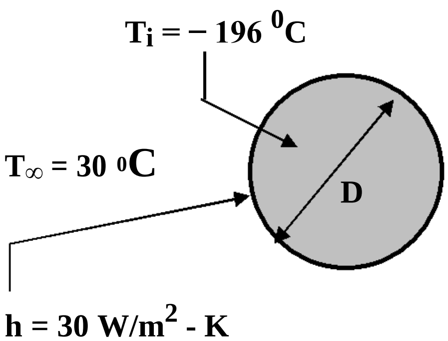
$$Bi = \frac{h L_s}{k} = \frac{(L_s / Ak)}{(1 / hA)} = \frac{\text{Thermal resistance for conduction}}{\text{Thermal resistance for convection}}$$

Hence a very low value of Biot number indicates that resistance for heat transfer by conduction within the solid is much less than that for heat transfer by convection and therefore a small temperature gradient within the body could be neglected.

4.2.3.Illustrative examples on lumped system analysis

Example 4.1: - A copper cylinder 10 cm diameter and 15 cm long is removed from a liquid nitrogen bath at – 196 ^{0} C and exposed to room temperature at 30 ^{0} C. Neglecting internal temperature gradients find the time taken by the cylinder to attain a temperature of 0 ^{0}C, with the following assumptions:
Surface heat transfer coefficient = 30 W / m^2 – K.
Density of the copper cylinder = 8800 kg / m^3.
Specific heat of the cylinder = 0.38 kJ/(kg-K) Thermal conductivity of the cylinder = 350 W / (m-K).

Solution: :



Other data:- D = 10 cm or R = 0.05 m; L = 0.15 m

k = 350 W / (m-K) ; ρ = 8800 kg / m^3 ;

c_p = 0.38 kJ / (kg-K) ; T(t) = 0

Let $\theta(t) = T(t) - T_\infty$

Biot Number = hR / k = 30 x 0.05 / 350 = 0.0043 which is << 0.1. Hence internal temperature gradients can be neglected. In that case we have
$\theta(t) = T(t) - T_i = \theta_0\, e^{-(hA/\rho Vcp)\, t}$, where $\theta_0 = T_i - T_\infty$

$$(hA/\rho Vc_p) = \frac{2\{\pi R^2 + \pi RL)h}{\pi R^2 L\, \rho c_p} = \frac{2\{R+L\}h}{\rho c_p RL} = \frac{2 \text{ x } \{0.05 + 0.15\} \text{ x} 30}{8800 \text{ x } 0.38 \text{ x } 1000 \text{ x } 0.05 \text{x } 0.15}$$

$$= 4.785 \text{ x } 10^{-4} \quad 1 / s$$

Now
$$\frac{T(t) - T_\infty}{T_i - T_\infty} = e^{-(hA/\rho Vcp)\, t}$$

Hence $$\frac{0 - 30}{-196 - 30} = \exp(-4.785 \times 10^{-4} \times t)$$

Solving for t we get t = 4226 s = 1 hr 10.43 mins.

Example 4.2:- A thin copper wire having a diameter D and length L (insulated at the ends) is initially at a uniform temperature of T_0. Suddenly it is exposed to a gas stream, the temperature of which changes with time according to the equation

$$\mathbf{T_g = T_f\,(1 - e^{-ct}) + T_0}$$

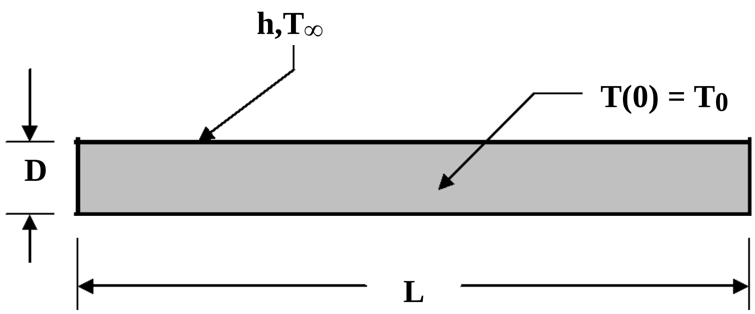
where T_f, T_0 and c are constants. The surface heat transfer coefficient is h. Obtain an expression for the temperature of the wire as a function of time t.

Solution:

Let T(t) be the temperature of the cylinder at any time t. Energy balance for the cylinder for a time interval dt is given by

$$hA\,[T_\infty - T(t)]\,dt = \rho V C_p\, dT$$

where dT is the increase in temperature of the cylinder in time dt.



Or $$dT / dt = (hA/\rho V C_p)\,[T_\infty - T(t)]$$

Putting $m = (hA/\rho V C_p)$, the above equation reduces to

$$dT / dt + m\,T(t) = m\,T_\infty$$

Substituting the given expression for T_∞ we have

$$dT / dt + m\,T(t) = m\,[T_0 + T_f\,(1 - e^{-ct})]$$

or $$dT / dt + m\,[T(t) - T_0] = mT_f\,(1 - e^{-ct})$$

Let $\theta(t) = T(t) - T_0$. Then the above equation reduces to

$$d\theta / dt + m\,\theta(t) = mT_f (1 - e^{-ct}) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

This equation is of the form $dy / dx + Py = Q$, which is solved by multiplying throughout by an integrating factor and then integrating. For equation (1) the integrating factor is $e^{\int m dt}$ e^{mt}. therefore multiplying equation (1) by e^{mt} we get

$$e^{mt} (d\theta / dt) + m\, e^{mt} \theta(t) = mT_f [e^{mt} - e^{(m-c)t}]$$

or $$d / dt (e^{mt}\theta) = mT_f [e^{mt} - e^{(m-c)t}]$$

Integrating with respect to t we have

$$e^{mt}\theta(t) = mT_f [(e^{mt} / m) - e^{(m-c)t} / (m-c)] + C_1$$

or $$\theta(t) = T_f - \frac{m}{(m-c)} T_f e^{-ct} + C_1 e^{-mt} \ldots\ldots\ldots\ldots\ldots(2)$$

When $t = 0$, $T(0) = T_0$ i.e., $\theta(0) = 0$. Substituting this condition in equation (2) we get

or $$0 = T_f - \frac{m}{(m-c)} T_f + C_1$$

Or $$C_1 = [c / (m-c)] T_f.$$

Substituting this expression for C_1 in equation (2) we get the temperature of the cylinder as

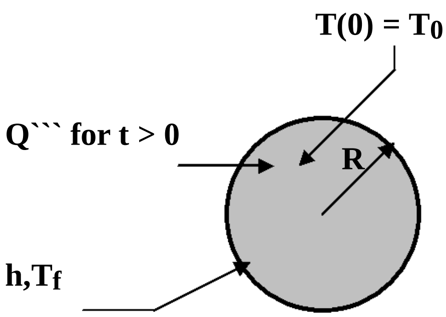
$$\theta(t) = T_f - \frac{m}{(m-c)} T_f e^{-ct} + \frac{c}{(m-c)} T_f e^{-mt}$$

Or $$T(t) - T_0 = T_f [1 - m / (m-c)\, e^{-ct} + c / (m-c)\, e^{-mt}]$$

Where $m = hA / (\rho VC_p) = \pi DLh / \{(\pi D^2/4)L\rho C_p\} = (4h) /(\rho DC_p)$.

Example 4.3:- A solid sphere of radius R is initially at a uniform temperature T_0. At a certain instant of time ($t = 0$), the sphere is suddenly exposed to the surroundings at a temperature T_f and the surface heat transfer coefficient, 'h'. In addition from the same instant of time, heat is generated within the sphere at a uniform rate of q''' units per unit volume. Neglecting internal temperature gradients, derive an expression for the temperature of sphere as a function of time

Solution:



Energy balance equation for the sphere at any time t can be written as

$$(4/3)\pi R^3 q^{''''''} + 4\pi R^2 h\,[T_f - T(t)] = (4/3)\pi R^3 \rho C_p\,(dT/dt)$$

$$\text{Or } (dT/dt) + (3h/\rho R C_p)[T(t) - T_f] = (q^{```}/\rho C_p)$$

Let $\theta(t) = T(t) - T_f$. Then the above equation reduces to

$$(d\theta / dt) + m\theta = q_0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(1)$$

Where $m = (3h/\rho R C_p)$ and $q_0 = (q^{```}/\rho C_p)$

Multiplying equation (1) by the integrating factor e^{mt} we

$$\text{have } e^{mt}(d\theta / dt) + e^{mt} m\theta = q_0 e^{mt}$$

or $$d / dt(\theta e^{mt}) = q_0 e^{mt}$$

Integrating throughout w.r.t. t we get

$$\theta e^{mt} = (q_0 / m)\, e^{mt} + C_1$$

or $$\theta = (q_0 / m) + C_1 e^{-mt} \quad \ldots\ldots\ldots\ldots\ldots\ldots...(2)$$

At t = 0 , $T = T_0$ or $\theta = T_0 - T_f = \theta_0$ (say). Substituting this condition in equation (2) we get $C_1 = (T_0 - T_f) - (q_0 / m)$. Therefore the temperature in the sphere as a function of time is given by

$$\theta(t) = [(T_0 - T_f) - (q_0 / m)]\, e^{-mt} + (q_0 / m)$$

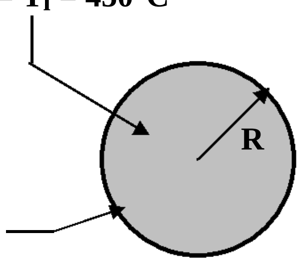
or $$\theta(t) = (q_0 / m)[\,1 - e^{-mt}\,] + (T_0 - T_f)\, e^{-mt}$$

$$\text{where } (q_0 / m) = \frac{q'''}{(\rho\, C_p)} \times \frac{(\rho\, R\, C_p)}{3h} = (q''' R / 3h)$$

Example 4.4:- A solid steel ball (ρ =8000 kg/m^3 ; c_p = 0.42 kJ/kg-K) 5 cm in diameter is at a uniform temperature of 450 ^{0} C. It is quenched in a controlled environment which is initially at 90 ^{0}C and whose temperature increases linearly with time at the rate of 10 ^{0}C per minute. If the surface heat transfer coefficient is 58 W/(m^2-K), determine the variation of the temperature of the ball with time neglecting internal temperature gradients. Find the value of the minimum temperature to which the ball cools and the time taken to reach this minimum temperature.

Solution:

$T(0) = T_i = 450^0C$



h, T_f

Other data:- $h = 58$ W / (m^2 – K) ;

$C_p = 0.42$ kJ / (kg – K) ; $\rho = 8000$ kg / m^3 ;

$T_f = a + bt$, where a and b are constants ;

at $t = 0$, $T_f = 90\ ^0C$;$(dT_f / dt) = 10\ ^0C$ / min

$= (1/6)\ ^0$ C / s

Therefore $a = 90\ ^0$ C and $b =$;$(dT_f / dt) = (1/6)\ ^0$ C / s.

Or $T_f = 90 + t / 6$, t in seconds……………….(1)

Energy balance equation for the sphere at any time t can be written as

$$\rho V C_p (dT / dt) = hA\, [T_f(t) – T(t)]$$

Or $$(dT / dt) = (hA / \rho V C_p)\, [T_f(t) – T(t)]$$

Letting $m = (hA / \rho V C_p)$ the above equation can be written as

$$(dT / dt) + mT(t) = m\, T_f(t)$$

Substituting for $T_f(t)$ from equation (1) we have

$$(dT / dt) + mT(t) = m\, [90 + t / 6\,]$$

Multiplying the above equation with the integrating factor e^{mt} we get

$$e^{mt}(dT / dt) + mT(t)\, e^{mt} = m\,[90 + t / 6]\, e^{mt}$$

or $$d / dt\,(T e^{mt}) = m\,[90 + t / 6]\, e^{mt}$$

Integrating throughout w.r.t t we have

$$(T e^{mt}) = m \int [90 + t / 6]\, e^{mt}\, dt + C_1$$

Or $$T(t) = m\, e^{-mt} \int [90 + t / 6]\, e^{mt}\, dt + C_1 e^{-mt}$$

$$= m\, e^{-mt} [(90e^{mt} /m) + (te^{mt}/6m) - (e^{mt}/6m^2)] + C_1 e^{-mt}$$

Or $$T(t) = [90 + (t / 6) - (1/6m)] + C_1 e^{-mt} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(2)$$

When t = 0 , T(t) = T_i. Substituting this condition in the above equation and solving for C_1 we get

$$C_1 = [T_i - 90 + 1 / 6m]$$

Therefore the temperature of sphere as a function of time is given by

$$\mathbf{T(t) = [\,90 + (t / 6) - (1/6m)] + [T_i - 90 + 1 / 6m]\, e^{-mt}} \quad \ldots\ldots\ldots\ldots(3)$$

For T(t) to be extremum (dT / dt) = 0.

Therefore we have $(dT / dt) = 1/6 + [T_i - 90 + 1 / 6m]\, e^{-mt} (-m) = 0$

Substituting $T_i = 450\ ^0C$ and simplifying we get

$$(360\, m + 1/6)\, e^{-mt} = 1/6$$

Or $$e^{mt} = (2160\, m + 1) \text{ -----------(4)}$$

Now $m = (hA / \rho VC_p) = \dfrac{4\pi R^2 h}{[(4/3)\pi R^3 \rho C_p]} = (3h/ \rho C_p R) = \dfrac{3 \times 58}{(8000 \times 0.025 \times 0.42 \times 10^3)}$

$= 2.07 \times 10^{-3}$.-------- (5)

Using (4) & (5) in (3),

$$T(t)= 90+ (t/6)- (1/(6x2.07x10^{-3})) + [\,240-90+(1/(6x2.07x10^{-3}))] \times \exp\{-2.07x10^{-3}\, t\}$$

$T(t)=9.4857+(230.5152)x(0.9979)^t$

$T(t) > 0$

Hence value of t will be minimum.

Therefore $e^{mt} = [\ 2160 \times 2.07 \times 10^{-3} + 1\ ] = 5.47$

$mt = 1.7$

Or $t = 1.7 / m = 1.7 / (2.07 \times 10^{-3}) = 821$ s = 13.7 min

Substituting this value of t in equation (3) we get the minimum temperature

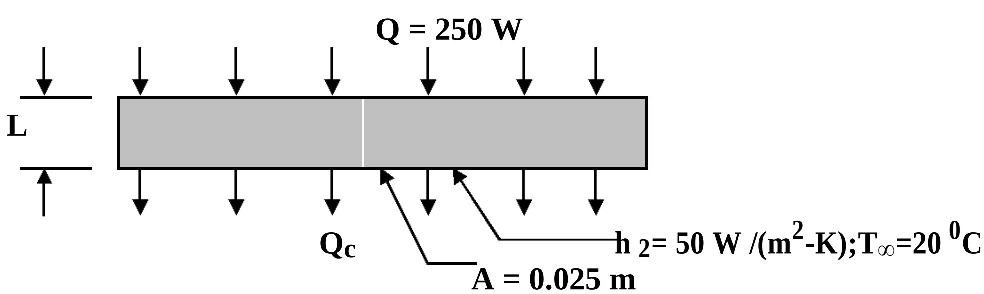
$$\text{as } T_{minimum} = [90 + (821/7) - \{1 / (6 \times 2.07 \times 10^{-3}) \}\ ]$$

$$+ [\ 450 - 90 + \{1 / (6 \times 2.07 \times 10^{-3}) \}\ ]\ e^{-1.7} = 226.7\ ^0C.$$

Example 4.5:- A house hold electric iron has a steel base [ρ =7840 kg/m^3 ; c_p = 450 J/(kg-K) ;k = 70 W/(m-K)] which weighs 1 kg. The base has an ironing surface area of 0.025 m^2 and is heated from the other surface with a 250 W heating element. Initially the iron is at a uniform temperature of 20 ^{0} C with a heat transfer coefficient of 50 W/(m^2-K).

(b) What would be the equilibrium temperature of the iron if the control of the iron box did not switch of the current?

Solution:



Other data:- ρ = 7840 kg / m^3 ; C_p = 450 J / (kg – K) ; k = 70 W /(m – K) ; m = 1 kg ; t = 5 min = 300 s.

$V = m / \rho = 1 / 7840 = 0.0001275\ m^3 = 1.275 \times 10^{-4}\ m^3.$

$$L = V / A = \frac{1.275 \times 10^{-4}}{0.025} = 0.005 \text{ m}$$

$$Bi = (hL / k) = \frac{50 \times 0.005}{70} = 0.00364$$

Since Bi < 0.1, it can be assumed that temperature gradients within the plate are negligible. Hence the temperature of the plate depends only on time till steady state condition is reached.

Energy balance at any time t for the plate can be written as

$$Q - Q_c = \rho V C_p\,(dT/dt)$$

Or $$Q - hA(T - T_\infty) = \rho V C_p\,(dT/dt)$$

Or $$(dT/dt) + m(T - T_\infty) = (Q / \rho V C_p) \quad \text{.....................................(1)}$$

Where m = $(hA / \rho V C_p)$. Letting $\theta = T - T_\infty$, equation (1) can be written as

$$(d\theta / dt) + m\,\theta = (Q / \rho V C_p)$$

Multiplying the above equation by the integrating factor e^{mt},($e^{\int m dt} = e^{mt}$) we get

$$(d\theta / dt)\, e^{mt} + m\,\theta e^{mt} = (Q / \rho V C_p)\, e^{mt}$$

Or $$d/dt\,(\theta e^{mt}) = (Q / \rho V C_p)\, e^{mt}$$

Or $$(\theta e^{mt}) = (Q / \rho V C_p)\, e^{mt}\,(1/m) + C_1$$

Or $$\theta = (Q / \rho V C_p m) + C_1 e^{-mt} \quad \text{.......................................(2)}$$

When t = 0, $T = T_i$ or $\theta = T_i - T_\infty = 20 - 20 = 0\,^0C$.

Substituting this condition in equation (2) we get

$$0 = (Q / \rho V C_p m) + C_1 \text{ or } C_1 = -(Q / \rho V C_p m)$$

Therefore the temperature in the plate as a function of time is given by

$$\theta = (Q / \rho V C_p m)\,[\,1 - e^{-mt}$$

] But $\rho V C_p m = hA$. Therefore

$$\theta = (Q / hA) [1 - e^{-mt}] \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

$$Q / hA = \frac{250}{50 \times 0.025} = 200 \; ; \; m = \frac{50 \times 0.025}{1 \times 450} = 2.8 \times 10^{-3}$$

Therefore $\theta = 200 [1 - e^{-0.028t}]$

When t = 300 s, $\theta = T - T_\infty = 200 \times [1 - e^{-0.028 \times 300}] = 113.7$

Or $T = 113.7 + 20 = 133.7\,^0 C.$

(b) When the control switch is not switched off and the iron is left in the ambient, steady state condition will be attained as t tends to ∞ so that the heat transferred to the baseplate will be convected to the ambient. i.e.,

$$Q = Q_c$$

Therefore $250 = 50 \times 0.025 \times [T - 20]$

Or $T = 220\,^0 C.$

This answer can also be obtained by putting $t = \infty$ in equation (3) and solving for T.

4.3 One-dimensional Transient Conduction (Use of Heissler's Charts): There are many situations where we cannot neglect internal temperature gradients in a solid while analyzing transient conduction problems. Then we have to determine the temperature distribution within the solid as a function of position and time and the analysis becomes more complex. However the problem of one-dimensional transient conduction in solids without heat generation can be solved readily using the method of separation of variables.The analysis is illustrated for solids subjected to convective boundary conditions and the solutions were presented in the form of transient – temperature charts by **Heissler.** These charts are now familiarly known as "Heissler‟s charts".

4.3.1.One-dimensional transient conduction in a slab:- Let us consider a slab of thickness 2L, which is initially at a uniform temperature T_i. Suudenly let the solid be exposed to an environment which is maintained at a uniform temperature of T_∞ with a surface heat transfer coefficient of h for time t > 0.Fig.4.3 shows the geometry , the coordinates and the boundary conditions for the problem. Because of symmetry in the problem with respect to the centre of the slab the „x‟ coordinate is measured from the centre line of the slab as shown in the figure.

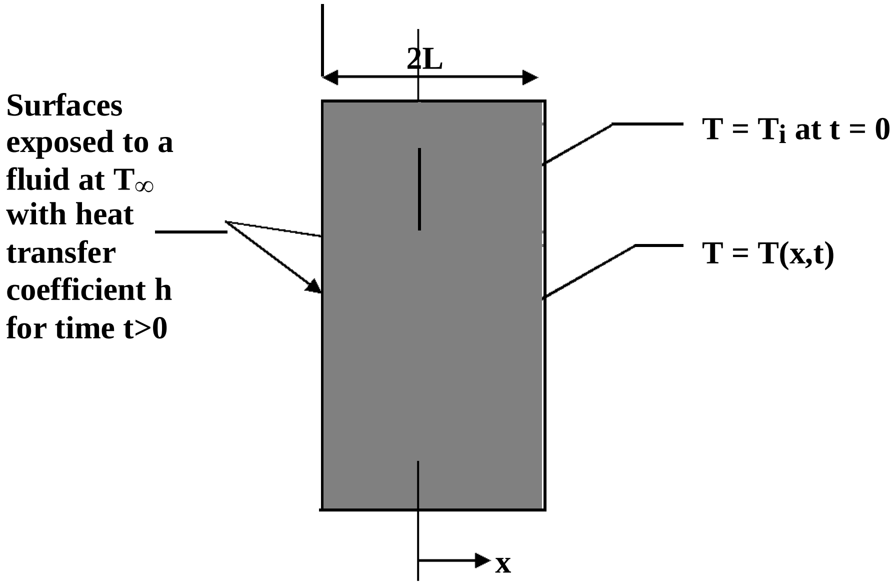


Fig.4.3: Geometry, coordinates and boundary conditions for transient conduction in a slab

The mathematical formulation of this transient conduction problem is given as follows:

Governing differential equation: $\partial^2T / \partial x^2 = (1/\alpha)\, \partial T / \partial t$ ………………………..(4.7a)

Initial condition : at $t = 0$, $T = T_i$ in $0 < x < L$ …………………………………….(4.7b)

Boundary conditions are :

(i) at $x = 0$, $\partial T / \partial x = 0$ (axis of symmetry) for all $t > 0$…………………………..(4.7c)

(ii) at $x = L$, $-k\, (\partial T / \partial x)|_{x = L} = h(T|_{x = L} - T_\infty)$ for all $t > 0$ ……………………….(4.7d)

It is more convenient to analyze the problem by using the variable $\theta(x,t)$, where

$\theta(x,t) = T(x,t) - T_\infty$. Then equations (4.7a) to (4.7d) reduce to the following forms:

$$\partial^2\theta / \partial x^2 = (1/\alpha)\, \partial\theta / \partial t \quad \text{………………………..(4.8a)}$$

Initial condition : at $t = 0$, $\theta = T_i - T_\infty = \theta_i$ in $0 < x < L$ ………………………….(4.8b)

Boundary conditions reduce to :

(i) at $x = 0$, $\partial\theta / \partial x = 0$ for all $t > 0$ ………………….……………………………..(4.8c)

(ii) at $x = L$, $-k\, (\partial\theta / \partial x)|_{x = L} = h\theta|_{x = L}$ for all $t > 0$ ……..……………………….(4.8d)

Eq.(4.8a) can be solved by the method of separation of variables as shown below.

Let $\theta(x,t) = X(x)\, Y(t)$ …………………………………………………………..(4.9)

Substituting this in Eq. (4.8a) we get

$$Y\,(d^2X / dx^2) = (X/\alpha)\,(dY / dt)$$

Or

$$\frac{1}{X}(d^2X / dx^2) = \frac{1}{(Y\alpha)}(dY / dt) \quad \text{……………………….(4.10)}$$

LHS of Eq. (4.10) is a function of x only and the RHS of Eq. (4.10) is a function of t only.They can be equal only to a constant say $-\lambda^2$.(The reason to choose the negative sign is to get a physically meaningful solution as explained later in this section).Hence we have two equations namely

$(1 / X)\,(d^2X / dx^2) = -\lambda^2$ and $[1/(Y\alpha)]\,((dY / dt) = -\lambda^2$

Or $\quad (d^2X / dx^2) + \lambda^2 X = 0$ ……………………………………………………(4.11)

and $\quad (dY/dt) = -\alpha\lambda^2\, Y$ ………………………………………………………(4.12)

Solution to Eq. (4.11) is $X(x) = C_1 \cos(\lambda x) + C_2 \sin(\lambda x)$ …………………….(4.13)

and solution to Eq. (4.12) is $\quad Y(t) = D \exp(-\alpha\lambda^2 t)$ ………………………..(4.14)

with C_1, C_2 and D as constants of integration. Substituting these solutions in Eq.(4.9) we have

$$\theta(x,t) = D \exp(-\alpha\lambda^2 t)\,[C_1 \cos(\lambda x) + C_2 \sin(\lambda x)]$$

or $\quad \theta(x,t) = \exp(-\alpha\lambda^2 t)\,[A_1 \cos(\lambda x) + A_2 \sin(\lambda x)]$………..(4.15)

Eq.(4.15) is the general solution involving the constants A_1, A_2 and λ which can be determined using the two boundary conditions and the initial condition as illustrated below.

Now from Eq. (4.15), $\partial\theta / \partial x = \lambda \exp(-\alpha\lambda^2 t)\,[-A_1 \sin(\lambda x) + A_2 \cos(\lambda x)]$

Substituting boundary condition (i) we have $0 = \lambda \exp(-\alpha\lambda^2 t)\,[0 + A_2]$ for all

t. Hence $A_2 = 0$. Therefore Eq. (4.15) reduce to

$$\theta(x,t) = A_1 \exp(-\alpha\lambda^2 t)\cos(\lambda x) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4.16)$$

Now $\theta(L,t) = A_1 \exp(-\alpha\lambda^2 t)\cos(\lambda L)$

and $\partial\theta / \partial x = \lambda \exp(-\alpha\lambda^2 t)\,[-A_1 \sin(\lambda x)]$

Hence $[\partial\theta / \partial x]_{x=L} = -\lambda A_1 \exp(-\alpha\lambda^2 t)\sin(\lambda L)$

Therefore boundary condition (ii) can be written as

$$k\lambda A_1 \exp(-\alpha\lambda^2 t)\sin(\lambda L) = h A_1 \exp(-\alpha\lambda^2 t)\cos(\lambda L)$$

or $\tan(\lambda L) = h / (k\lambda)$

or $\lambda L \tan(\lambda L) = Bi \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4.17)$

where $Bi = hL / k$.

Equation (4.17) is called the *"characteristic equation"* and has infinite number of roots namely $\lambda_1, \lambda_2, \lambda_3$, ...Corresponding to each value of λ we have one solution and hence there are infinite number of solutions. Sum of all these solutions will also be a solution as the differential equation is linear. Therefore the solution $\theta(x,t)$ can be written as follows.

$$\boldsymbol{\theta(x,t) = \sum A_n \exp(-\alpha\lambda_n{}^2 t)\cos(\lambda_n x) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(4.18)}$$

To find A_n:- The constants A_n in Eq. (4.18) can be found using the orthogonal property of trigonometric functions as shown below. Substituting the initial condition we have

$$\theta_i = \sum A_n \cos(\lambda_n x)$$

Multiplying both sides of Eq.(4.18) by $\cos\lambda_m x$ and integrating w.r.t „x" between the limits 0 and L we have

$$\int_1^L \theta_i \cos(\lambda_m x)\,dx = \int_1^L \sum A_n \cos(\lambda_m x)\cos(\lambda_n x)\,dx$$

Using the orthogonal; property

$$\int A_n \cos(\lambda_m x)\cos(\lambda_n x)\,dx = 0 \text{ for } \lambda_n \neq \lambda_m$$

The above equation reduce to

$$\int_0^L \theta_i \cos(\lambda n x)\,dx = \int_0^L A_n \cos^2(\lambda_n x)\,dx$$

Or

$$A_n = \frac{\theta_i \int_0^L \cos(\lambda_n x)\,dx}{\int_0^L \cos^2(\lambda_n x)\,dx}$$

It is very convenient to express Eq. (4.18) in dimension less form as follows:

$$\mathbf{\frac{\theta(x,t)}{\theta i} = \sum (A_n^* \exp(-\lambda_n^{*2} Fo) \cos(\lambda_n^* x / L)} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4.19)$$

where $A_n^* = A_n / \theta_i$; $\lambda_n^* = \lambda_n L$; Fo = Fourier Number = $\alpha t / L^2$;

4.3.2. Heissler's Charts for transient conduction:- For values of Fo > 0.2 the above series solution converges rapidly and the solution will be accurate within 5 % if only the first term in the series is used to determine the temperature. In that case the solution reduces to

$$\frac{\theta(x,t)}{\theta i} = A_1^* \exp(-\lambda_1^{*2} Fo) \cos(\lambda_1^* x / L) \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4.20)$$

From the above equation the dimensionless temperature at the centre of the slab (x =0) can be written as

$$\frac{\theta(0,t)}{\theta i} = A_1^* \exp(-\lambda_1^{*2} Fo) \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4.21)$$

The values of A_1^* and λ_1^* for different values of Bi are presented in the form of a table (See Table 4.1). These values are evaluated using one term approximation of the series solution.It can also be concluded from Eq.(4.20) at any time „t" the ratio θ(x,t) / θ(0,t) will be independent of temperature and is given by

$$\frac{\theta(x,t)}{\theta(0,t)} = \cos(\lambda_1^* x / L) \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4.22)$$

Heissler has represented Eq. (4.21) and (4.22) in the form of charts and these charts are normally referred to as Heissler"s charts. Eq. (4.21) is plotted as Fourier number Fo versus dimensionless centre temperature $\theta(0,t) / \theta_i$ using [Fig.4.4(b)].
reciprocal of Biot number1 / Bi as the parameter [Fig.4.4(a)], where as Eq. (4.22) is plotted as θ(x,t) / θ(0,t) versus reciprocal of Biot number using the dimensionless distance x / L as the parameter.In Fig.[4.4(a)], the curve for 1/Bi = 0 corresponds to the case
$h \rightarrow \infty$, or the outer surfaces of the slab are maintained at the ambient temperature T_∞. For large values of 1 / Bi, the Biot number is small, or the internal conductance is large in comparison with the surface heat transfer coefficient. This in turn, implies that the temperature distribution within the solid is sufficiently uniform and hence lumped system analysis becomes applicable.

Fig. (4.5) shows the dimensionless heat transferred Q / Q_0 as a function of dimensionless time for different values of the Biot number for a slab of thickness 2L. Here Q represents the total amount of thermal energy which is lost by the slab up to any time t during the transient conduction heat transfer. The quantity Q_0, defined as

$$Q_0 = \rho V C_p[T_i - T_\infty] \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4.23)$$

represents the initial thermal energy of the slab relative to the ambient temperature.

4.3.3.Transient-Temperature charts for Long cylinder and sphere: The dimensionless transient-temperature distribution and the heat transfer results for infinite cylinder and sphere can also be represented in the form of charts as in the case of slab. For infinite cylinder and sphere the radius of the outer surface R is used as the characteristic length so that the Biot number is defined as Bi = hR / k and the dimensionless distance from the centre is r/R where r is any radius ($0 \le r \le R$).These charts are illustrated in Figs. (4.6) to (4.9).

4.3.4.Illustrative examples on the use of Transient Temperature Charts:-Use of the transient temperature charts for slabs, infinite cylinders and spheres is illustrated in the following examples.

4.3.1. Transient conduction in semi-infinite solids:- A semi-infinite solid is an idealized body that has a *single plane surface* and extends to infinity in all directions.The transient conduction problems in semi-infinite solids have numerous practical applications in engineering. Consider, for example, temperature transients in a slab of finite but large thickness, initiated by a sudden change in the thermal condition at the boundary surface. In the initial stages, the temperature transients near the boundary surface behave similar to those of semi-infinite medium, because some time is required for the heat to penetrate the slab before the other boundary condition begins to influence the transients.The earth for example, can be considered as a semi-infinite solid in determining the variation of its temperature near its surface

We come across basically three possibilities while analyzing the problem of one-dimensional transient conduction in semi-infinite solids.These three problems are as follows:

Problem 1:- The solid is initially at a uniform temperature T_i and suddenly at time t>0 The boundary-surface temperature of the solid is changed to and maintained at a uniform temperature T_0 which may be greater or less than the initial temperature T_i.

Problem 2:- The solid is initially at a uniform temperature T_i and suddenly at time t>0 the boundary surface of the solid is subjected to a uniform heat flux of q_0 W/m^2.

Problem 3:- The solid is initially at a uniform temperature T_i. Suddenly at time t>0 the boundary surface is exposed to an ambience at a uniform temperature T_∞ with the surface heat transfer coefficient h. T_∞ may be higher or lower than T_i.

Solution to Problem 1:- The schematic for problem 1 is shown in Fig. 4.10. The mathematical formulation of the problem to determine the unsteady temperature distribution in an infinite solid T(x,t) is as follows:

The governing differential equation is

$$\partial^2 T / \partial x^2 = (1/\alpha)\,(\partial T / \partial t) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \; 4.24(a)$$

The initial condition is at time t = 0, $T(x,0) = T_i$ ……………………………….4.24(b)

and the boundary condition is at x = 0, $T(0,t) = T_0$... 4.24(c)

It is convenient to solve the above problem in terms of the variable θ(x,t), where θ(x,t) is defined as

$$\theta(x,t) = \frac{T(x,t) - T_\infty}{\quad} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 4.25$$

$T_i - T_\infty$

The governing differential equation in terms of $\theta(x,t)$ will be

$$\partial^2\theta / \partial x^2 = (1/\alpha)\,(\partial\theta /\partial t) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 4.26(a)$$

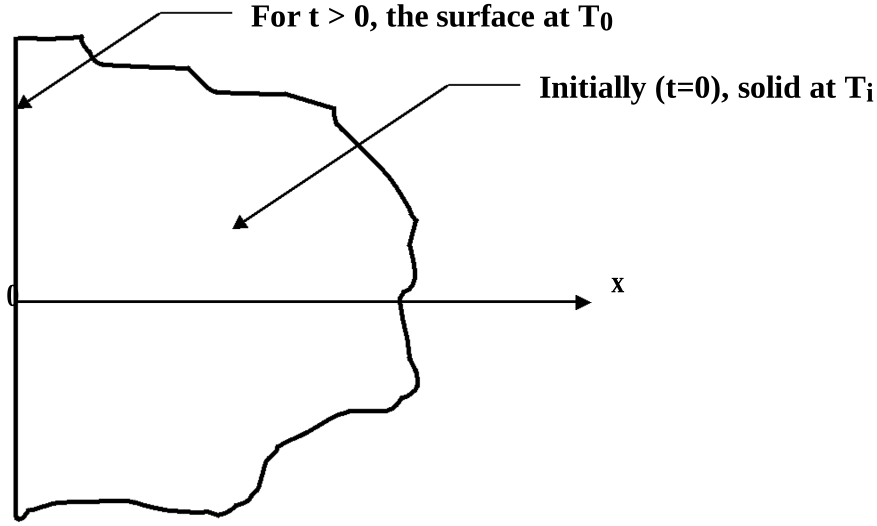


Fig. 4.10: Semi-infinite solid with specified surface temperature T_0 for t > 0

The initial condition will be at time t = 0, $\theta(x,0) = T_i - T_\infty$ …………………….4.26(b)

And the boundary condition will be at x = 0, $\theta(0,t) = T_0 - T_\infty$ …………………4.26(c)

This problem has been solved analytically and the solution $\theta(x,t)$ is represented graphically as $\theta(x,t)$ as a function of the dimensionless variable $x / [2\sqrt{(\alpha t)}]$ as shown in Fig. 4.11.

In engineering applications, the heat flux at the boundary surface x = 0 is also of interest. The analytical expression for heat flux at the surface is given by

$$q_s(t) = \frac{k(T_0 - T_i)}{\sqrt{(\pi\alpha t)}} \quad \text{……………………..4.27}$$

Solution to problem 2:- The schematic for this problem is shown in Fig. 4.12.

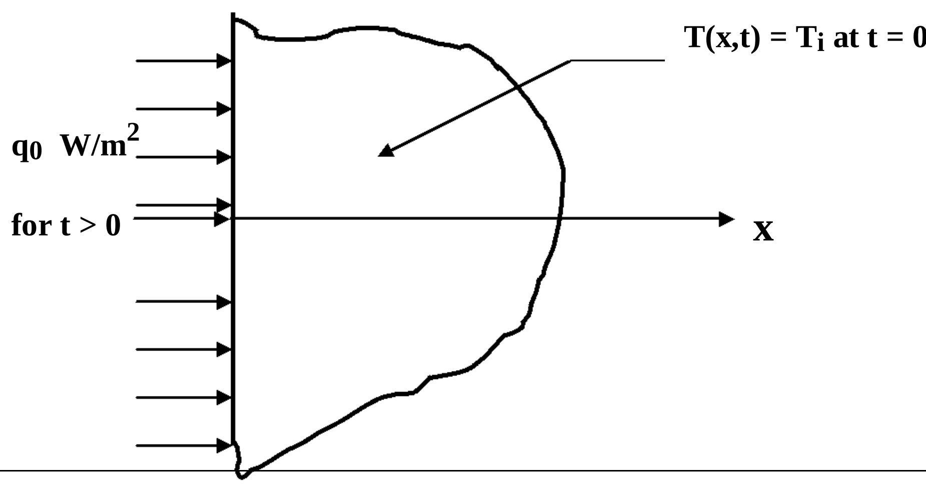


Fig. 4.12: An infinite solid subjected to a constant heat flux at x = 0 for t > 0

Governing differential equation in terms of T(x,t) and the initial condition are same that for problem 1[i.e. equations 4.26(a) and 4.26(b)].

The boundary condition is : at x = 0, $-k\,(\partial\theta / \partial x)|_{x=0} = q_0$.

The temperature distribution within the solid T(x,t) is given by

$$T(x, t) = T_i + \frac{2q_0}{k} (\alpha t)^{1/2} \left[(1/\sqrt{\pi}) \exp(-\xi^2) + \xi \operatorname{erf}(\xi) - \xi \right] \quad \text{……………………..(4.28 a)}$$

$$\text{where } \xi = x / (2\sqrt{\alpha t}) \text{ and } \operatorname{erf}(\xi) = \frac{2}{\sqrt{\pi}} \int_0^{\xi} \exp(-y^2)\, dy \quad \text{………………………...(4.28b)}$$

Here erf (ξ) is called the *"error function"* of argument ξ and its values for different values of ξ are tabulated.

Solution to Problem 3 :- The solid is initially at a uniform temperature T_i and suddenly for t >0 the surface at x = 0 is brought in contact with a fluid at a uniform temperature T_∞ with a surface heat transfer coefficient h. For this problem the solution is represented in the form of a plot where the dimensionless temperature [1 − θ(x,t)] is plottedagainst dimensionless distance x / √(αt), using h√(αt) / k as the parameter. It can be noted that the case h → ∞ is equivalent to the boundary surface ay x = 0 maintained at a constant temperature T_∞.

4.3.2. Illustrative examples on Transient Conduction in Semi – Infinite solids

Example 4.9:-A thick stainless steel slab [$\alpha = 1.6 \times 10^{-5}\ m^2/s$ and k = 61 W/(m-K)] is initially at a uniform temperature of 150 ^{0} C. Its surface temperature is suddenly lowered to 20 ^{0} C. By treating this as a one-dimensional transient conduction problem in a semi-infinite medium, determine the temperature at a depth 2 cm from the surface and the heat flux 1 minute after the surface temperature is lowered

Solution:

$T_i = 150\ ^0 C$; $T_0 = T|_{x=0} = 20\ ^0 C$; $\alpha = 1.6 \times 10^{-5}\ m^2 / s$; k = 61 W/(m – K) ; x = 0.02 m ;

T = 1 min = 60 s

$$\xi = \frac{x}{2\sqrt{(\alpha t)}} = \frac{0.02}{2 \times \sqrt{(1.6 \times 10^{-5} \times 60)}} = 0.323$$

From chart,
$$\frac{T(x,t) - T_0}{T_i - T_0} = 0.35$$

Therefore $T(x,t) = T_0 + 0.35\,(T_i - T_0) = 20 + 0.35 \times (150 - 20) = 65.5\ ^0 C$.

$$q_s(t) = \frac{k(T_0 - T_i)}{\sqrt{(\pi \alpha t)}} = \frac{61 \times (20-150}{\sqrt{(\pi \times 1.6 \times 10^{-5} \times 60)}} = -435.5\ W / m^2$$

Example 4.10:- A semi-infinite slab of copper ($\alpha = 1.1 \times 10^{-4}\ m^2/s$ and k = 380 W/(m-K) is initially at a uniform temperature of 10 ^{0} C. Suddenly the surface at x = 0 is raised to 100 ^{0}C. Calculate the heat flux at the surface 5 minutes after rising of the surface temperature . How long will it take for the temperature at a depth of 5 cm from the surface to reach 90 ^{0} C?

Solution:

$T_i = 10\ ^0 C$; $T_0 = 100\ ^0 C$; k = 380 W / (m – K) ; $\alpha = 1.1 \times 10^{-4}\ m^2 / s$; t = 300 s ;

$$q_s(t) = \frac{k(T_0 - T_i)}{\sqrt{(\pi \alpha t)}} = \frac{380 \times (100 - 10)}{\sqrt{(\pi \times 1.1 \times 10^{-4} \times 300)}} = 11012\ W / m^2 = 11.012\ kW/m^2$$

$$\theta(x,t) = \frac{T(x,t) - T_0}{T_i - T_0} = \frac{90 - 100}{10 - 100} = 0.11 \text{ . From chart } \xi = 0.1$$

$x \qquad x^2 \qquad 0.05^2$

$$\xi = \frac{}{2\sqrt{(\alpha t)}} \quad \text{or } t = \frac{}{4\,\alpha\,\xi^{2}} = \frac{}{4 \times 1.1 \times 10^{-4} \times (0.1)^{2}}$$

$$= 586 \text{ s} = 9.46 \text{ min}$$

Example 4.11:-A thick bronze [$\alpha = 0.86 \times 10^{-5}$ m^2/s and k = 26 W/(m-K)] is initially at 250 ^{0} C. Suddenly the surface is exposed to a coolant at 25 ^{0} C. If the surface heat transfer coefficient is 150 W/(m^2-K), determine the temperature 5 cm from the surface 10 minutes after the exposure.

Solution:

$T_i = 250\,^0$ C; $T_\infty = 25\,^0$ C; h = 150 W/(m^2 – K) ; k = 26 W /(m – K) ; $\alpha = 0.86 \times 10^{-5}$ m^2/s

t = 600 s ; x = 0.05 m ;

$$\xi = \frac{x}{2\sqrt{(\alpha t)}} = \frac{0.05}{2 \times \sqrt{(0.86 \times 10^{-5} \times 600)}} = 0.35$$

$$\frac{h\sqrt{(\alpha t)}}{K} = \frac{150 \times \sqrt{[0.86 \times 10^{-5} \times 600]}}{26} = 0.414$$

Therefore from chart $\quad 1 - \dfrac{[T(x,t) - T_\infty]}{(T_i - T_\infty)} = 0.15$

Solving for T(x,t) we have $\quad T(x,t) = T_\infty + (1 - 0.15)(T_i - T_\infty)$

$$= 25 + 0.85 \times (250 - 25) = 216.25\,^0\text{ C}.$$

UNIT-III
Basic Concepts of Convective Heat Transfer

5.1. Definition of Convective Heat Transfer:- When a fluid flows over a body or inside a channel and if the temperatures of the fluid and the solid surface are different, heat transfer will take place between the solid surface and the fluid due to the macroscopic motion of the fluid relative to the surface. This mechanism of heat transfer is called as "*convective heat transfer"*. If the fluid motion is due to an external force (by using a pump or a compressor) the heat transfer is referred to as "*forced convection"*. If the fluid motion is due to a force generated in the fluid due to buoyancy effects resulting from density difference (density difference may be caused due to temperature difference in the fluid) then the mechanism of heat transfer is called as *"natural or free convection"*. For example, a hot plate suspended vertically in quiescent air causes a motion of air layer adjacent to the plate surface because the temperature gradient in the air gives rise to a density gradient which in turn sets up the air motion.

5.2. Heat Transfer Coefficient:- In engineering application, to simplify the heat transfer calculations between a hot surface say at temperature T_W and a cold fluid flowing over it at a bulk temperature T_∞ as shown in Fig. 5.1 a term called "*heat transfer coefficient,h"* is defined by the equation

$$q = h(T_W - T_\infty) \qquad \ldots 5.1(a)$$

where q is the heat flux (expressed in W / m^2) from the surface to the flowing fluid. Alternatively if the surface temperature is lower than the flowing fluid then the heat transfer takes place from the hot fluid to the cold surface and the heat flux is given by

$$q = h(T_\infty - T_W) \qquad \ldots 5.1(b)$$

The heat flux in this case takes place from the fluid to the cold surface.If in equations 5.1(a) and 5.1(b) the heat flux is expressed in W / m^2, then the units of heat transfer coefficient will be $W /(m^2 – K)$ or $W / (m^2 – {}^0C)$.

The heat transfer coefficient is found to vary with (i) the geometry of the body, (ii) the type of flow (laminar or turbulent), (iii) the transport properties of the fluid (density, viscosity and thermal conductivity),(iv) the average temperature, (v) the position along the surface of the body, and (vi) whether the heat transfer is by forced convection or free convection. For convection problems involving simple geometries like flow over a flat plate or flow inside a circular tube, the heat transfer coefficient can be determined analytically

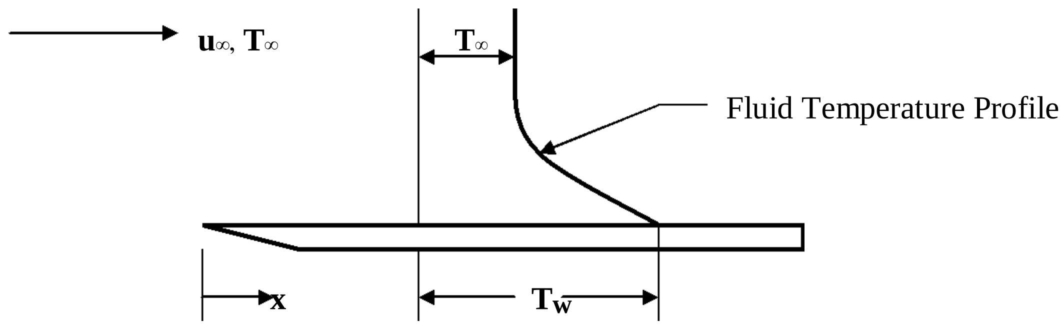


Fig. 5.1: Temperature distribution of the fluid at any x for $T_W > T_\infty$

But for flow over complex configurations, experimental / numerical approach is used to determine h. There is a wide difference in the range of values of h for various applications. Typical values of heat transfer coefficients encountered in some applications are given in Table 5.1.

Table 5.1: Typical Values of heat transfer coefficients

Type of flow		**h [W /(m^2 – K)]**
Free convection	air	5 – 15
-----do ---------	oil	25 – 60
-----do---------	water	400 –800
Forced Convection	air	15 –300
-------do------------	oil	50 –1700
-------do-----------	water	300 – 12000
Boiling	water	3000 – 55000
Condensing	steam	5500 – 120000

5.3. Basic concepts for flow over a body:- When a fluid flows over a body, the velocity and temperature distribution at the vicinity of the surface of the body strongly influence the heat transfer by convection. By introducing the concept of boundary layers (velocity boundary layer and thermal boundary layer) the analysis of convective heat transfer can be simplified.

5.3.1. Velocity Boundary Layer:- Consider the flow of a fluid over a flat plate as shown in Fig. 5.2. The fluid just before it approaches the leading edge of the plate has a velocity u_∞ which is parallel to the plate surface. As the fluid moves in x-direction along the plate,

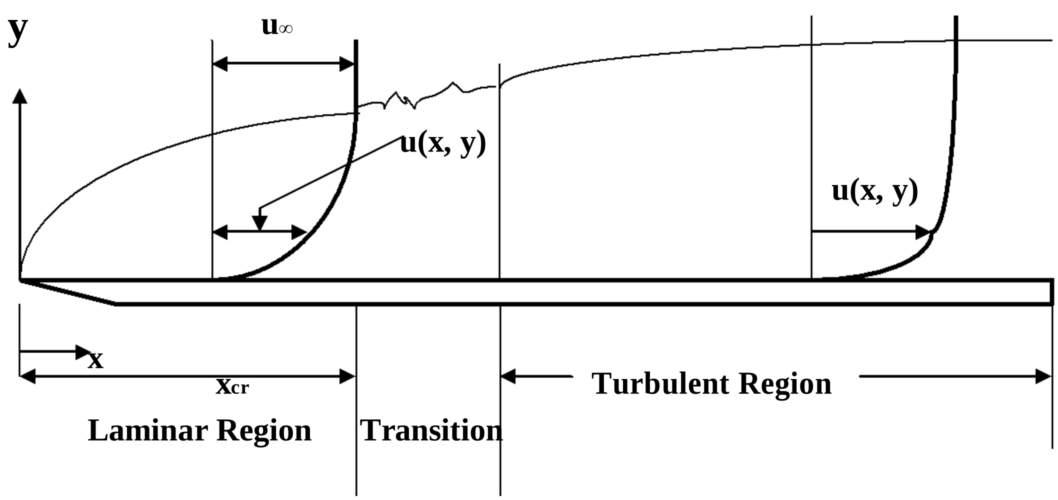


Fig. 5.2: Velocity boundary layer for flow over a flat plate

those fluid particles that makes contact with the plate surface will have the same velocity as that of the plate. Therefore if the plate is stationary, then the fluid layer sticking to the plate surface will have zero velocity.But far away from the plate ($y = \infty$) the fluid will have the velocity u_∞.Therefore starting from the plate surface ($y = 0$) there will be retardation of the fluid in x-direction component of velocity u(x,y).This retardation effect is reduced as we move away from the plate surface.At distances sufficiently long from the plate($y = \infty$) the retardation effect is completely reduced: i.e. $u \rightarrow u_\infty$ as $y \rightarrow \infty$. This means that there is a region surrounding the plate surface where the fluid velocity changes from zero at the surface to the velocity u_∞ at the outer edge of the region. This region is called ***the velocity boundary layer***. The variation of the x-component of velocity u(x,y) with respect to y at a particular location along the plate is shown in Fig. 5.2.The distance measured normal to the surface from the plate surface to the point at which the fluid attains 99% of u_∞ is called "***velocity boundary layer thickness***" and denoted by ***δ(x)*** Thus for flow over a flat plate, the flow field can be divided into two distinct regions, namely, (i) ***the boundary layer region*** in which the axial component of velocity u(x,y) varies rapidly with y with the result the velocity gradient ($\partial u / \partial y$) and hence the shear stress are very large and (ii) ***the potential flow region*** which is outside the boundary layer region, where the velocity gradients and shear stresses are negligible.

The flow in the boundary layer, starting from the leading edge of the plate will be initially laminar in which the fluid particles move along a stream line in an orderly manner. In the laminar region the retardation effect is due to the viscosity of the fluid and therefore the shear stress can be evaluated using Newton"s law of viscosity. The laminar flow continues along the plate until a critical distance „x_{cr}" is reached. After this the small disturbances in the flow begin to grow and fluid fluctuations begin to develop. This characterizes the end of the laminar flow region and the beginning of transition from laminar to ***turbulent boundary layer.*** A dimensionless parameter called ***Reynolds number*** is used to characterize the flow as laminar or turbulent. For flow over a flat plate the Reynolds number is defined as

$$Re_X = \frac{u_\infty x}{\nu} \quad \ldots\ldots 5.2$$

where u_∞ = free-stream velocity of the fluid, x = distance from the leading edge of the plate and ν = kinematic viscosity of the fluid.

For flow over a flat plate it has been found that the transition from laminar flow to turbulent flow takes place when the Reynolds number is $\approx 5 \times 10^5$.This number is called as the critical Reynolds number Re_{cr} for flow over a flat plate. Therefore

$$Re_{cr} = \frac{u_\infty x_{cr}}{\nu} = 5 \times 10^5 \quad \ldots\ldots 5.3$$

The critical Reynolds number is strongly dependent on the surface roughness and the turbulence level of the free stream fluid. For example, with very large disturbances in the free stream, the transition from laminar flow to turbulent flow may begin at Re_X as low as 1×10^5 and for flows which are free from disturbances and if the plate surface is smooth transition may not take place until a Reynolds number of 1×10^6 is reached. But it has been found that for flow over a flat plate the boundary layer is always turbulent for $Re_X \geq 4 \times 10^6$.In the turbulent boundary layer next to the wall there is a very thin layer called *"the viscous sub-layer"*, where the flow retains its viscous flow character. Next to the viscous sub-layer is a region called *"buffer layer"* in which the effect of fluid viscosity is of the same order of magnitude as that of turbulence and the mean velocity rapidly increases with the distance from the plate surface. Next to the buffer layer is *"the turbulent layer"* in which there is large scale turbulence and the velocity changes relatively little with distance.

5.3.2. Drag coefficient and Drag force:- If the velocity distribution u(x,y) in the boundary layer at any „x" is known then the viscous shear stress at the wall can be determined using Newton"s law of viscosity. Thus if $\eta_W(x)$ is the wall-shear stress at any location x then

$$\eta_W(x) = \mu(\partial u / \partial y)_{y=0} \text{ ....5.4}$$

where μ is the absolute viscosity of the fluid. The drag coefficient is dimensionless wall shear stress. Therefore *the local drag coefficient, C_X* at any „x" is defined as

$$C_X = \frac{\eta_W(x)}{(1/2)\,\rho u_\infty^2} \text{ ....5.5}$$

Substituting for $\eta_W(x)$ in the above equation from Eq. 5.4 and simplifying we get

$$C_X = \frac{2\nu\,(\partial u / \partial y)_{y=0}}{u_\infty^2} \text{ ....5.6}$$

Therefore if the velocity profile u(x,y) at any x is known then the local drag coefficient C_X at that location can be determined from Eq. 5.6.The average value of C_X for a total length L of the plate can be determined from the equation

$$C_{av} = (1/L) \int_0^L C_X\, dx \text{ ....5.7}$$

Substituting for C_X from Eq. 5.5 we have

$$C_{av} = \frac{\int^L \eta_W(x)dx}{L\,(1/2)\,\rho u_\infty^2}$$

Or $$C_{av} = \frac{\bar{\eta}_W}{(1/2)\,\rho u_\infty^2} \text{ ....5.8}$$

Where $\bar{\eta}_W$ is the average wall-shear stress for total length L of the plate.

The total drag force experienced by the fluid due to the presence of the plate can be written as

$$F_D = A_s\, \bar{\eta}_W \text{ ....5.9}$$

Where A_s is the total area of contact between the fluid and the plate. If „W" is the width of the plate then A_s = LW if the flow is taking place on one side of the plate and A_s = 2LW if the flow is on both sides of the plate.

5.3.3. Thermal boundary layer:- Similar to the velocity boundary layer one can visualize the development of a thermal boundary layer when a fluid flows over a flat plate with the temperature of the plate being different from that of the free stream fluid.Consider that a fluid at a uniform temperature T_∞ flows over a flat plate which is maintained at a uniform temperature T_W.Let T(x,y) is the temperature of the fluid at any location in the flow field.Let the dimensionless temperature of the fluid θ(x,y) be defined as

$$\theta(x,y) = \frac{T(x,y) - T_W}{T_\infty - T_W} \quad \ldots\ldots 5.10$$

The fluid layer sticking to the plate surface will have the same temperature as the plate surface $[T(x,y)_{y=0} = T_W]$ and therefore θ(x,y) = 0 at y = 0.Far away from the plate the fluid temperature is T_∞ and hence $\theta(x,y) \to 1$ as $y \to \infty$. Therefore at each location x along the plate one can visualize a location $y = \delta_t(x)$ in the flow field at which θ(x,y) = 0.99. $\delta_t(x)$ is called ***"the thermal boundary layer thickness"*** as shown in Fig. 5.3. The locus of such points at which θ(x,y) = 0.99 is called the edge of the thermal boundary layer. The relative thickness of the thermal boundary layer $\delta_t(x)$ and the velocity

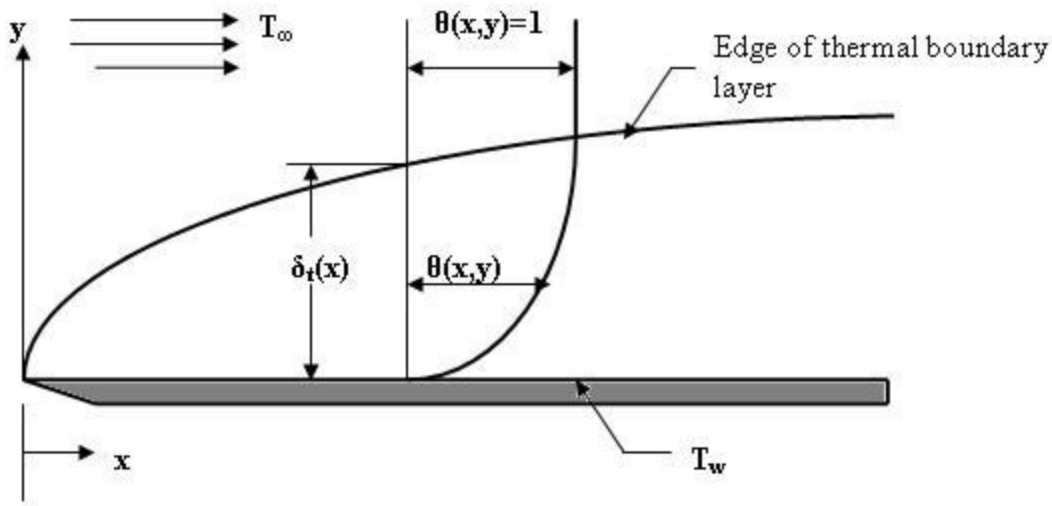


Fig. 5.4: Growth of thermal boundary layer for flow over a flat plate

boundary layer δ(x) depends on a dimensionless number called ***"Prandtl number"*** of the fluid.It is denoted by Pr and is defined as

$$Pr = \frac{\mu C_p}{k} = \frac{(\mu/\rho)}{(k/\rho C_p)} = \frac{\nu}{\alpha} \quad \ldots\ldots 5.11$$

Where μ is the absolute viscosity of the fluid, C_p is the specific heat at constant pressure

and k is the thermal conductivity of the fluid. The Prandtl number for fluids range from 0.01 for liquid metals to more than 100,000 for heavy oils. For fluids with Pr = 1 such as gases $\delta_t(x) = \delta(x)$, for fluids with Pr << 1such as liquid metals $\delta_t(x) >> \delta(x)$ and for fluids with Pr >> 1, like oils $\delta_t(x) << \delta(x)$.

5.3.4. General expression for heat transfer coefficient:- Let us assume that $T_w > T_\infty$. Then heat is transferred from the plate to the fluid flowing over the plate.Therefore at any „x" the heat flux is given by

$$q = -k\,(\partial T/\partial y)_{y=0} \quad \ldots\ldots 5.12(a)$$

In terms of the local heat transfer coefficient h_x, the heat flux can also be written as

$$q = h_x\,(T_w - T_\infty) \quad \ldots\ldots 5.12(b)$$

From equations 5.12(a) and 5.12(b) it follows that

$$h_x = \frac{-k\,(\partial T/\partial y)_{y=0}}{(T_w - T_\infty)} \quad \ldots\ldots 5.13$$

From equation 5.10 we have $(\partial T/\partial y)_{y=0} = [T_\infty - T_w]\,(\partial\theta/\partial y)_{y=0}$. Substituting this expression in Eq.5.13 and simplifying we get the general expression for h_x as

$$h_x = k\,(\partial\theta/\partial y)_{y=0} \quad \ldots\ldots 5.14$$

The same expression for h_x could be obtained even when $T_w < T_\infty$. Equation 5.14 can be used to determine the local heat transfer coefficient for flow over a flat plate if the dimensionless temperature profile $\theta(x,y)$ is known.

Average heat transfer coefficient:- For a total length L of the plate the average heat transfer coefficient is given by

$$h_{av} = (1/L)\int_0^L h_x\,dx \quad \ldots\ldots 5.15$$

Substituting for h_x from Eq. 5.14 we get

$$h_{av} = (1/L)\int_0^L k\,(\partial\theta/\partial y)_{y=0}\,dx \quad \ldots\ldots 5.15$$

Since $(\partial\theta/\partial y)_{y=0}$ at any x depends on whether the flow at that section is laminar or turbulent the expression for h_{av} can be written as

$$h_{av} = (1/L)\left\{\int^{x_{cr}} k\,[(\partial\theta/\partial y)_{y=0}]_{laminar}\,dx + \int^{L} k\,[(\partial\theta/\partial y)_{y=0}]_{turbulent}\,dx\right\} \ldots\ldots 5.16$$

Example 5.1:- *Assuming the transition from laminar to turbulent flow takes place at a Reynolds number of 5 x 10 ^{5}, determine the distance from the leading edge of a flat plate at which transition occurs for the flow of each of the following fluids with a velocity of 2 crm/s at 40 ^{0} C.(i) Air at atmospheric pressure;(ii)Hydrogen at atmospheric pressure;(iii) water;(iv) Engine oil;(v) mercury. Comment on the type of flow for the 5 fluids if the total length of the plate is 1 m.*

Solution: Data:- $Re_{cr} = 5 \times 10^{5}$; $u_\infty = 2$ m/s ; $T_\infty = 40\,^{0}$ C

(i)Air at atmospheric pressure :- At 40 ^{0} C, $\nu = 17 \times 10^{-6}$ m^2/s.

$$Re_{cr} = \frac{u_\infty x_{cr}}{\nu} \quad \text{or} \quad x_{cr} = \frac{Re_{cr}\,\nu}{u_\infty} = \frac{5 \times 10^{5} \times 17 \times 10^{-6}}{2} = 4.25 \text{ m.}$$

(ii) Hydrogen :- For hydrogen at 40 ^{0} C, $\nu = 117.9 \times 10^{-6}$ m^2/s.

Therefore $$x_{cr} = \frac{5 \times 10^{5} \times 117.9 \times 10^{-6}}{2} = 29.5 \text{ m}$$

(iii) Water :- For water at 40 ^{0} C, $\nu = 0.658 \times 10^{-6}$ m^2/s.

Therefore $$x_{cr} = \frac{5 \times 10^{5} \times 0.658 \times 10^{-6}}{2} = 0.1645 \text{ m}$$

(iv) Engine oil :- For engine oil at 40 ^{0} C, $\nu = 0.24 \times 10^{-3}$ m^2/s.

Therefore $$x_{cr} = \frac{5 \times 10^{5} \times 0.24 \times 10^{-3}}{2} = 60 \text{ m}$$

(v) Mercury :- For mercury at 40 ^{0} C, $\nu = 0.107 \times 10^{-6}$ m^2/s.

Therefore $$x_{cr} = \frac{5 \times 10^{5} \times 0.107 \times 10^{-6}}{2} = 0.027 \text{ m}$$

Comments on the type of flow

Sl.No	Type of fluid	x_{cr}	x_{cr} vs L	Type of Flow
1	Air	4.25	$x_{cr} > L$	Flow is Laminar for entire length
2	Hydrogen	29.5	$x_{cr} >> L$	Flow is laminar for entire length
3	Water	0.1645	$x_{cr} < L$	Flow is partly Laminar & Partly Turbulent
4	Engine oil	60	$x_{cr} >> L$	Flow is laminar for entire length
5	Mercury	0.027	$x_{cr} << L$	Flow is turbulent for almost entire length

Example 5.2:- *An approximate expression for the velocity profile u(x,y) for laminar boundary layer flow along a flat plate is given by*

$$u(x, y)/ u_\infty = 2[y / \delta(x)] - 2[y / \delta(x)]^3 + [y / \delta(x)]^4$$

where $\delta(x)$ *is the velocity boundary layer thickness given by the expression*

$$\delta(x) / x = 5.83 / (Re_x)^{1/2}$$

(a) *Develop an expression for the local drag coefficient.*
(b) *Develop an expression for the average drag coefficient for a length L of the plate.*
(c) *Determine the drag force acting on the plate 2 m x 2 m for flow of air with a free stream velocity of 4 m /s and a temperature of 80* ^0C.

Solution:- (a) The velocity profile u(x,y) is given as

$$u(x, y) = u_\infty \{2[y / \delta(x)] - 2[y / \delta(x)]^3 + [y / \delta(x)]^4 \}$$

Therefore $$(\partial u / \partial y)_{y=0} = 2u_\infty / \delta(x)$$

$$\eta_w(x) = \mu (\partial u / \partial y)_{y=0} = (2 \mu u_\infty) / \delta(x)$$

$$= \frac{(2 \mu u_\infty) Re_x}{5.83 x} = \frac{(2 \mu u_\infty) [(u_\infty x) / \nu]^{1/2}}{5.83 x} = 0.343 (\mu u_\infty) [u_\infty /(x \nu)]^{1/2} \quad ....(1)$$

The local drag coefficient at any x is given by

$$C_x = \frac{\eta_w(x)}{(1/2)\,\rho u_\infty^2} = \frac{0.343\,(\mu u_\infty)\,[u_\infty/(x\,\nu)]^{1/2}}{(1/2)\,\rho u_\infty^2}$$

$$= \frac{0.686}{\{(u_\infty x)/\nu\}^{1/2}} = \frac{0.686}{(Re_x)^{1/2}}$$

(b) The average drag coefficient is given by

$$C_{av} = (1/L)\int_0^L C_x\,dx = (1/L)\int_0^L 0.686\,(Re_x)^{-1/2}\,dx$$

$$= \frac{\{0.686\,(u_\infty/\nu)^{-1/2}\}}{L}\int_0^L x^{-1/2}\,dx$$

Or

$$C_{av} = \frac{2 \times 0.686}{(u_\infty L/\nu)^{1/2}} = \frac{1.372}{(Re_L)^{1/2}}$$

(c) At 80 ^{0}C for air $\nu = 20.76 \times 10^{-6}$ m^2 / s ; $\rho = 1.00$ kg / m^3

$$Re_L = \frac{u_\infty L}{\nu} = \frac{4 \times 2}{21.09 \times 10^{-6}} = 3.793 \times 10^5$$

Average drag coefficient = $C_{av} = \dfrac{1.372}{Re_L^{0.5}} = \dfrac{1.372}{(3.793 \times 10^5)^{0.5}} = 2.228 \times 10^{-3}$

Drag force assuming that the flow takes place on one side of the plate is given by

$F_D = \eta_w LW = (1/2)\rho\, u_\infty^2\, C_{av}\, LW$ for flow over one side of the plate

$= (1/2) \times 1.00 \times 4^2 \times 2.2228 \times 10^{-3} \times 2 \times 2 = 0.071$ N

Example 5.3:- *An approximate expression for temperature profile θ(x,y) in the thermal boundary layer region is given by*

$$\theta(x,y) = 2y / \delta_t - [y / \delta_t]^2$$

where the thermal boundary layer thickness δ_t is given by

$$\delta_t / x = \frac{5.5}{Re_x^{0.5} Pr^{1/3}}$$

; Re_x is the Reynolds number based on „x" and Pr is the Prandtl number of the fluid. Develop an expression for (i) the local heat transfer coefficient h_x and (ii) the average heat transfer coefficient for total length L of the plate.

Solution: (i) The local heat transfer coefficient h_x is given by

$$h_x = k (\partial\theta / \partial y)|_{y = 0}.$$

Now $$\theta(x,y) = 2y / \delta_t - [y / \delta_t]^2$$

Hence $$(\partial\theta / \partial y)|_{y = 0} = 2 / \delta_t = \frac{2 Re_x^{0.5} Pr^{1/3}}{5.5.x}$$

Or $$h_x = \frac{2 k Re_x^{0.5} Pr^{1/3}}{5.5.x} = 0.364 (k / x) Re_x^{0.5} Pr^{1/3}$$

Or $$\frac{h_x x}{K} = 0.364 Re_x^{0.5} Pr^{1/3}$$

$\frac{h_x x}{k}$ is a dimensionless number involving local heat transfer coefficient and is called "*local Nusselt number*".

(ii) The average heat transfer coefficient for a total length L of the plate is given by

$$h_{av} = (1 / L) \int_0^L h_x \, dx = (1 / L) \int 0.364 (k / x) \quad Re_x^{0.5} Pr^{1/3} dx$$

Or $$= (1 / L) 0.364 Pr^{1/3} k (U_\infty / \nu)^{0.5} \int_0^L x^{-0.5} \, dx$$

$$= (1 / L) \frac{L^{0.5}}{0.5} 0.364 Pr^{1/3} k (U_\infty / \nu)^{0.5}$$

$$= 0.728\,(k / L)\,(U_\infty L / \nu)^{0.5}\,Pr^{1/3}$$

Or $$h_{av} L / k = 0.728\, Re_L^{0.5}\, Pr^{1/3}$$

$h_{av} L / k$ is a dimensionless number involving the average heat transfer coefficient and is called the "average Nusselt number".

Example 5.4:- *The heat transfer rate per unit width from a longitudinal section $x_2 - x_1$ of a flat plate can be expressed as $q_{12} = h_{12} (x_2 - x_1)(T_s - T_\infty)$, where h_{12} is the average heat transfer coefficient for the section length of $(x_2 - x_1)$. Consider laminar flow over a flat plate with a uniform temperature T_S. The spatial variation of the local heat transfer coefficient is of the form $h_x = C\, x^{-0.5}$, where C is a constant.*
(a) Derive an expression for h_{12} in terms of C, x_1 and x_2.
(b) Derive an expression for h_{12} in terms of x_1, x_2, and the average coefficients h_1 and h_2 corresponding to lengths x_1 and x_2 respectively.

Solution:

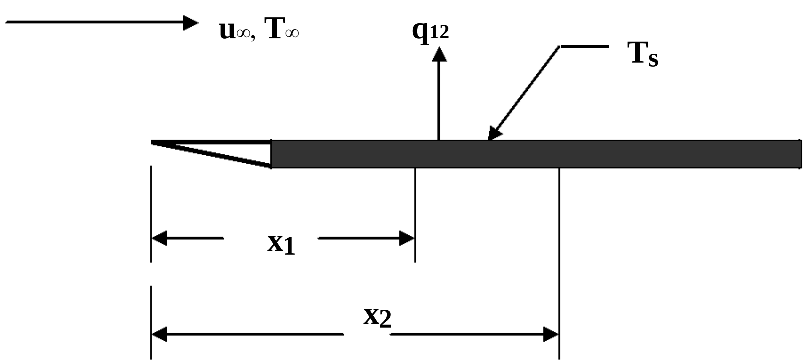


Fig. P5.5: Schematic for problem 5.5

(a) $$h_x = C\, x^{-0.5}$$

Therefore $$\bar{h}_{12} = \frac{1}{(x_2 - x_1)} \int_{x_1}^{x_2} h_x\, dx$$

$$= \frac{1}{(x_2 - x_1)} \int_{0}^{x_2} C\, x^{-0.5}\, dx$$

$$= \frac{2C}{(x_2 - x_1)} [x_2^{0.5} - x_1^{0.5}]$$

(b) $$\bar{h}_1 = (1/x_1) \int_0^{x_1} C\, x^{-0.5}\, dx$$

$$= 2C / \sqrt{x_1}$$

Similarly $$\bar{h}_2 = 2C / \sqrt{x_2}$$

Therefore Since $\int_0^{x_1} h_x dx = x_1 \bar{h}_1$, $$\bar{h}_{12} = \frac{1}{(x_2 - x_1)} \left[\int_0^{x_2} h_x dx - \int_0^{x_1} h_x dx \right]$$

$$\bar{h}_{12} = \frac{\bar{h}_2 x_2 - \bar{h}_1 x_1}{x_2 - x_1}$$

5.4. Basic Concepts For Flow Through Ducts :- The basic concepts developed on the development of velocity and thermal boundary layers for flow over surfaces are also applicable to flows at the entrance region of the ducts.

5.4.1. Velocity Boundary Layer:- Consider the flow inside a circular tube as shown in Fig.5.4. Lat u_o be the uniform velocity with which the fluid approaches the tube. As the fluid enters the tube, a *"velocity boundary layer"* starts to develop along the wall-surface. The velocity of the fluid layer sticking to the tube-surface will have zero velocity and the fluid layer slightly away from the wall is retarded. As a result the velocity in the central portion of the tube increases to satisfy the continuity equation (law of conservation of mass).The thickness of the velocity boundary layer δ(z) continuously grows along the tube-surface until it fills the entire tube. The region from the tube inlet up to little beyond the hypothetical location where the boundary layer reaches the tube centre is called *"hydrodynamic entrance region or hydrodynamically developing region"* and the corresponding length is called *"hydrodynamic entrance length L_h"*. In the hydrodynamically developing region the shape of the velocity profile changes both in axial and radial direction, i.e., u = u(r,z). The region beyond the hydrodynamic entry length is called *"Hydrodynamically developed region"*, because in this region the velocity profile is invariant with distance along the tube,i.e., u = u(r).

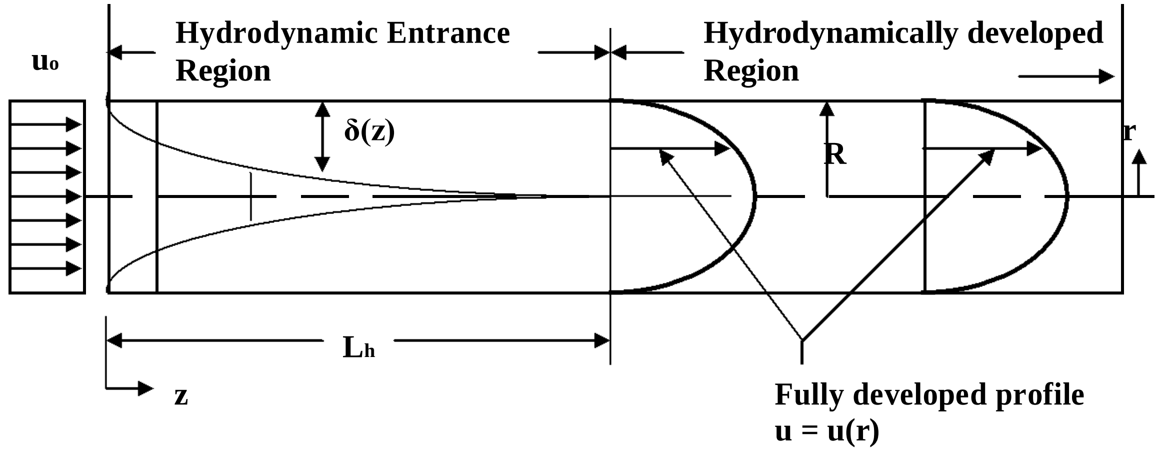


Fig. 5.4: development of velocity boundary layer at entrance region of a tube

If the boundary layer remains laminar until it fills the tube, then laminar flow will prevail in the developed region. However if the boundary layer changes to turbulent before its thickness reaches the tube centre, fully developed turbulent flow will prevail in the hydrodynamically developed region. The velocity profile in the turbulent region is flatter than the parabolic profile of laminar flow. The Reynolds number, defined as

$$Re_d = (u_m D_h) / \nu \quad \text{.......................................}(5.17)$$

is used as a criterion for change from laminar flow to turbulent flow. In this definition, u_m is the average velocity of the fluid in the tube, D_h is the hydraulic diameter of the tube and ν is the kinematic viscosity of the fluid. The hydraulic diameter is defined as

$$D_h = \frac{4 \text{ x Area of flow}}{\text{Wetted Perimeter}} \quad \text{........................}(5.18)$$

For flows through ducts it has been observed that turbulent flow prevails for

$$Re_d \geq 2300 \quad \text{..}(5.19)$$

But this critical value is strongly dependent on the surface roughness, the inlet conditions and the fluctuations in the flow. In general, transition may occur in the range $2000 < Re_d < 4000$. It is a common practice to assume a value of 2300 fro transition from laminar flow to turbulent flow.

5.4.2. Friction Factor and Pressure Drop Relations For Hydrodynamically Developed Laminar Flow

In engineering applications, the pressure gradient (dp / dz) associated with the flow is a quantity of interest, because this decides the pumping power required to overcome thr frictional losses in the pipe of a given length.

Consider a differential length dz of the tube at a distance z from the entrance and let this length be in the fully developed region. The various forces acting on the fluid element in the direction of flow are shown in Fig.5.5.

Resultant force in the direction of motion = $F = (pA)_z - (pA)_{z+dz} - \eta_w \, S dz$

where S is the perimeter of the duct.
Using Taylor"s series expansion and neglecting higher order terms we can write

$$(pA)_{z+dz} = (pA)_z + \frac{d}{dz}(pA)\, dz$$

Therefore $F = \frac{d}{dz}(pA)\, dz - \eta_w \, S dz$

Rate of change of momentum in the direction of flow = 0 because the velocity u does not vary with respect to z in the fully developed region.

Hence $$\frac{d}{dz}(pA)\, dz - \eta_w \, S dz = 0$$

For duct of uniform cross section A is constant. Therefore the above equation reduces to

$$\mathbf{dp/dz = -\eta_w \, S / A} \quad \text{……………………………….}\mathbf{(5.20)}$$

For laminar flow $\eta_w = -\mu\, (du / dr)|_{wall}$. Hence Eq. (5.20) reduces to

$$\frac{dp}{dz} = \frac{\mu S}{A} (du/dr)|_{wall} \quad \text{……………………….(5.21)}$$

Eq.(5.21) is not practical for the determination of (dp/dz), because it requires the evaluation f the velocity gradient at the wall. Hence for engineering applications a parameter called *"friction factor, f "* is defined as follows:

$$f = -\frac{(dp/dz)\, D_h}{\frac{1}{2}(\rho u_m^2)} \quad \text{………………………….(5.22a)}$$

Substituting for (dp/dz) from Eq. (5.21) we have

$$f = -\frac{(\mu S/A)\, (du/dr)|_{wall}\, D_h}{\frac{1}{2}(\rho u_m^2)} \quad \text{………………….(5.22b)}$$

For a circular tube $S = \pi D_i$, and $A = \pi D_i^2 / 4$. Hence $D_h = D_i$

Hence for a circular tube Eq. (5.22b) reduces to

$$f = -\frac{8\mu}{(\rho u_m^2)} (du/dr)|_{wall} \quad \text{.............................(5.22c)}$$

Also from Eq. (5.22a) we have

$$dp = -\frac{(½)(\rho u_m^2) f}{D_h} dz$$

Integrating the above equation over a total length L of the tube we have

$$\int_{p_1}^{p_2} dp = -\frac{(½)(\rho u_m^2) f}{D_h} \int_0^L dz$$

or pressure drop = $\mathbf{\Delta p = (p_1 - p_2) = (½)(L/D_h) f \rho u_m^2}$ **.............................(5.23)**

Pumping power is given by $P = \dot{V} \Delta p$ ....(5.24)

where $\dot{V}$ = volume flow rate of the fluid.

<u>5.4.3. Thermal Boundary Layer:</u> In the case of temperature distribution in flow inside a tube, it is more difficult to visualize the development of thermal boundary layer and the existence of thermally developed region. However under certain heating or cooling conditions such as *constant wall-heat flux* or *constant wall-temperature* it is possible to have thermally developed region.

Consider a laminar flow inside a circular tube subjected to uniform heat flux at the wall. Let „r" and „z" be the radial and axial coordinates respectively and T(r,z) be the local fluid temperature. A dimensionless temperature θ(r,z) is defined as

$$\theta(r,z) = \frac{T(r,z) - T_w(z)}{T_m(z) - T_w(z)} \quad \text{..................................(5.25a)}$$

where $T_w(r,z)$ = Tube wall-temperature and $T_m(z)$ = Bulk mean temperature of the fluid. The bulk mean temperature at any cross section „z" is defined as follows:

$$T_m(z) = \frac{\int \rho(2\pi r dr)\, u(r,z)\, C_p\, T(r,z)}{\int \rho(2\pi r dr)\, u(r,z)\, C_p} = \frac{\int r dr\, u(r,z) T(r,z)}{\int r dr)\, u(r,z)} \quad \text{..................(5.25b)}$$

At the tube wall it is clear that $\theta(r,z) = 0$ and attains some finite value at the centre of the tube. Thus we can visualize the development of thermal boundary layer along the tube surface as shown in Fig. 5.5.The thickness of the thermal boundary layer δ_t continuously grows along the tube surface until it fills the entire tube. The region from the tube inlet to the hypothetical location where the thermal boundary layer thickness reaches the tube centre is called the *"thermal entry section"*. In this region the shape of the dimensionless temperature profile $\theta(r,z)$ changes both in axial and in radial directions. The region beyond the thermal entry section is called as the *"thermally developed region"*, because in this region the dimensionless temperature profile θ remains invariant with respect to z. That is in this region $\theta = \theta(r)$.It is difficult to explain qualitatively why θ should be independent of z even though the temperature of the fluid T depends both on r and z. However it can be shown mathematically that, for both constant wall-heat flux and constant wall-temperature conditions, θ depends only on r for large values of z.For constant wall-heat flux condition the wall-temperature $T_W(z)$ increases with z.

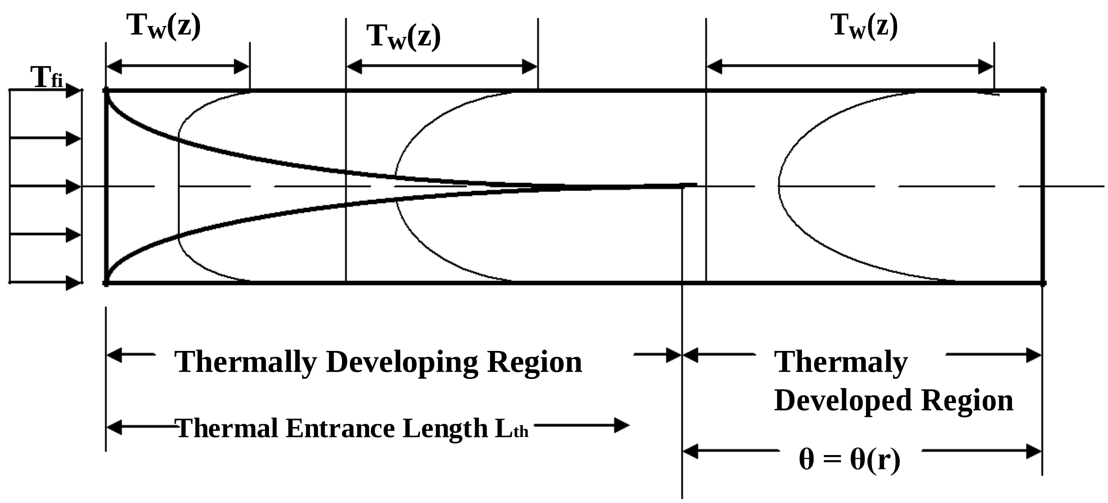


Fig. 5.5: Development of Thermal Boundary Layer In a Flow Through A Tube Subjected to Constant Wall-Heat Flux Condition

The variation of wall-temperature and the bulk fluid temperature as we proceed along the length of the tube for constant wall-heat flux conditions is shown in Fig. 5.6.

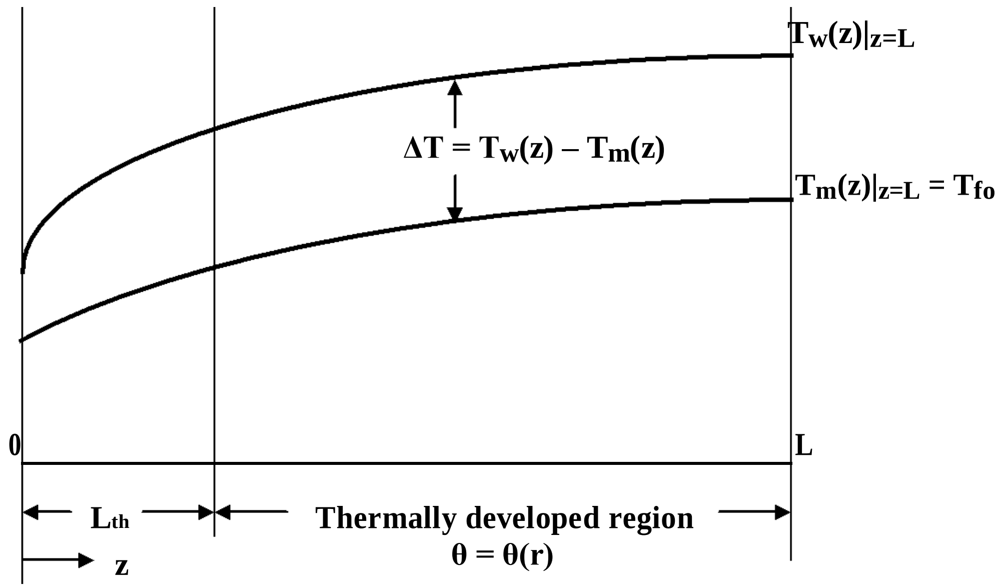


Fig. 5.6: Variation of tube wall-temperature and bulk fluid temperature along the length of the tube

It can be shown that for constant wall-heat flux condition the temperature difference ΔT between the tube wall and the bulk fluid remains constant along the length of the tube.

The growth of the thermal boundary layer for constant wall-temperature conditions is similar to that for constant wall-heat flux condition except that the wall temperature does not vary with respect to z. Therefore the temperature profile T(r,z) becomes flatter and flatter as shown in Fig. 5.7 as we proceed along the length of the tube and eventually the fluid temperature becomes equal to the wall temperature. Since the

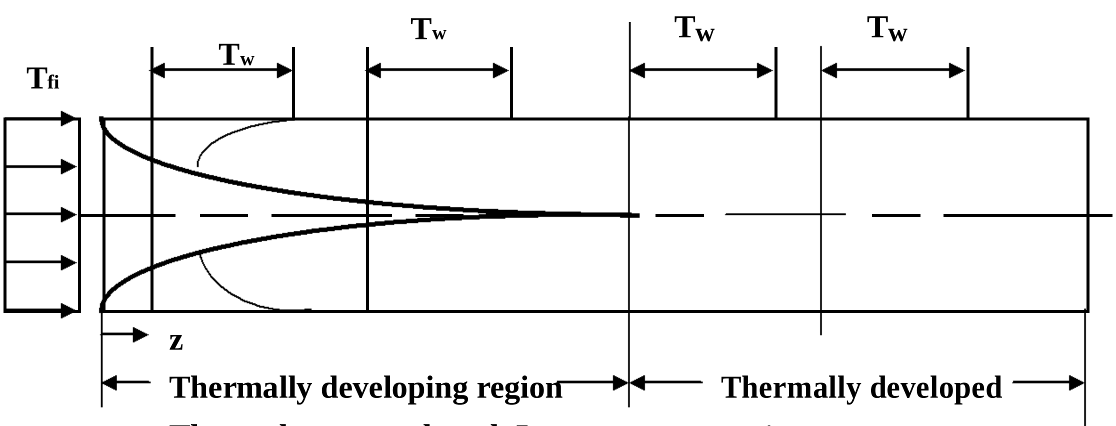


Fig.5.7: Growth of thermal boundary layer for flow through a tube with constant wall-temperature

wall-temperature remains constant and the bulk fluid temperature varies along the length

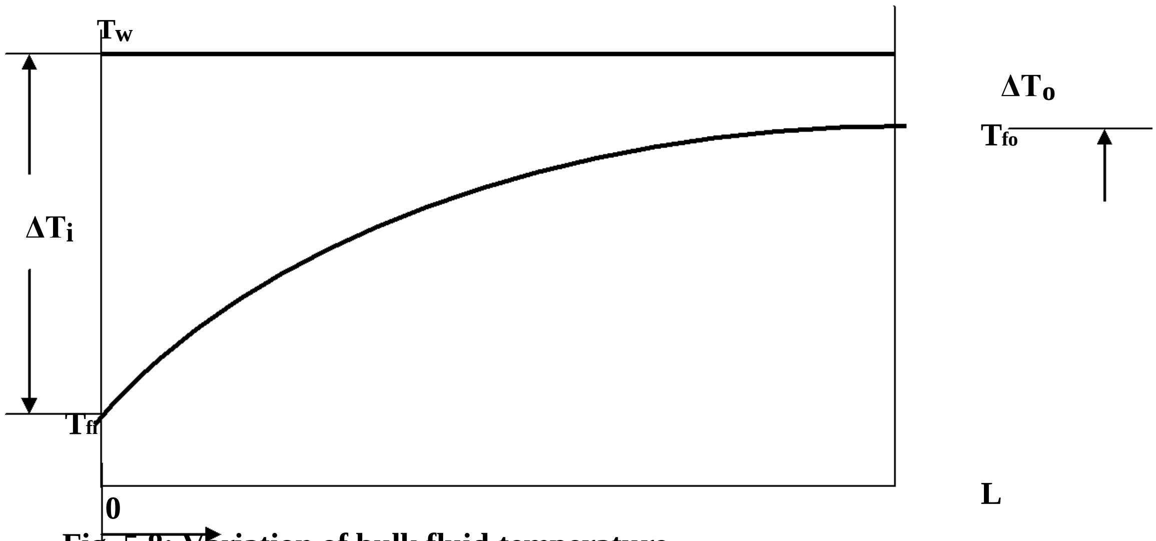


Fig. 5.8: Variation of bulk fluid temperature along the length of the tube for tube with constant wall-temperature

the temperature difference between the tube wall and the bulk fluid varies along the length of the tube as shown in Fig. 5.8.

5.4.4. Mean Temperature Difference, ΔT_m: If Q is the total heat transfer rate between the fluid and the tube surface , A_s is the area of contact between the fluid and the surface, h_m is the average heat transfer coefficient for the total length of the tube then we can write

$$Q = h_m A_s \Delta T_m \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(5.26)$$

Where ΔT_m = mean temperature difference between the tube wall and the bulk fluid. For a tube with constant wall-heat flux condition, since the temperature difference between the fluid and the tube surface remains constant along the length of the tube it follows that

$$\Delta T_m = [T_w(z)|_{z=0} - T_{fi}] = [T_w(z)|_{z=L} - T_{fo}] \ldots\ldots\ldots\ldots\ldots\ldots(5.27a)$$

For a tube with constant wall-temperature condition the mean temperature difference is given by

$$\Delta T_m = \frac{\Delta T_i - \Delta T_o}{\ln(\Delta T_i / \Delta T_o)} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5.27b)$$

Free Convective Heat Transfer

A. Free convection from/to plane surfaces:

7.1. A vertical plate 30 cm high and 1 m wide and maintained at a uniform temperature of 120 ^{0} C is exposed to quiescent air at 30 ^{0} C.Calculate the average heat transfer coefficient and the total heat transfer rate from the plate to air.

7.2. An electrically heated vertical plate of size 25 cm x 25 cm is insulated on one side and dissipates heat from the other surface at a constant rate of 600 W/m^2 by free convection into quiescent atmospheric air at 30 ^{0} C. Determine the surface temperature of the plate.

7.3. Determine the heat transfer by free convection from a plate 30 cm x 30 cm whose surfaces are maintained at 100 ^{0} C and exposed to quiescent air at 20 ^{0} C for the following conditions: (a) the plate is vertical. (b) Plate is horizontal

7.4. A circular plate of 25 cm diameter with both surfaces maintained at a uniform temperature of 100 ^{0} C is suspended in horizontal position in atmospheric air at 20 ^{0} C. Determine the heat transfer from the plate.

7.5. Consider an electrically heated plate 25 cm x 25 cm in which one surface is thermallt insulated and the other surface is dissipating heat by free convection into atmospheric air at 30 ^{0} C. The heat flux over the surface is uniform and results in a mean surface temperature of 50 ^{0} C. The plate is inclined making an angle of 50 ^{0} from the vertical. Determine the heat loss from the plate for (i) heated surface facing up and (ii) heated surface facing down.

7.6. A thin electric strip heater of width 20 cm is placed with its width oriented vertically. It dissipates heat by free convection from both the surfaces into atmospheric air at 20 ^{0} C. If the surface temperature of the heater is not to exceed 225 ^{0} C, determine the length of the heater required in order to dissipate 1 kW of energy into the atmospheric air.

7.7. A plate 75 cm x 75 cm is thermally insulated on the one side and subjected to a solar radiation flux of 720 W/m^2 on the other surface. The plate makes an angle of 60 ^{0} with the vertical such that the hot surface is facing upwards. If the surface is exposed to quiescent air at 25 ^{0} C and if the heat transfer is by pure free convection determine the equilibrium temperature of the plate.

B. Free convection from/to Cylinders:

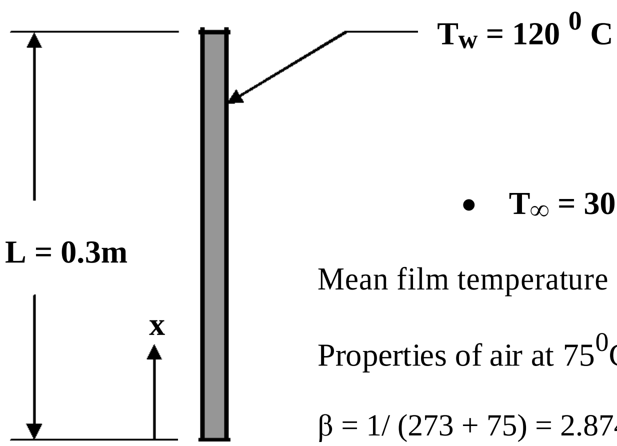
7.8. A 5 cm diameter, 1.5 m long vertical tube at a uniform temperature of 100 ^{0}C is exposed to quiescent air at 20 ^{0} C. calculate the rate of heat transfer from the surface to air. What would be the heat transfer rate if the tube were kept horizontally?

7.9. A horizontal electrical cable of 25 mm diameter has a heat dissipation rate of 30 W/m. If the ambient air temperature is 27 ^{0} C, estimate the surface temperature of the cable.

7.10. An electric immersion heater, 10 mm in diameter and 300 mm long is rated at 550 W. If the heater is horizontally positioned in a large tank of water at 20 ^{0}C, estimate its surface temperature. What would be its surface temperature if the heater is accidentally operated in air.

A.Free Convection to or from plane surfaces

7.1. Solution:



- **T_∞ = 30 ^{0}C**

Mean film temperature of air = 0.5 x (120 +30) = 75^{0}C

Properties of air at 75^{0}C are:

β = 1/ (273 + 75) = 2.874 x 10 $^{-3}$ 1/K; Pr = 0.693

k = 0.03 W/(m-K) ; ν = 20.555 x 10 $^{-6}$ m ^{2}/s ;

\

First we have to establish whether the flow become turbulent within the given length of the plate by evaluating the Rayleigh number at x = L.

$$Gr_L = (g\beta\Delta T L^3) / \nu^2 = \frac{9.81 \times 2.874 \times 10^{-3} \times (120 - 30) \times 0.3^3}{20.555 \times 10^{-6}}$$

$$= 1.62 \times 10^8$$

Rayleigh number = $Ra_L = Gr_L Pr$ = 1.62 x 10 ^{8} x 0.693 = 1.12 x 10 ^{8}.

Since $Ra_L < 10^9$ flow is laminar for the entire height of the plate. Hence the average Nusselt number is given by (from data hand book)

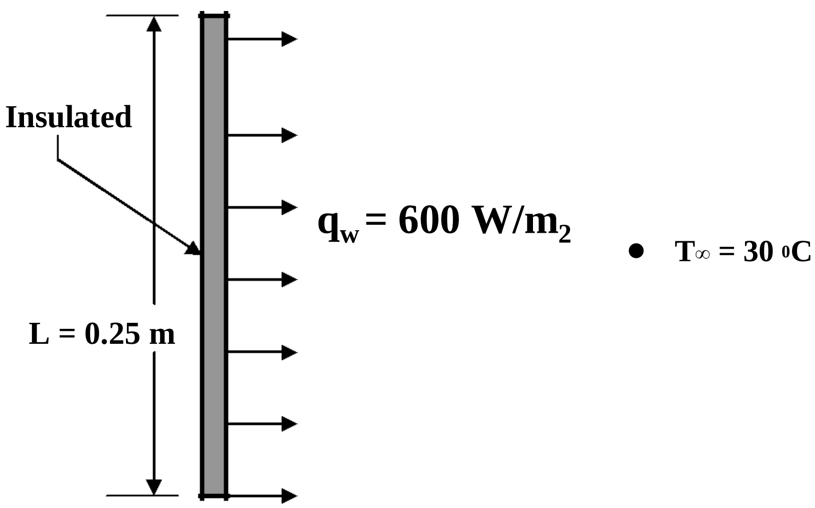
$$Nu_{av} = 0.59 \times (Ra_L)^{0.25} = 0.59 \times (1.12 \times 10^8)^{0.25} = 60.695$$

Therefore $h_{av} = Nu_{av}\, k / L = \dfrac{60.6 \times 0.03}{0.3} = 6.069$ W /(m^2 – K).

Total heat transfer fro both sides of the plate per unit width of the plate is given by

$$Q_{total} = h_{av}(2LW)\ T = 6.06 \times (2 \times 0.3 \times 1) \text{ X } (120 - 30) = 327.726 \text{ W/m.}$$

7.2. Solution:



Since T_W is not known, it is not possible to determine the mean film temperature at which fluid properties have to be evaluated. Hence this problem requires a trial and error solution either by assuming T_W and then calculate T_W by using the heat balance equation and check for the assumed value or assume a value for h_{av} ,calculate T_W and then calculate h_{av} and check for the assumed value of h_{av}.Since it is difficult to guess a reasonable value for T_W to reduce the number of iterations, it is preferable to guess a reasonable value for h_{av} for air as we know that for air h_{av} varies anywhere between 5 and 15 W/(m^2-K).

Trial 1:- Assume h_{av} = 10 W/(m^2-K).

Now $q_W = h_{av}[T_W - T_\infty]$ or $T_W = T_\infty + q_W / h_{av} = 30 + 600 / 10 = 90^0$C.

Hence mean film temperature = 0.5 x [90 + 30] = 60 ^{0}C.

Properties of air at 60 ^{0}C are: $\beta = 1 / (60 + 273) = 3.003 \times 10^{-3}$ 1/K; Pr = 0.696;

k = 0.02896 W/(m-K); $\nu = 18.97 \times 10^{-6}$ m^2/s.

$$Ra_L^* = Gr_L^* Pr = [(g\beta q_w x^4)/(k\nu^2)]Pr = \frac{9.81 \times 3.003 \times 10^{-3} \times 600 \times (0.25)^4}{0.02896 \times (18.97 \times 10^{-6})^2} \times 0.696$$

Or $Ra_L^* = 4.61 \times 10^9$.

Since $Ra_L^* > 10^9$ flow is turbulent for the entire length of the plate

Hence $Nu_{av} = 1.25\, Nu_x|_{x=L} = 1.25 \times 0.17 \times (4.61 \times 10^9)^{0.2} = 55.37$

Therefore $h_{av} = 55.37 \times 0.02896 / 0.25 = 6.41$ W/(m^2 – K)

Since the calculated value of h_{av} deviates from the assumed value by about 34 %, one more iteration is required.

Trial 2:- Assume h_{av} = 6.41 W/(m^2-K)

Hence $T_w = 30 + 600 / 6.41 = 123.6\ ^0C \cong 120\ ^0C$

Mean film temperature = 0.5 x (120 + 30) = 75 ^{0}C

Properties of air at 75 ^{0}C are:- $\beta = 1/(75 + 273) = 2.873 \times 10^{-3}$ 1/K. Pr =

0.686 k = 0.03338 W /(m-K); $\nu = 25.45 \times 10^{-6}$ m^2/s.

$$Ra_L^* = \frac{9.81 \times 2.873 \times 10^{-3} \times 600 \times 0.25^4}{0.03338 \times (25.45 \times 10^{-6})^2} \times 0.686 = 2.06 \times 10^9$$

Flow is turbulent for the entire length of the plate.

Hence $Nu_{av} = 1.25\, Nu_x|_{x=L} = 1.25 \times 0.17 \times (2.06 \times 10^9)^{0.25} = 45.27$

Therefore $h_{av} = 45.27 \times 0.03338 / 0.25 = 6.04$ W/(m^2 – K).

Since the calculated value of h_{av} is very close to the assumed value, the iteration is stopped. The surface temperature of the plate is therefore given by

$$T_w = 30 + 600 / 6.04 = 129.3\ ^0C.$$

7.3. Solution:- *Case(i) When the plate is vertical*

Data:- Characteristic length = L = height of the plate = 0.3 m; T_w = 100 ^{0}C; T_∞ = 20 ^{0}C; Mean film temperature = 0.5 x (100 + 20) = 60 ^{0}C.

Properties of air at 60 ^{0}C are: $\beta = 1 / (60 + 273) = 3.003 \times 10^{-3}$ 1/K; Pr = 0.696;

k = 0.02896 W/(m-K); $\nu = 18.97 \times 10^{-6}$ m^2/s.

$$Ra_L = Gr_L Pr = (g\beta\Delta T L^3 / \nu^2)\, Pr$$

$$= \frac{9.81 \times 3.003 \times 10^{-3} \times (100 - 20) \times (0.3)^3}{(18.97 \times 10^{-6})^2} \times 0.696$$

$$= 1.23 \times 10^8$$

From data hand book corresponding to this value of Ra_L have

$$Nu_{av} = 0.59 \times (1.23 \times 10^8)^{0.25} = 62.13$$

Therefore $h_{av} = 62.13 \times 0.02896 / 0.3 = 5.99$ W/(m^2-K).

Rate of heat transfer = $Q = h_{av}(2LW)(\Delta T) = 5.99 \times (2 \times 0.3 \times 0.3) \times (100 - 20)$

$= 86.256$ W

UNIT-IV
Condensation & Boiling

Introduction: Knowledge of heat transfer occurring during change of phase i.e. during condensation and boiling is very useful in a number of ways. For example in all power and refrigeration cycles, it is necessary to convert a liquid into a vapour and vice-versa. This is accomplished in boilers or evaporators and condensers.

Heat transfer coefficients in both condensation and boiling are generally much higher than those encountered in single phase processes. Values greater than 1000 W/(m^2-K) are almost always obtained. This fact has been used in several recent applications where it is desired to transfer high heat fluxes with modest temperature differences. An example is the "heat pipe" which is a device capable of transferring a large quantity of heat with very small temperature differences.

8.2. Film-wise and Drop-wise condensation:- Condensation occurs whenever a vapour comes into contact with a surface at a temperature lower than the saturation temperature of the vapour corresponding to its vapour pressure. The nature of condensation depends on whether the liquid thus formed wets the solid surface or does not wet the surface. If the liquid wets the surface, the condensate flows on the surface in the form of a film and the process is called "*film-wise condensation"*. If on the other hand, the liquid does not wet the surface, the condensate collects in the form of droplets, which either grow in size or coalesce with neighboring droplets and eventually roll of the surface under the influence of gravity. This type of condensation is called "*drop-wise condensation"*.

The rate of heat transfer during the two types of condensation processes is quite different. For the same temperature difference between the vapour and the surface, the heat transfer rates in drop-wise condensation are significantly higher than those in film-wise condensation. Therefore it is preferable to have drop-wise condensation from the designer‟s point of view if the thermal resistance on the condensing side is a significant part of the total thermal resistance. However it is generally observed that, although drop-wise condensation may be obtained on new surfaces, it is difficult to maintain drop-wise condensation continuously and prolonged condensation results in a change to film-wise condensation. Therefore it is still the practice to design condensers under the conservative assumption that the condensation is of film type.

8.3. Nusselt's theory for laminar film-wise condensation on a plane vertical surface:-The problem of laminar film-wise condensation on a plane vertical surface was first analytically solved by Nusselt in 1916.He made the following simplifying assumptions in his analysis.

(i) The fluid properties are constant.

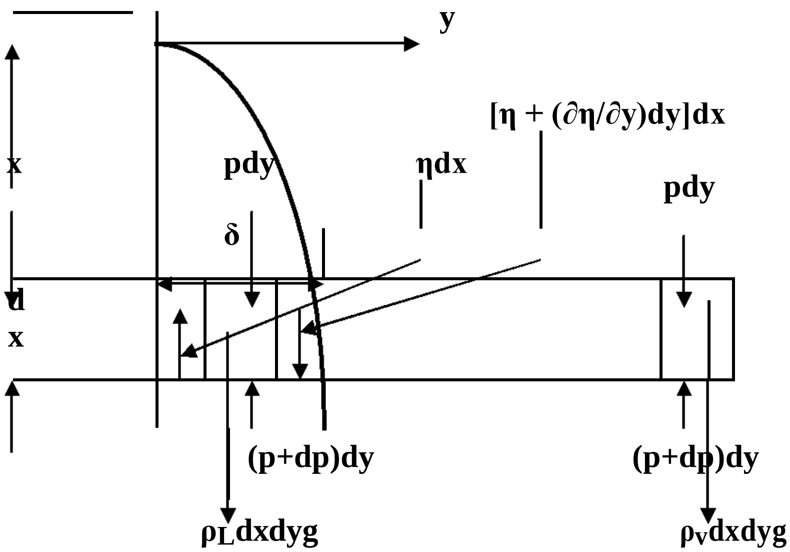
(ii) The plane surface is maintained at a uniform temperature, T_W which is less than the saturation temperature T_V of the vapour.

(iii) The vapour is stationary or has a very low velocity and so it does not exert any drag on the motion of the condensate: i.e., the shear stress at the liquid-vapour interface is zero.

(iv) The flow velocity of the condensate layer is so low that the acceleration of the condensate is negligible.

(v) The downward flow of the condensate under the action of gravity is laminar.

(vi) Heat transfer across the condensate layer is purely by conduction; hence the liquid temperature distribution is linear.



(a) Force balance on a condensate element

(b) Force balance on a vapour element at the same distance x from top

Fig. 8.1: Laminar film condensation on a vertical plate

Consider the film-wise condensation on a vertical plate as illustrated in Fig.8.1. Here „x‟ is the coordinate measured downwards along the plate, and „y‟ is the coordinate measured normal to the plate from the plate surface. The condensate thickness at any x is represented by δ [$\delta = \delta(x)$]. The velocity distribution u(y) at any location x can be determined by making a force balance on a condensate element of dimensions dx and dy in x and y directions as shown in Fig. 8.1(a). Since it is assumed that there is no acceleration of the liquid in x direction, Newton‟s second law in x direction gives

$$\rho_L dxdyg + pdy + [\eta + (\partial\eta/\partial y)dy]dx - \eta dx - (p + dp)dy = 0$$

or $$(\partial\eta/\partial y) = (dp/dx) - \rho_L g \quad \ldots\ldots(8.1)$$

Expression for (dp/dx) in terms of vapour density ρ_V can be obtained by making a force

balance for a vapour element as shown in Fig. 8.1(b). The force balance gives

$$\rho_V dxdyg + pdy = (p + dp)\, dy$$

or $(dp/dx) = \rho_V g$ Substituting this expression for dp/dx in

Eq. (8.1) we have

$$(\partial\eta/\partial y) = (\rho_V - \rho_L)g$$

Since the flow is assumed to be laminar, $\eta = \mu_L(\partial u/\partial y)$

Therefore $$\partial/\partial y\{\mu_L (\partial u/\partial y)\} = (\rho_V - \rho_L)g$$

Integrating with respect to y we have $$\mu_L (\partial u/\partial y) = (\rho_V - \rho_L)g\, y + C_1$$

Or $$(\partial u/\partial y) = \frac{(\rho_V - \rho_L)g\, y}{\mu_L} + \frac{C_1}{\mu_L} \quad \ldots\ldots(8.2)$$

Integrating once again with respect to y we get

$$u(y) = \frac{(\rho_V - \rho_L)g\, y^2}{2\,\mu_L} + \frac{C_1\, y}{\mu_L} + C_2 \quad \ldots\ldots (8.3)$$

The boundary conditions for the condensate layer are: (i) at y = 0, u = 0;

(ii) at $y = \delta$, $(\partial u/\partial y) = 0$.

Condition (i) in Eq. (8.3) gives $C_2 = 0$ and condition (ii) in Eq. (8.2) gives

$$0 = \frac{(\rho_V - \rho_L)g\, \delta}{2\,\mu_L} + \frac{C_1}{\mu_L}$$

Therefore $$C_1 = -\frac{(\rho_V - \rho_L)g\, \delta}{2}$$

Substituting for C_1 and C_2 in Eq.(8.3) we get the velocity distribution in the condensate layer as

$$u(y) = \frac{g(\rho_L - \rho_v)}{\mu_L}\,[\,\delta y - (y^2/2)] \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(8.4)$$

If „m" is the mass flow rate of the condensate at any x then

$$m = \int_0^{\delta} \rho_L u\, dy$$

$$m = \int_0^{\delta} \rho_L\{\, g(\rho_L - \rho_v) / \mu_L\}[\,\delta y - (y^2/2)]dy$$

$$= \frac{g \rho_L (\rho_L - \rho_v) \delta^3}{3 \mu_L} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8.5)$$

Hence $$dm = \frac{g \rho_L (\rho_L - \rho_v) \delta^2 \, d\delta}{\mu_L}$$

Amount of heat transfer across the condensate element = dq = dm h_{fg}

Or $$dq = \frac{g \rho_L (\rho_L - \rho_v) \delta^2 \, d\delta \, h_{fg}}{\mu_L} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8.6)$$

Energy balance for the condensate element shown in the figure can be written as

$$dq = k_L(T_v - T_w)dx / \delta$$

Or $$\frac{g \rho_L (\rho_L - \rho_v) \delta^2 \, d\delta \, h_{fg}}{\mu_L} = k_L(T_v - T_w)dx / \delta \ldots\ldots\ldots\ldots (8.6)$$

or $$\delta^3 d\delta = \frac{k_L \mu_L (T_v - T_w)dx}{g \rho_L (\rho_L - \rho_v) h_{fg}}$$

Integrating we get

$$\frac{\delta^4}{4} = \frac{k_L \mu_L (T_v - T_w)x}{g \rho_L (\rho_L - \rho_v) h_{fg}} + C_3$$

At x = 0, δ = 0. Hence C_3 = 0.

Therefore $$\frac{\delta^4}{4} = \frac{k_L \mu_L (T_v - T_w)x}{g \rho_L (\rho_L - \rho_v) h_{fg}}$$

or $$\boldsymbol{\delta = \left[\frac{4 k_L \mu_L (T_v - T_w)x}{g \rho_L (\rho_L - \rho_v) h_{fg}} \right]^{1/4}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8.7)$$

Now
$$\frac{k_L (T_v - T_w) dx}{\delta} = h_x \, dx \, [T_v - T_w]$$

Therefore
$$h_x = \frac{k_L}{\delta} = \left[\frac{g \rho_L (\rho_L - \rho_v) h_{fg} k_L^3}{4 \mu_L (T_v - T_w) x}\right]^{1/4}$$

Or
$$\mathbf{h_x = 0.707\left[\frac{g \rho_L (\rho_L - \rho_v) h_{fg} k_L^3}{\mu_L (T_v - T_w) x}\right]^{1/4}} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (8.8)$$

The local Nusselt number Nu_x can therefore be written as

$$\mathbf{Nu_x = \frac{h_x x}{k_L} = 0.707\left[\frac{g \rho_L (\rho_L - \rho_v) h_{fg} x^3}{\mu_L (T_v - T_w) k_L}\right]^{1/4}} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (8.8)$$

The average heat transfer coefficient for a length L of the plate is given by

$$h_{av} = (1/L) \int_0^L h_x dx \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8.9)$$

It can be seen from Eq. (8.8) that $h_x = C\, x^{-1/4}$, where C is a constant given by

Or
$$C = 0.707\left[\frac{g \rho_L (\rho_L - \rho_v) h_{fg} k_L^3}{\mu_L (T_v - T_w)}\right]^{1/4} \quad \ldots\ldots\ldots\ldots\ldots\ldots (8.10)$$

Hence
$$h_{av} = (1/L)\, C \int_0 x^{-1/4} dx = (C / L)\,(4/3)\, L^{-1/4} = (4/3) C\, L^{-1/4}$$

Substituting for C from Eq. (8.10) we have

$$\mathbf{h_{av} = 0.943\left[\frac{g \rho_L (\rho_L - \rho_v) h_{fg} k_L^3}{\mu_L (T_v - T_w) L}\right]^{1/4} = (4/3) h_x|_{x = L} \ldots\ldots\ldots\ldots\ldots\ldots (8.11)}$$

8.4. Condensation on Inclined Surfaces : Nusselt,s analysis given above can readily be extended to inclined plane surfaces making an angle θ with the horizontal plane as shown in Fig. 8.2.

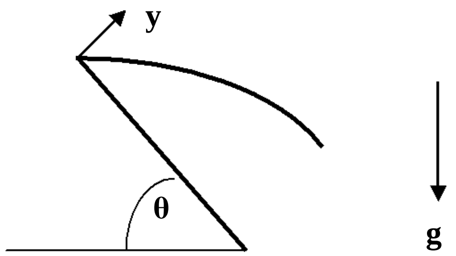


Fig. 8.2 : Condensation on an inclined plane surface

The component of the gravitational force along the length of the pate is g sin θ.The expressions for local and average heat transfer coefficients can therefore be written as

$$h_x = 0.707\left[\frac{g \sin\theta \rho_L (\rho_L - \rho_v) h_{fg} k_L^3}{\mu_L (T_v - T_w) x}\right]^{1/4} \quad \ldots\ldots\ldots\ldots(8.12)$$

and

$$h_{av} = 0.943\left[\frac{g \sin\theta \rho_L (\rho_L - \rho_v) h_{fg} k_L^3}{\mu_L (T_v - T_w) L}\right]^{1/4} = (4/3) h_x|_{x = L} \quad \ldots\ldots\ldots\ldots(8.13)$$

8.5. Condensation on a horizontal tube: The analysis of heat transfer for condensation on the outside surface of a horizontal tube is more complicated than that for a vertical surface. Nusselt,s analysis for laminar film-wise condensation on the surface of a horizontal tube gives the average heat transfer coefficient as

$$h_{av} = 0.725\left[\frac{g \rho_L (\rho_L - \rho_v) h_{fg} k_L^3}{\mu_L (T_v - T_w) D}\right]^{1/4} \quad \ldots\ldots\ldots\ldots(8.14)$$

where D is the outside diameter of the tube. A comparison of equations (8.11) and (8.14) for condensation on a vertical tube of length L and a horizontal tube of diameter D gives

$$\frac{[h_{av}]_{vertical}}{[h_{av}]_{horizontal}} = \frac{0.943}{0.725}(D/L)^{1/4} = 1.3\,(D/L)^{1/4} \quad \ldots\ldots\ldots\ldots(8.15)$$

This result implies that for a given value of ($T_V - T_W$), the average heat transfer coefficient for a vertical tube of length L and a horizontal tube of diameter D becomes equal when L = 2.856 D.For example when L = 100 D, theoretically $[h_{av}]_{horizontal}$ would be 2.44 times $[h_{av}]_{vertical}$. Therefore horizontal tube arrangements are generally preferred to vertical tube arrangements in condenser design.

8.6. Condensation on horizontal tube banks: Condenser design generally involves horizontal tubes arranged in vertical tiers as shown in Fig. 8.3 in such a way that the

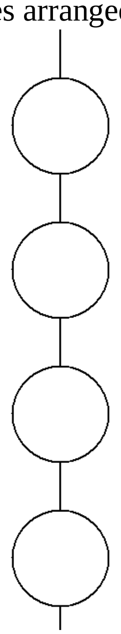


Fig. 8.3 : Film-wise condensation on horizontal tubes arranged in a vertical tier.

condensate from one tube drains on to tube just below. If it is assumed that the drainage from one tube flows smoothly on to the tube below, then for a vertical tier of N tubes each of diameter D, the average heat transfer coefficient for N tubes is given by

$$[h_{av}]_{N\ tubes} = 0.725 \left[\frac{g\rho_L(\rho_L - \rho_V)h_{fg}\, k_L^3}{\mu_L(T_V - T_W)\, N\, D} \right]^{1/4} = \frac{1}{N^{1/4}} [h_{av}]_{1\ tube} \quad \text{..............(8.16)}$$

This relation generally gives a conservative value for the heat transfer coefficient. Since some turbulence and some disturbance of condensate are unavoidable during drainage, the heat transfer coefficient would be more than that given by the above equation.

8.7. Reynolds number for condensate flow: Although the flow hardly changes to turbulent flow during condensation on a single horizontal tube, turbulence may start at the lower portions of a vertical tube. When the turbulence occurs in the condensate film, the average heat transfer coefficient begins to increase with the length of the tube in contrast to its decrease with the length for laminar film condensation. To establish a criterion for transition from laminar to turbulent flow, a *"Reynolds number for condensate flow"* is defined as follows.

$$Re = \frac{\rho_L\, u_{av}\, D_h}{\mu_L} \quad \text{.......................(8.17)}$$

where u_{av} is the average velocity of the condensate film and D_h is the hydraulic diameter for the condensate flow given by

$$D_h = \frac{4 \text{ x (Cross sectional area for condensate flow)}}{\text{Wetted Perimeter}} = \frac{4A}{P}$$

Therefore $$Re = \frac{4A\, \rho_L\, u_{av}}{P\, \mu_L} = \frac{4M}{P\, \mu_L} \quad \text{..................(8.18)}$$

where M is mass flow rate of condensate at the lowest part of the condensing surface in kg/s. The wetted perimeter depends on the geometry of the condensing surface and is given as follows.

πD .....For vertical tube of outside diameter D ....(8.19 a)

P = 2L ......For horizontal tube of length L ....(8.19 b)

W ..... For vertical or inclined plate of width W...........(8.19 c)

Experiments have shown that the transition from laminar to turbulent condensation takes place at a Reynolds number of 1800. The expression for average heat transfer coefficient for a vertical surface [Eq.(8.11)] can be expressed as follows.

$$h_{av} = 0.943 \left[\frac{g \rho_L(\rho_L - \rho_v) k_L^3 h_{fg}}{\mu_L(T_v - T_w)} \right]^{1/4}$$

Generally $\rho_L >> \rho_v$. Therefore

$$h_{av} = 0.943 \left[\frac{g \rho_L^2 k_L^3 h_{fg}}{\mu_L(T_v - T_w)} \right]^{1/4} \quad \text{……………….(8.20)}$$

The above equation can be arranged in the form

$$h_{av} [\nu_L^2 / (g k_L^3)]^{1/3} = 1.47 \, Re_L^{-1/3} \quad \text{………………………(8.21)}$$

The above equation is valid for $Re_L < 1800$.
It has been observed experimentally that when the value of the film Reynolds number is greater than 30, there are ripples on the film surface which increase the value of the heat transfer coefficient. Kutateladze has proposed that the value of the local heat transfer coefficient be multiplied by $0.8(RE / 4)^{0.11}$ to account for the ripples effect. Using this correction it can be shown that

$$(h_{av} / k_L)(\nu_L^2 / g)^{1/3} = \frac{Re_L}{[1.08 \, Re_L^{1.22} - 5.2]} \quad \text{………………(8.22)}$$

8.8. Turbulent film condensation: For turbulent condensation on a vertical surface, Kirkbride has proposed the following empirical correlation based on experimental data.

$$h_{av} [\nu_L^2 / (g k_L^3)]^{1/3} = 0.0077 \, (Re_L)^{0.4} \quad \text{……………………(8.23)}$$

In the above correlation the physical properties of the condensate should be evaluated at the arithmetic mean temperature of T_v and T_w.

8.9. Film condensation inside horizontal tubes: In all the correlations mentioned above, it is assumed that the vapour is either stationary or has a negligible velocity. In practical applications such as condensers in refrigeration and air conditioning systems, vapour condenses on the inside surface of the tubes and so has a significant velocity. In such situations the condensation phenomenon is very complicated and a simple analytical treatment is not possible. Consider, for example, the film condensation on the inside surface of a long vertical tube.

The upward flow of vapour retards the condensate flow and causes thickening of the condensate layer, which in turn decreases the condensation heat transfer coefficient. Conversely the down ward flow of vapour decreases the thickness of the condensate film and hence increases the heat transfer coefficient.

Chato recommends the following correlation for condensation at low vapour velocities inside horizontal tubes:

$$h_{av} = 0.555 \left[\frac{g\, \rho_L(\rho_L - \rho_v)\, k_L^3\, h^*_{fg}}{\mu_L(T_v - T_w)\; D} \right]^{1/4} \quad \text{……………..(8.24 –a)}$$

where $h^*_{fg} = h_{fg} + (3/8)c_{p,L}(T_v - T_w)$ ………………………….(8.24 –b)

This result has been developed for the condensation of refrigerants at low Reynolds number [$Re_v = (\rho_v u_v D) / \mu_v < 35{,}000$; Re_v should be evaluated at the inlet conditions.]

For higher flow rates, Akers, Deans and Crosser propose the following correlation for the average condensation heat transfer coefficient on the inside surface of a horizontal tube of diameter D:

$$\frac{h_{av}\, D}{k} = 0.026\, Pr^{1/3} \left[Re_L + Re_v(\rho_L / \rho_v)^{1/2}\right]^{0.8} \quad \text{………..(8.25)}$$

where $Re_L = (4M_L) / (\pi D \mu_L)$: $Re_v = (4M_v) / (\pi D \mu_v)$ …………………….(8.26)

The above equation correlates the experimental data within 50 % for $Re_L > 5000$

and $Re_v > 20{,}000$.

8.10. Illustrative examples on film wise condensation:

Example 8.1: *Saturated steam at 1.43 bar condenses on a 1.9 cm OD vertical tube which is 20 cm long. The tube wall is at a uniform temperature of 109 ^{0}C . Calculate the average heat transfer coefficient and the thickness of the condensate film at the bottom of the tube.*

Solution: Data:- T_v = Saturation temperature at 1.43 bar = 110 ^{0} C (from steam tables)

T_w = 109 ^{0}C ; Characteristic length = L = 0.2 m ; D = 0.019 m ;

To find : (i) h_{av} ; (ii) $\delta(x)|_{x=L}$;

Mean film temperature of the condensate (water) = 0.5 x (110 + 109) = 109.5 ^{0}C.

Properties of water at 109.5 ^{0}C are: ρ_L = 951.0 kg/m^3; μ_L = 258.9 x 10^{-6} N-s / m^2; k = 0.685 W/(m-K); ν = 0.2714 x 10^{-6} m^2/s; h_{fg} = 2230 kJ/kg. Also $\rho_L \ggg \rho_v$.

Let us assume that the condensate flow is laminar and later check for this assumption.

$$h_{av} = 0.943 \left[\frac{g \rho_L^2 k_L^3 h_{fg}}{\mu_L (T_V - T_W) L} \right]^{1/4}$$

$$\text{Hence } h_{av} = 0.943 \times \left[\frac{9.81 \times (951)^2 \times (0.685)^3 \times 2230 \times 10^3}{258.9 \times 10^{-6} \times (110 - 109) \times 0.2} \right]^{1/4}$$

$$= 17{,}653 \text{ W / (m}^2\text{-K)}$$

(ii) $h_{av} = (4 / 3) h_x|_{x=L}$ or $h_x|_{x=L} = ¾ \times h_{av} = 0.75 \times 17{,}653 = 13{,}240$ W/(m^2-K).

Therefore $\delta(x)|_{x=L} = k_L / h_x|_{x=L} = 0.685 / 13240 = 5.174 \times 10^{-5}$ m = 0.0517 mm.

Check for Laminar flow assumption:- The relation between h_{av} and Reynolds number at the bottom of the tube is given by

$h_{av} [\nu_L^2 / (g k_L^3)]^{1/3} = 1.47 \, Re_L^{-1/3}$ or $Re_L = (1.47 / h_{av})^3 (g k_L^3 / \nu_L^2)$

Hence $Re_L = (1.47 / 17{,}653)^3 [9.81 \times 0.685^3 / \{0.2714 \times 10^{-6}\}^2]$

$= 24.72$

Since $Re_L < 1800$, our assumption that condensate flow is laminar is correct.

Example 8.2:- *Saturated steam at 80 ^{0}C condenses as a film on a vertical plate 1 m high. The plate is maintained at a uniform temperature of 70 ^{0}C. Calculate the average heat transfer coefficient and the rate of condensation. What would be the corresponding values if the effect of ripples is taken into consideration.*

Solution: Data:- $T_V = 80\ ^0$C; $T_W = 70\ ^0$C; Mean film temperature =0.5 x (80 + 70) = 75 ^{0}C.

Properties of condensate (liquid water) at 75 ^{0}C are: $\rho_L = 974.8$ kg/m^3;

$k_L = 0.672$ W /(m-K) ; $\mu_L = 381 \times 10^{-6}$ N-s/m^2; h_{fg} at 80 ^{0}C = 2309 kJ/kg-K;

$\nu_L = 0.391 \times 10^{-6}$ m^2/s.Charecteristic length = L = 1.0 m.

Assuming laminar film condensation the average heat transfer coefficient is given by

$$h_{av} = 0.943 \left[\frac{g\,\rho_L^2\,k_L^3\,h_{fg}}{\mu_L (T_v - T_w)} \right]^{1/4}$$

$$= 0.943 \times \left[\frac{9.81 \times (974.8)^2 \times (0.672)^3 \times 2309 \times 10^3}{381.6 \times 10^{-6} \times (80 - 70) \times 1.0} \right]^{1/4} = 6066.6\ W/(m^2 - K).$$

Condensate rate = $M = \dfrac{h_{av}\,L\,(T_v - T_w)}{h_{fg}} = \dfrac{6066.6 \times 1.0 \times (80 - 70)}{2309 \times 10^3}$ 0.0263 kg/s.

Check for laminar flow assumption :- $Re_L = \dfrac{4M}{\mu_L P}$, where P = width of the plate for vertical flat plate. Hence $Re_L = \dfrac{4 \times 0.0263}{381 \times 10^{-6}} = 276$

Since $Re_L < 1800$, the condensate flow is laminar.

Since $Re_L > 30$, it is clear that the effects of ripples have to be considered.

Now $Re_L = \dfrac{4M}{\mu_L P} = \dfrac{4\,h_{av}\,L\,(T_v - T_w)}{\mu_L\,P\,h_{fg}}$

Hence $h_{av} = \dfrac{Re_L\,\mu_L\,P\,h_{fg}}{4L(T_v - T_w)}$ ……………………………………………….(1)

When the effects of ripples are considered the relation between Re_L and h_{av} is given by Eq.(8.22) as follows:

$1.08\,Re_L^{1.22} - 5.2 = \dfrac{Re_L}{(h_{av}/k_L)(\nu_L^2/g)^{1/3}}$ Substituting for h_{av} from Eq.(1) we have

$$1.08\,Re_L^{1.22} - 5.2 = \frac{4L\,(T_v - T_w)\,k_L\,(g/\nu_L^2)^{1/3}}{\mu_L\,P\,h_{fg}}$$

$$1.08\,Re_L^{1.22} - 5.2 = \frac{4 \times 1 \times (80 - 70) \times 0.672 \times \{9.81/(0.391 \times 10^{-6})^2\}^{1/3}}{381.6 \times 10^{-6} \times 1.0 \times 2309 \times 10^3}$$

$1.08\ Re_L^{1.22} - 5.2 = 1221.3$. Or $Re_L = 319.4$

Hence from Eq.(1) we have $h_{av} = \dfrac{319.4 \times 381.6 \times 10^{-6} \times 1.0 \times 2309 \times 10^{3}}{4 \times 1.0 \times (80 - 70)}$

$= 7036\ W/(m^2 - K)$.

Hence $M = \dfrac{h_{av} L (T_v - T_w)}{h_{fg}} = \dfrac{7036 \times 1.0 \times (80 - 70)}{2309 \times 10^{3}} = 0.03047\ kg/s$.

[It can be seen that the ripples on the surface increase the heat transfer coefficient by about 15 %].

Example 8.3:- *Air free saturated steam at 65 ^{0}C condenses on the surface of a vertical tube of OD 2.5 cm. The tube surface is maintained at a uniform temperature of 35 ^{0}C. Calculate the length of the tube required to have a condensate flow rate of 6×10^{-3} kg/s.*

Solution: Data:- $T_v = 65\ ^0C$; $T_w = 35\ ^0C$; $D_0 = 0.025$ m; $M = 6 \times 10^{-3}$ kg/s.

To find length of the tube, L.

Mean film temperature = 0.5 x (65 + 35) = 50 ^{0}C.Properties of condensate

(liquid water) at 50 ^{0}C are: $k_L = 0.640$ W/(m-K); $\mu_L = 0.562 \times 10^{-3}$ N-s/m^2; $\rho_L = 990$

kg/m^3; At 65 ^{0}C, $h_{fg} = 2346 \times 10^{3}$ J/(kg-K).

$$\text{Reynolds number} = Re = \frac{4M}{\mu_L \pi D_o} = \frac{4 \times 6 \times 10^{-3}}{0.562 \times 10^{-3} \times \pi \times 0.025} = 544$$

Since Re < 1800 flow is laminar. It is more convenient to use Eq.(8.21)

$$h_{av} [\nu_L^2 / (g k_L^3)]^{1/3} = 1.47\ Re_L^{-1/3}$$

or

$$h_{av} = 1.47\ Re_L^{-1/3} \left[\frac{(g k_L^3)}{\nu_L^2}\right]^{1/3} = \frac{1.47 \times (544)^{-1/3} \times [9.81 \times 0.64^3]^{1/3}}{(0.562 \times 10^{-3} / 990)^2}$$

$= 3599\ W/(m^2 - K)$

Heat balance equation gives M $h_{fg} = h_{av} \pi D_o L [T_v - T_w]$

Therefore
$$L = \frac{M h_{fg}}{h_{av} \pi D_o [T_v - T_w]} = \frac{6 \times 10^{-3} \times 2346 \times 10^{3}}{3599 \times \pi \times 0.025 \times (65 - 35)}$$

$$= 1.66 \text{ m}$$

Example 8.4:- *Air free saturated steam at 85 ^{0}C condenses on the outer surfaces of 225 horizontal tubes of 1.27 cm OD, arranged in a 15 x 15 array. Tube surfaces are maintained at a uniform temperature of 75 ^{0}C. Calculate the total condensate rate per one metre length of the tube.*

Solution: Data:- $T_v = 85\ ^0C$; $T_w = 75\ ^0C$; $D_o = 0.0127$ m; L = 1 m;

Number of tubes in vertical tier = N = 15 ; Total number of tubes = n = 225;

Mean film temperature = 0.5 x (85 + 75) = 80 ^{0}C. Properties of the condensate (liquid water) are: $k_L = 0.668$ W/(m-K); $\mu_L = 0.355 \times 10^{-3}$ N-s/m^2; $\rho_L = 974$ kg/m^3;

At 85 ^{0}C, $h_{fg} = 2296 \times 10^{3}$ J/(kg-K).

For N horizontal tubes arranged in a vertical tier, h_{av} is given by

$$h_{av} = 0.725 \left[\frac{g \rho_L^2 h_{fg} k_L^3}{\mu_L (T_v - T_w) N D_o} \right]^{1/4}$$

$$h_{av} = \frac{0.725 \times [9.81 \times (974)^2 \times (0.668)^3]^{1/4}}{[0.355 \times 10^{-3} \times (85 - 75) \times 15 \times 0.0127]^{1/4}} = 7142 \text{ W/(m}^2 - \text{K)}$$

$$Q = h_{av} A_{total} (T_v - T_w) = h_{av}\, n\, \pi\, D_o L\, (T_v - T_w)$$

$$= 7142 \times 225 \times \pi \times 0.0127 \times 1 \times (85 - 75) = 641.14 \times 10^{3} \text{ W}$$

Mass flow rate of condensate = $M = Q / h_{fg} = 641.14 \times 10^{3} / 2296 \times 10^{3} = 0.28$ kg/ (s-m)

Example 8.5:- *Superheated steam at 1.43 bar and 200 ^{0}C condenses on a 1.9 cm OD vertical tube which is 20 cm long. The tube wall is maintained at a uniform temperature of 109 ^{0}C. Calculate the average heat transfer coefficient and the thickness of the condensate at the bottom of the tube. Assume c_p for super heated steam as 2.01 kJ/(kg-K).*

Solution: With a superheated vapour, condensation occurs only when the surface temperature is less than the saturation temperature corresponding to the vapour pressure. Therefore for a superheated vapour, the amount of heat to be removed per unit mass to condense it is given by

$$Q / M = h_{fg} + c_{pv}(T_v - T_{sat})$$

Where c_p is the specific heat of superheated steam and T_{sat} is the saturation temperature corresponding to the vapour pressure. If it is assumed that the liquid – vapour interphase is at the saturation temperature, then Eq.(8.20) still holds good with h_{fg} replaced by $h_{fg} + c_{pv}(T_v - T_{sat})$.

Hence

$$h_{av} = 0.943 \left[\frac{g \rho_L^2 k_L^3 \{ h_{fg} + c_{pv}(T_v - T_{sat})\}}{\mu_L(T_{sat} - T_w)L} \right]^{1/4}$$

At 1.43 bar, T_{sat} = 110 ^{0}C.Mean film temperature = 0.5 x (110 + 109) = 109.5 ^{0}C.

Properties of the condensate at 109.5 ^{0}C are: k_L = 0.685W/(m-K); $\mu_L = 0.259 \times 10^{-3}$ N-s/m^2; ρ_L= 951 kg/m^3;At 1.43 bar, $h_{fg} = 2230 \times 10^3$ J/(kg-K).

$$h_{av} = 0.943 \times \left[\frac{9.81 \times (951)^2 \times (0.685)^3 \times \{2230 \times 10^3 + 2010 \times (200 - 110)\}}{0.259 \times 10^{-3} \times (110 - 109) \times 0.2} \right]^{1/4}$$

= 18,000 W /(m^2 – K).

Hence $h_x|_{x=L}$ = (¾) x 18000 = 13,500 W / (m^2 – K).

$\delta(x)|_{x=L} = k_L / h_x|_{x=L} = 0.685 / 13{,}500 = 5.07 \times 10^{-5}$m

Example 8.6:- *Air free saturated steam at 70 ^{0}C condenses on the outer surface of a 2.5 cm OD vertical tube whose outer surface is maintained at a uniform temperature of 50 ^{0}C. What length of the tube would produce turbulent film condensation?*

Solution: Data:- T_v = 70 ^{0}C; T_w = 50 ^{0}C; D_o = 0.025 m; Vertical tube.

To find L such that Re = 1800.

Mean film temperature = 0.5 x (70 + 50) = 60 ^{0}C. Properties of the condensate (liquid water) are : k_L = 0.659W/(m-K); $\mu_L = 0.4698 \times 10^{-3}$ N-s/m^2; ρ_L= 983.2 kg/m^3;

At 70 ^{0}C h_{fg} = 2358 x 10 ^{3} J/(kg-K).

$$Re = 4M / (\mu_L \pi D_o) \text{ or } M = \frac{Re\,(\mu_L \pi D_o)}{4} = \frac{1800 \times 0.4698 \times 10^{-3} \times \pi \times 0.025}{4}$$

$$= 0.0166 \text{ kg / s.}$$

For turbulent flow $h_{av}\,[\nu_L^2 / (g k_L^3)]^{1/3} = 0.0077\,(Re_L)^{0.4}$

Or $h_{av} = 0.0077\,(Re_L)^{0.4} [\nu_L^2 / (g k_L^3)]^{-1/3}$

Hence $h_{av} = 0.0077 \times (1800)^{0.4} \times [\,(0.4698 \times 10^{-3}/983.2)^2 / (9.81 \times 0.659^3)\,]^{-1/3} = 3563.4$ W / (m^2 – K).

Heat balance equation is $M\,h_{fg} = h_{av}\,\pi\,D_o L\,(T_v - T_w)$

Hence
$$L = \frac{M\,h_{fg}}{h_{av}\,\pi\,D_o\,(T_v - T_w)} = \frac{0.0166 \times 2358 \times 10^3}{3563.4 \times \pi \times 0.025 \times (70 - 50)}$$

$$= 7 \text{ m}$$

Example 8.7:- *Saturated steam at 100 ^{0}C condenses on the outer surface of a 2 m long vertical plate. What is the temperature of the plate below which the condensing film at the bottom of the plate will become turbulent?*

Solution: Data:- T_v = 100 ^{0}C; L = 2 m. Since T_w is not known, properties of the condensate at the mean film temperature cannot be determined and therefore the problem has to be solved by trial and error procedure as follows:

Trial 1:- The properties of the condensate are read at T_v = 100 ^{0}C. The properties

are k_L = 0.683 W/(m-K); μ_L = 0.2824 x 10 $^{-3}$ N-s/m^2; ρ_L= 958.4 kg/m^3; At 100 ^{0}C,

h_{fg} = 2257 x 10 ^{3} J/kg-K.

Since the flow has to become turbulent at the bottom of the plate we have

$$h_{av} = 0.0077\,(Re_L)^{0.4} [\nu_L^2 / (g k_L^3)]^{-1/3} \text{ with } Re_L = 1800$$

Hence $$h_{av} = 0.0077 \times (1800)^{0.4} \times \left[\frac{9.81 \times 0.683^3}{(0.2824 \times 10^{-3} / 958.4)^2} \right]^{1/3}$$

$$= 5098 \text{ W / (m}^2 - \text{K)}$$

Now $M h_{fg} = h_{av} L W (T_v - T_w)$

Or $T_w = T_v - (M/W)h_{fg} / (h_{av} L)$. But $Re_L = 4M / (\mu_L W)$ or $M/W = Re_L \mu_L / 4$.

Therefore $$T_w = T_v - \frac{Re_L \mu_L h_{fg}}{4 h_{av} L} = 100 - \frac{1800 \times 0.2824 \times 10^{-3} \times 2257 \times 10^3}{4 \times 5098 \times 2}$$

$$= 72\ ^0C$$

Trial 2:- Assume $T_w = 72\ ^0C$. Mean film temperature = 0.5 x (100 + 72) = 86 ^0C. Properties of the condensate at 86 ^0C ; $k_L = 0.677$ W/(m-K); $\mu_L = 0.3349 \times 10^{-3}$ N-s/m^2; ρ_L= 968.5kg/m^3; At 100 ^0C, $h_{fg} = 2257 \times 10^3$ J/(kg-K).

Hence $$h_{av} = 0.0077 \times (1800)^{0.4} \times \left[\frac{9.81 \times 0.677^3}{(0.3349 \times 10^{-3} / 968.5)^2} \right]^{1/3}$$

$$= 4541 \text{ W /(m}^2 - \text{K).}$$

Therefore $$T_w = 100 - \frac{1800 \times 0.3349 \times 10^{-3} \times 2257 \times 10^3}{4 \times 4541 \times 2} = 60\ ^0C$$

Since the calculated value of T_w is quite different from the assumed value, one more iteration is required.

Trial 3:- Assuming $T_w = 60\ ^0C$ and proceeding on the same lines as shown in trial 2 we get $h_{av} = 4365$ W /(m^2 – K) and hence $T_w = 59^0C$. This value is very close to the value assumed (difference is within 2 %). The iteration is stopped. Hence $T_w = 59\ ^0C$.

Example 8.8:- *Air free saturated steam at 90 ^0C condenses on the outer surface of a 2.5 cm OD, 6 m long vertical tube, whose outer surface is maintained at a uniform temperature of 60 ^0C. Calculate the total rate of condensation of steam at the tube surface.*

Solution: Data:- $T_v = 90\ ^0C$; $T_w = 30\ ^0C$; $D_o = 0.025$ m; L = 6 m. Vertical tube.

Mean film temperature = 0.5 x (90 + 60) = 75 ^{0}C. Properties of the condensate at 75 ^{0}C are: $k_L = 0.671$ W/(m-K); $\mu_L = 0.3805 \times 10^{-3}$ N-s/m^2; $\rho_L = 974.8$ kg/m^3; At 90 ^{0}C, $h_{fg} = 2283 \times 10^3$ J/(kg-K).

We do not know whether the condensate flow is laminar or turbulent. We start the calculations assuming laminar flow and then check for laminar flow condition. For laminar flow

$$h_{av} = 0.943 \left[\frac{g\,\rho_L^2\, k_L^3\, h_{fg}}{\mu_L (T_v - T_w)\, L} \right]^{1/4}$$

Hence $$h_{av} = 0.943 \times \left[\frac{9.81 \times (974.8)^2 \times (0.671)^3 \times 2283 \times 10^3}{0.3805 \times 10^{-3} \times (90 - 60) \times 6} \right]^{1/4}$$

$$= 2935.3 \text{ W}/(\text{m}^2 - \text{K}).$$

For laminar flow $\quad h_{av} [\nu_L^2 / (g k_L^3)]^{1/3} = 1.47\, Re_L^{-1/3}$

Or $\quad Re_L = (1.47 / h_{av})^3 (g k_L^3 / \nu_L^2)$

$$= (1.47 / 2935.3)^3 \times [9.81 \times 0.671^3 \times 974.8^2 / (0.3805 \times 10^{-3})^2]$$

$$= 2443$$

Since $Re_L > 1800$, flow is turbulent.

For turbulent flow $\quad h_{av} [\nu_L^2 / (g k_L^3)]^{1/3} = 0.0077 (Re_L)^{0.4}$

Or $\quad h_{av} = 0.0077 (Re_L)^{0.4} / [\nu_L^2 / (g k_L^3)]^{1/3}$ ……….(1)

$Re_L = 4M / (\mu_L \pi D_o)$. But $M h_{fg} = h_{av} \pi D_o L (T_v - T_w)$ or $M / (\pi D_o) = h_{av} L (T_v - T_w) / h_{fg}$

Therefore $$Re_L = \frac{4\, h_{av} L (T_v - T_w)}{h_{fg}\, \mu_L}$$

Substituting this expression for Re_L in equation (1) we have

$$h_{av} = 0.0077 \left[\frac{4\, h_{av} L\, (T_v - T_w)}{h_{fg}\, \mu_L}\right]^{0.4} \left[\nu_L^2 / (g k_L^3)\right]^{-1/3}$$

$$(h_{av})^{0.6} = 0.0077 \left[\frac{4\, L\, (T_v - T_w)}{h_{fg}\, \mu_L}\right]^{0.4} \left[\nu_L^2 / (g k_L^3)\right]^{-1/3}$$

$$= 0.0077 \times \left[\frac{4 \times 6 \times (90 - 60)}{2283 \times 10^{3} \times 0.3805 \times 10^{-3}}\right]^{0.4} \times \left[\frac{(0.3805 \times 10^{-3})^2}{974.8^2 \times 9.81 \times (0.671)^3}\right]^{-1/3}$$

$= 192$. Hence $h_{av} = [192]^{1/0.6} = 6390 \text{ W}/(\text{m}^2 - \text{K})$.

$$\text{Therefore } M = \frac{h_{av} \pi D_o L\, (T_v - T_w)}{h_{fg}} = \frac{6390 \times \pi \times 0.025 \times 6 \times (90 - 60)}{2283 \times 10^{3}} = 0.0396 \text{ kg/s}$$

8.11. Dropwise Condensation: Experimental investigations on condensation have indicated that, if traces of oil are present in steam and the condensing surface is highly polished, the condensate film breaks into droplets. This type of condensation is called "*drop wise condensation*". The droplets grow, coalesce and run off the surface, leaving agreater portion of the condensing surface exposed to the incoming steam. Since the entire condensing surface is not covered with a continuous layer of liquid film, the heat transfer rate for ideal drop wise condensation is much higher than that for film wise condensation.

The heat transfer coefficient may be 2 to 3 times greater for drop wise condensation than for film wise condensation. Hence considerable research has been done with the objective of producing long lasting drop wise condensation. Various types of chemicals have been tried to promote drop wise condensation. Continuous drop wise condensation, obtainable with different promoters varies between 100 to 300 hours with pure steam and are shorter with industrial steam. Failure occurs because of fouling or oxidation of the surface, or by the flow of the condensate or by a combination of these effects.

It is unlikely that long lasting drop wise condensation can be produced under practical conditions by a single treatment of any of the promoters currently available. Therefore in the analysis of a heat exchanger involving condensation of steam, it is recommended that film wise condensation be assumed for the condensing surface.

8.12. Boiling Types: When evaporation occurs at a solid-liquid interface, it is called as *"boiling"*. The boiling process occurs when the temperature of the surface T_w exceeds the saturation temperature T_{sat} corresponding to the liquid pressure. Heat is transferred from the solid surface to the liquid, and the appropriate form of Newton"s law of cooling is

$$q_w = h\, [T_w - T_{sat}] = h\, \Delta T_e \quad \text{..............................(8.27)}$$

Where $\Delta T_e = [T_w - T_{sat}]$ and is termed as the *"excess temperature"*. The boiling process is characterized by the formation of vapour bubbles which grow and subsequently detach from the surface. Vapour bubble growth and dynamics depend, in a complicated manner, on the excess temperature ΔT_e, the nature of the surface, and the thermo-physical properties of the fluid, such as its surface tension. In turn the dynamics of vapour bubble growth affect fluid motion near the surface and therefore strongly influence the heat transfer coefficient.

Boiling may occur under varying conditions. For example if the liquid is quiescent and if its motion near the surface is due to free convection and due to mixing induced by bubble growth and detachment, then such a boiling process is called *"pool boiling"*. In contrast in *"forced convection boiling"*, the fluid motion is induced by an external means as well as by free convection and bubble induced mixing. Boiling may also be classified as *"sub-cooled boiling"* and *"saturated boiling"*. In sub-cooled boiling, the temperature of the liquid is below the saturation temperature and the bubbles formed at the surface may condense in the liquid. In contrast, in saturated boiling, the temperature of the liquid slightly exceeds the saturation temeperature, Bubbles formed at the surface are then propelled through the liquid by buoyancy forces, eventually escaping from a free surface.

8.13. Pool Boiling Regimes: The first investigator who established experimentally the different regimes of pool boiling was Nukiyama. He immersed an electric resistance wire into a body of saturated water and initiated boiling on the surface of the wire by passing electric current through it. He determined the heat flux as well as the temperature from the measurements of current and voltage. Since the work of Nukiyama, a number of investigations on pool boiling have been reported. Fig. 8.4 illustrates the characteristics of pool boiling for water at atmospheric pressure. This boiling curve illustrates the variation of heat flux or the heat transfer coefficient as a function of excess temperature ΔT_e. This curve pertains to water at 1 atm pressure.From Eq. (8.27) it can seen that q_w depends on the heat transfer coefficient h and the excess temperature ΔT_e.

Free Convection Regime(up to point A):- Free convection is said to exist if $\Delta T_e \leq 5\ ^0$ C. In this regime there is insufficient vapour in contact with the liquid phase to cause boiling at the saturation temperature. As the excess temperature is increased, the bubble inception will eventually occur, but below point A (referred to as onset of nucleate boiling,ONB), fluid motion is primarily due to free convection effects.Therefore,according to whether the flow is laminar or turbulent, the heat transfer coefficient h varies as $\Delta T_e^{1/4}$ or as $\Delta T_e^{1/3}$ respectively so that q_w varies as $\Delta T_e^{5/4}$ or as $\Delta T_e^{4/3}$.

Nucleate Boiling Regime(Between points A and C):- Nucleate boiling exists in the range $5\ ^0\,C \leq \Delta T_e \leq 30\ ^0$ C. In this range, two different flow regimes may be distinguished. In the region A – B, isolated bubbles form at nucleation sites and separate from the surface, substantially increasing h and q_w. In this regime most of the heat exchange is through direct transfer from the surface to liquid in motion at the surface, and not through vapour bubbles rising from the surface. As ΔT_e is increased beyond 10 ^{0}C (Region B-C), the nucleation sites will be numerous and the bubble generation rate is so high that continuous columns of vapour appear. As a result very high heat fluxes are obtainable in this region. In practical applications, the nucleate boiling regime is most desirable, because large heat fluxes are obtainable with small temperature differences. In the nucleate boiling regime, the heat increases rapidly with increasing excess temperature

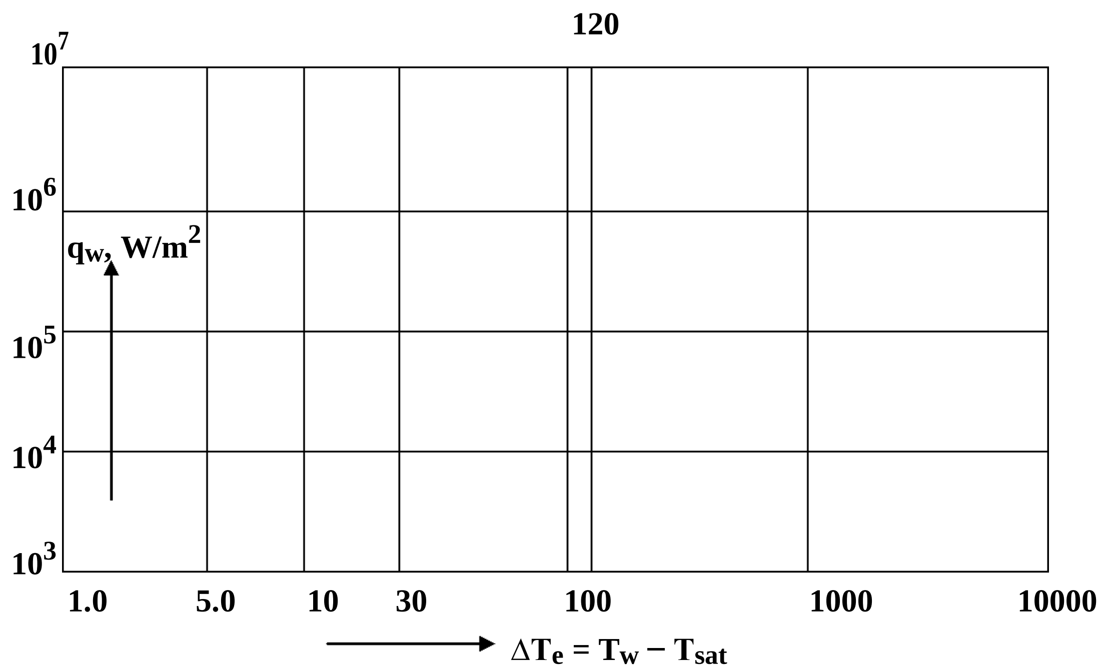


Fig. 8.4: Typical boiling curve for water at 1 atm; surface heat flux q_w as function of excess temperature ΔT_e

ΔT_e until the peak heat flux is reached. The location of this peak heat flux is called the *burnout point,* or *departure from nucleate boiling* (DNB), or the *critical heat flux* (CHF). The reason for calling the critical heat flux the burnout point is apparent from the Fig.
8.4. Such high values of ΔT_e may cause the burning up or melting away of the heating element.

Film Boiling Regime:- It can be seen from Fig. 8.4 that after the peak heat flux is reached, any further increase in ΔT_e results in a reduction in heat flux. The reason for this curious phenomenon is the blanketing of the heating surface with a vapour film which restricts liquid flow to the surface and has a low thermal conductivity. This regime is called the *film boiling regime.* The film boiling regime can be separated into three distinct regions namely (i) the *unstable film boiling region,* (ii) the *stable film boiling region* and
(iii) *radiation dominating region.* In the unstable film boiling region, the vapour film is unstable, collapsing and reforming under the influence of convective currents and the
iv) surface tension. Here the heat flux decreases as the surface temperature increases, because the average wetted area of the heater surface decreases. In the stable film boiling region, the heat flux drops to a minimum, because a continuous vapour film covers the heater surface.In the radiation dominating region, the heat flux begins to increase as the excess temperature increases, because the temperature at the heater surface is sufficiently high for thermal radiation effects to augment heat transfer through the vapour film.

8.14. Pool Boiling Correlations:

Correlation for The Nucleate Boiling Regime:- The heat transfer in the nucleate boiling regime is affected by the nucleation process, the distribution of active nucleation sites on the surface, and the growth and departure of bubbles.Numerous experimental investigations have been reported and a number of attempts have been made to correlate the experimental data corresponding to nucleate boiling regime.The most successful and widely used correlation was developed by Rohsenow. By analyzing the significance of various parameters in relation to forced - convection effects. He proposed the following empirical relation to correlate the heat flux in the entire nucleate boiling regime:

$$\frac{C_{pl}\,\Delta T_e}{h_{fg}\,Pr_l^{\,n}} = C_{sf}\left[\frac{q_w}{(\mu_l\, h_{fg})}\sqrt{\zeta^* / \{g\,(\rho_l - \rho_v)\}}\right]^{0.33} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ (8.28)$$

where C_{pl} = specific heat of saturated liquid, J /(kg -^{0}C)

C_{sf} = constant to be determined from experimental data depending upon Heating surface – fluid combination

h_{fg} = latent heat of vapourization, J / kg

g = acceleration due to gravity, m / s^2

Pr_l = Prandtl number of saturated liquid

q_w = boiling heat flux, W / m^2

ΔT_e = excess temperature as defined in Eq. (8.27)

μ_l = viscosity of saturated liquid, kg / (m – s)

ρ_l, ρ_v = density of liquid and saturated vapour respectively, kg /

m^3 ζ^* = surface tension of liquid – vapour interface, N / m.

In Eq. (8.28) the exponent n and the coefficient C_{sf} are the two provisions to adjust the correlation for the liquid – surface combination. Table 8-1 gives the experimentally determined values C_{sf} for a variety of liquid – surface combinations. The value of n *should be taken as 1 for water and 1.7 for all other liquids* shown in Table 8 – 1.

Example 8.10:- *A metal clad heating element of 6 mm diameter and emissivity equal to unity is horizontally immersed in a water bath. The surface temperature of the metal is 255 ^{0}C under steady state boiling conditions. If the water is at atmospheric pressure estimate the power dissipation per unit length of the heater.*

Solution: *Given:-* $T_w = 255\ ^0C$; T_{sat} = saturation temperature of water at 1 atm = 100 ^{0}C;

$\Delta T_e = 255 - 100 = 155\ ^0C$. Since $\Delta T_e > 120\ ^0C$, film boiling conditions will prevail.
The heat transfer in this regime is given by Eq.(8.33) namely

$$h_o = 0.62 \left[\frac{k_v^3\, \rho_v\, (\rho_l - \rho_v)\, g\, h_{fg}^*}{D\, \mu_v\, \Delta T_e} \right]^{1/4}$$

Properties of water at 100 ^{0}C are: $\rho_l = 957.9\ kg/m^3$; $h_{fg} = 2257 \times 10^3\ J/kg$;

$\rho_v = 4.808\ kg/m^3$; $C_{pv} = 2.56 \times 10^3\ J/(kg\text{-}K)$; $k_v = 0.0331\ W/(m\text{-}^0C)$;

$\mu_v = 14.85 \times 10^{-6}\ kg/(m\text{-}s)$.

Substituting these values in the expression for h_o we have

$$h_o = 0.62 \times \left[\frac{(0.0331)^3 \times 4.808 \times (957.9 - 4.808) \times 9.81 \times \{2257 \times 10^3 + 0.8 \times 2.56 \times 10^3 \times 155\}}{14.85 \times 10^{-6} \times 0.006 \times 155} \right]^{1/4}$$

$= 460\ W/(m^2 - K)$

$$h_r = \frac{1}{[1/\varepsilon + 1/\alpha - 1]} \times \frac{\zeta \{T_w^4 - T_{sat}^4\}}{\{T_w - T_{sat}\}}$$

$$= \frac{1}{[1/1 + 1/1 - 1]} \times \frac{5.67 \times 10^{-8} \times \{528^4 - 373^4\}}{\{528 - 373\}}$$

$= 21.3\ W/(m^2\text{-}K)$.

Now $h \approx h_o + ¾\ h_r = 460 + ¾ \times 21.3 = 476\ W/(m^2 - K)$.

Hence $Q = h\, A\, \Delta T_e = 476 \times (\pi \times 0.006 \times 1) \times 155 = 1.36 \times 10^3\ W/m$.

Example 8.11:- *A vessel with a flat bottom and 0.1 m^2 in area is used for boiling water at atmospheric pressure. Find the temperature at which the vessel must be maintained if a boiling rate of 80 kg/h is desired. Assume that the vessel is made of copper and the boiling is nucleate boiling. Take $\rho_v = 0.60\ kg/m^3$.*

Solution: *Given:*- $A = 0.1\ m^2$; $T_{sat} = 100\ ^0C$; $M = 80$ kg/h $= 0.022$ kg/s; $Pr_l = 1.75$

$h_{fg} = 2257 \times 10^3$ J /kg; $C_{pl} = 4216$ J/(kg-K); $\rho_l = 960.6\ kg/m^3$; $\zeta^* = 58.8 \times 10^{-3}$ N/m;

$\mu_l = 282.4 \times 10^{-6}$ kg / (m-s); n = 1; For water-copper combination $C_{sf} = 0.0130$;

$$q_w = Q / A = \frac{M\, h_{fg}}{A} = \frac{0.022 \times 2257 \times 10^3}{0.1} = 4.965 \times 10^3\ W/m^2$$

For nucleate boiling Eq.(8.28) is used to calculate the excess temperature .ΔT_e

$$\frac{C_{pl}\,\Delta T_e}{h_{fg}\,Pr_l^n} = C_{sf}\left[\frac{q_w}{(\mu_l\, h_{fg})}\sqrt{\zeta^* / \{g\,(\rho_l - \rho_v)\}}\right]^{0.33}$$

$$\frac{4216 \times \Delta T_e}{2257 \times 10^3 \times 1.75} = 0.013 \times \{4.965 \times 10^5/(282.4 \times 10^{-6} \times 2257 \times 10^3) \times \sqrt{58.8 \times 10^{-3} / [9.81 \times (960.6 - 0.6)]}\ \}^{0.33}$$

Or $\quad \Delta T_e = 15.2\ ^0C$

Hence $T_w = 100 + 15.2 = 115.2\ ^0C$.

Example 8.12:- *Calculate the heat transfer coefficient during stable film boiling of water from a 0.9 cm diameter horizontal carbon tube. The water is saturated and at 100* ^0C *and the tube surface is at 1000* ^0C. *Take the emissivity of the carbon surface to be 0.8 and assume that at the average film temperature, the steam has the following properties.*
$k_v = 0.0616$ *W/(m-K);* $\rho_v = 0.266$ *kg/m*^{3}*;* $\mu_v = 28.7 \times 10^{-6}$ *kg/(m-s);* $C_{pv} = 2168$ *J/(kg-K);* $\rho_l = 958.4$ *kg/m*^{3}

Solution: *Given:*- D = 0.009 m; $\Delta T_e = T_w - T_{sat} = 1000 - 100 = 900\ ^0C$; $\varepsilon = 0.8$; $\alpha =$

1.0 $h_{fg}^* = h_{fg} + 0.8\, C_{pv}\, \Delta T_e = 2257 \times 10^3 + 0.8 \times 2168 \times 900 = 3818 \times 10^3$ J/kg.

For stable film boiling the convection coefficient is given by Eq.(8.33)

$$h_o = 0.62 \left[\frac{k_v^3\, \rho_v\, (\rho_l - \rho_v)\, g\, h_{fg}^*}{D\, \mu_v\, \Delta T_e} \right]^{1/4}$$

$$h_o = 0.62 \times \left[\frac{(0.0616)^3 \times 0.266 \times (958.4 - 0.266) \times 9.81 \times 3818 \times 10^3}{0.009 \times (28.7 \times 10^{-6}) \times 900} \right]^{1/4}$$

$$= 194 \text{ W/(m}^2 - \text{K)}$$

Radiation heat transfer coefficient is given by

$$h_r = \frac{1}{[\,1/\varepsilon + 1/\alpha - 1\,]} \; \frac{\zeta\,(T_w^4 - T_{sat}^4)}{(T_w - T_{sat})}$$

$$h_r = \frac{1}{[\,1/0.8 + 1/1 - 1\,]} \times \frac{5.67 \times 10^{-8}\,(1273^4 - 373^4)}{(1273 - 373)}$$

$$= 131.4 \text{ W/(m}^2 - \text{K).}$$

Hence $h = h_o + ¾\, h_r = 194 + ¾ \times 131.4 = 292.5$

RADIATION HEAT TRANSFER

10.1. INTRODUCTORY CONCEPTS AND DEFINITIONS

10.1.1 *THERMAL RADIATION*

When a body is placed in an enclosure whose walls are at temperatures below that of the body, the temperature of the body will decrease even if the enclosure is evacuated. This process by which heat is transferred from a body by virtue of its temperature, without the aid of any intervening medium is called "THERMAL RADIATION". The actual mechanism of radiation is not yet completely understood. There are at present two theories by means of which radiation propagation is explained. According to Maxwell‟s electromagnetic theory, Radiation is treated as electromagnetic waves, while Max Planck‟s theory treats radiation as "Photons" or "Quanta of energy". Neither theory completely describes all observed phenomena. It is however known that radiation travels with the speed of light, c (c = 3×10^8 m/s) in a vacuum. This speed is equal to the product of the frequency of the radiation and the wavelength of this radiation,

OR $$\mathbf{c = \lambda\nu} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\mathbf{(10.1)}$$

Where λ = wavelength of radiation (m) and ν = frequency (1/s).

Usually, it is more convenient to specify wavelength in micrometer, which is equal to 10^{-6} m.

From the viewpoint of electromagnetic wave theory, the waves travel at the speed of light, while from the quantum theory point of view, energy is transported by photons which travel at the speed of light. Although all the photons have the same velocity, there is always a distribution of energy among them. The energy associated with a photon, $\mathbf{e_p = h\nu}$ where h is the Planck‟s constant equal to 6.6256 x 10^{-34} Js. The entire energy spectrum can also be described in terms of the wavelength of radiation.

Radiation phenomena are usually classified by their characteristic wavelength, λ. At temperatures encountered in most engineering applications, the bulk of the thermal energy emitted by a body lies in the wavelengths between λ= 0.1 and 100 **μm**. For this reason, the portion of the wavelength spectrum between λ= 0.1 and 100 μm is generally referred to as "THERMAL RADIATION". The wavelength spectrum in the range λ= 0.4 and 0.7 μm is visible to the naked eye, and this is called „light rays‟. The wavelength spectrum of radiation is illustrated in Fig 10.1

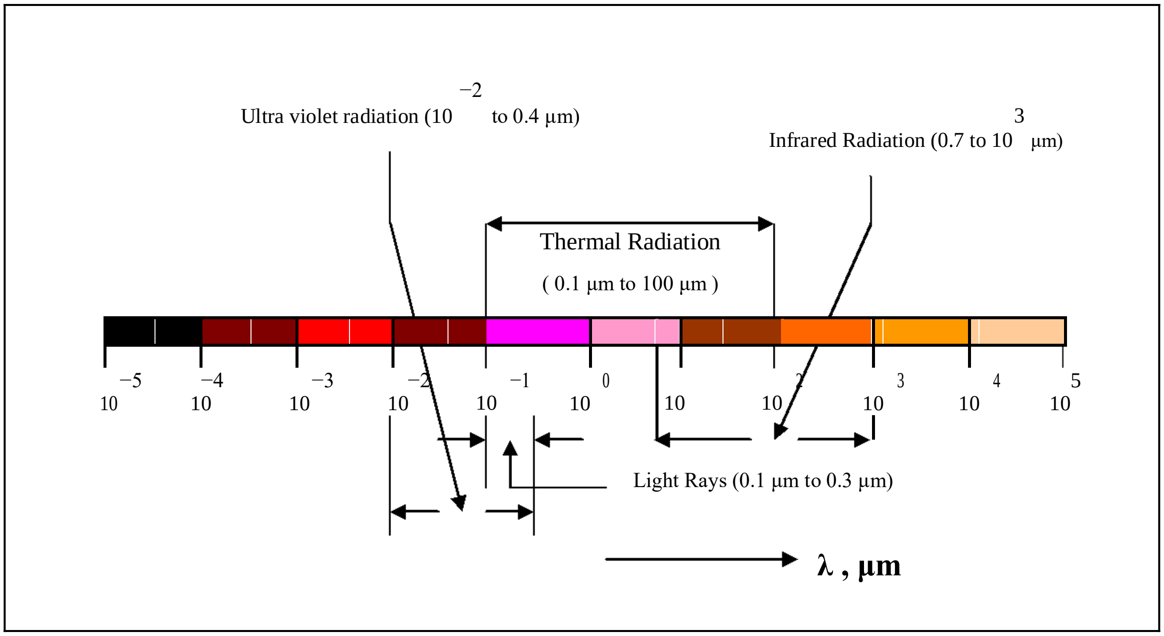


Fig. 10.1 Typical Spectrum of electromagnetic radiation

In the study of radiation transfer, a distinction should be made between bodies which are "semi-transparent" to radiation and those which are "opaque". If the material is semitransparent to radiation, such as glass, salt crystals, and gases at elevated temperatures, then the radiation leaving the body from its outer surfaces results from emissions at all depths within the material. The emission of radiation for such cases is a "BULK" or a "VOLUMETRIC PHENOMENON". If the material is opaque to thermal radiation, such as metals, wood, rock etc. then the radiation emitted by the interior regions of the body cannot reach the surface. In such cases, the radiation emitted by the body originates from the material at the immediate vicinity of the surface (i.e. within about 1**µ**m) and the emission is regarded as a "SURFACE PHENOMENON". It should also be noted that a material may behave as a semi transparent medium for certain temperature ranges, and as opaque for other temperatures. Glass is a typical example for such behaviour. It is semi transparent to thermal radiation at elevated temperatures and opaque at intermediate and low temperatures.

10.1.2 *DEFINITIONS OF TERMS USED IN THERMAL RADIATION*

- ***Monochromatic Emissive Power (E_λ):*** The monochromatic emissive power of a surface at any temperature **T** and wavelength **λ** is defined as the quantity which when multiplied by **dλ** gives the radiant flux in the wavelength range - **λ** to **λ+dλ**.

- ***Emissive Power (E):*** The emissive power of a surface is the energy emitted by a surface at a given temperature per unit time per unit area for the entire wavelength range, from **λ = 0** to **λ = ∞**.

$$E = \int_0^{\infty} E_\lambda d\lambda \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(10.2)$$

- ***Absorptivity, Reflectivity and Transmissibility of a body:***

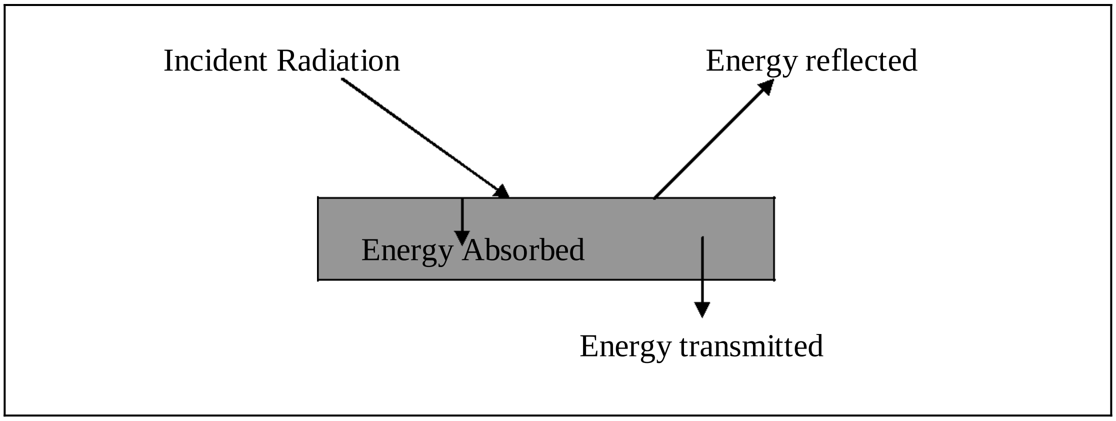


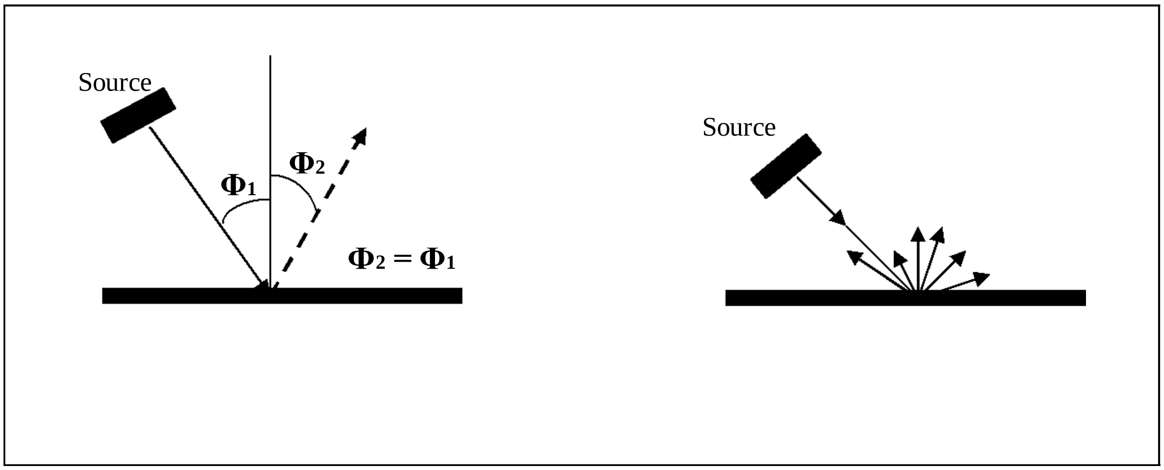
Fig.10.2: Effects of radiation incident on a surface

When a radiant energy strikes a material surface, part of the radiation is reflected, part is absorbed, and part is transmitted, as shown in Fig. 10.2. Reflectivity (ρ) is defined as the fraction of energy which is reflected, Absorptivity (α) as the fraction absorbed, and Transmissivity (η) as the fraction transmitted. Thus, $\boldsymbol{\rho + \alpha + \eta = 1}$.

Most solid bodies do not transmit thermal radiation, so that for many applied problems, the transmissivity may be taken as zero. Then

$$\rho + \alpha = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(10.3)$$

- ***Specular Radiation and Diffuse Radiation*:**



(a) Specular Radiation **(b) Diffuse Radiation**

Fig.10.3: Specular and Diffuse Radiation

When radiation strikes a surface, two types of reflection phenomena may be observed. If the angle of incidence is equal to the angle of reflection, the radiation is called Specular. On the other hand, when an incident beam is distributed uniformly in all directions after reflection, the radiation is called Diffuse Radiation. The two types of radiation are depicted in Fig. 10.3. Ordinarily, no real surface is either specular or diffuse. An ordinary mirror is specular for visible light, but would not necessarily be specular over the entire wavelength range. A rough surface exhibits diffuse behaviour better than a highly polished surface. Similarly, a highly polished surface is more specular than a rough surface.

- ***Black Body:***

A body which absorbs all incident radiation falling on it is called a blackbody. For a blackbody, $\alpha = 1$, $\rho = \eta = 0$. For a given temperature and wavelength, no other body at the same temperature and wavelength, can emit more radiation than a blackbody. Blackbody radiation at any temperature T is the maximum possible emission at that temperature. A blackbody or ideal radiator is a theoretical concept which sets an upper limit to the emission f radiation. It is a standard with which the radiation characteristics of other media are compared.

- ***Emissivity of a Surface (ε):***

The emissivity of a surface is the ratio of the emissive power of the surface to the emissive power of a black surface at the same temperature. It is denoted by the symbol **ε**.

$$\text{i.e. } \boldsymbol{\varepsilon} = [E/E_b]_T.$$

- ***Monochromatic Emissivity of a Surface (ελ):***

The monochromatic emissivity of a surface is the ratio of the monochromatic emissive power of the surface to the monochromatic emissive power of a black surface at the same temperature and same wavelength.

$$\varepsilon_\lambda = [E_\lambda / E_{b\lambda}]_{\lambda,T}.$$

- ***Gray Body:***

A gray body is a body having the same value of monochromatic emissivity at all wavelengths. i.e.

$$\varepsilon = \varepsilon_\lambda, \text{ for a gray body.}$$

- ***Radiosity of a Surface (J):***

This is defined as the total energy leaving a surface per unit time per unit area of the surface. This definition includes the energy reflected by the surface due to some radiation falling on it.

- ***Irradiation of a surface(G):***

This is defined as the radiant energy falling on a surface per unit time, per unit area of the surface.

Therefore if E is the emissive power, J is the radiosity, ε is the irradiation and ρ the reflectivity of a surface, then,

$$\mathbf{J = E + \rho G}$$

For an opaque surface, $\boldsymbol{\rho + \alpha = 1}$ or $\boldsymbol{\rho = (1 - \alpha)}$

$$\mathbf{J = E + (1-\alpha)G} \quad \ldots\ldots \quad \mathbf{(10.4)}$$

10.2 LAWS OF RADIATION

10.2.1 *STEFAN – BOLTZMANN LAW:*

This law states that the emissive power of a blackbody is directly proportional to the fourth power of the absolute temperature of the body.

i.e., $\mathbf{E_b \ \alpha \ T^4}$

Or $\mathbf{E_b = \zeta T^4}$ ---------------------------------- **(10.5)**

where ζ is called the Stefan – Boltzmann constant.

In SI units $\zeta = 5.669\text{x}10^{-8}$ W/(m^2-K^4).

10.2.2 *PLANCK'S LAW:*

This law states that the monochromatic power of a blackbody is given by

$$E_{b\lambda} = \frac{C_1}{\lambda^5 \left[e^{(C_2 / \lambda T)} - 1 \right]} \quad \ldots\ldots (10.6)$$

where C_1 and C_2 are constants whose values are found from experimental data; $C_1 = 3.7415 \times 10^{-16}$ Wm^2 and $C_2 = 1.4388 \times 10^{-2}$ m-K.

λ is the wavelength and T is the absolute temperature in K.

10.2.3 *WEIN'S DISPLACEMENT LAW:*

It can be seen from Eq. 10.6 that at a given temperature, $E_{b\lambda}$ depends only on $\boldsymbol{\lambda}$**.** Therefore the value of $\boldsymbol{\lambda}$ which gives maximum value of $E_{b\lambda}$ can be obtained by differentiating Eq(10.6) w.r.t $\boldsymbol{\lambda}$ and equating it to zero.

Let $C_2/\lambda T = y$. Then Eq. (10.6) reduces to

$$E_{b\lambda} = \frac{C_1}{[C_2/(yT)]^5[e^y - 1]}$$

Then $$\frac{dE_{b\lambda}}{dy} \quad \frac{C_1\, d/dy\, \{[C_2/(yT)]^5 [e^y - 1]\}}{\{[C_2/(yT)]^5 [e^y - 1]\}^2}$$

or $$d/dy\, \{[C_2/(yT)]^5 (e^y - 1)\} = 0$$

Or $$e^y(5 - y) = 5$$

By trial and error, $y = 4.965$

Therefore, if λ_m denotes the value of λ which gives maximum $E_{b\lambda}$, then

$$C_2/\lambda_m T = 4.965$$

Or $$\lambda_m T = C_2/4.965 = 1.4388 \times 10^{-2}/4.965$$

$$\boldsymbol{\lambda_m T = 0.002898 \text{ m-K}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \textbf{(10.7)}$$

Equation (10.7) is called the Wein"s displacement law. From this equation it can be seen that the wavelength at which the monochromatic emissive power is a maximum decreases with increasing temperature. This is also illustrated in Fig 10.4(a). Fig 10.4(b) gives a comparison of monochromatic emissive powers for different surfaces at a particular temperature for different wavelengths.

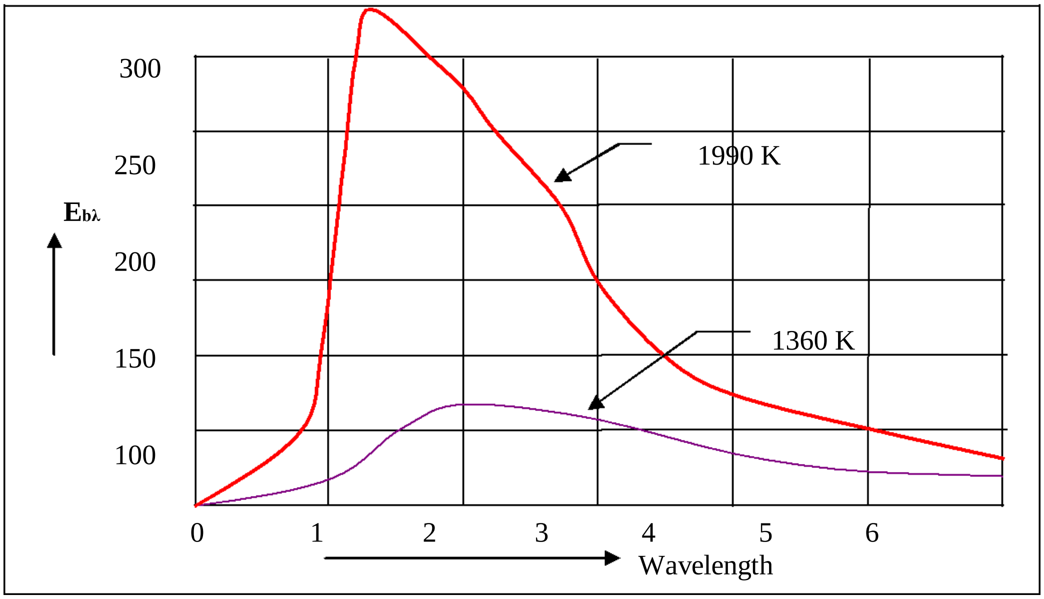


Fig. 10.4 (a) Black body emissive power as a function of wave length and Temperature

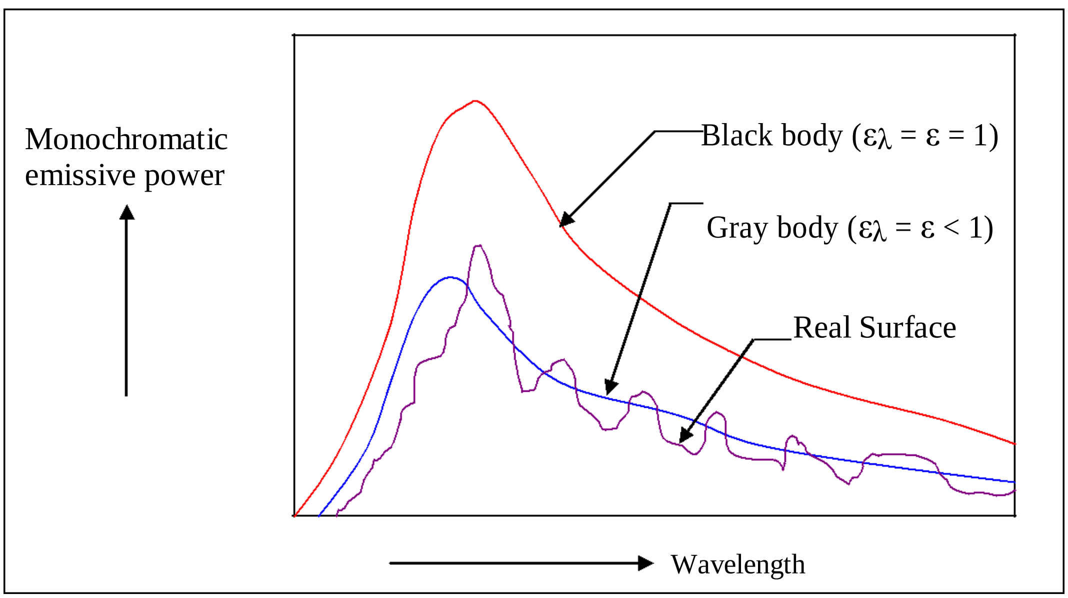


Fig. 10.4 (b) Comparison of emissive powers of different types o surfaces as a function of wavelength at a given temperature

10.2.4 *KIRCHOFF'S LAW:*

This law states that the emissivity of a surface is equal to its absorptivity when the surface is in thermal equilibrium with the surroundings.

Proof: Consider a perfect black enclosure i.e. the one which absorbs all the incident radiation falling on it (see Fig 10.5). Now let the radiant flux from this enclosure per unit area arriving at some area be $\mathbf{q_i}$ W/m^2.

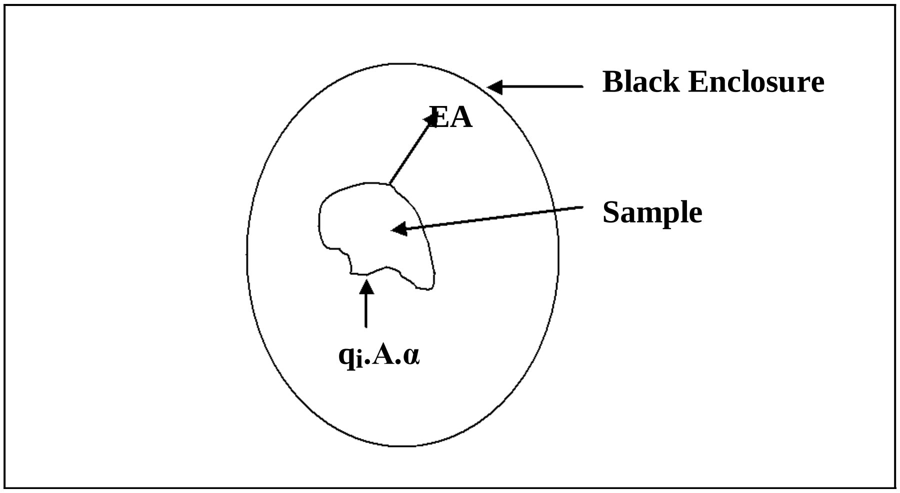


Fig. 10.5 : Model used for deriving Kirchoff law

Now suppose that a body is placed inside the enclosure and allowed to come to thermal equilibrium with it. At equilibrium, the energy absorbed by the body must be equal to the energy emitted; otherwise there would be an energy flow into or out of the body, which would raise or lower its temperature. At thermal equilibrium we may write

$$\mathbf{EA = q_i\, A\, \alpha \ldots\ldots\ldots\ldots(10.8)}$$

If we now replace the body in the enclosure with a black body of the same size and shape and allow it to come to thermal equilibrium with the enclosure,

$$\mathbf{E_bA = q_i\, A \ldots\ldots\ldots\ldots(10.9)}$$

Since $\alpha = 1$ for a blackbody.

If Eq. 10.8 is divided by Eq. 10.9 we get

$$\mathbf{E/E_b = \alpha}$$

But by definition $E/E_b = \varepsilon$, the emissivity of the body, so that $\boldsymbol{\varepsilon = \alpha}$**…………(10.10)**

Equation 10.10 is called Kirchoff‟s law and is valid only when the body is in thermal equilibrium with the surroundings. However, while analyzing radiation problems in practice we assume that Kirchoff‟s law holds good even if the body is not in thermal equilibrium with the surroundings, as the error involved is not very significant.

10.3 ILLUSTRATIVE EXAMPLES ON BASIC CONCEPTS

Example 10.1: *The emission of radiation from a surface can be approximated as blackbody radiation at 1000K.*

(a) What fraction of the total energy emitted is below λ = 5μm
(b) What is the wavelength below which the emission is 10.5% of the total emission at 1000K.
(c) What is the wavelength at which the maximum spectral emission occurs at 1000K.

Solution: The radiation flux emitted by the blackbody over the wavelength interval 0 – λ is given by

$$[E_b]_{0-\lambda} = \int_0^{\lambda} E_{b\lambda} d\lambda$$

The integration required in the above equation has been done numerically and the results are presented in the form of a table. The table gives the value of $D_{0-\lambda}$ where

$$D_{0-\lambda} = \frac{\int_0^{\lambda} E_{b\lambda} d\lambda}{\int_0^{\infty} E_{b\lambda} d\lambda} = \frac{1}{\zeta T^4} \int_0^{\lambda} E_{b\lambda}\, d\lambda$$

(a) From Table of Radiation properties, for λT = 5 x 1000 = 5000, $D_{0-\lambda}$ = 0.6337. This means that 63.37 % of the total emission occurs below λ = 5 μm.

(b) From the same table, for $D_{0-\lambda}$ = 0.105, λT = 2222. Hence λ = 2222/1000 = 2.222 μm.

(c) From Wein‟s displacement law, $\lambda_m T$ = 0.002898.
Hence for T = 1000 K, λ_m =0.002898 / 1000 = 2.898 x 10^{-6} m = 2.898 μm.

Example 10.2: *The monochromatic emissivity of a surface varies with the wavelength in the following manner:*

$$\varepsilon_\lambda = 0 \quad \text{for } \lambda < 0.3\mu m$$
$$= 0.9 \text{ for } 0.3\mu m < \lambda < 1\mu m$$
$$= 0 \text{ for } \lambda > 1\mu m$$

Calculate the heat flux emitted by the surface if it is at a temperature of 1500 K

Solution:

$$E_\lambda = \varepsilon_\lambda \; E_{b\lambda}$$

Therefore $$E = \int_0^\infty \varepsilon_\lambda \; E_{b\lambda} = \int_0^{0.3\,\mu m} 0.0\, E_{b\lambda}\, d\lambda + \int_{0.3\,\mu m}^{1\,\mu m} 0.9\, E_{b\lambda}\, d\lambda + \int_{1\,\mu m}^{\infty} 0.0\, E_{b\lambda}\, d\lambda$$

$$= 0.9 \int_{0.3\,\mu m}^{1\,\mu m} E_{b\lambda}\, d\lambda = 0.9 \left[\int_0^{1\,\mu m} E_{b\lambda}\, d\lambda - \int_0^{0.3\,\mu m} E_{b\lambda}\, d\lambda \right]$$

$$= 0.9\, \zeta\, T^4 \,[\, D_{0\text{-}1} - D_{0\,-\,0.3}]$$

For $\lambda = 1\mu m$, $\lambda T = 1500\ \mu m$-K, therefore $D_{0\text{-}1} = ½\,(0.01972 + 0.00779) =$

0.93755 For $\lambda = 0.3\mu m$, $\lambda T = 450\ \mu m$-K, therefore $D_{0\text{-}3} = 0$

Thus $E = 0.9 x 5.67 x 10^{-8} x 1500^4\,[0.013755 - 0] = 3553\ W/m^2$

Example 10.3: *Calculate the heat flux emitted due to thermal radiation from a black surface at 6000^0 C. At what wavelength is the monochromatic emissive power maximum and what is the maximum value?*

Solution: Temp of the black surface = 6273K

Heat Flux emitted = $E_b = \zeta T^4 = 5.67 x 10^{-8} x 6273^4 = 87798\ KW/m^2$

Wavelength corresponding to max monochromatic emissive power is given

by $\lambda_m T = 0.002898$ m-K

$\lambda_m = 0.002898/6273 = 4.62 x 10^{-7}$ m

The maximum monochromatic emissive power is given by

$$(E_{b\lambda})_{max} = \frac{2\,\pi\, C_1}{\lambda_{max}\,[\,\exp\{C_2 / (\lambda_{max} T)\} - 1]}$$

$$= \frac{2 \times \pi \times 0.596 \times 10^{-16}}{(4.62 \times 10^{-7})^5 \times [\,\exp\{\,0.014387 / 0.002898\} - 1]}$$

$$= 1.251 \times 10^{14}\ W / m^2$$

Example 10.4: *The spectral hemispherical emissivity (monochromatic emissivity) of fire brick at 750K as a function of wavelength is as follows:*

$\varepsilon_1 = 0.1$ for $0 \le \lambda \le 2\mu m$
$\varepsilon_2 = 0.6$ for $2\mu m \le \lambda \le 14\mu m$
$\varepsilon_3 = 0.8$ for $14 \le \lambda \le \infty$
Calculate the hemispherical emissivity, ε for all wavelengths.

Solution:

$$\varepsilon = \frac{E}{E_b} = \frac{\int_0^\infty \varepsilon_\lambda E_{b\lambda}\, d\lambda}{\zeta T^4} = \frac{1}{\zeta T^4}\left[\varepsilon_1 \int_0^{\lambda 1} E_{b\lambda}\, d\lambda + \varepsilon_2 \int_{\lambda 1}^{\lambda 2} E_{b\lambda}\, d\lambda + \varepsilon_3 \int_{\lambda 2}^{\lambda 3} E_{b\lambda}\, d\lambda\right]$$

Where $\lambda_1 = 2\mu m$, $\lambda_2 = 14\mu m$, $\lambda_3 = \infty$

Thus $= \varepsilon_1 D_{0-\lambda 1} + \varepsilon_2[D_{0-\lambda 2} - D_{0-\lambda 1}] + \varepsilon_3[D_{0-\infty} - D_{0-\lambda 2}]$

Now, $\lambda_1 T = 2x750 = 1500$; $D_{0-\lambda 1} = 0.013$

$\lambda_2 T = 14x750 = 10500$; $D_{0-\lambda 2} = 0.924$ $\lambda_3 T = \infty$; $D_{0-\lambda 3} = 1$

Hence $\varepsilon = 0.1 \text{ x } 0.013 + 0.6 \text{ x } [0.924 - 0.013] + 0.8 \text{ x } [1 - 0.924] = 0.609$

Example 10.5: *the filament of a light bulb is assumed to emit radiation as a black body at 2400K. if the bulb glass has a transmissivity of 0.90 for radiation in the visible range, calculate the percentage of the total energy emitted by the filament that reaches the ambient as visible light.*

Solution: The wavelength range corresponding to the visible range is taken as $\lambda_1 = 0.38\mu m$ to $\lambda_2 = 0.76\mu m$. Therefore the fraction F of the total energy emitted in this range is given by

$$F = \eta \left[\frac{\int_{\lambda 1}^{\lambda 2} E_{b\lambda}\, d\lambda}{E_b(T)}\right] = \eta\left[\int_0^{\lambda 2} E_{b\lambda}\, d\lambda - \int_0^{\lambda 1} E_{b\lambda}\, d\lambda\right] / E_b$$

$$= \eta\, [D_{0-\lambda 2} - D_{0-\lambda 1}].$$

Now $\lambda_1 T = 0.38 \text{ x } 2400 = 912$. Hence $D_{0-\lambda 1} = 0.0002$

and $\lambda_2 T = 0.76 \text{ x } 2400 = 1824$. Hence $D_{0-\lambda 2} = 0.0436$

Therefore $F = 0.9 \text{ x } [0.0436 - 0.0002] = 0.039$.

Only 3.9 % of the total energy enters the ambient as light. The remaining energy produces heating.

10.4 RADIATION HEAT EXCHANGE BETWEEN INFINITE PARALLEL SURFACES IN THE PRESENCE OF NON PARTICIPATING MEDIUM

Assumptions:

(i) The medium does not participate in radiation heat exchange between the two surfaces.

(ii) The surfaces are flat and are at specified uniform temperatures.

10.4.1: *RADIATION EXCHANGE BETWEEN TWO PARALLEL BLACK SURFACES*

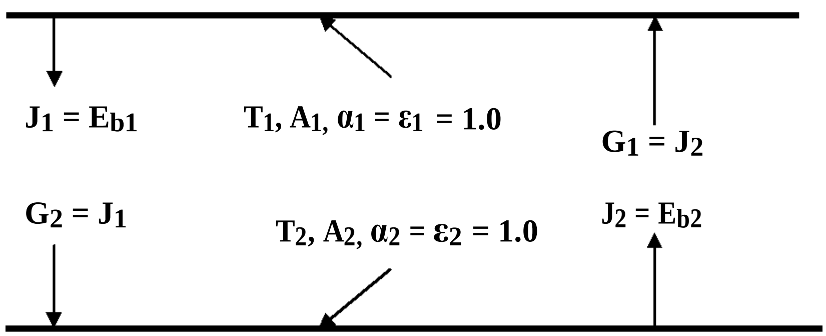


Fig: 10.6 Radiation heat exchange between two parallel black surfaces.

Since both surfaces are parallel, flat and infinite, radiosity of surface 1 = irradiation of surface 2 and vice versa. i.e. $J_1 = G_2$ and $J_2 = G_1$. Since both the surfaces are black, $\mathbf{J_1 = E_{b1} = \zeta T_1^4}$ and $\mathbf{J_2 = E_{b2} = \zeta T_2^4}$

Net radiation leaving $A_1 = Q_{r1} = A_1(J_1 - G_1)$ All this energy will reach A_2.

Net radiation leaving A_1 and reaching A_2 is given by

$Q_{1-2} = Q_{r1} = A_1(J_1 - G_1) = A_1[J_1 - J_2]$

Or $Q_{1-2} = A_1[E_{b1} - E_{b2}]$

Or $$\mathbf{Q_{1-2} = \zeta A_1[T_1^4 - T_2^4]} \qquad \textbf{(10.11)}$$

10.4.2 RADIATION HEAT EXCHANGE BETWEEN TWO PARALLEL INFINITE GRAYSURFACES:

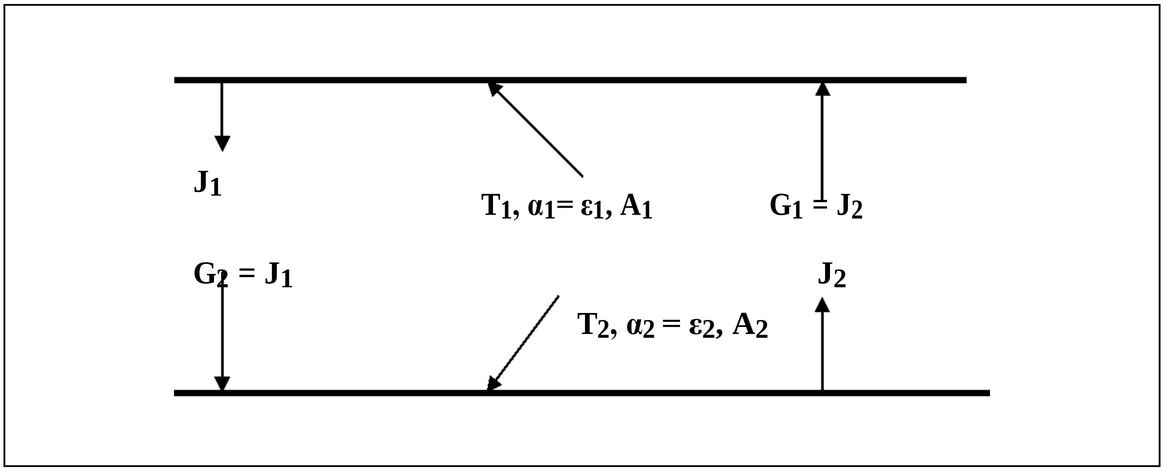


Fig: 10.7 Radiation Heat Exchange Between 2 Parallel Infinite Gray Surfaces.

Since the net radiation leaving A_1 will reach

A_2, $Q_{1-2} = Q_{r1} = A_1[J_1 - G_1]$ $J_1 = E_1 + (1-$ (10.12a)

$\alpha_1)G_1$ (10.12b)

$J_2 = E_2 + (1-\alpha_2)G_2$ (10.12c)

$J_1 = G_2$ (10.12d)

$J_2 = G_1$ (10.12e)

Equation (10.12b) can be written as

$$J_1 - (1 - \alpha_1)G_1 = E_1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(4.12f)$$

Equation (4.12c) with the help of Eqns. (10.12d) and Eqns. (10.12e) can be rewritten as

$$-(1 - \alpha_2)J_1 + G_1 = E_2 \qquad (10.12g)$$

Solving for J_1 and G_1 from Eq. (10.12f) and (10.12g) we get

$$J_1 = \frac{E_1 + (1 - \alpha_1) E_2}{1 - (1 - \alpha_1)(1 - \alpha_2)}$$

$$\text{Or } J_1 = \frac{\varepsilon_1 E_{b1} + (1 - \alpha_1)\varepsilon_2 E_{b2}}{1 - (1 - \alpha_1)(1 - \alpha_2)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(10.13a)$$

$$\text{and } G_1 = \frac{\varepsilon_2 E_{b2} + (1 - \alpha_2)\varepsilon_1 E_{b1}}{1 - (1 - \alpha_1)(1 - \alpha_2)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(10.13b)$$

Substituting these expressions for J_1 and G_1 in Eq.(10.12a) we get

$$Q_{1-2} = \frac{A_1}{[1 - (1 - \alpha_1)(1 - \alpha_2)]}[\varepsilon_1 E_{b1} + (1 - \alpha_1)\varepsilon_2 E_{b2} - \varepsilon_2 E_{b2} - (1 - \alpha_2)\varepsilon_1 E_{b1}]$$

$$\text{Or } Q_{1-2} = \frac{A_1 [\alpha_2 \varepsilon_1 E_{b1} - \alpha_1 \varepsilon_2 E_{b2}]}{[1 - (1 - \alpha_1)(1 - \alpha_2)]}$$

Substituting for E_{b1} and E_{b2} in terms of temperatures we get

$$\text{Or } Q_{1-2} = \frac{\zeta A_1 [\alpha_2 \varepsilon_1 T_1^4 - \alpha_1 \varepsilon_2 T_2^4]}{[1 - (1 - \alpha_1)(1 - \alpha_2)]} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(10.14)$$

If Kirchoff‟s law holds good then $\alpha_1 = \varepsilon_1$ and $\alpha_2 = \varepsilon_2$.

$$\text{Hence } Q_{1-2} = \frac{\zeta A_1 [\varepsilon_1 \varepsilon_2 T_1^4 - \varepsilon_1 \varepsilon_2 T_2^4]}{[1 - (1 - \varepsilon_1)(1 - \varepsilon_2)]}$$

$$\text{Or} \quad \mathbf{Q_{1-2} = \frac{\zeta A_1 (T_1^4 - T_2^4)}{[1/\varepsilon_1 + 1/\varepsilon_2 - 1]}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\mathbf{(10.15)}$$

__10.4.3 PLANE RADIATION SHIELDS:__ It is possible to reduce the net radiation heat exchange between two infinite parallel gray surfaces by introducing a third surface in between them. If the third surface, known as the radiation shield is assumed to be very thin, then both sides of this surface can be assumed to be at the same temperature.

Fig.10.8 shows a scheme for radiation heat exchange between two parallel infinite gray surfaces at two different temperatures T_1 and T_2 in presence of a radiation shield at a uniform temperature, T_3.

Now $$\frac{Q_{1-3}}{A_1} = \frac{\zeta\,(T_1^4 - T_3^4)}{[\,1/\varepsilon_1 + 1/\varepsilon_{13} - 1]} \quad \text{…………………………..(10.16a)}$$

And $$\frac{Q_{3-2}}{A_1} = \frac{\zeta\,(T_3^4 - T_2^4)}{[\,1/\varepsilon_{32} + 1/\varepsilon_2 - 1]} \quad \text{…………………………..(10.16b)}$$

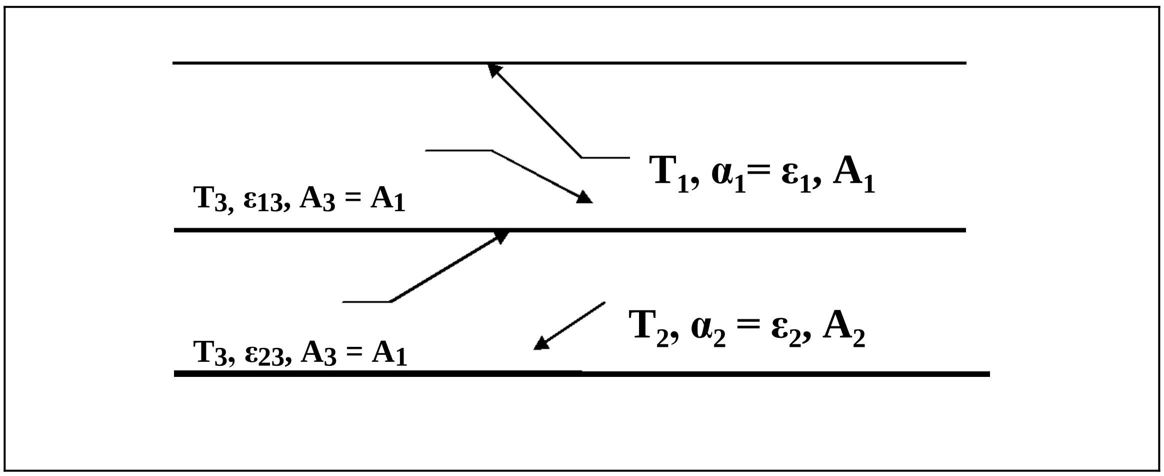


Fig: 10.8 Radiation Heat Exchange Between Two Parallel Infinite Gray surfaces in presence of a radiation shield

For steady state conditions, these two must be equal..Therefore we have

$$\frac{(T_1^4 - T_3^4)}{[\,1/\varepsilon_1 + 1/\varepsilon_{13} - 1]} = \frac{(T_3^4 - T_2^4)}{[\,1/\varepsilon_{32} + 1/\varepsilon_2 - 1]}$$

Let $X = [\,1/\varepsilon_1 + 1/\varepsilon_{13} - 1]$

and $Y = [\,1/\varepsilon_{32} + 1/\varepsilon_2 - 1]$

Then,

$$\frac{(T_1^4 - T_3^4)}{X} = \frac{(T_3^4 - T_2^4)}{Y}$$

Solving for T_3 we get

$$\mathbf{T_3 = \left[\frac{T_1^4 + (X/Y)T_2^4}{(1 + X/Y)}\right]^{1/4}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(\mathbf{10.16c})$$

Substituting this value of T_3 in Eq. (10.16a) we get

$$\mathbf{Q_{1\text{-}3} / A_1 = Q_{3\text{-}2} / A_1 = (Q_{1\text{-}2} / A)_{1\ Rad.Shield} = \zeta\ \{ T_2^{\ 4} - [\{T_1^4 + (X/Y)T_2^4\}/(1 + X/Y)]\ \} / X}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\mathbf{10.17a})$$

Special case:

When $\varepsilon_1 = \varepsilon_2 = \varepsilon_{13} = \varepsilon_{32} = \varepsilon$, then $X = Y = (2/\varepsilon) - 1$

Hence, $$T_3 = [(T_1^4 + T_2^4) / 2\ ]^{1/4} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(10.18a)$$

and $$[Q_{1\text{-}2} / A\ ]_{1\ rad\ shield} = \frac{\zeta\{\ T_1^4 - [(T_1^4 + T_2^4) / 2\ ]}{[2/\varepsilon - 1]}$$

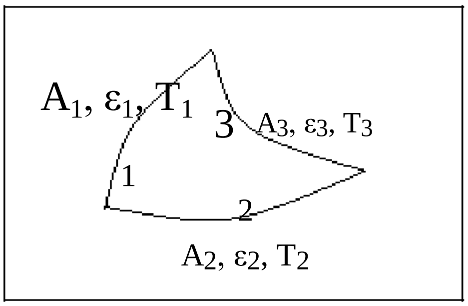
$$= \frac{\zeta\ [T_1^4 - T_2^4]}{2\ [2/\varepsilon - 1]} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(10.18b)$$

It can be seen from the above equation that when the emissivities of all surfaces are equal, the net radiation heat exchange between the surfaces in the presence of single radiation shield is 50% of the radiation heat exchange between the same two surfaces without the presence of a radiation shield. This statement can be generalised for N radiation shields as follows:

$$\mathbf{[Q_{1\text{-}2} / A]_{N\ shields} = \frac{1}{(N + 1)}\ [Q_{1\text{-}2} / A]_{without\ shield}} \quad \ldots\ldots\ldots\ldots(\mathbf{10.18c})$$

10.8.6: *NETWORK METHOD FOR THREE ZONE ENCLOSURE*

The network method described above can be readily generalised to enclosures involving three or more zones. However when there are more than three zones, the analysis becomes more involved and it is preferable to use the more direct "Radiosity Matrix" method. The radiation network for a three zone enclosure shown in Fig 4.20(a) is shown in Fig 4.20(b)



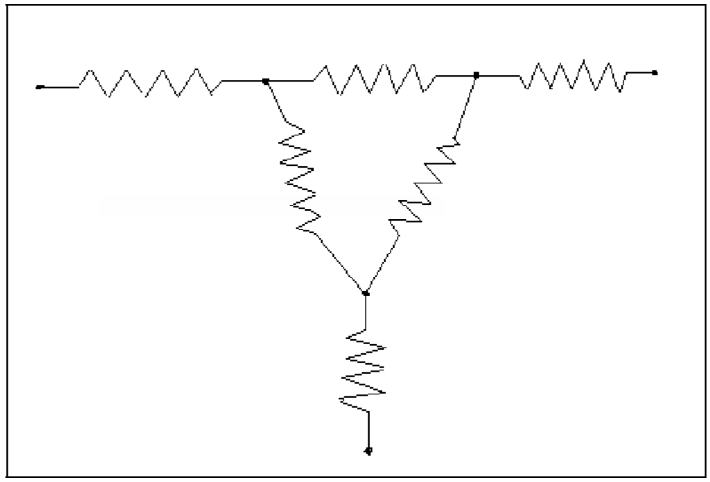
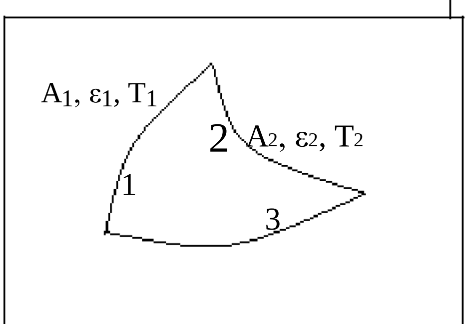


Fig 10.20: Radiation network for a three zone enclosure.

Reradiating Surface: In many practical situations one of the zones may be thermally insulated. In such a case, the net radiation heat flux in that particular zone is zero, because that surface emits as much energy as it receives by radiation from the surrounding zones. Such a zone is called a "RERADIATION ZONE" or an "ADIABATIC ZONE". Fig 10.21(a) represents a three zone enclosure with surface (3) being the reradiating surface and Fig 10.21(b) the corresponding radiation network.



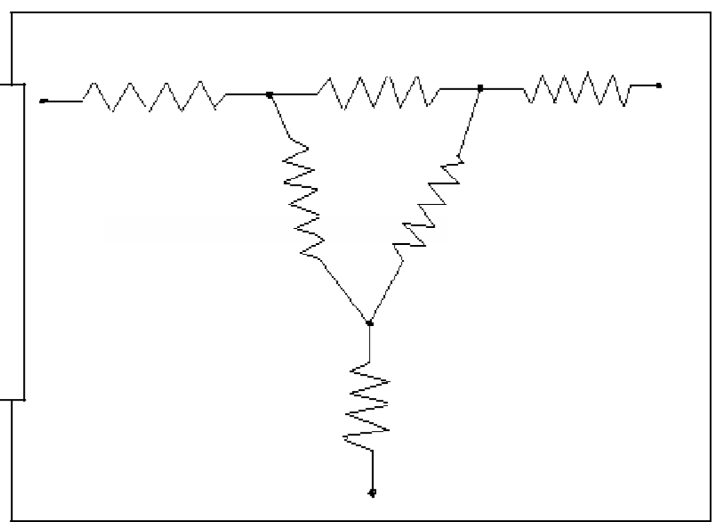


Fig 10.21: Radiation Heat Exchange in a 3 zone enclosure with one reradiating surface

For a three zone enclosure under steady state conditions, by I law of thermodynamics.

$$Q_{r1} + Q_{r2} + Q_{r3} = 0 \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_} (4.3$$

2) A_3 is reradiating, $Q_{r3} = 0$

$$\therefore Q_{r_1} = -Q_{r_2} = \frac{E_{b1} - E_{b2}}{R_{eq}} \text{\_\_\_\_\_\_\_\_\_\_\_\_\_} (4.33a)$$

$$\text{where } R_{eq} = R_1 + \left[\frac{1}{R_{12}} + \frac{1}{R_{13} + R_{23}}\right]^{-1} + R_2$$

$$\text{or } R_{eq} = \frac{1-\varepsilon_1}{A_1\varepsilon_1} + \left[A_1F_{1-2} + \frac{1}{\left(\frac{1}{A_1F_{13}} + \frac{1}{A_2F_{2-3}}\right)}\right]^{-1} + \frac{1-\varepsilon_2}{A_2\varepsilon_2} \text{\_\_\_\_\_\_} (4.33b)$$

4.9 ILLUSTRATIVE EXAMPLES ON NETWORK METHOD:

Example 4.24: Two square plates 1m x 1m are parallel to and directly opposite to each other at a distance of 1m. The hot plate is at 800K and has an emissivity of 0.8. The clod plate is at 600K and also has an emissivity of 0.8. The radiation heat exchange takes place between the plates as well as the ambient at 300K through the opening between the plates. Calculate the net radiation at each plate and the ambient.

Solution:

Data:-

T_1 = 800K, ϵ_1 = 0.8

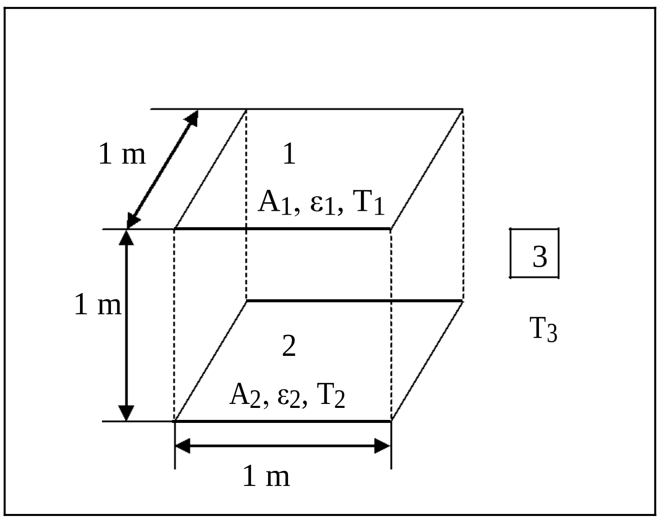
T_2 = 600K, ϵ_2 = 0.8

T_3 = 300K

To find:-

i) Q_{r1}, Q_{r2},

ii) Q_{r3}



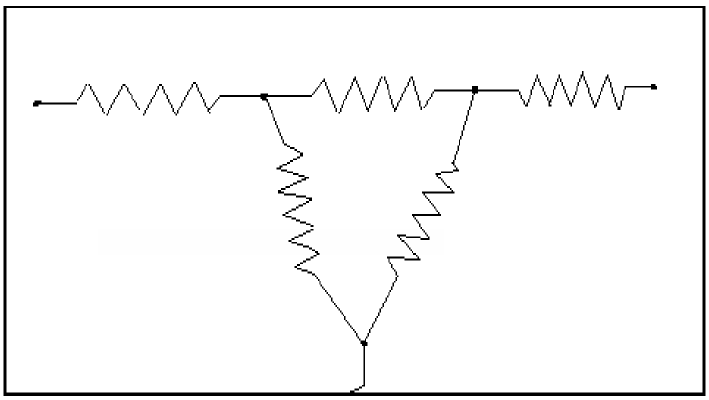
$$R_1 = \frac{1-\varepsilon_1}{A_1\varepsilon_1} = \frac{1-0.8}{1\times1\times0.8} = 0.25$$

$$R_2 = \frac{1-\varepsilon_2}{A_2\varepsilon_2} = \frac{1-0.8}{1\times1\times0.8} = 0.25$$

$$R_3 = \frac{1-\varepsilon_3}{A_3\varepsilon_3} : A_3 \text{ is the area of the surroundings } \therefore A_3\ 1$$

$$\therefore R_3 = 0$$

$\therefore$ The radiation network for this problem will be as shown below



$$\text{From chart } F_{1-2} = F_{2-1} = 0.20$$

$$\text{But } F_{1-1} + F_{1-2} + F_{1-3} = 1 \text{ and } F_{1-1} = 0$$

$$\therefore F_{1-3} = 1 - F_{1-2} = 1 - 0.2 = 0.8 = F_{2-3}$$

$$R_{12} = \frac{1}{A_1F_{1-2}} = \frac{1}{(1\times1)\,0.2} = 5$$

$$R_{13} = \frac{1}{A_1F_{1-3}} = \frac{1}{(1\times1)\,0.8} = 1.25$$

$$R_{23} = \frac{1}{A_1F_{2-3}} = \frac{1}{(1\times1)\,0.8} = 1.25$$

$$E_{b_1} = \sigma T_1^4 = 5.67\times10^{-8}\times800^4 = 23224\,W/m^2 = 23.224\,KW/m^2$$

$$E_{b_2} = \sigma T_2^4 = 5.67\times10^{-8}\times600^4 = 7348\,W/m^2 = 7.348\,KW/m^2$$

$$E_{b_3} = \sigma T_3^4 = 5.67\times10^{-8}\times300^4 = 459\,W/m^2 = 0.459\,KW/m^2$$

For steady state radiation, radiation energy cannot accumulate at nodes J_1, J_2 and J_3

$$\therefore Q_{r1} = Q_{12} + Q_{13}$$

$$\text{or } Q_{r1} - Q_{12} - Q_{13} = 0$$

$$\therefore \frac{E_{b_1} - J_1}{R_1} - \frac{(J_1 - J_2)}{R_{12}} - \frac{J_1 - J_3}{R_{13}} = 0$$

$$\text{or } \frac{23.224 - J_1}{0.25} - \frac{(J_1 - J_2)}{5} - \frac{(J_1 - 0.459)}{1.25} = 0 \_\_\_\_\_\_\_ (a)$$

$$\text{similarly } Q_{r_2} = Q_{21} + Q_{23} \Rightarrow Q_{r_2} - Q_{21} - Q_{23} = 0$$

$$\frac{E_{b_2} - J_2}{R_2} - \frac{(J_2 - J_1)}{R_{12}} - \frac{J_2 - E_{b_3}}{R_{23}} = 0$$

$$\frac{7.348 - J_2}{0.25} - \frac{(J_2 - J_1)}{5} - \frac{(J_2 - 0.459)}{1.25} = 0 \_\_\_\_\_\_\_ (b)$$

Solving Eqn (a) and (b) simultaneously we get

$$J_1 = 18.921 KW/m^2;\ J_2 = 6.709 KW/m^2$$

$$\therefore Q_{r_1} = \frac{E_{b_1} - J_1}{R_1} = \frac{23.224 - 18.921}{0.25} = 17.212 KW$$

$$Q_{r_2} = \frac{E_{b_2} - J_2}{R_2} = \frac{7.348 - 6.709}{0.25} = 2.557 KW$$

$$\text{But } Q_{r1} + Q_{r2} + Q_{r3} = 0 \Rightarrow Q_{r_3} = -\left[Q_{r_1} + Q_{r_2}\right] = -[17.212 + 2.557]$$

$$= -19.769 KW$$

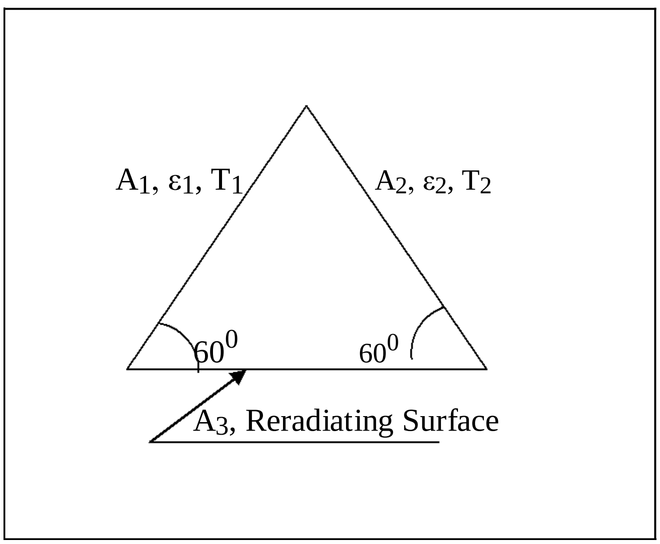
Example 4.25 The configuration of a furnace can be approximated as an equilateral triangular duct which is sufficiently long that the end efforts are negligible. The hot wall is at 900K with an emissivity of 0.8 and the cold wall is at 400K with emissivity of 0.8. The third wall is a reradiating wall. Determine the net radiation flux leaving the hot wall.

Solution:

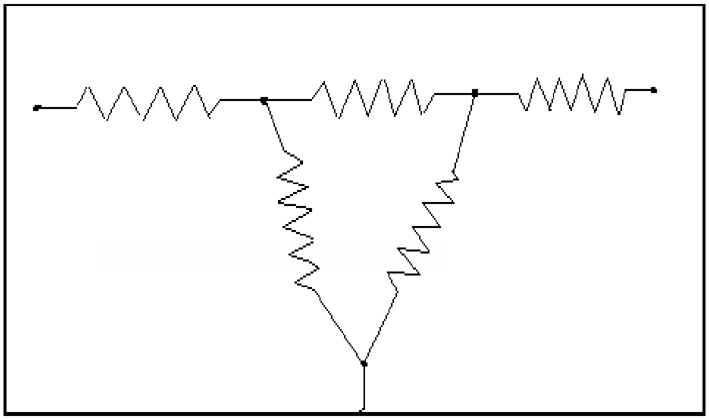
$A_1 = A_2 = A_3 = 1m^2$ (assumed)

T_1 =900K, $\epsilon_1 = 0.8$

T_2 = 400K, $\epsilon_2 = 0.8$



The radiation network for the above problem will be as shown below



$$R_1 = \frac{1-\varepsilon_1}{A_1\varepsilon_1} = \frac{1-0.8}{1\times 0.8} = 0.25$$

$$R_2 = \frac{1-\varepsilon_2}{A_2\varepsilon_2} = \frac{1-0.8}{1\times 0.8} = 0.25$$

Using Hottel's crossstring formula, we have

$$F_{1-2} = \frac{(A_1 + A_2) - A_3}{2A_1} = \frac{(1+1)-1}{2\times 1} = 0.5 = F_{1-3} = F_{2-3}$$

$$\therefore R_{12} = \frac{1}{A_1 F_{1-2}} = \frac{1}{1\times 0.5} = 2 = R_{23} = R_{13}$$

$$R_{eq} = R_1 + \left[\frac{1}{R_{12}} + \frac{1}{R_{13} + R_{23}}\right]^{-1} + R_2$$

$$= 0.25 + \left[\frac{1}{2} + \frac{1}{2+2}\right]^{-1} + 0.25 = 1.833$$

$$Q_{r_1} = \frac{E_{b_1} - E_{b_2}}{R_{eq}} = \frac{\sigma(T_1^4 - T_2^4)}{R_{eq}} = \frac{5.67\times 10^{-8}\times\left[900^4 - 400^4\right]}{1.833}$$

$$= 19503\ W/m^2$$

$$Q_{r_1} + Q_{r_2} + Q_{r_3} = 0 \text{ and } Q_{r_3} = 0 \Rightarrow Q_{r_2} = -Q_{r_1} = -19503\ W/m^2$$

Example 4.26 A short cylindrical enclosure is maintained at the temperatures as shown in Fig P4.26. Assuming $\epsilon_2 = \epsilon_3 = 1$; $\epsilon_1 = 0.8$ determine Q_{r1} and Q_{r2}

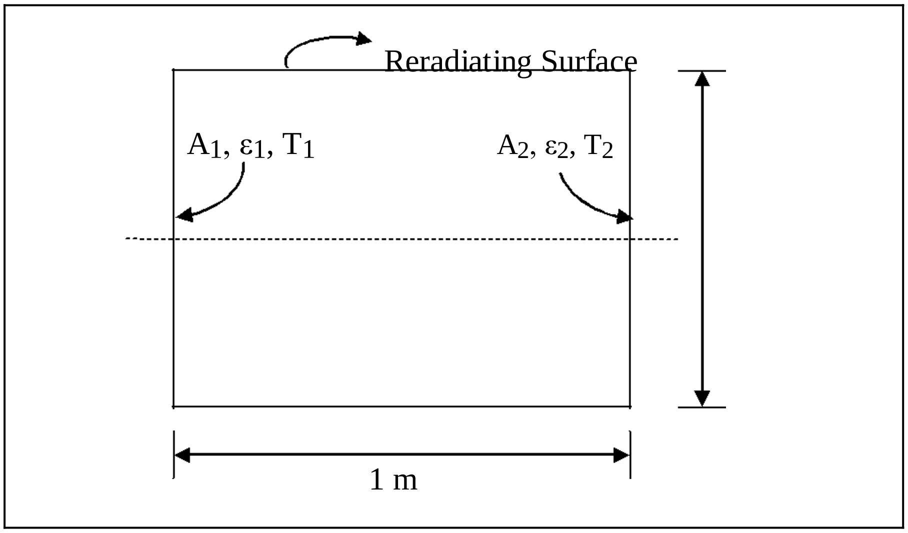
Solution:

From chart, $F_{1-2} = 0.175 = F_{2-1}$ $(A_2 = A_1)$

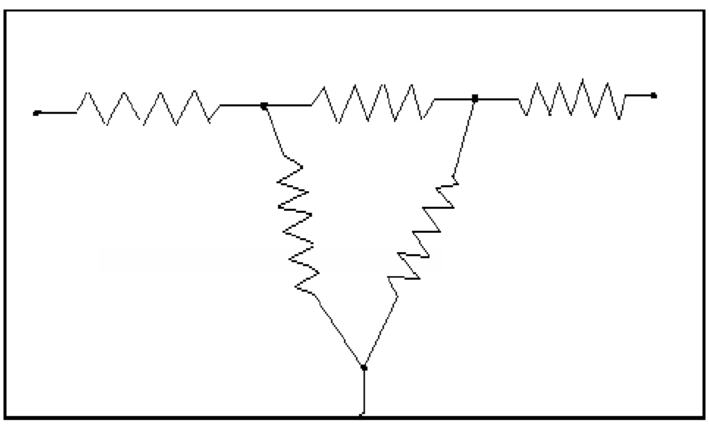
Also, $F_{1-1} + F_{1-2} + F_{1-3} = 1$ and $F_{1-1} = 0$

So, $F_{1-3} = 1 - F_{1-2} = 1 - 0.175$ or

$F_{1-3} = 0.825 = F_{2-3}$



The radiation network for the above problem will be as shown below



$$R_1 = \frac{1-\varepsilon_1}{A_1\varepsilon_1} = \frac{1-0.8}{\pi\left(0.5^2\right)0.8} = 0.318$$

$$R_2 = \frac{1-\varepsilon_2}{A_2\varepsilon_2} = \frac{1-1}{A_2 \times 1} = 0$$

$$R_{12} = \frac{1}{A_1 F_{1-2}} = \frac{1}{\pi\left(0.5^2\right)0.175} = 7.3$$

$$R_{13} = \frac{1}{A_1 F_{1-2}} = \frac{1}{\pi\left(0.5^2\right)0.825} = 1.54$$

$$R_{23} = \frac{1}{A_2 F_{2-3}} = \frac{1}{\pi\left(0.5^2\right)0.825} = 1.54$$

$$R_{eq} = R_1 + \left[\frac{1}{R_{12}} + \frac{1}{R_{13}+R_{23}}\right]^{-1} + R_2$$

$$= 0.318 + \left[\frac{1}{7.3} + \frac{1}{1.54+1.54}\right]^{-1} + 0 = 2.484$$

$$\therefore Q_{r_1} = \frac{E_{b1}-E_{b2}}{R_{eq}} = \frac{\sigma\left(T_1^4 - T_2^4\right)}{R_{eq}} = \frac{5.67\times10^{-8}\times\left[2000^4-1000^4\right]}{2.484}$$

$$= 329.14\times10^3\ W = 329.14\ KW$$

$$Q_{r1} + Q_{r2} + Q_{r3} = 0 \ and\ Q_{r3} = 0$$

$$\therefore Q_{r2} = -Q_{r1} = -329.14\ KW$$

<u>Example 4.27</u> A spherical tank with diameter 40cm fixed with a cryogenic fluid at 100K is placed inside a spherical container of diameter 60cm and is maintained at 300K. The emissivities of the inner and outer tanks are 0.15 and 0.2 respectively. A spherical radiation shield of diameter 50cm and having an emissivity of 0.05 on both sides is placed between the spheres. Calculate the rate of heat loss from the system by radiation and find also the rate of evaporation of the cryogenic liquid if the latent heat of vaporization of the fluid is $2.1\text{x}10^5$ W-s/Kg

<u>Solution:</u> The schematic and the corresponding network for the problem will be as shown in Fig P.10.27

$T_1 = 100$ K
$D_1 = 40$ cm
$\varepsilon_1 = 0.15$

$T_2 = 300$ K
$D_2 = 60$cm
$\varepsilon_2 = 0.2$

$D_3 = 50$cm
$\varepsilon_3 = 0.05$

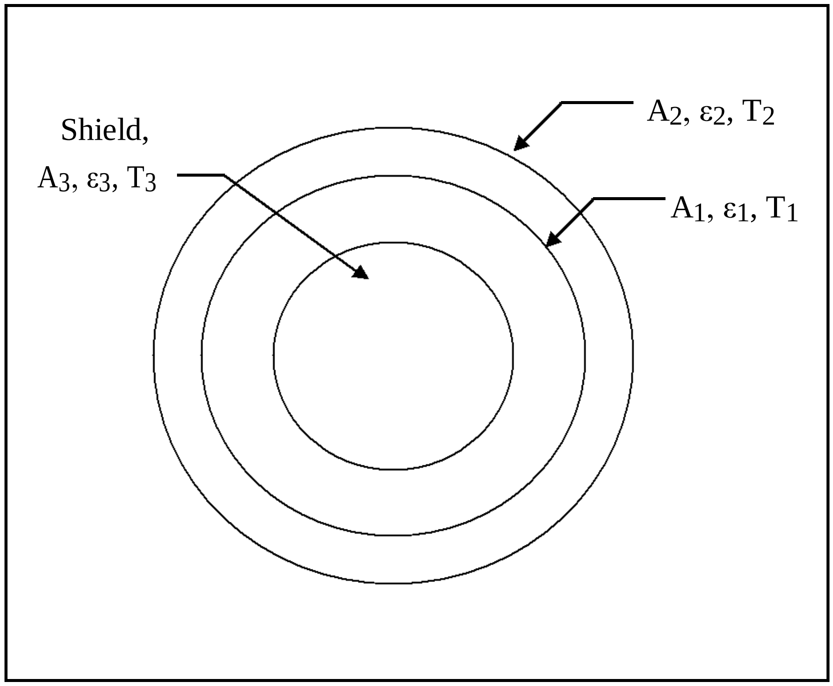


Fig P 10.27

$$Q_{r1} = Q_{12} = \frac{E_{b1} - E_{b2}}{R_1 + R_{13} + 2R_3 + R_{32} + R_2}$$

$$= \frac{\sigma\left(T_1^4 - T_2^4\right)}{\frac{1-\varepsilon_1}{A_1\varepsilon_1} + \frac{1}{A_1F_{1-3}} + 2\frac{1-\varepsilon_3}{A_3\varepsilon_3} + \frac{1}{A_3F_{3-2}} + \frac{1-\varepsilon_2}{A_2\varepsilon_2}}$$

$$= \frac{\sigma A_1\left(T_1^4 - T_2^4\right)}{\frac{1}{\varepsilon_1} + \frac{A_1}{A_3}\left(\frac{2}{\varepsilon_3} - 1\right) + \frac{A_1}{A_2}\left(\frac{1}{\varepsilon_2} - 1\right)}$$

$$= \frac{5.67\times10^{-8} \times 4\pi\times (0.2)^2 \times\left[100^4 - 300^4\right]}{\frac{1}{0.15} + \left(\frac{40}{50}\right)^2\left(\frac{2}{0.05} - 1\right) + \left(\frac{40}{60}\right)^2\left[\frac{1}{0.2} - 1\right]} = -6.83\ W$$

$$\text{Evaporation Ratio} = \frac{Q_{r1}}{hfg} = \frac{6.83}{2.1\times10^5} = 3.25\times10^{-5}\ Kg/s$$