Group Theory
Lecture Notes for MTH 912/913
04/05

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Contents

Chapter 4

Linear Algebra

4.1 Bilinear Forms

Definition 4.1.1 [**def:bilinear form**] *Let R be a ring, V an R-module and W a right R-module and $s : V \times W \to R, (v, w) \to (v \mid w)$ a function. Let $A \subseteq V$ and $B \subseteq W$. Suppose that s is R-bilinear, that is $(\sum_{i=1}^n r_i v_i \mid \sum_{j=1}^m w_j s_j) = \sum_{i=1}^n \sum_{j=1}^m r_i (v_i \mid w_j) s_j$ for all $v_i \in V, w_j \in W$ and $r_i, s_j \in R$. Then*

(a) [**a**] *s is called a* bilinear form.

(b) [**b**] *s is called* symmetric *if $V = W$ and $(v \mid w) = (w \mid v)$ for all $v, w \in V$.*

(c) [**z**] *s is called* symplectic *if $V = W$ and $(v \mid v) = 0$ for all $v \in V$.*

(d) [**c**] *Let $v \in V$ and $w \in W$ we say that v and w are* perpendicular *and write $v \perp w$ if $(v \mid w) = 0$.*

(e) [**d**] *We say that A and B are perpendicular and write $A \perp B$ if $a \perp b$ for all $a \in A$, $b \in B$.*

(f) [**e**] *$A^\perp = \{w \in W \mid A \perp w\}$ and ${}^\perp B = \{v \in V \mid v \perp B\}$. $A^\perp$ is called the right* perp *of A and ${}^\perp B$ the left perp of B.*

(g) [**f**] *If A is an R-submodule of V, define $s_A : W \to A^*$ by $s_A(w)(a) = (a \mid w)$ for all $a \in A, w \in W$.*

(h) [**g**] *If B is an R-submodule of W, define $s_B : V \to B^*$ by $s_B(v)(b) = (v \mid b)$ for all $v \in V, b \in B$.*

(i) [**h**] *s is called non-degenerate if $V^\perp = 0$ and ${}^\perp W = 0$.*

(j) [**i**] *If V is free with basis $\mathcal{V}$ and W is free with basis $\mathcal{W}$, then the $\mathcal{V} \times \mathcal{W}$ matrix $M_{\mathcal{V}}^{\mathcal{W}}(s) = (\,(v \mid w)\,)_{v \in \mathcal{V}, w \in \mathcal{W}}$ is called the* Gram Matrix *of s with respect to $\mathcal{V}$ and $\mathcal{W}$. Observe that the Gram Matrix is just the restriction of s to $\mathcal{V} \times \mathcal{W}$.*

Let I be a set, R a ring, $W = \prod_I R$ and $V = \bigoplus_I R$. Define $s : V \times W \to R$, $(v \mid w) = \sum_{i \in I} v_i w_i$. Note that this is well defined since almost all v_i are zero. Note also that if we view v and w as $I \times 1$ matrices we have $(v \mid w) = v^{\mathrm{T}} w$.

As a second example let V be any R-module and $W = V^*$ and define $(v \mid w) = w(v)$. If V is a free R-module this example is essentially the same as the previous:

Lemma 4.1.2 [**dual basis**] *Let V be a free R module with basis $\mathcal{V}$. For $u \in V$ define $u^* \in V^*$ by $u^*(v) = \delta_{uv}$. Define*

$$\phi_{\mathcal{V}} : V \to \bigoplus_{\mathcal{V}} R, v \to (w^*(v))_{w \in \mathcal{V}}$$

and

$$\phi_{\mathcal{V}*} : V^* \to \prod_{\mathcal{V}} R, \alpha \to (\alpha(v))_{v \in \mathcal{V}}$$

(a) [**a**] *Both $\phi_{\mathcal{V}}$ and $\phi_{\mathcal{V}*}$ are R-isomorphisms.*

(b) [**b**] *Let $w \in V^*$ and $v \in V$ and put $\tilde{v} = \phi_{\mathcal{V}}(v)$ and $\tilde{w} = \phi_{\mathcal{V}*}(w)$. Then $w(v) = \tilde{v}^{\mathrm{T}} \tilde{w}$.*

Proof: (a) Since V is free with basis $\mathcal{V}$, the map $\oplus_{\mathcal{V}} R \to V, (r_v) \to \sum_{v \in \mathcal{V}} r_v v$ is an R-isomorphism. Clearly $\phi_{\mathcal{V}}$ is the inverse of this map and so $\phi_{\mathcal{V}}$ is an R-isomorphism. To check that $\phi_{\mathcal{V}*}$ is an R-linear map of right R-modules recall first that V^* is a right R-module via $(wr)(v) = w(v)r$. Also $\prod_{\mathcal{V}} R$ is a right R-module via $(r_v)_v r = (r_v r)_v$. We compute

$$\phi_{\mathcal{V}*}(wr) = ((wr)(v))_v = (w(v)r)_v = (w(v))_v r$$

and so $\phi_{\mathcal{V}*}$ is R-linear. Given $(r_v)_v \in \prod_{\mathcal{V}} R$, then $w : V \to R, \sum_{v \in \mathcal{V}} s_v v \to \sum_{v \in \mathcal{V}} s_v r_v$ is the unique element of V^* with $w(v) = r_w$ for all $v \in \mathcal{V}$, that is with $\phi_{\mathcal{V}*}(w) = (r_v)_v$. So $\phi_{\mathcal{V}*}$ is a bijection.

(b) For $u \in \mathcal{V}$ let $s_u = u^*(v)$ and $r_u = w(u)$. Then $v = \sum_{u \in \mathcal{V}} s_u u$ and so $w(v) = \sum_{u \in \mathcal{V}} s_u w(u) = \sum_{u \in \mathcal{V}} s_u r_u = \tilde{v}^{\mathrm{T}} \tilde{w}$. □

Definition 4.1.3 [**dual map**] *Let R be a ring and $\alpha : V \to W$ an R-linear map. Then the R-linear map $\alpha^* : W^* \to V^*, \phi \to \phi \circ \alpha$ is called the* dual *of α.*

Lemma 4.1.4 [**matrix of dual**] *Let R be a ring and V and W free R modules with basis $\mathcal{V}$ and $\mathcal{W}$, respectively. Let $\alpha : V \to W$ be an R-linear map and M its matrix with respect to $\mathcal{V}$ and $\mathcal{W}$. Let $\delta \in W^*$. Then*

$$\phi_{\mathcal{V}*}(\alpha^*(\delta)) = M^{\mathrm{T}} \phi_{\mathcal{W}*}(\delta)$$

Proof: Let $v \in \mathcal{V}$. Then the v-coordinate of $\phi_{\mathcal{V}*}(\alpha^*(\delta))$ is $\alpha^*(\delta)(v) = (\delta \circ \alpha)(v) = \delta(\alpha(v))$. By definition of $M = (m_{wv})_{w \in \mathcal{W}, v \in \mathcal{V}}$, $\alpha(v) = \sum_{w \in \mathcal{W}} m_{wv} w$ and so

$$\phi_{\mathcal{V}*}(\alpha^*(\delta)) = (\delta(\alpha(v)))_v = (\sum_{w\in\mathcal{W}} m_{wv}\delta(w)) = M^{\mathrm{T}}\phi_{\mathcal{W}*}(\delta)$$

□

Lemma 4.1.5 [**associated non-deg form**] *Let R be a ring and $s : V \times W \to R$ an R-bilinear form. Let A be an R-subspace of V and B an R-subspace of W. Then*

$$\overline{s}_{AB} : A/A\cap{}^{\perp}B \times B/B\cap A^{\perp}, (a + (A\cap{}^{\perp}B), b + (B\cap A^{\perp}) \to (a \mid b)$$

is a well-defined non-degenerate R-bilinear form.

Proof: Readily verified. □

Lemma 4.1.6 [**basic bilinear**] *Let R be a ring and let $s : V \times W \to R$ be an R-bilinear form.*

(a) [**a**] *Let A be an R-subspace of V, then $A^{\perp} = \ker s_A$.*

(b) [**b**] *Let B be an R-subspace of W then ${}^{\perp}B = \ker s_B$.*

(c) [**c**] *s is non-degenerate if and only if s_V and s_W are 1-1.*

Proof: (a) and (b) are obvious and (c) follows from (a) and (b). □

Lemma 4.1.7 [**finite dim non-deg**] *Let $\mathbb{F}$ be a division ring and $s : V \times W \to \mathbb{F}$ a non-degenerate $\mathbb{F}$-bilinear form. Suppose that one of V or W is finite dimensional. Then both V and W are finite dimensional, both s_V and s_W are isomorphisms and $\dim_{\mathbb{F}} V = \dim_{\mathbb{F}} W$.*

Proof: Without loss $\dim_{\mathbb{F}} V < \infty$ and so $\dim V = \dim V^*$. By 4.1.6(c), s_V and s_W are 1-1 and so $\dim W \leq \dim V^* = \dim V$. So also $\dim W$ is finite and $\dim V \leq \dim W^* = \dim W$. Hence $\dim V = \dim W = \dim W^* = \dim V^*$. Since s_V and s_W are 1-1 this implies that s_V and s_W are isomorphisms. □

Corollary 4.1.8 [**dual s-basis**] *Let $\mathbb{F}$ be a division ring, $s : V \times W \to \mathbb{F}$ a non-degenerate $\mathbb{F}$-bilinear form, $\mathcal{B}$ a basis for V. Suppose that $\mathcal{B}$ is finite. Then for each $b \in \mathcal{B}$ there exists a unique $\tilde{b} \in W$ with $s(a, \tilde{b}) = \delta_{ab}$ for all $a, b \in B$. Moreover, $(\tilde{b} \mid b \in \mathcal{B})$ is an $\mathbb{F}$-basis for W.*

Proof: By 4.1.7 $s_V : W \to V^*$ is an isomorphism. Let $b^* \in V^*$ with $b^*(a) = \delta_{ab}$ and define $\tilde{b} = s_V^{-1}(b^*)$. □

Definition 4.1.9 [**def:s-dual basis**] *Let $\mathbb{F}$ be a division ring, $s : V \times W \to \mathbb{F}$ a non-degenerate $\mathbb{F}$-bilinear form, $\mathcal{B}$ a basis for V. A tuple $(\tilde{b} \mid b \in \mathcal{B})$ such that for all $a, b \in \mathcal{B}$, $\tilde{b} \in W$ $(a \mid \tilde{b}) = \delta_{ab}$ and $(\tilde{b} \mid b \in \mathcal{B})$ is basis for W is called the basis for W dual to $\mathcal{B}$ with respect to s.*

Definition 4.1.10 [**def:adjoint**] *Let R be ring , $s_i, V_i \times W_i \to R$ $(i = 1, 2)$ R-bilinear forms and $\alpha : V_1 \to V_2$ and $\beta : W_2 \to W_1$ R-linear maps. We say that α and β are adjoint (with respect to s_1 and s_2) or that β is an adjoint of α provided that*

$$(\alpha(v_1) \mid w_2)_2 = (v_1 \mid \beta(w_2))_1$$

for all $v_1 \in V_1$, $w_2 \in W_2$.

Lemma 4.1.11 [**basic adjoint**] *Let R be a ring , $s_i : V_i \times W_i \to R, (v, w) \to (v \mid w)_i$ $(i = 1, 2)$ R-bilinear forms and $\alpha : V_1 \to V_2$ and $\beta : W_2 \to W_1$ R-linear maps. Then α and β are adjoint iff $s_{1V_1} \circ \beta = \alpha^* \circ s_{2V_2}$.*

Proof: Let $v_1 \in V_1$ and $w_2 \in W_2$. Then

$$(\alpha v_1 \mid w_2)_2 = s_{2V_2}(w_2)(\alpha)(v_1) = (\alpha^*(s_{2V_2}(w_2))(v_1) = (\alpha^* \circ s_{2V_2})(w_2)(v_1)$$

and

$$(v_1 \mid \beta(w_2))_1 = s_{1V_1}(\beta(w_2))(v_1) = (s_{1V_1} \circ \beta)(w_2)(v_1)$$

and the lemma holds. □

Lemma 4.1.12 [**kernel of adjoint**] *Let R be a ring , $s_i : V_i \times W_i \to R$ $(i = 1, 2)$ R-bilinear forms and $\alpha : V_1 \to V_2$ and $\beta : W_2 \to W_1$ R-linear maps. Suppose α and β are adjoint. Then $\ker \alpha \leq {}^\perp \operatorname{Im} \beta$ with equality if ${}^\perp W_2 = 0$.*

Proof: Let $v_1 \in V_1$. Then

$$\begin{array}{cc} & v_1 \in \ker \alpha \\ \iff & \alpha(v_1) = 0 \\ \Longrightarrow (\iff \text{ if } W_2^\perp = 0) & (\alpha(v_1) \mid w_2) = 0 \, \forall w_2 \in W_2 \\ \iff & (v_1 \mid \beta(w_2)) = 0 \, \forall w_2 \in W_2 \\ \iff & v_1 \in {}^\perp \operatorname{Im} \beta \end{array}$$

□

Lemma 4.1.13 [**unique adjoint**] *Let R be a division ring, $s_i : V_i \times W_i \to R$ $(i = 1, 2)$ R-bilinear forms and $\alpha : V_1 \to V_2$ and $\beta : W_2 \to W_1$ R-linear maps. Suppose s_1 is non-degenerate and V_1 is finite dimensional over R.*

(a) [**a**] *There exists a unique adjoint α^{ad} of α with respect to s_1 and s_2.*

(b) [**b**] *Suppose that also s_2 is non-degenerate and V_2 is finite dimensional. Let $\mathcal{V}_i$ be a basis for V_i and $\tilde{\mathcal{V}}_i = (\tilde{v} \mid v \in \mathcal{V}_i)$ the basis W_i dual to $\mathcal{V}_i$ with respect to s_i. If M is the matrix of α with respect to $\mathcal{V}_1$ and $\mathcal{V}_2$, then M^{T} is the matrix for α^{ad} with respect to $\tilde{\mathcal{V}}_2$ and $\tilde{\mathcal{V}}_1$.*

Proof: (a) By 4.1.7 s_{1V_1} is an isomorphism and so by 4.1.11 $s_{1V_1}^{-1} \circ \alpha^* \circ s_{2V_2}$ is the unique adjoint of α. □

(b) Let $v_i \in \mathcal{V}_i$. Then the (v_1, v_2)-coefficient of M is $(\alpha(v_1) \mid \tilde{v}_2)_2$. By definition of the adjoint $(\alpha(v_1) \mid \tilde{v}_2)_2 = (v_1 \mid \alpha^{\mathrm{ad}}(\tilde{v}_2))_1$ and so (b) holds.

Corollary 4.1.14 [**dual basis for subspace**] *Let $\mathbb{F}$ be a field, V a finite dimensional $\mathbb{F}$-space and $s : V \times V \to \mathbb{F}$ an non-degenerate symmetric $\mathbb{F}$-bilinear form on V. Let W be an s-non-degenerate $\mathbb{F}$-subspace of V. Let $\mathcal{V}$ be an $\mathbb{F}$-basis for V and $\mathcal{W}$ an $\mathcal{W}$-basis for W. Let $\tilde{\mathcal{V}} = (\tilde{v} \mid v \in \mathcal{V}$ and $\tilde{\mathcal{W}} = (\tilde{w} \mid w \in \mathcal{W})$ be the corresponding dual basis for W and V, respectively. Let $M = (m_{vw})$ be the $\mathcal{V} \times \mathcal{W}$ matrix over $\mathbb{F}$ defined by*

$$v + W^{\perp} = \sum_{w \in \mathcal{W}} m_{vw} w + W^{\perp}$$

for all $v \in \mathcal{V}$. Then

$$\tilde{w} = \sum_{v \in \mathcal{V}} m_{vw} \tilde{w}$$

Proof: Since W is non-degenerate, $V = W \oplus W^{\perp}$. Let $\alpha : V \to W$ be the orthogonal projection onto W, that is if $v = w + y$ with $w \in W$ and $y \in W^{\perp}$, then $w = \alpha(v)$. Observe that the matrix of α with respect to $\mathcal{V}$ and $\mathcal{W}$ is M^{T}. Let $\beta : W \to V, w \to w$, be the inclusion map. Then for all $v \in V, w \in W$:

$$(\alpha(v) \mid w) = (v \mid w) = (v \mid \beta w)$$

and so β is the adjoint of α. Thus by 4.1.13(b) the matrix for β with respect to $\tilde{\mathcal{W}}$ and $\tilde{\mathcal{V}}$ is $M^{\mathrm{TT}} = M$. So

$$\tilde{w} = \beta(\tilde{w}) = \sum_{v \in \mathcal{V}} m_{vw} \tilde{w}.$$

□

Lemma 4.1.15 [**gram matrix**] *Let R be a ring, V a free R-module with basis $\mathcal{V}$ and W a free right R-module with basis $\mathcal{W}$. Let $\phi_{\mathcal{V}} : V \to \bigoplus_{\mathcal{V}} R$, $\phi_{\mathcal{W}} : V \to \bigoplus_{\mathcal{W}} R$, $\phi_{\mathcal{V}*} V^* \to \bigoplus_{\mathcal{V}} R$ and $\phi_{\mathcal{W}*} W^* \to \bigoplus_{\mathcal{V}} R$ be the associated isomorphisms. Let $s : V \times W \to R$ be bilinear form and M its Gram Matrix with respect to $\mathcal{V}$ and $\mathcal{W}$. Let $v \in V$, $w \in W$, $\tilde{v} = \phi_{\mathcal{V}}(v)$ and $\tilde{w} = \phi_{\mathcal{W}}(w)$,* *(a)* [**a**] $(v \mid w) = \tilde{v}^{\mathrm{T}} M \tilde{w}$.

(b) [**b**] $\phi_{\mathcal{V}}(V^{\perp}) = \mathrm{Null}(M)$, *the Null space of* M.

(c) [**c**] $\phi_{\mathcal{V}}(^{\perp}W) = \mathrm{Null}\, M^{\mathrm{T}}$

(d) [**d**] $\phi_{\mathcal{W}*}(s_W(v)) = M^{\mathrm{T}}\tilde{v}$.

(e) [**e**] $\phi_{\mathcal{V}*}(s_V(w)) = M\tilde{w}$.

Proof: (a) We have $v = \sum_{a\in\mathcal{V}} \tilde{v}_a a$, $w = \sum_{b\in\mathcal{W}} b\tilde{w}_b$ and $M = ((a \mid b))_{ab}$. Since s is R-bilinear,

$$(v \mid w) = \sum_{a\in\mathcal{V}, b\in\mathcal{W}} \tilde{v}_a (a \mid b)\tilde{w}_b = \tilde{v}^{\mathrm{T}} M \tilde{w}$$

(b) By (a) $w \in V^{\perp}$ iff $\tilde{v}^{\mathrm{T}} M\tilde{w} = 0$ for all $\tilde{v}$, iff $M\tilde{w} = 0$ and iff $\tilde{w} \in \mathrm{Null}(M)$.
(c) $v \in {}^{\perp}W$ iff $\tilde{v}^{\mathrm{T}} M = 0$, iff $M^{\mathrm{T}}\tilde{v} = 0$ iff $\tilde{v} \in \mathrm{Null}\, M^{\mathrm{T}}$.
(d) Let $u = s_W(v)$ and $\tilde{u} = \Phi_{\mathcal{W}*}(v)$. Then by "right-module" version of 4.1.2

$$u(w) = \tilde{w}^{\mathrm{T}} \cdot_{\mathrm{op}} \tilde{u} = \tilde{u}^{\mathrm{T}} \cdot \tilde{w}.$$

On the other hand

$$u(w) = s_W(v)(w) = (v \mid w) = \tilde{v}^{\mathrm{T}} M \cdot \tilde{w} =$$

Thus $\tilde{u}^{\mathrm{T}} = \tilde{v}^{\mathrm{T}} M$ and so $\tilde{u} = M^{\mathrm{T}} v$ and (d) holds.
(e) Let $u = s_V(w)$ and $\tilde{u} = \Phi_{\mathcal{V}*}(u)$. Then by 4.1.2

$$u(v) = \tilde{v}^{\mathrm{T}} \cdot \tilde{u}.$$

On the otherhand

$$u(v) = s_V(w)(v) = (v \mid w) = \tilde{v}^{\mathrm{T}} \cdot M\tilde{w}.$$

So $\tilde{u} = M\tilde{w}$ and (e) holds. □

Lemma 4.1.16 [**gram matrix of dual basis**] *Let* $\mathbb{F}$ *be a division ring and* $s : V \times W \to \mathbb{F}$ *a non-degenerate* $\mathbb{F}$*-bilinear form. Let* $\mathcal{V}$ *and* $\mathcal{W}$ *be* $\mathbb{F}$*-basis for* V *and* W *respectively and* $\tilde{\mathcal{V}}$ *and* $\tilde{\mathcal{W}}$*, the corresponding dual basis for* W *and* V*. Let* M *be the Gram matrix for* s *with respect to* $\mathcal{V}$ *and* $\mathcal{W}$*. Let* N *the Gram matrix for* s *with respect to* $\tilde{\mathcal{W}}$ *and* $\tilde{\mathcal{V}}$*. Then*

(a) [**a**] M^{T} *is the matrix for* id_V *with respect to* $\mathcal{V}$ *and* $\tilde{\mathcal{W}}$.

(b) [**b**] N *is the matrix for* id_W *with respect to* $\mathcal{W}$ *and* $\tilde{\mathcal{V}}$

(c) [**c**] M *and* N *are inverse to each other.*

Proof: (a) We have $\mathrm{id}_V : V \xrightarrow{s_W} W^* \xrightarrow{s_W^{-1}} V$. By 4.1.15(d), the matrix of s_W with respect to $\mathcal{V}$ and $\mathcal{W}^*$ is M. By definiton of $\tilde{\mathcal{W}}$ the matrix of s_W^{-1} with respect to $\mathcal{W}^*$ and $\tilde{\mathcal{W}}$ is the identity matrix. So (a) holds.

(b) Similar to (a), use s_V and 4.1.15(e).

(c) By (b) N^{-1} is the matrix of id_W with respect to $\tilde{\mathcal{V}}$ and $\mathcal{W}$. Note that id_V is the adjoint of id_W. So by (a) and 4.1.13(b), $N^{-1} = M^{\mathrm{TT}} = M$. □

Lemma 4.1.17 [**circ and bilinear**] *Let R be a commutative ring, G a group and let V and W be RG-modules. Let $s : V \times W \to R$ be R-bilinear form.*

(a) [**a**] *s is G-invariant iff $(a^\circ v \mid w) = (v \mid aw)$ for all $a \in inRG$.*

(b) [**b**] *Let $a \in RG$. Then $\mathrm{A}_W(a) \leq (a^\circ V)^\perp$ with equality if $V^\perp = 0$.*

Proof: (a) Recall first for $a = \sum_{g\in G} a_g g \in Rg$, $a^\circ = \sum_{g\in G} a_g g^{-1}$. Thus

$$\begin{array}{rl} & s \text{ is } G \text{ invariant} \\ & \iff (gu \mid gw) = (u \mid w) \quad \forall g \in G, u \in V, w \in W \\ (u \to v = gu \text{ is a bijection}) & \iff (v \mid gw) = (g^{-1}v \mid w) \ \ \forall g \in G, v \in V, w \in W \\ (s \text{ is } R \text{ bilinear}) & \iff (v \mid aw) = (a^\circ v \mid w) \ \ \forall a \in RG, v \in V, w \in W \end{array}$$

(b) By (a) a and a° are adjoints. So (b) follows from 4.1.12 □

Lemma 4.1.18 [**extending scalars and bilinear**] *Let $R \leq \tilde{R}$ be an extensions of rings and $s : V \times W \to R$ an R-bilinear form. There exists a unique $\tilde{R}$-bilinear form*

$$\tilde{s} : \tilde{R} \otimes_R V \times W \otimes_R \tilde{R} \to \tilde{R}, (a \otimes v, w \otimes b) = a((\mid v), w)b$$

for all $a, b \in \tilde{R}, v \in V, w \in V$.

Proof: Observe that the map

$$\tilde{R} \times V \times W \times \tilde{R}\ to\tilde{R}, (a, v, b, w) \to a((\mid v), w)b$$

is R-balanced in (a, v) and (b, w). The universal property of the tensor product now shows the existence of the map $\tilde{s}$. A simple calculation shows that $\tilde{s}$ is $\tilde{R}$-bilinear. □

Lemma 4.1.19 [**extending scalars and intersections**] *Let $\mathbb{F} \leq \mathbb{K}$ be an extension of division rings and V an $\mathbb{F}$ space.*

(a) [**a**] *Let $\mathcal{W}$ be a set of $\mathbb{F}$-subspaces of V. Then*

$$\bigcap_{W\in\mathcal{W}} \mathbb{K} \otimes W = \mathbb{K} \otimes \bigcap_{W\in\mathcal{W}} W$$

(b) [**b**] *Let $s : V \otimes W \to \mathbb{F}$ be an $\mathbb{F}$-bilinear form and extend s to a bilinear form $\tilde{s} : \mathbb{K} \otimes_{\mathbb{F}} V \times W \otimes_{\mathbb{F}} \mathbb{K} \to \mathbb{K}$ (see 4.1.18). Let X an $\mathbb{F}$-subspace of V. Then $\mathbb{K} \otimes_{\mathbb{F}} X^{\perp} = (\mathbb{K} \otimes X)^{\perp}$.*

Proof: (a) Suppose first that $\mathcal{W} = \{W_1, W_2\}$. Then there exists $\mathbb{F}$-subspaces X_i of W_i with $W_i = X_i \oplus (W_1 \cap W_2)$. Observe that $W_1 + W_2 = (W_1 \cap W_2) \oplus X_1 \oplus X_2$. For X an $\mathbb{F}$-subspace of V let $\overline{X} = \mathbb{K} \otimes_{\mathbb{F}} X \leq \mathbb{K} \otimes_{\mathbb{F}} V$. Then $\overline{W_i} = \overline{W_1 \cap W_2} \oplus \overline{X_i}$ and $\overline{W_1 + W_2} = \overline{W_1 \cap W_2} \oplus \overline{X_1} \oplus \overline{X_2}$ and so $\overline{W_1} \cap \overline{W_2} = \overline{W_1 \cap W_2}$. So (a) holds if $|\mathcal{W}| = 2$. By induction it holds if $\mathcal{W}$ is finite.

In the general case let $\overline{v} \in \overline{V}$. Then there exists a finite dimensional $U \leq V$ with $\overline{v} \in \overline{U}$ Moreover, there exists a finite subset $\mathcal{X}$ of $\mathcal{W}$ with $\overline{U} \cap \bigcap_{X \in \mathcal{X}} \overline{X} = \overline{U} \cap \bigcap_{X \in \mathcal{W}} \overline{X}$. By the finite case, $\overline{U} \cap \bigcap_{X \in \mathcal{X}} \overline{X} = \overline{U \cap \bigcap_{X \in \mathcal{X}} X}$ and so (a) is proved.

(b) Note that $X^{\perp} = \bigcap_{x \in X} x^{\perp}$. So by (a) we may assume that $X = \mathbb{F}x$ for some $x \in X$. If $X \perp V$, then also $\overline{X} \perp \overline{V}$ and we are done. Otherwise $\dim V/X^{\perp} = 1$ and so also $\dim \overline{V}/\overline{X^{\perp}} = 1$. From $\overline{X^{\perp}} \leq \overline{X}^{\perp} < \overline{V}$ we conclude that $\overline{X^{\perp}} = \overline{X}^{\perp}$. $\square$

Lemma 4.1.20 [**symmetric form for p=2**] *Let $\mathbb{F}$ be a field with $\operatorname{char} \mathbb{F} = 2$. Define $\sigma : \mathbb{F} \to \mathbb{F}, f \to f^2$ and let $\mathbb{F}^{\sigma}$ by the $\mathbb{F}$-space with $\mathbb{F}^{\sigma} = \mathbb{F}$ as abelian group scalar multiplication $f \cdot_{\sigma} k = f^2 l$. Let s a symmetric form on V and define $\alpha : V \to \mathbb{F}^{\sigma} : v \to (v \mid v)$. Then α is $\mathbb{F}$-linear, $W := \ker \alpha = \{v \in V \mid (v \mid v) = 0\}$ is an $\mathbb{F}$-subspace, $s \mid_W$ is a symplectic form and $\dim_{\mathbb{F}} V/W \leq \dim_{\mathbb{F}} \mathbb{F}^{\sigma} = \dim_{\mathbb{F}^2} \mathbb{F}$.*

Proof: Since $(v+w \mid v+w) = (v \mid v)+(v \mid w)+(w \mid v)+(w \mid w) = (v \mid v)+2(v \mid w)+(w \mid w) = (v \mid v) + (w \mid w)$ and $(fv \mid fv) = f^2(v \mid v) = f \cdot_{\sigma} (v \mid v)$ conclude that α is $\mathbb{F}$-linear. Thus $W = \ker \alpha$ is an $\mathbb{F}$-subspace of V and $V/W \cong \operatorname{Im} \alpha$. Also $\dim_{\mathbb{F}} \operatorname{Im} \alpha \leq \dim_{\mathbb{F}} \mathbb{F}^{\sigma}$. The map $(\sigma, \mathrm{id}_{\mathbb{F}} : \mathbb{F} \times \mathbb{F}^{\sigma} \to \mathbb{F}^2 \times \mathbb{F}, (f, k) \to (f^2, k)$ provides an isomorphism of the $\mathbb{F}$ space $\mathbb{F}^{\sigma}$ and the $\mathbb{F}^2$-space $\mathbb{F}$. So $\dim_{\mathbb{F}} \mathbb{F}^{\sigma} = \dim_{\mathbb{F}^2} \mathbb{F}$.

Cleary $s \mid_W$ is a symplectic form. $\square$

Lemma 4.1.21 [**symplectic forms are even dimensional**] *Let $\mathbb{F}$ be a field, V a finite dimensional $\mathbb{F}$-space and s a non-degenerate symplectic $\mathbb{F}$-form on V. Then there exists an $\mathbb{F}$-basis $v_i, i \in \{\pm 1, \pm 2, \dots \pm n\}$ for V with $(v_i \mid v_j) = \delta_{i,-j} \cdot \operatorname{sgn}(i)$. In particular $\dim_{\mathbb{F}} V$ is even.*

Proof: Let $0 \neq v_1 \in V$. Since $v_1 \notin 0 = V^{\perp}$, there exists $v \in V$ with $(v_1 \mid v) \neq 0$. Let $v_{-1} = (v_1 \mid v)^{-1} v$. Then $(v_1 \mid v_{-1}) = 1 = -(v_{-1} \mid v_1)$. Let $W = \mathbb{F}\langle v_1, v_{-1} \rangle$. The Gram Matrix of s on W with respect to (v_1, v_{-1}) is $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. So the Gram matrix has determinant $1 \neq 0$. Thus W is non-degenerate and so $V = W \oplus W^{\perp}$. Hence also $W^{\perp}$ is non-degenerate and the theorem follows by induction on $\dim_{\mathbb{F}} V$. $\square$

Lemma 4.1.22 [**selfdual and forms**] *Let $\mathbb{F}$ be field, G a group and V simple $\mathbb{F}G$ module. Suppose that V is self-dual (that is $V^* \cong V$ as $\mathbb{F}G$-module).*

(a) [**a**] *There exists a non-degenerate G-invariant symplectic or symmetric form s on V.*

(b) [**b**] *Suppose that* $\operatorname{char} \mathbb{F} = 2$ *and* $\mathbb{F}$ *is perfect. Then either* $V \cong \mathbb{F}_G$ *or* s *is symplectic.*

(a) Let $\alpha : V \to V^*$ be an $\mathbb{F}G$-isomorphism and $t : V \times V \to \mathbb{F}, (v, w) \to \alpha(v)(w)$, the corresponding G-invariant $\mathbb{F}$-bilinear form. Since V is a simple $\mathbb{F}G$-module any non-zero G-invariant bilinear form on V is non-degenerate.

Define $r(v, w) = t(v, w) + t(w, v)$. Then r is a symmetric form. If $r \neq 0$, then (a) holds with $s = r$. If $r = 0$ then $t(v, w) = -t(w, v)$ for all $v, w \in V$. If $\operatorname{char} \mathbb{F} = 2$, then t is symmetric and (a) holds with $s = t$. If $\operatorname{char} \mathbb{F} \neq 2$, then $t(v, v) = -t(v, v)$ implies that t is symplectic. So again (a) holds with $s = t$.

(b) Let s be as in (a) and observe that in either case of (a), s is symmetric. Let $\alpha : V \to \mathbb{F}\sigma$ be as in 4.1.20. View $\mathbb{F}^\sigma$ as an $\mathbb{F}G$-module with G acting trivially. Then by 4.1.20 α is $\mathbb{F}$ linear and since S is G-invariant also $\mathbb{F}G$-linear. Since $\mathbb{F}$ is perfect, $\dim_{\mathbb{F}} F^\sigma = 1$. So $\mathbb{F}^\sigma \cong \mathbb{F}_G$ has $\mathbb{F}G$-modulo and either $\alpha = 0$ or α is onto. If $\alpha = 0$, s is symplectic. If α is onto $\ker \alpha \neq V$ is an $\mathbb{F}G$-submodule of V. Since V is simple, $\ker \alpha = 0$ and so $V \cong \operatorname{Im} \alpha = F^\sigma \cong \mathbb{F}_G$. □

Chapter 5

Representations of the Symmetric Groups

5.1 The Symmetric Groups

For $n \in \mathbb{Z}^+$ let $\Omega_n = \{1, 2, 3 \ldots, n\}$ and $\mathrm{Sym}(n) = \mathrm{Sym}(\Omega_n)$. Let $g \in \mathrm{Sym}(n)$ and let $O(g) = \{O_1, \ldots 0_k\}$ be the sets of orbits for g on Ω_n. Let $|O_i| = n_i$ and choose notation such that $n_1 \geq n_2 \geq n_3 \geq \ldots n_k$. Define $n_i = 0$ for all $i > 1$. Then the sequence $(n_i)_{i=1}^{\infty}$ is called the cycle type of g. Pick $a_{i0} \in O_i$ and define $a_{ij} = g^j(a_{i0})$ for all $j \in \mathbb{Z}$. Then $a_{ij} = a_{ik}$ if and only if $j \equiv k \pmod{n}_i$. The denote the element g by

$$g = (a_{11}, a_{12}, \ldots a_{1n_1})(a_{21}, a_{22}, \ldots, a_{2n_2}) \ldots (a_{k1}, a_{k2}, \ldots a_{kn_k}).$$

Lemma 5.1.1 [**conjugacy classes in sym(n)**] *Two elements in $Sym(n)$ are conjugate if and only if they have the same cycle type.*

Proof: Let g be as above and $h \in Sym(n)$. Then

$$hgh^{-1} = \\ (h(a_{11}), h(a_{12}), \ldots h(a_{1n_1}))(h(a_{21}), h(a_{22}), \ldots, h(a_{2n_2})) \ldots (h(a_{k1}), h(a_{k2}), \ldots h(a_{kn_k}))$$

and the lemma is now easily proved. □

Definition 5.1.2 [**def:partition of n**] *A* partition *of $n \in \mathbb{N}$ is a non decreasing sequence $\lambda = (\lambda_i)_{i=1}^{\infty}$ of non-negative intergers with $n = \sum_{i=1}^{\infty} \lambda_i$.*

Note that if λ is a partion of n the necessarily $\lambda_i = 0$ for almost all i. For example $(4, 4, 4, 3, 3, 1, 1, 1, 1, 0, 0, 0, \ldots)$ is a partition of 22. We denote such a partition by $(4^3, 3^2, 1^4)$.

Observe that the cycle type of $g \in \mathrm{Sym}(n)$ is a partition of n. Together with 3.1.3(f) we conclude

Lemma 5.1.3 [**number of partitions**] *Let $n \in \mathbb{Z}^+$. The follwing numbers are equal:*

(a) [**a**] *The numbers of partitions of n.*

(b) [**b**] *The numbers of conjugacy classes of* $\mathrm{Sym}(n)$*.*

(c) [**c**] *The number of isomorphism classes of simple* $\mathbb{C}\mathrm{Sym}(n)$*-modules.* □

Our goal now is to find an explicit 1-1 correspondence between the set of partions of n and the simple $\mathbb{C}\mathrm{Sym}(n)$-modules. We start by associating a $\mathrm{Sym}(n)$-module M^λ to each partition λ of n. But this modules is not simple. In later section we will determine a simple section of M^λ.

Definition 5.1.4 [**def:lambda partition**] *Let I be a set of size n and λ a partition of n. A λ-partition of I is a sequence $\Delta = (\Delta_i)_{i=1}^\infty$ of subsets of Δ such that*

(a) [**a**] $I = \bigcup_{i=1}^\infty \Delta_i$

(b) [**b**] $\Delta_i \cap \Delta_j = \emptyset$ *for all* $1 \leq i < j < \infty$.

(c) [**c**] $|\Delta_i| = \lambda_i$.

For example $(\{1,3,5\},\{2,4\},\{6\},\emptyset,\emptyset,\dots)$ is a $(3,2,1)$ partition of I_6 where $I_n = \{1,2,3,\dots n\}$. we will write such a partition as

$$\begin{array}{l} \hline 1\,3\,5 \\ \hline 2\,4 \\ \hline 1 \\ \hline \end{array}$$

The lines in this array are a remainder that the order of the elements in the row does not matter. On the otherhand since sequences are ordered

$$\begin{array}{l} \hline 1\,3\,5 \\ \hline 2\,4\,6 \\ \hline \end{array} \neq \begin{array}{l} \hline 2\,4\,6 \\ \hline 1\,3\,5 \\ \hline \end{array}$$

Let $\mathcal{M}^\lambda$ be the set of all λ-partions of I_n. Note that $\mathrm{Sym}(n)$ acts on λ via $\pi\Delta = (\pi(\Delta_i))_{i=1}^\infty)$. Let $\mathbb{F}$ be a fixed field and let $M^\lambda = M^\lambda_{\mathbb{F}} = \mathbb{F}\mathcal{M}(\lambda)$. Then M^λ is an $\mathbb{F}\mathrm{Sym}(n)$-module. Note that for $M^{(n-1,1)} \cong \mathbb{F}I_n$. Let $(\cdot \mid \cdot)$ the unique bilinear form on M^λ with orthonormal basis $\mathcal{M}^\lambda$. Then by $(\cdot \mid \cdot)$ is $\mathrm{Sym}(n)$-invariant and non-degenerate.

5.2 Diagrams,Tableaux and Tabloids

Definition 5.2.1 [**def:diagram**] *Let $D \subseteq \mathbb{Z}_+ \times \mathbb{Z}_+$*

(a) [**z**] *Let $(i,j),(k,l) \in \mathbb{Z}^+ \times \mathbb{Z}^+$. Then $(i,j) \leq (k,l)$ provided that $i \leq k$ and $j \leq l$*

(b) [**a**] *D is called a* diagram *i if for all $d \in D$ and $e \in \mathbb{Z}_+ \times \mathbb{Z}_+$ with $e \leq d$ one has $e \in D$.*

(c) [**b**] *The elements of diagram are called the* nodes *of the diagram.*

(d) [**c**] *$r : \mathbb{Z}^+ \times \mathbb{Z}^+ \times (i, j) \to i$ and $c : \mathbb{Z}^+ \times \mathbb{Z}^+ \times (i, j) \to j$.*

(e) [**e**] *The i-th* row *of D is $D_i := D \cap \{i\} \times \mathbb{Z}^+$ and the j-*column *of D is $D^j := \mathbb{Z}^+ \times \{j\}$.*

(f) [**d**] *$\lambda(D) = (|D_i|)_{i=1}^{\infty}$ and $\lambda'(D) = (|D^j|)_j^{\infty}$*

Definition 5.2.2 [**def:diagram2**] *$\lambda \in \mathbb{Z}_+^{\infty}$ define*

$$[\lambda] = \{(i, j) \in \mathbb{Z}_+ \times \mathbb{Z}_+ \mid 1 \leq j \leq \lambda_i\}.$$

Lemma 5.2.3 [**basic diagram**] *Let $n \in \mathbb{N}$. Then the map $D \to \lambda_D$ is a bijection between the Diagram of size n and the partitions of n. The inverse is is by $\lambda \to [\lambda]$.*

Proof: Let D be a diagram of size n and put $\lambda = \lambda(D)$. Let $i \in \mathbb{N}$ and let j be maximal with $(i, j) \in D$. By maximality of j and the definition of a diagram, $(i, k) \in D$ iff $k \leq j$. Thus $j = |D_i| = \lambda_i$ and $D = [\lambda]$. Let $k \leq i$. Since $(i, \lambda_i) \in D$, the defintion of a diagram implies (k, λ_i) and so $\lambda_i \leq \lambda_k$. Thus λ is non-increasing. Clearly $\sum_{i=1}^{\infty} \lambda_i = |D| = n$ and so λ is a partition of n.

Conversely suppose that λ is a partition of n. Let $(i, j) \in D$ and $(a, b) \in \mathbb{Z}_+ \times \mathbb{Z}_+$ with $a \leq i$ and $b \leq j$. Then $a \leq i \leq \lambda_j \leq \lambda_b$ and so $(a, b) \in [\lambda]$. Thus $[\lambda]$ is a diagram. Clearly $|[\lambda]_i| = \lambda_i$, that is $\lambda([\lambda]) = \lambda$. □

We draw diagams as in the following example:

$$[5, 3^3, 2^2, 1] = \begin{array}{l} x\,x\,x\,x\,x \\ x\,x\,x \\ x\,x\,x \\ x\,x\,x \\ x\,x \\ x\,x \\ x \end{array}$$

Definition 5.2.4 [**def:dominates**] *Let λ and μ be partitions of $n \in \mathbb{Z}^+$. We say that λ dominates μ and write $\lambda \trianglerighteq \mu$ if*

$$\sum_{i=1}^{j} \lambda_i \geq \sum_{i=1}^{j} \mu_i$$

for all $j \in \mathbb{Z}^+$.

Note that “dominates” is a partial ordering but not a total ordering. For $n = 6$ we have

$$\begin{tikzcd}[column sep=small]
 & (6) \arrow[d, dash] & \\
 & (5,1) \arrow[d, dash] & \\
 & (4,2) \arrow[dl, dash] \arrow[dr, dash] & \\
(3,3) \arrow[dr, dash] & & (4,1^2) \arrow[dl, dash] \\
 & (3,2,1) \arrow[dl, dash] \arrow[dr, dash] & \\
(3,1^3) \arrow[dr, dash] & & (2^3) \arrow[dl, dash] \\
 & (2^2,1^2) \arrow[d, dash] & \\
 & (2,1^4) \arrow[d, dash] & \\
 & (1^6) &
\end{tikzcd}$$

On rare occasions it will be useful to have a total ordering on the partition.

Definition 5.2.5 [**def:lexiographic ordering**] *Let λ and μ be partitions of $n \in \mathbb{Z}^+$. We write $\lambda > \mu$ provided that there exists $i \in \mathbb{Z}^+$ with $\lambda_i > \mu_i$ and $\lambda_j = \mu_j$ for all $1 \leq j < i$.*

Observe that $'' <''$ is a total ordering on the partitions of n, called the *lexiographic* ordering. If $\lambda \rhd \mu$ and i is minimal with $\lambda_i \neq mu_i$, then $\sum_{j=1}^{i-1} \lambda_j = \sum_{j=1}^{i-1} \mu_i$ and $\sum_{j=1}^{i} \lambda_j \geq \sum_{j=1}^{i} \mu_i$. Thus $\lambda_i \geq \mu_i$ and so $\lambda > \mu$.

Definition 5.2.6 [**def:conjugate partition**]

(a) [**a**] *Let $D \subseteq \mathbb{Z}^+ \times \mathbb{Z}^+$. Then $D' = \{(j,i) \mid (i,j) \in D\}$. D' is called the* conjugate *of D.*

(b) [**b**] *Let λ be a partition of n. Then $\lambda' = (|[\lambda]^i|)$ is the number of nodes in the i'th column of $[\lambda]$.*

Lemma 5.2.7 [**basic conjugate**]

(a) [**a**] *The conjugate of a diagram is a diagram.*

(b) [**b**] *Let D be a diagram. Then the rows of D' are the conjugates of the columns of D: $D'_i = (D^i)'$.*

(c) [**c**] *Let λ be a partition of n. Then λ' is a partition of n and $[\lambda]' = [\lambda']$.*

Proof: (a) follows immediately from the definition of a diagram.

(b) is obvious.

(c) By (b) $|[\lambda]'_i| = |[\lambda^i] = \lambda'_i$. Thus $\lambda' = \lambda([\lambda]')$. So (c) follows from 5.2.3. $\square$

Lemma 5.2.8 [**reverse ordering**] *Let λ and μ be partitions of n. Then $\lambda \trianglerighteq \mu$ if and only if $\lambda' \trianglelefteq \mu'$.*

Proof: Let $j \in \mathbb{Z}^+$ and put $i = \mu'_j$.Define the following subsets of $\mathbb{Z}^+ \times \mathbb{Z}^+$

$$Top = \{(a,b) \mid a \leq i\} \ Bottom = \{(a,b) \mid a > i\}$$
$$Left = \{(a,b) \mid b \leq j\} \ \ Right = \{(a,b) \mid b > i\}$$

Since λ dominates μ:

(1) $$|Top \cap [\lambda]| \geq |Top \cap [\mu]|$$

By definition of $i = \mu'_j$, $\lambda_i \geq j$ and $\lambda_{i+1} > j$. Thus

$$Top \cap Left \subseteq [\mu] \text{ and } Bottom \cap Right \cap [\mu] = \emptyset$$

Hence

(2) $$|Top \cap Left \cap [\lambda]| \leq |Top \cap Left \cap [\mu]|$$

and

(3) $$|Bottom \cap Right \cap [\lambda]| \geq |Bottom \cap Right \cap [\mu]$$

From (1) and (2) we conclude

(4) $$|Top \cap Right \cap [\lambda]| \geq |Top \cap Right \cap [\mu]|$$

(3) and (4) imply:

$$|Right \cap [\lambda]| \geq |Bottom \cap [\mu]$$

Since $|[\lambda]| = n = |[\mu]|$ we conclude

$$|Left \cap [\lambda]| \geq Left \cap [\mu]$$

Thus $\sum_{c=1}^{j} \lambda'_c \leq \sum_{c=1}^{j} \mu'_c$ and $\lambda' \trianglelefteq \mu'$. $\square$

Definition 5.2.9 [**def:tableau**] *Let λ be a partition of n. A* λ-tableau *is a function* $t : [\lambda] \to I_n$.

We denote tableaux as in the following example

$$\begin{array}{l} 5\,1\,4 \\ 2\,3 \end{array}$$

denotes the $[3,2]$-tableau $t : (1,1) \to 4, (1,2) \to 1, (1,3) \to 4, (2,1) \to 2, (2,2) \to 3.$

Definition 5.2.10 [**def:partition of tableau**] *Let $t : D \to I_n$ be a tableau. Then $\Delta(t) = (t(D_i))_{i=1}^{\infty})$ and $\Delta'(t) = (t(D^i))_{i=1}^{\infty}$. $\Delta(t)$ is called the* row partition *of t and $\Delta'(t)$ the* column partition *of t.*

Note that if t is a λ-tableau, then $\Delta(t)$ is a λ partition of I_n and $\Delta'(t)$ is a λ-partition of I_n. For example

$$\text{if } t = \begin{array}{l} 2\,4\,3 \\ 6\,1 \\ 5 \end{array} \quad \text{then } \Delta(t) = \begin{array}{l} \hline 2\,4\,3 \\ \hline 6\,1 \\ \hline 5 \\ \hline \end{array}$$

Definition 5.2.11 [**def:tabloids**] *Let s, t be λ-tableaux.*

(a) [**a**] *s and t are called* row-equivalent *if $\Delta(t) = \Delta(s)$. An equivalence class of this relations is called a* tabloid *and the tabloid containing t is denoted by $\overline{\underline{t}}$.*

(b) [**b**] *s and t are called* column-equivalent *if $\Delta'(t) = \Delta'(s)$. The equivalence class of this relations containing t is denoted by $|t|$.*

For example if $t = \begin{array}{l} 1\,4 \\ 2\,3 \end{array}$ then

$$\overline{\underline{t}} = \left\{ \begin{array}{l} \hline 1\,4 \\ \hline 2\,3 \\ \hline \end{array} \;,\; \begin{array}{l} \hline 4\,1 \\ \hline 2\,3 \\ \hline \end{array} \;,\; \begin{array}{l} \hline 1\,4 \\ \hline 3\,2 \\ \hline \end{array} \;,\; \begin{array}{l} \hline 4\,1 \\ \hline 3\,2 \\ \hline \end{array} \right\}$$

Lemma 5.2.12 [**action on tableaux**] *Let λ be partition of n. Let $\pi \in \mathrm{Sym}(n)$ and s, t be λ tableaux.*

(a) [**a**] $\mathrm{Sym}(n)$ *acts transitively on the set of λ-tableaux via $\pi t = \pi \circ t$.*

(b) [**b**] $\pi\Delta(t) = \Delta(\pi t))$.

(c) [**c**] *s and t are row equivalent iff πs and πt are row equivalent. In particular,* $\mathrm{Sym}(n)$ *acts on the set of λ-tabloids via $\pi\overline{\underline{t}} = \overline{\underline{\pi t}}$.*

Proof: (a) Clearly $\pi t = \pi \circ t$ defines an action of $\mathrm{Sym}(n)$ on the set of λ tableaux. Since s, t a bijections from $[\lambda] \to I_n$, $\rho := s \circ t^{-1} \in \mathrm{Sym}(n)$. Then $\rho \circ t = s$ and so the action is transitive.

(b) Let $D = [\lambda]$. Then $\Delta(t) = (D_i)_{i=1}^{\infty})$ and so

$$\pi\Delta(t) = \pi(t(D_i)_{i=1}^{\infty}) = (\pi(t(D_i)_{i=1}^{\infty}) = ((\pi t)(D_i))_{i=1}^{\infty} = \Delta(\pi t)$$

(c) s is row-equivalent to t iff $\Delta(s) = \Delta(t)$ and so iff $\pi\Delta(s) = \pi\Delta(t)$. So by (b) iff $\Delta(\pi s) = \Delta(\pi t)$ and iff πt and πs are row-equivalent. □

Let $\Delta = (\Delta_i)_{i=1}^{\infty}$ be λ-partition of I_n. Let $\pi \in \mathrm{Sym}(n)$. Recall that $\pi \in C_G(\Delta)$ means $\pi\Delta = \Delta$ and so $\pi(\Delta_i) = \Delta_i$ for all i.

$C_{\mathrm{Sym}(n)}(\Delta) = \bigcap_{i=1}^{\infty} N_{\mathrm{Sym}(n)}(\Delta_i)) = \textcircled{\text{T}}_{i=1}^{\infty} \mathrm{Sym}(\Delta_i)$. So $C_{\mathrm{Sym}(n)}(\Delta)$ has order $\lambda! := \prod_{i=1}^{\infty} \lambda_i!$.

Definition 5.2.13 [**def: row stabilizer**] *Let t be a tableau. The $R_t = C_{\mathrm{Sym}(n)}(\Delta(t)$ and $C_t = C_{\mathrm{Sym}}(t)(\Delta'(t)$. R_t is called the* row stabilzer *and C_t the* column stablizer *of t.*

Lemma 5.2.14 [**char row equiv**] *Let s and t be λ-tableaux. The s and t are row equivalent iff $s = \pi t$ for some $\pi \in R_t$.*

Proof: Then by 5.2.12(a), $s = \pi t$ for some $\pi \in \mathrm{Sym}(n)$. Then s is row-equivalent to t if and only if $\Delta(t) = \Delta(\pi t)$. By 5.2.12(b), $\Delta(\pi)t) = \pi\Delta(t)$ and so s and t are row equivalent iff $\pi \in R_t$. □

Lemma 5.2.15 [**basic combinatorical lemma**] *Let λ and μ be partions of n, t a λ-tableau and s a μ-tableau. Suppose that for all i, j, $|\Delta(t)_i \cap |\Delta'(s)_j| \leq 1$ (That is no two entrees from the same row of t lie in the same column of s). Then $\lambda \trianglelefteq \mu$. Moreover if $\lambda = \mu$, then there exists λ-tableau r such that r is row equivalent to t and r is column equivalent to s.*

Proof: Fix a column C of Changing the order the entrees of C neither effects the assumptions nor the conclusions of the lemma. So we may assume that if i appears before j in C, then i also lies earlier row than j in the tableau t. We do this for all the columns of s. It follows that an entree in the k-row of t must lie in one of the first k-rows of s. Thus $\sum_{r=1}^{k} \lambda_i \leq \sum_{r=1}^{l} \mu_i$ and μ dominates λ.

Suppose now that $\lambda = \mu$. Since $\lambda_1 = \mu_1$ and the firs row of t is contained in the first row of s, the first row of $\Delta(t)_1 = \Delta(s)_1$. Proceeding by induction we see that $\Delta_(t)_k = \Delta(s)_t$ for all s and t. So s and t are row equivalent. □

5.3 The Specht Module

Definition 5.3.1 [**def:fh**] *Let G be a group, $H \subseteq G$, R a ring and $f \in RG$. Then $f_H = \sum_{h\in H} f_h h$.*

Lemma 5.3.2 [**basic fh**] *Let G be a group, R a ring and $f \in RG$. Suppose that f view as a function is a multiplicative homomorphism.*

(a) [**a**] *Let $A, B \subseteq G$ such that the maps $A \times B \to G, (a, b) \to G$ is $1-1$, then $f_{AB} = f_A f_B$.*

(b) [**b**] *Let $A \leq B \leq G$ and T a left-transversal to A in B. Then $f_B = f_T f_A$.*

(c) [**c**] *Let $A_1, A_2, A_n \leq G$ and $A = \langle A_i \mid 1 \leq i \leq n\rangle$ Suppose $A = \textcircled{\scriptsize{T}}_{i=1}^n A_i$, then $f_A = f_{A_1} f_{A_2} \dots f_{A_n}$.*

(d) [**d**] *Suppose f is a class function, then for all $g \in G$ and $H \subseteq G$, $gf_H g^{-1} = f_{gHg^{-1}}$.*

Proof: (a) Since the map $(a, b) \to ab$ is $1-1$, every element in AB can be uniquely written has ab with $a \in A$ and $b \in B$. Thus

$$\begin{aligned} f_A f_B = & \sum_{a\in A} f_a a \cdot \sum_{b\in B} f_b b & = \sum a \in A, b \in B f_a f_b ab \\ = & \sum_{a\in A, b\in B} f_{ab} ab = \sum_{c\in AB} f_c c & \\ & & = \qquad f_{AB} \end{aligned}$$

(b) is a special case of (a).
(c) follows from (a) and induction on n.
(d) Readily verified.

Since the map $\underline{\overline{t}} \to \Delta(t)$ is a well defined bijection between the λ tabloids and the the λ partitions of I_n we will often identify $\underline{\overline{t}}$ with $\Delta(t)$. In particular, we have $\underline{\overline{t}} \in M^\lambda$.

Definition 5.3.3 [**polytabloid**] *Let t be λ-tableau.*

(a) [**a**] $k_t = \mathrm{sgn}_{C_t} = \sum_{\pi\in C_t} \mathrm{sgn}\pi\pi \in F\mathrm{Sym}(n)$.

(b) [**b**] $e_t = k_t \underline{\overline{t}} = \sum_{\pi\in C_t} \mathrm{sgn}\pi \underline{\overline{\pi t}} \in M^\lambda$. *$e_t$ is called a* polytabloid.

(c) [**c**] *S^λ is the F-subspace of M^λ spanned by the λ-polytabloids. S^λ is called a* Specht module.

(d) [**d**] *F^λ is the left ideal in $F\mathrm{Sym}(n)$ generated by the k_t, t a λ-tableau.*

As a first example consider $t = \begin{matrix} 3\,2\,5 \\ 1\,4 \end{matrix}$.

The $C_t = \mathrm{Sym}(\{1, 3\}) \times \mathrm{Sym}(\{\{2, 4\}$,

$k_t = (1 - (13) \cdot (1 - (24)) = 1 - (13) - (24) + (13)(24)$ and

$e_t = \dfrac{\overline{3\,2\,5}}{\underline{1\,4}} - \dfrac{\overline{1\,2\,5}}{\underline{3\,4}} - \dfrac{\overline{3\,4\,5}}{\underline{1\,2}} + \dfrac{\overline{1\,4\,5}}{\underline{3\,2}}$

As a second example consider $\lambda = (n-1,1)$ and $t = \begin{matrix} i & \dots \\ j & \end{matrix}$. Then $C_i = \mathrm{Sym}(\{i,j\} = \{1,(i,j)\}$ $k_t = 1-(i,j)$ and

$$e_t = \frac{\overline{i \dots}}{\underline{j}} - \frac{\overline{j \dots}}{\underline{i}}$$

For $i \in I_n$ put $x_i := (I_n \setminus, \{i\}) = \frac{\overline{1\,2 \dots i-1\, i+1 \dots n}}{\underline{i}}$

Then $M^{(n-1,1)}$ is the $\mathbb{F}$ space with basis $(x_i, i \in I_n)$ and $e_t = x_j - x_i$. Thus

$$S^{(n-1,1)} = F\langle x_j - x_i \mid i \neq j \in I_n \rangle = \{\sum_{i=1}^n f_i x_i \mid f_i \in F \mid \sum_{i=1}^n f_i = 0\} = (x_1 + x_2 + \ldots + x_n)^\perp$$

The reader should convince herself that if $\mathrm{char}\,\mathbb{F} \nmid n$, then $S^{(n-1,1)}$ is a simple $\mathbb{F}\mathrm{Sym}(n)$-module and if $\mathrm{char}\,\mathbb{F} \mid n$, then $x := \sum_{i=1}^n x_i \in S^{(n-1,1)}$ and $S^{(n-1,1)}/\mathbb{F}x$ is a simple $\mathbb{F}\mathrm{Sym}(n)$-module.

Lemma 5.3.4 [**transitive on polytabloids**] *Let $\pi \in \mathrm{Sym}(n)$ and t a tableau.*

(a) [**z**] $\pi k_t \pi^{-1} = k_{\pi t}$

(b) [**a**] $\pi e_t = e_{\pi t}$.

(c) [**b**] $\mathrm{Sym}(n)$ *acts transitively on the set of λ-polytabloids.*

(d) [**c**] S^λ *is a $F\mathrm{Sym}(n)$-submodule of M^λ.*

(e) [**d**] *If $\pi \in C_t$, then $k_{\pi t} = k_t = \mathrm{sgn}\pi k_t$ and $e_{\pi t} = \mathrm{sgn}\pi e_t$.*

Proof:

(a) We have $C_{\pi t} = \pi C_t \pi^{-1}$ and so by 5.3.2(d) applied to the class function sgn on $\mathrm{Sym}(n)$,

$$k_{\pi t} = \mathrm{sgn}_{C_{\pi t}} = \mathrm{sgn}_{\pi C_t \pi^{-1}} = \pi \mathrm{sgn}_{C_t} \pi^{-1} = \pi k_t \pi^{-1}$$

(b) Using (b), $e_{\pi t} = k_{\pi t}\underline{\overline{\pi t}} = \pi k_t \pi^{-1} \pi \underline{\overline{t}} = \pi k_t \underline{\overline{t}} = \pi e_t$

(c) and (d) follow from (b).

(e) Since $\pi \in C_t$, $C_{\pi t} = C_t = C_t \pi$. Thus $k_t = k_{\pi t}$ and

$$\begin{aligned} k_t = & \textstyle\sum_{\alpha \in C_t} \mathrm{sgn}\alpha \cdot \alpha &= \textstyle\sum_{\beta \in C_t} \mathrm{sgn}(\beta\pi) \cdot (\beta\pi) \\ = \mathrm{sgn}\pi & \textstyle\sum_{\beta \in C_{\pi t}} \mathrm{sgn}\beta \cdot \beta = & \mathrm{sgn}\pi k_t \pi \end{aligned}$$

The second statement follows from the first and $\pi\underline{\overline{t}} = \underline{\overline{\pi t}}$. □

Lemma 5.3.5 [**action of es on ml**] *Let λ and μ be partitions of n.*

(a) [**a**] *If $F^\mu M^\lambda \neq 0$, then $\lambda \trianglelefteq \mu$.*

(b) [**b**] *If t and s are λ-tableau with $k_s\underline{\overline{t}} \neq 0$, then then $k_s\underline{\overline{t}} = \pm e_s$.*

Proof: Let s be a μ tableau and t and λ-tableau with $k_s\underline{\overline{t}} \neq 0$.

Suppose first that there exists a $i \neq j \in I_n$ such that i and j are on the same row of t and in the same column of s. Let $H = \mathrm{Sym}(\{i,j\} = \{1,(i,j)\}$. Then

$$\mathrm{sgn}_H\underline{\overline{t}} = \underline{\overline{t}} + \mathrm{sgn}((i,j))(i,j)\underline{\overline{t}} = \underline{\overline{t}} = \underline{\overline{b}} = 0.$$

Since i,j are in the same column of s, $H \leq C_s$ and we can choose a transversal $\mathcal{T}$ to H in C_s. Then

$$k_s\underline{\overline{t}} = (\mathrm{sgn}\mathcal{T})\mathrm{sgn}H\underline{\overline{t}} = 0,$$

contrary to our assumption. Thus no such i,j exists. So by 5.2.15 $\lambda \trianglelefteq \mu$. Moreover, if $\lambda = \mu$, there exists a λ tableau r which is row equivalent to t an columns equivalent to s. Hence $k_r = k_s$ and $\underline{\overline{r}} = \underline{\overline{s}}$. Moreover $\pi s = r$ for some $\pi \in C_s$ and so by 5.3.4(e),

$$k_s\underline{\overline{t}} = e_r = \mathrm{sgn}\pi e_s$$

□

Lemma 5.3.6 [**es self dual**] *Let λ and μ be partitions of n and s an μ-tableau. Then*

(a) [**a**] $k_S = k_S^\circ$

(b) [**b**] $(k_S M^\lambda)^\perp = \mathrm{A}_{M^\lambda}(k_s)$.

(c) [**c**] *$k_s M^\mu = Fe_s$ and $\mathrm{A}_{M^\mu}(k_s) = e_s^\perp$.*

(d) [**d**] *$k_s v = (v \mid e_s)e_s$ for all $v \in M^\mu$.*

Proof: (a) If $\pi \in C_s$ then also $\pi^{-1} \in C_s$. Moreover $\mathrm{sgn}\pi = \mathrm{sgn}\pi^{-1}$ and (a) holds.

(b) Follows from (a) and 4.1.17

(c) By 5.3.5 $e_S M^\lambda = Fe_s$ and so by (b) $\mathrm{A}_{M^\lambda}(k_s) = e_s^\perp$.

(d) By (c) $k_s v = fe_s$ for some $f \in F$. Hence

$$(v \mid e_s) = (v \mid k_s\underline{\overline{t}}) = (k_s v \mid \underline{\overline{t}}) = (fe_t \mid \underline{\overline{t}}) = f$$

□

Lemma 5.3.7 [**fl and ml**] $F^\lambda M^\lambda = S^\lambda$ *and* $\mathrm{A}_{M^\Lambda}(F^\lambda) = S^{\lambda\perp}$.

Proof: This follows immediately from 5.3.6(b) and 5.3.6(c). □

Lemma 5.3.8 [**submodules of ml**] *Supp F is a field and let λ be a partition of n and V be an $F\mathrm{Sym}(n)$-submodule of M^λ. Then either $F^\lambda V = S^\mu$ and $S^\mu \leq V$ or$F^\lambda V = 0$ and $S^\lambda \leq V$.*

Proof: If $F^\lambda V = 0$, then by 5.3.7, $V \leq S^{\lambda\perp}$.

So suppose $F^\lambda V \neq 0$. Then $k_s V \neq 0$ for some λ-tableau s. So 5.3.6 implies $k_s V = Fe_s = k_s M^\lambda$. Since by 5.3.4(a) implies $k_s V = k_s M^\lambda$ for all λ-tableaux s. Thus $F^\lambda V = F^\lambda M^\lambda = S^\lambda$ and $S^\lambda \leq V$. □

If $\mathbb{F} \leq \mathbb{K}$ is a field extensions we view $M^\lambda = M^\lambda_{\mathbb{F}}$ has a subset of S^μ. Note also that $M^\lambda_{\mathbb{K}}$ is canonically isomorphic to $\mathbb{K} \otimes_{\mathbb{F}} M^\lambda$. Put $D\lambda = S^\lambda/(S^\lambda \cap S^{\lambda\perp})$.

Lemma 5.3.9 [**dl=fldl**] *Let λ be a partition of n. If F is a field then $F^\lambda D^\lambda = D^\lambda$.*

Proof: By 5.3.8 either $F^\lambda S\lambda = S^\lambda$ or $S^\lambda \leq S^{\lambda\perp}$. In the first case $F^\lambda D^\lambda = D^\lambda$ and in the second $D^\lambda = 0$ and again $F^\lambda D^\lambda = D^\lambda$.

Proposition 5.3.10 [**dl=du**] *Let λ and μ be partitions of n with $D^\lambda = 0$. Suppose F is a field. If D^λ is isomorphic to an $F\mathrm{Sym}(n)$-section of M^μ, then $\lambda \trianglelefteq \mu$. In particular, $D^\lambda \cong D^\mu$ then $\lambda = \mu$.*

Proof: By 5.3.9 $F^\lambda D^\lambda = D^\lambda \neq 0$. Hence also $F^\lambda D^\mu \neq 0$ and $F^\lambda M^\mu \neq 0$. So by 5.3.5(a), $\lambda \trianglelefteq \mu$. If $D^\lambda \cong D^\mu$, the D^μ is a section of M^λ and so $\mu \trianglelefteq \lambda$ and $\mu = \lambda$. □

Lemma 5.3.11 [**scalar extensions of ml**] *Let λ be a partition of n and $\mathbb{F} \leq \mathbb{K}$ a field extension.*

(a) [**a**] $S^\lambda_{\mathbb{K}} = \mathbb{K}S^\lambda \cong K \otimes_{\mathbb{F}} S^\lambda$.

(b) [**b**] $S^{\lambda\perp}_{\mathbb{K}} = \mathbb{K}(S^{\lambda\perp}) \cong \mathbb{K} \otimes_{\mathbb{F}} S^{\lambda\perp}$.

(c) [**d**] $S^\lambda_{\mathbb{K}} \cap S^{\lambda\perp}_{\mathbb{K}} = \mathbb{K}(S^\lambda \cap S^{\lambda\perp}) = \mathbb{K} \otimes_{\mathbb{F}} S^\lambda \cap S^{\lambda\perp})$.

(d) [**c**] $D^\lambda_{\mathbb{K}} \cong \mathbb{K} \otimes_{\mathbb{F}} D^\lambda$.

Proof: (a) is obvious.

(b) follows from (a) and 4.1.19(b)

(a) follows from (a), (b) and 4.1.19(a).

(d) follows from (a) and (c). □

Lemma 5.3.12 [**dl absolutely simple**] *Let λ be a partition of n and suppose $D^\lambda \neq 0$. Then D^λ is an absolutely simple $\mathbb{F}Sym(n)$-module.*

Proof: By 5.3.11(d) it suffices to show that D^λ is simple. So let V be an $\mathbb{F}Sym(n)$-submodule of S^λ with $S^\lambda \cap S^{\lambda\perp} \leq V$. By 5.3.8 either $S^\lambda \leq V$ or $V \leq S^{\lambda\perp}$. In the first case $V = S^\lambda$ and in the second $V \leq S^\lambda \cap S^{\lambda\perp}$ and $V = S \cap S^{\lambda\perp}$. Thus $D^\lambda = S^\lambda/(S^\lambda \cap S^{\lambda\perp})$ is simple. □

5.4 Standard basis for the Specht module

Proposition 5.4.1 [**garnir relations**] *Let t be a λ-tableau, $i < j \in \mathbb{Z}^+$, $X \subseteq \Delta'(t)_i$ and $Y \subseteq \Delta'(t)_j$. Let $\mathcal{T}$ be any transversal to* $\mathrm{Sym}(X) \times \mathrm{Sym}(Y)$ *in* $\mathrm{Sym}(X \cup Y)$.

(a) [**a**] $\mathrm{sgn}_{\mathcal{T}}\, e_t$ *is independent from the choice of the tranversal $\mathcal{T}$.*

(b) [**b**] *If $|X \cup Y| > \lambda'_i$. Then*
$$\mathrm{sgn}_{\mathcal{T}} e_t = 0$$

Proof: (a) Let $\pi \in \mathrm{Sym}(X \cup Y)$ and $\rho \in \mathrm{Sym}(X) \times \mathrm{Sym}(Y) \leq C_t$. Then

$$\mathrm{sgn}(\pi\rho) \cdot \pi\rho \cdot e_t = \mathrm{sgn}(\pi)\pi \cdot \mathrm{sgn}(\rho)\rho e_t \overset{5.3.4(e)}{=} \mathrm{sgn}(\pi)\pi e_t$$

and so (a) holds.

(b) Since $|X \cap Y| > \lambda'_i \geq \lambda'_j$, there exists $i \in X$ and j in Y such that i and j are in the same row of t. So $(1 - (ij))\overline{\pi t} = 0$. If $\pi \in \mathrm{Sym}(X \cup Y)$, then π and $\pi \cdot (ij)$ lie in differen cosets of $\mathrm{Sym}(X) \times \mathrm{Sym}(Y)$. Hence we can choose $\mathcal{R} \subseteq \mathrm{Sym}(X \cup Y)$ such that $\mathcal{R} \cap \mathcal{R} \cdot (i,j) = \emptyset$ and $\mathcal{R} \cup \mathcal{R} \cdot (ij)$ is a transversal to $\mathrm{Sym}(X) \cup \mathrm{Sym}(Y)$. By (a) we may assume $\mathcal{T} = \mathcal{R} \cup \mathcal{R} \cdot (ij)$ and so

$$\mathrm{sgn}_{\mathcal{T}} = \mathrm{sgn}_{\mathcal{R}}\mathrm{sgn}_{\{1,(ij)\}} = \mathrm{sgn}_{\mathcal{R}} \cdot (1 - (ij))$$

and
$$\mathrm{sgn}_{\mathcal{T}} e_t = \mathrm{sgn}_{\mathcal{R}} \cdot (1 - (ij))e_t = 0.$$

□

Definition 5.4.2 [**def:garnir**] *Let t be a λ-tableau, $i < j \in \mathbb{Z}^+$, $X \subseteq \Delta'(t)_i$ and $Y \subseteq \Delta'(t)_j$.*

(a) [**a**] *$\mathcal{T}_{XY}$ is the set of all $\pi \in$ $\mathrm{Sym}(X \cup \mathrm{Sym} Y)$ such that the restrictions of $\pi \circ t$ to $\pi^{-1}(X)$ and $\pi^{-1}(Y)$ are increasing.*

(b) [**b**] *$G_{XYt} = \mathrm{sgn}_{\mathcal{T}_{XY}}$. G_{XYt} is called a* Garnir element *in $F\mathrm{Sym}(n)$.*

Lemma 5.4.3 [**basic garnir**] *Let t be a λ-tableau, $i < j \in \mathbb{Z}^+$, $X \subseteq \Delta'(t)_i$ and $Y \subseteq \Delta'(t)_j$.*

(a) [**a**] *$\mathcal{T}_{XY}$ is a transvsersal to $Sym(X) \times$ $\mathrm{Sym}(Y)$ *in* $\mathrm{Sym}(X \cup Y)$.

(b) [**b**] *If* $|X \cup Y| > \lambda'_i$. *Then*

$$G_{XYt}e_t = 0.$$

Proof: (a) Just observe that if $\pi \in \mathrm{Sym}(X \cup \mathrm{Sym}(Y)$, then there exists a unique element $\rho \in \mathrm{Sym}(X) \cup \mathrm{Sym}(Y)$ such that the restriction of $\pi\rho$ to $t^{-1}(X)$ and to $t^{-1}(Y)$ are increasing. (b) follows from (a) and 5.4.1(b). $\square$

Consider $n = 5$, $\lambda = (3,2)$, $t = \begin{array}{l}\hline 1\,2\,3\\ \hline 4\,5\\ \hline\end{array}$, $X = \{2,5\}, Y = \{3\}$

Then $G_{XY}e_t = 0$ gives

$$\begin{array}{l}\hline 1\,2\,3\\ \hline 4\,5\\ \hline\end{array} - \begin{array}{l}\hline 1\,3\,2\\ \hline 4\,5\\ \hline\end{array} - \begin{array}{l}\hline 1\,2\,5\\ \hline 4\,3\\ \hline\end{array} = 0$$

Definition 5.4.4 [**def:increasing tableau**] *Let* λ *be a partion of* n *and* t *a* λ*-tableau.*

(a) [**a**] $r_t = r \circ t^{-1}$ *and* $c_t = s \circ t^{-1}$. *So* $i \in I_n$ *lies in row* $r_t(i)$ *and column* $c_t(i)$ *of* t.

(b) [**b**] *We say that* t *is row-increasing* c_t *is increasing on each row* $\Delta_i(t)$ *of* t

(c) [**c**] *We say that* t *is column-increasing if* r_t *is increasing on column* $\Delta'_i(t)$.

Note that r_t only depends on $\overline{\underline{T}}$ and so we will also write $r_{\overline{\underline{t}}}$ for r_t. Indeed $\overline{\underline{r}} = \overline{\underline{s}}$ iff $r_t = r_s$.

Lemma 5.4.5 [**basic increasing**] *Let* λ *be a partion of* n *and* t *a* λ*-tableau.*

(a) [**a**] $\overline{\underline{t}}$ *contains a unique row-increasing tableau.*

(b) [**b**] $|t|$ *contains a unique column-increasing tableau.*

(c) [**c**] *Let* $\pi \in \mathrm{Sym}(n)$ *and* $i \in I$. *Then* $r_t(i)) = r_{\pi t}(\pi i)$.

Proof: (a) and (b) are readily verfied.
(c) $r_{\pi t} \circ \pi = r \circ (\pi \circ t)^{-1} \circ \pi = r \circ t^{-1} = r_t$. $\square$

Definition 5.4.6 [**def:standart tableau**] *Let* λ *be a partition of* n *and* t *a* λ*-tableau. A* standard tableau *is row- and column-increasing tableau. A tabloid is called standard if it contains a standard tableau. If* t *is a standard tableau, then* e_t *is called* standard polytabloid.

By 5.4.5(a), a standard tabloid contains a unique standard tableau.
We will show that the standard polytabloids form a basis of S^λ for any ring F.
For this we need to introduce a total order on the tabloids

Definition 5.4.7 [**def:order tabloids**] *Let* $\overline{\underline{t}}$ *and* $\overline{\underline{s}}$ *be the distinct* λ*-tabloids. Let* $i \in I_n$ *be maximal with* $r_{\overline{\underline{t}}}(i) \neq r_{\overline{\underline{s}}}(i)$. *Then* $\overline{\underline{t}} < \overline{\underline{s}}$ *provided that* $r_{\overline{\underline{t}}}(i) < r_{\overline{\underline{s}}}(i)$.

Lemma 5.4.8 [**basic order tabloids**] *$<$ is a total ordering on the set of λ tabloids.*

Proof: Any tabloid $\underline{\overline{t}}$ is uniquely determined by the tuple $(r_{\underline{\overline{t}}}(i))_{i=1}^{n}$. Moreover the ordering is just a lexiographic ordering in terms of it associated tuple. $\square$

Lemma 5.4.9 [**proving maximal I**] *Let A and B be totally ordered sets amd $f : A \to B$ be a function. Suppose A is finite and $\pi \in \mathrm{Sym}(A)$ with $f \neq f \circ pi$. Let $a \in A$ be maximal with $f(a) \neq f(\pi(a))$. If f is non-decreasing then $f(a) > f(\pi(a))$ and if f is non-increasing then $f(a) < f(\pi(a))$.*

Proof: Reversing the ordering on F if necessary we may assume that f is non-decreasing. Let $J = \{j \in J \mid f(j) > f(a)\}$ and let $j \in J$. Since f is non-decreasing, $j > a$ and so by maximality of f, $f(\pi j) = f(j) > f(a)$. Hence $\pi(J) \subseteq J$. Since J is finite this implies $\pi(J) = J$ andso since π is $1-1$, $\pi(I \setminus J) \subseteq I \setminus J$. Thus $\pi(a) \notin J$, $f(\pi(a) \leq f(a)$ and since $f(\pi(a)) \neq f(a)$, $f(\pi(a)) < f(a)$. $\square$

The above lemma is false if I is not finite (even if there exists a maximal a): Define $f : \mathbb{Z}^+ \to \{0, 1\}$ by $f(i) = 0$ if $i \leq 0$ and $f(i) = 1$ otherwise. Define $\pi : \mathbb{Z}^+\mathbb{Z}^+, i \to i + 1$. Then f is non-decreasing and $a = 0$ is the unique element with $f(a) \neq f(\pi(a))$. But $f(a) = 0 < 1 = f(\pi(a))$.

Allthough the lemma stays true if there exists a maximal a and f is increasing (decreasing). Indeed in thus case $J = C_I(\pi)$ and so $\pi(I \setminus J) = I \setminus J$.

Lemma 5.4.10 [**proving maximal**] *Let t be a λ-tableau and $X \subseteq I_n$.*

(a) [**a**] *Suppose that r_t is non-decreasing on X. Then $\underline{\overline{\pi t}} \leq \underline{\overline{t}}$ for all $\pi \in \mathrm{Sym}(X)$.*

(b) [**b**] *Suppose that r_t is non-increasing on X. Then $\underline{\overline{\pi t}} > \underline{\overline{t}}$ for all $\pi \in \mathrm{Sym}(X)$.*

Proof: (a) Suppose that $\underline{\overline{\pi t}} \neq \underline{\overline{t}}$. Let i be maximal in I_n with $r_t(i) \neq r_{\pi t}(i)$. Note that $r_{\pi t}(i) = r_t(\pi^{-1}(i)$ Since r_t is non-decreasing 5.4.9 gives $r_t(i) < r_t(\pi^{-1}i) = r_{\pi t}(i)$. Thus $\underline{\overline{t}} < \overline{\underline{\pi}}t$.

(b) Similar to (a). $\square$

Lemma 5.4.11 [**maximal in et**] *Let t be column-increasing λ tableau. Then $\underline{\overline{t}}$ is the maximal tabloid involved in e_t.*

Proof: Any tabloid involved in e_t is of the form $\underline{\overline{\pi t}}$ with $\pi \in C_t$. Since r_t is increasing on each column, we can apply 5.4.10 to the restriction of π to each of the columns. So the result holds. $\square$

Lemma 5.4.12 [**linear independent and order**] *Let $\mathbb{F}$ be ring, V a vector space with a totally ordered basis $\mathcal{B}$ and $\mathcal{L}$ a subset of V. Let $b \in \mathcal{B}$ and $v \in V$. We say that b is involved in v if the b-coordinate of v is non-zero. Let b_v be maximal element of $\mathcal{V}$ involved in v. Suppose that the $b_l, l \in \mathcal{L}$ are pair wise distinct and the coefficient f_l of b_l in l is not a left zero divisor.*

(a) [**a**] *$\mathcal{L}$ is linearly independent.*

(b) [**b**] *Suppose in addition that each $f_l, l \in \mathcal{L}$ is a unit and $\mathcal{L}$ is finite. Put $\mathcal{C} = \{b_l \mid l \in \mathcal{L}\}$ and $\mathcal{D} = \mathcal{B} \setminus \mathcal{C}$.*

- *(a)* [**a**] *$\mathcal{L} \cup \mathcal{D}$ is an R-basis for M.*
- *(b)* [**b**] *Suppose R is commutative and $(\cdot \mid \cdot)$ be the unique R bilinar form on M with orthormal basis $\mathcal{B}$. Then*
  - *(a)* [**a**] *For each $d \in \mathcal{D}$ there exists a unique $e_d \in d + R\mathcal{C}$ with $e_d \in \mathcal{L}^\perp$.*
  - *(b)* [**b**] *$(e_d \mid d \in \mathbb{D}$ is an R-basis for $\mathcal{L}^\perp$.*
  - *(c)* [**c**] *$\mathcal{L}^{\perp\perp} = R\mathcal{L}$.*

Proof: (a) Let $0 \neq (f_l) \in \bigoplus_{\mathcal{L}} F$. Choose $l \in \mathcal{L}$ with b_l maximal with respect to $f_l \neq 0$. Then $b_l > b_k$ for $l \neq k \in \mathcal{L}$ with $f_k \neq 0$. So b_l is involved in $f_l l$, but in not other $f_k k$. Thus $\sum_{l \in \mathcal{L}} f_l l \neq 0$ and $\mathcal{L}$ is linearly independent.

(b) We assume without loss that $f_l = 1$ for all $l \in \mathcal{L}$.

(b:a) Let $m = \sum_{b \in \mathcal{B}} m_b b \in M$. We need to show that $m \in R(\mathcal{D} \cup \mathcal{L}$. If $m_b = 0$ for all $b \in \mathcal{B}_{\mathcal{L}}$, this is obvious. Otherwise pick $b \in \mathcal{B}_{\mathcal{L}}$ maximal with $m_b \neq 0$ and let $l \in \mathcal{L}$ with $b = b_l$. Then by induction on b, $m - m_b l \in R(\mathcal{D} \cup \mathcal{L})$.

(b:b) We will first show that

$$(*) \qquad R \cap C \cap \mathcal{L}^\perp = 0$$

Let $0 \neq m = \sum_{l \in \mathcal{L}} m_l b_l$ and choose l with $m_l \neq 0$ and b_l minimal. Then $(m \mid l) = m_l \neq 0$ and $m \notin \mathcal{L}^\perp$.

(b:b:a) This is just the Gram Schmidt process. For completeness here are the details. Let $\mathcal{L} = \{l_1, l_2, \ldots l_n\}$ and $b_i = b_{l_i}$ with $b_1 < b_2 < \ldots b_n\}$. Put $e_0 = d$ and suppose inductively that we have found $e_i \in d + Rb_1 + \ldots + Re_i$ with $e_i \perp l_j$ for all $1 \leq j \leq e_i$. If $i < n$ put $e_{i+1} = e_i - (e_i \mid l_{i+1}) b_{l+1}$. Then $(e_{i+1} \mid l_{i+1} = 1$ and since $b_{i+1} \perp l_j$ for all $j \leq i$. Put $e_d = e_n$. By (*), e_d is unique.

(b:b:b)) Clearly $(e_d \mid d \in \mathcal{D})$ is R-linearly independent. Moreover if $m = \sum_{b \in caB} m_b b \in \mathcal{L}^\perp$, then $\tilde{m} := m - \sum_{d \in \mathcal{D}} m_d e_d \in R\mathcal{C} \cap \mathcal{L}^\perp$. So (*) implies $\tilde{m} = 0$ and (b:b:b) holds.

(b:b:c) $m = \sum_{b \in caB} m_b b \in \mathcal{L}^{\perp\perp}$. By (b:a) there exists $\tilde{m} \in R\mathcal{L}$ with $m = \tilde{m} \in R\mathcal{D}$ and so we may assume that $m_c = 0$ for all $c \in \mathcal{C}$. Then $0 = (m \mid e_d) = m_d$ for all $d \in \mathcal{D}$ and so $m = 0$. $\square$

Theorem 5.4.13 [**standard basis**] *Let F be a ring and λ a partition of n. The standard polytabloids form a basis of S^λ. Moreover, $S^{\lambda\perp\perp} = S^\lambda$ and there exists an R-basis for S^λ indexed by the nonstandard λ-polytabloids.*

By 5.4.10(a) and 5.4.12 the standard polytabloids are linearly independent. Let t be λ-tableau. Let $|t|$ be the column equivalence class of t. Total order the column euqivalence classes analog to 5.4.7 We show by downwards induction that e_t is a F-linear combination of the standard polytableaux. Since $e_t = \pm e_s$ for any s column-equivalent to t we may assume that t is column increasing. If t is also row-increasing, t is standard tableaux and we are done. So suppose t is not row-increasing so there exists $(i,j) \in \mathbb{Z}^+\times$ such that $t(i,j) > t(i,j+1)$. Let $X = \{t(k,j) \mid i \leq k \leq \lambda'_i$ and $Y = \{t(k,j+1) \mid 1 \leq k \leq j$. Then $|X \cup Y| = \lambda'_j + 1$ and so by 5.4.1

$$\sum_{\pi \in \mathcal{T}_{XY}} \operatorname{sgn}\pi e_{\pi t} = 0$$

Since c_t is increasing on X and on Y and since $t(i,j) > t(i,j+1)$, r_t is non-increasing on $X \cup Y$. So by 5.4.10 $|\pi t| > |$— for all $1 \neq \pi \in Sym(X\cup)$. Thus by downwards induction $e_{\pi t}$ is an R-linear combination of the standard polytabloids. Hence the same is true for $e_t = -\sum_{1\neq\pi\mathcal{T}} \operatorname{sgn}\pi e_{\pi t}$.

The remaining statements now follow from 5.4.12. □

5.5 The number of simple modules

Definition 5.5.1 [**def:p-regular class**] *Let p be an integer. An element g in a group G is called p-singular if p divides $|g|$. Otherwise g is called p-regular. A conjugacy class is called p-regular if its elements are p-regular.*

The goal of this section is to show that if $\mathbb{K}$ is an algebraicly closed field, G is a finite group and $p = \operatorname{char} K$ then the number of isomorpism classes of simle $\mathbb{K}G$-modules equals the number of p-regular conjugacy classes.

Lemma 5.5.2 [**cyclic permutation**]

(a) [**a**] *Let G be a group, $n \in \mathbb{Z}^+$ and $a_1, \dots a_n \in G$. Then for all $i \in \mathbb{N}$ $a_{i+1}a_{i+2}\dots a_{i+n}$ is conjugate $a_1a_2\dots a_n$ in G.*

(b) [**b**] *Let R be a group, $n \in \mathbb{Z}^+$ and $a_1, \dots a_n \in R$. Then for all $i \in \mathbb{N}$, $a_{i+1}a_{i+2}\dots a_{i+n} \equiv a_1a_2\dots a_n \pmod{} S(R)$*

Proof: (a) We have $a_1^{-1} \cdot a_1a_2\dots a_n \dots a_1 = a_2\dots a_na_1$. So (a) follows by induction on n.

(b) $a_1 \cdot a_2 \dots a_n - a_2 \dots a_n \cdot a_1 \in \mathrm{S}(R)$ So (b) follows by induction on n. □

Definition 5.5.3 [**def: sr**] *Let R be ring and $p = \operatorname{char} R$. Then $\mathrm{S}(R) = \langle xy - yx \mid x, y \in R\rangle_{\mathbb{Z}}$. Let $\tilde{p} = p$ if $p \neq 0$ and $\tilde{p} = 1$ if $p = 0$. $\mathrm{T}(R) = \{r \in R \mid r^{\tilde{p}^m} \in \mathrm{S}(R)$ for some $m \in \mathbb{N}\}$.*

Lemma 5.5.4 [**sr for group rings**] *Let R be a commutative ring and G a group. Then $\mathrm{S}(RG)$ consists of all $a = \sum_{r_g g} \in RG$ with $\sum_{g \in \mathbb{C}} r_g = 0$ for all conjugacy classes C of G.*

Proof: Let U consists of $a = \sum_{r_g g} \in RG$ with $\sum_{g \in \mathbb{C}} r_g = 0$ for all conjuagacy classes C of G. Note that both $\mathrm{S}(R)$ and U are R-submodules. As an R-modules $\mathrm{S}(R)$ is spaned by the $gh - hg$ wth $g, h \in G$. By 5.5.2 gh and hg are conjugate in G. Thus $gh = hg \in U$ and $\mathrm{S}(R) \subseteq U$. U is spanned by the $g - h$ where g, h in G are conjuagte. Then $h = aga^{-1}$ and $g - h = a^{-1} \cdot ag = ag \cdot a^{-1}$ and so $g - h \in \mathrm{S}(R)$ and $U \subseteq \mathrm{S}(R)$. □

Lemma 5.5.5 [**basic sr**] *Let R be a ring with $p := \operatorname{char} R$ a prime.*

(a) [**a**] $(a+b)^{p^m} \equiv a^{p^m} + b^{p^m} \mod \mathrm{S}(R)$ *for all $a, b \in R$ and $m \in \mathbb{N}$.*

(b) [**b**] $\mathrm{T}(R)$ *is an additive subgroup of R.*

(c) [**c**] *Suppose that $R = \bigoplus_{i=1}^{s} R_i$. Then $S(R) = \bigoplus_{i=1}^{r} S_i$ and $T(R) = \bigoplus T(R_i)$.*

(d) [**d**] *Let I be an ideal in R. Then $S(R/I) = S(R) + I/I$.*

(e) [**e**] *Let I be a nilpotent ideal in R. Then $I \leq T(R)$, $T(R/I) = T(R)/I$ and $R/T(R) \cong (R/I)/T(R/I)$.*

Proof: (a) Let $A = \{a, b\}^p$ and let $H = \langle h \rangle$ be a cyclic group of order p acting on A via $h(a_i) = (a_{i+1})$. Then H has two fixed points on A namely the constant sequence (a) and (b). Since the length of any orbit of H divises $|H|$, all other orbits have lenghth p. Let C be an orbit of length p for H on A. For $a = (a_1, a_2, \ldots a_p) \in A$ puy $\prod a = a_1 a_2 \ldots a_p/$ Then by 5.5.2 $\prod a \equiv \prod b \pmod{} S(R)$ for all $a, b \in C$ and so $\sum_{b \in C} \prod b \equiv p \prod a = 0 \mod \mathrm{S}(R)$. Hence for $(a+b)^p = \sum_{\alpha in A} \prod a \equiv a^p + b^p \mod \mathrm{S}(R)$. (a) now follows by induction on m.

(b) Follows from (a).

(c) Obvious.

(d) Obvious.

(e) Since I is nilpotent, $I^k = 0$ for some integer k. Choose m with $p^m \geq k$. Then for all $i \in I$, $i^{p^m} = 0 \in \mathrm{S}(R)$ and so $i \in \mathrm{T}(R)$. Thus $I \leq \mathrm{T}(R)$. Since $S(R) + I/I = S(T/I)$ we have $T(R)/I \leq T(R/I)$. Conversely if $t + I \in T(R/I)$, then $t^{p^l} \in S(R) + I$. Since bith $\mathrm{S}(R)$ and I are in $T(R)$, (b) implies $t^{p^l} \in T(R)$ and so also $t \in T(R)$. □

Lemma 5.5.6 [**tr for group rings**] *Let $\mathbb{F}$ be an integral domain with $\operatorname{char} \mathbb{F} = p$. Let G be a periodic group and let $\mathcal{C}_p$ be the set of p-regular conjugacy classes of G. For $C \in \mathcal{C}_p$ let $g_C \in C$. Then $(g_C + \mathrm{S}(\mathbb{F}G) \mid C \in \mathcal{C}_p)$ is a F-basis for $\mathbb{F}G/\mathrm{S}(\mathbb{F}G)$.*

Proof: Let $g \in G$ and write $g = ab$ with $[a,b] = 1$, $a^{p^m} = 1$ and b, p-regular. Then $g^{p^m} - b^{p^m} = 0$ and so by 5.5.5(b), $g \equiv \mod \mathrm{T}(\mathbb{F}G)$. Also by 5.5.4 $b \equiv g_C$ where $C = {}^G b$. $(g_C + (\mathbb{F}G) \mid C \in \mathcal{C}_p)$ is a spanning set for $\mathbb{F}G/\,\mathrm{S}(\mathbb{F}G)$. Now let $r_C \in R$ with

$$\sum_{C \in \mathcal{C}_r} r_c g_C \in \mathrm{T}(\mathbb{F}G)$$

Then there exists $m \in \mathbb{N}$ with $(\sum_{C \in \mathcal{C}_p} r_c g_C)^{p^m} \in \mathrm{S}(\mathbb{F}G)$. Since g_C is p-regular, $p \nmid g_C$ and so p is invertible in $\mathbb{Z}/|g_C|\mathbb{Z}$. Hence there exists $m_C \in \mathbb{Z}$ with $|g_C| \mid p^{m_C} - 1$. Put $k = m \prod_{C \in \mathcal{C}_p} m_C$. Then $g_C^{p^k} = g_C$ and $(\sum_{C \in \mathcal{C}_p} r_c g_C)^{p^k} \in \mathrm{S}(\mathbb{F}G)$. By 5.5.5(b),

$$\sum_{C \in \mathcal{C}_p} r_C^{p^k} g_C = \sum_{C \in \mathcal{C}_p} r_C^{p^k} g_C^p \in \mathrm{S}(\mathbb{F}G)$$

Thus 5.5.4 shows that $r_C^{p^k} = 0$ for all $C \in \mathcal{C}_p$. So also $r_C = 0$ and $(g_C + (\mathbb{F}G) \mid C \in \mathcal{C}_p)$ is a linearly independent. □

Lemma 5.5.7 [**sr for matrix ring**] *Let R be a commutative ring and $p = \operatorname{char} R$.*

(a) [**a**] $\mathrm{S}(\mathrm{M}_n(R))$ *consists of the trace zero matrices and* $M_n(R)/\,\mathrm{S}(M_n(R)) \cong R$.

(b) [**b**] $p = \operatorname{char} \mathbb{K}$ *is a prime, then* $\mathrm{T}(\mathrm{M}_n(R)) = \{a \in \mathrm{M}_n(R) \mid \operatorname{tr}(a)^{\tilde{p}^m} = 0$ *for some* $m \in \mathbb{N}\}\}$.

(c) [**c**] *If R is a field, then* $\mathrm{S}(\mathrm{M}_n(R)) = \mathrm{T}(\mathrm{M}_n(R))$ *and* $M_n(R)/\,\mathrm{T}(M_n(R)) \cong R$.

Proof: Since $\operatorname{tr}(xy) = \operatorname{tr}(yx)$ and so $\mathrm{S}(\mathrm{M}_n(R)) \leq \ker \operatorname{tr}$. $\ker \operatorname{tr}$ is generted by the matrices E_{ij} and $E_{ii} - E_{jj}$ with $i \neq j$. $E_{ij} = E_{ii}E_{ij} - E_{ij}E_{ii}$ and so $E_{ij} \in \mathrm{S}(\mathrm{M}_n(R)$. $E_{ii} - E_{jj} = E_{ij}E_{ji} - E_{ji}E_{ij}$ and so $E_{ii} - E_{jj} \in \ker \operatorname{tr}$.

Suppose now that p is a prime and let $a \in M_n(R)$. Let $b = \operatorname{tr}(a)E_1 1$ and $c = a - b$. Then $\operatorname{tr} c = 0$, $c \in \mathrm{S}(\mathrm{M}_n(R))$ and so by 5.5.5 $a \in T(M_n(R)0$ if and only if $b \in \mathrm{T}(\mathrm{M}_n(R))$. Since $\operatorname{tr}(b^{p^m}) = \operatorname{tr}(a)^{p^m}$ the lemma is proved. □

Theorem 5.5.8 [**pmodular simple**] *Let G be a finite group, $\mathbb{F}$ an algebraicly closed field and $p = \operatorname{char} F$. Then the number of isomorphism classes of simple $\mathbb{F}G$-modules equals the number of p-regular conjugacy classes.*

Proof: By 5.5.6 the number of p' conjugacy classes is $\dim_{\mathbb{F}} \mathbb{F}G/\,\mathrm{T}(\mathbb{F}G)$.

Let $A = \mathbb{F}G/\mathrm{J}(\mathbb{F}G)$. By 6.3.4 $\mathrm{J}(\mathbb{F}G)$ is nilpotent and so by 5.5.5(e), $\mathbb{F}G/\,\mathrm{T}(\mathbb{F}G) \cong A/\,\mathrm{T}(A)$.

By 2.5.24 $R \cong \bigoplus_{i=1}^n \mathrm{M}_{d_i}(\mathbb{F})$, where n is the number of isomorphism classes of simple $\mathbb{F}G$-modules.

Thus by 5.5.5(c) and 5.5.7(c), $R/T(R) \cong \mathbb{F}^n$. So $\dim_{\mathbb{F}} \mathbb{F}G/\,\mathrm{T}(\mathbb{F}G)$ is the number of isomorphism classes of simple $\mathbb{F}G$-modules. □

5.6 p-regular partitions

Definition 5.6.1 [**def:p-regular partition**] *Let p and n be positive integers with p being a prime. A partition λ of n is called* p-singular, *if there eixsts $i \in \mathbb{N}$ with $\lambda_{i+1} = \lambda_{i+2} = \ldots = \lambda_{i+p}$. Otherwise λ is called* p-regular.

Lemma 5.6.2 [**p-regular=p-regular**] *Let p, n be positive integers with p beieng a prime. The number of p-regular conjugacy classes of* $\mathrm{Sym}(n)$ *equals the number of p-regular partitions of* $\mathrm{Sym}(n)$.

Proof: Let $g \in G$ and μ its cycle-type. Then g is p-regular iff none of the μ_i is divisible by p. Any such partions we can uniquely determined by a sequence $(z_i)_{p \nmid i}$ of non-negative integers with $\sum i z_i = n$, where j_i is the number of $k's$ with $\mu_k = i$. Any p-regular partion we can write as a sequence $(z_i)_{i=1}^{\infty}$ with $0 \leq j_i < p$.

Let $f = \frac{\prod_{i=1}^{\infty}(1-x^{pi})}{\prod_{i=1}^{\infty}(1-x^{i})}$ viewed as an element of $\mathbb{Z}(x))$, the ring of formal integral power series.

We compute f in two different ways:

(i) [**1**] Let $A = \mathbb{N} \setminus p\mathbb{N}$. For each i cancel the factor $1 - x^{pi}$ in the numerator and denumerator of f to obtain:

$$\begin{aligned} f &= \prod_{p \in A} \frac{1}{1-x^i} &&= \prod_{p \in A} \sum_{j=0}^{\infty} x^{ij} \\ &= \sum_{(j_i) \in \oplus_A \mathbb{N}} \prod_{i \in A} x^{ij_i} &&= \sum_{(j_i) \in \oplus_A \mathbb{N}} x^{\sum_{i \in A} ij_i} \end{aligned}$$

Thus the coefficent of x^n is the number of partions of n, none of whose parts is divisible by p. So the coefficent of x^n is the number of p-regular conjugacy classes in $\mathrm{Sym}(n)$.

(ii) [**2**] Let $B = \{0, 1, \ldots p-1\}$.

$$\begin{aligned} f &= \prod_{i=1}^{\infty} \frac{1-x^{pi}}{1-x^i} &&= \prod_{i=1}^{\infty} \sum_{j=0} p - 1 x^j \\ &= \sum_{(j_i) \in \oplus_\infty B} \prod x^{j_i} &&= \sum_{(j_i) \in \oplus_\infty B} x^{\sum_{i=1}^{\infty} ij_i} \end{aligned}$$

So the coefficient of x^n in f is the number of p-regular partitions.

□

Definition 5.6.3 [**def:glambda**] *Let λ be a partition of n and $F = \mathbb{Z}$. Then*

$$g^{\lambda} = \gcd\{(e_t \mid e_s) \mid t, s\lambda - \textit{tableaux}\}$$

Lemma 5.6.4 [**glambda and dlambda**] *Let λ be a partition of n. Then $D^{\lambda} = 0$ iff* $\mathrm{char}\, F \mid g^{\lambda}$.

Proof: Since S^λ is spanned by the λ-polytabloid we have

$$
\begin{array}{rcl}
 & D^\lambda = 0 & \\
\iff & S^\lambda = S^\lambda \cap S^{\lambda\perp} & \\
\iff & S^\lambda \perp S^\lambda & \\
\iff & e_t \perp e_s & \forall\lambda - \text{tableaux} s, t \\
\iff & (e_t \mid e_s) & \forall\lambda\text{-tableaux} s, t \\
\iff & \operatorname{char} F \mid (e_t \mid e_s)_{\mathbb{Z}} & \forall\lambda\text{-tableaux} s, t \\
\iff & \operatorname{char} F \mid g^\lambda &
\end{array}
$$

□

Lemma 5.6.5 **[glambda]** *Let λ be a partition of n and for $F = \mathbb{Z}$ define*

$$g^\lambda = \gcd\{(e_t \mid e_s) \mid t, s\lambda - \textit{tableaux}\}$$

Let $z_j = |\{i \mid \lambda_i = j|\}$. Then g^λ divides $\prod_{j=1}^\infty (z_j!)^j$ and $\prod_{j=1}^\infty z_j!$ divides g^λ.

Define two λ-tabloids $\overline{\underline{s}}$ and $\overline{\underline{t}}$ to be equivalent $\{\Delta_i(t) \mid i \in \mathbb{Z}^+ = \{\Delta_i(s) \mid i \in \mathbb{Z}\}$, that is if $\overline{\underline{t}}$ and $\overline{\underline{s}}$ have the rows but in possible different orders. Define $Z_j = \{i \in \mathbb{Z}^+ \mid \lambda_i = j$ and $Z = (Z_j)_{j=1}^\infty$. Then Z is partition of $\mathbb{Z}^+$. Note that $\overline{\underline{t}}$ and $\overline{\underline{s}}$ $\overline{\underline{s}}$ are this is the case if and only if there exists $\pi = \pi(\overline{\underline{r}}, \overline{\underline{s}}) \in \operatorname{Sym}(\mathbb{Z}^+)$ with $\Delta_{\pi i}(t) = \Delta_i(s)$. Then $\lambda_{\pi t} = |\Delta_{\pi t}| = |\Delta_i(s)| = \lambda_i$ and so $\pi Z = Z$. Conversely if $\pi \in \operatorname{Sym}(Z) := C_{\operatorname{Sym}(\mathbb{Z}^+)}(Z) = \mathop{\textcircled{Π}}_{j \in \mathbb{Z}^+} \operatorname{Sym}(Z_j)$, then there exists a unique tabloid $\overline{\underline{s}}$ with $\Delta_i(s) = \Delta_{\pi i}(t)$ and $\overline{\underline{s}}$ is equivalent to $\overline{\underline{s}}$.

Hence

1° **[1]** *Each equivalence class contains $|Sym(Z) = z! := \prod_{j=1}^\infty z_j!$ tabloids.*

For a tabloid $\overline{\underline{r}}$ and a tableau t let $\epsilon_t(\overline{\underline{r}})$ be the coefficient of $\overline{\underline{r}}$ in e_t. So $e_t = \sum \epsilon_t(\overline{\underline{r}})\overline{\underline{r}}$.

2° **[2]** *Let $\overline{\underline{r}}$ and $\overline{\underline{s}}$ are equivalent λ-tableaux. Then there exists $\epsilon = \epsilon(\overline{\underline{r}}, \overline{\underline{s}}) \in \{\pm 1\}$ such that for any λ-tableaux t, $\epsilon_t(\overline{\underline{s}}) = \epsilon \cdot \epsilon_t(\overline{\underline{r}})$.*

Let $\pi = \pi(\overline{\underline{r}}, \overline{\underline{s}})$. Let π_j be the restriction of π to Z_j and define $\epsilon = \prod_j \operatorname{sgn}\pi^j$. We may assume that $\overline{\underline{r}}$ is involved in e_t and so $\overline{\underline{r}} = \overline{\underline{\rho t}}$ for some $\rho \in C_t$. Without loss $r = \rho t$. Define $\pi^* \in \operatorname{Sym}(n$ by $\pi^*(r(i,j) = r(\pi(i), j)$. Then $\pi^* \in C_t$, $\operatorname{sgn}\pi^* = \epsilon$ and $\overline{\underline{\pi^* r}} = \overline{\underline{s}}$. Thus $\overline{\underline{s}} = \overline{\underline{\pi^* \rho}}$, the coefficent of $\overline{\underline{r}}$ in e_t is $\operatorname{sgn}\rho$ and the coefficent of $\overline{\underline{s}}$ is $\operatorname{sgn}(\pi * \operatorname{sgn}\rho) = \epsilon\operatorname{sgn}\rho$.

3° **[3]** *$z!$ divides g^λ.*

Let t, u be λ tableaux. Let A be an equivalence class of tabloids and $\underline{\overline{r}} \in A$. Let $\underline{\overline{s}} \in A$ and choose ϵ as in (2°). Then

$$\epsilon_t(\underline{\overline{s}})\epsilon_u(\underline{\overline{s}}) = \epsilon \cdot \epsilon_t(\underline{\overline{s}}) \cdot \epsilon \cdot \epsilon_s(\underline{\overline{r}}) = \epsilon_t(\underline{\overline{r}})\epsilon_t(\underline{\overline{s}})$$

Thus $\sum_{s \in \mathcal{A}} \epsilon_t(\underline{\overline{s}})\epsilon_u(\underline{\overline{s}}) = |A|\epsilon_t(\underline{\overline{r}})\epsilon_u(\underline{\overline{r}})$

By (1°), $|A| = z!$. Summing over all the A's we conclude that $z!$ divides $(e_t \mid e_s)$. Thus (3°) holds.

Let t be λ-tableau. Define $\sigma \in \mathrm{Sym}(n)$ by $\sigma(t(i,j)) = t(i, \lambda_i + 1 - j)$ and put $\tilde{t} = \sigma t$. So $\tilde{t}$ is the tableaux obtained by reversing the rows of t. We will show that $(e_t \mid () \mid e_{\tilde{t}}) = \prod_{i=1}^{\infty}(z_i!)^j$.

Put $U_i := U_i(t) := \bigcup_{k \in Z_i} \Delta_k(t)$, the union of the rows of t of size i. Note that $U_i = U_i(\tilde{t})$ and $U = (U_i)$ is partion of I_n. Also put $U_i^j := U_i^j(t) = U_i \cap \Delta'_j$, the part of column j of t lying in U_i. Then $U_i^j(\tilde{t}) = U_i^{i+1-j} = \sigma(U_i^j)$. Let $P = (U_i^j) \mid i, j \in \mathbb{Z})$. Then P is a partition of I_n refining both U and column partition. $\Delta'(t)$. Hence $\mathrm{Sym}(U) \le C_t$. Also σ permutes the U_{i^j} and so σ normalizes $\mathrm{Sym}(U)$ and so $\mathrm{Sym}(U) \le \sigma C_t \sigma^{-1} = C_{\tilde{t}}$. Observe $|U_i^j(t)| = z_j$ if $j \le i$ and $U_i^j(t) = \emptyset$ otherwise. Thus

4° [**4**] $|\mathrm{Sym}(U)| = \prod_{i,j} |U_i^j(t)|! = \prod_{i=1}^{\infty}(z_i!)^i$.

We show next

5° [**5**] *Let* $\pi \in \mathrm{Sym}(U)$. *Then* $\epsilon_t(\underline{\overline{\pi t}}) = \epsilon_{\tilde{t}}(\underline{\overline{\pi t}}) = \mathrm{sgn}\pi$.

Since $\pi \in C_t$ we have $\epsilon_t(\underline{\overline{\pi t}}) = \mathrm{sgn}\pi$.
Since $\pi \in C_{\tilde{t}}$ we have $\epsilon_t(\underline{\overline{\pi \tilde{t}}}) = \mathrm{sgn}\pi$.
Since σ fixes the rows of t, $\pi\sigma\pi^{-1}$ fixes the rows of πt. Thus

$$\underline{\overline{\pi t}} = \underline{\overline{\pi\sigma\pi^{-}1\pi t}} = \underline{\overline{\pi\sigma t}} = \underline{\overline{\pi\tilde{t}}}$$

and so (5°) holds.

6° [**6**] *Let* $\pi \in C_t$ *such that* $\underline{\overline{\pi t}}$ *is involved in* $e_{\tilde{t}}$. *Then* $\pi \in \mathrm{Sym}(U)$.

Since $\underline{\overline{\pi t}}$ is involved in $e_{\tilde{t}}$ there exists $\tilde{\pi} \in C_{\tilde{t}}$ with $\underline{\overline{\pi t}} = \underline{\overline{\tilde{\pi}\tilde{t}}}$. Hence for all $k \in I_n$, $r_{\pi t)}(k) = r_{\tilde{\pi}\tilde{t}}(k)$ and so $r_t(\pi^{-1}k) = r_{\tilde{t}}(\tilde{\pi}{-}1k)$. Put $\alpha = \pi^{-1}$ and $\tilde{\alpha} = \pi^{*-1}$. Then for all $k \in I$.

$$(*) \qquad \alpha \in C_t, \quad \tilde{\alpha} \in C_{\tilde{t}} \quad \text{and} \quad r_t(\alpha(k)) = r_{\tilde{t}}(\tilde{\alpha}(k))$$

We need to show that $\alpha(U_i^j) = U_i^j = \tilde{\alpha}(U_i^j)$ for all i, j. The proof uses double induction. First on j and then downwards on i.

For $I, J \subset \mathbb{Z}^+$ let $U_I^J = \bigcup\{U_i^j \mid i \in I, j \in J\}$. If $I = \mathbb{Z}^+$ or $J = \mathbb{Z}^+$ we drop the subscript I, respectively superscript. For example $U^{\leq j} = \bigcup U_i^k \mid i, k \in \mathbb{Z}^+ \mid k \leq j\}$ consists ofthe first j columns of t.

Suppose that $\alpha(U_k^l) = U_k^l = \tilde{\alpha}(U_k^l)$ whenever $l < j$ or $l = j$ and $k > i$. Then $\alpha(U_{>i}^j) = \alpha(U_{>i}^j)$ and $\alpha(U^j) = U^j$ implies $\alpha(U_i^j) \subseteq U_{\leq i}^j$. Hence by (*) also

$$(**) \qquad \tilde{a}lpha(U_i^j) \subseteq U_{\leq i}$$

Let $c = i + 1 - j$. Then $U_i^j = \tilde{U}_i^c$ and

$$\tilde{U}_{<i}^c = \bigcup_{k<i} U_k^{c+1-k}$$

and so by induction $\tilde{\alpha}\tilde{U}_{<i}^c = U_{<i}^c$. Hence $\tilde{\alpha}(U_i^j) \subseteq \tilde{\alpha}(\tilde{U}_{\geq i}^c) = \tilde{U}_{\geq i}^c \subseteq \tilde{U}_{\geq i} = U_{\geq i}$. So by (**) $\tilde{\alpha}(U_i^j) \subseteq U_i \cap \tilde{U}^c = \tilde{U}_i^c = U_i^j$ and $\tilde{a}(U_i^j) = U_{i^j}$. Hence by (*) also $\alpha(U_i^J \leq U_i \cap U^j = U_i^j$ and $\alpha(U_i^j) = U_i^j$.

So (6°) is proved.

From (5°) and (6°) we conclude that $(e_t \mid e_{\tilde{t}}) = |\mathrm{Sym}(U)| = \prod_{i=1}^{\infty}(z_i!)^i$. Since g^λ divides $(e_t \mid e_{\tilde{t}})$ the lemma is proved. □

Proposition 5.6.6 [**dlambda not zero**] *Suppose F is an integral domain and λ is a partition of n. Let $p = \mathrm{char}\, F$. Then $D^\lambda \neq 0$ iff λ is p-regular.*

Proof: Since F is an integral domain, $p = 0$ or p is a prime. Let $\lambda = (i_i^z)_{i=1}$. Then $p \mid \prod_i z_i!$ iff $p \leq z_i$ for some i, iff $p \mid \prod_i (z_i!)^i$ and iff λ is p-singular.

So 5.6.5 implies that $p \mid g_\lambda$ iff λ is p-singular. And so by 5.6.4, $D_\lambda = 0$ iff λ is p-singular. □

Theorem 5.6.7 [**all simple sym(n)-modules**] *Let F be a field, n a postive integer and $p = \mathrm{char}\, F$.*

(a) [**a**] *Let λ be a p-regular partition of n. Then D_λ is an absolutely simple, selfdual $F\mathrm{Sym}(n)$-module.*

(b) [**b**] *Let I be a simple $F\mathrm{Sym}(n)$-module. Then there exists a unique p-regular partition λ of n with $I \cong D^\lambda$.*

Proof: (a) By 5.6.6 $D^\lambda \neq 0$. By 4.1.5, s induces a non-degenerate G-invariant form on D^λ and so by 4.1.6(c), D^λ is isomorphic to its dual. By 5.3.12, D^λ is absolutely simple.

(b) If λ and μ are distinct p-regular partition then by 5.3.10 and (a), D^λ and D^μ are non-isomorphic simple $F\mathrm{Sym}(n)$-modules. The number of simple $F\mathrm{Sym}(n)$-modules is less or equal to the number simple $Sym(n)$-modules over the algebraic closure of $\mathbb{F}$. The latter number is by 5.5.8 equal to to the number of p'-conjuagacy classes and so by 5.6.2 equal to the number of p-regular partitions of n. So (b) holds. □

5.7 Series of R-modules

Definition 5.7.1 [**def:series**] *Let R be a ring and M and R-module. Let $\mathcal{S}$ be a set of R-submodules of M. Then $\mathcal{S}$ is called an R-*series *on M provided that:*

(a) [**a**] $0 \in \mathcal{S}$ *and* $M \in \mathcal{S}$.

(b) [**b**] $\mathcal{S}$ *is totally ordered with respect to inclusion.*

(c) [**c**] *For all* $\emptyset \neq T \subset \mathcal{S}$, $\bigcap \mathcal{T} \in \mathcal{S}$ *and* $\bigcup \mathcal{T} \in \mathcal{S}$.

For example $\mathbb{Z} > 2\mathbb{Z} > 6\mathbb{Z} > 30\mathbb{Z} > 210\mathbb{Z} > \ldots > 0$ is an $\mathbb{Z}$-series on $\mathbb{Z}$.

Definition 5.7.2 [**def:jumps**] *Let R be a ring, M an R-module and $\mathcal{S}$ an R-series on M. For $0 \neq A \in \mathcal{S}$ put $A^- = \bigcup\{B \in \mathcal{S} \mid B \subset A\}$. If $A \neq A^-$ then (A^-, A) is called a* jump *of $\mathcal{S}$ and A/A^- a* factor *of $\mathcal{S}$. $\mathcal{S}$ is called a composition series for R on $\mathcal{S}$ provided that all its factors are simple R-modules.*

The above example is composition series and its sets of factors is isomorphic to $\mathbb{Z}/p\mathbb{Z}$, p a prime.

Lemma 5.7.3 [**basic series**] *Let R be a ring, M an R-module, $\mathcal{S}$ an R-series on M.*

(a) [**a**] *Let $A, B \in \mathcal{S}$ with $B \subset A$. Then (B, A) is a jump iff $A = C$ or $B = C$ for all $C \in \mathcal{S}$ with $B \subseteq C \subseteq A$.*

(b) [**b**] *Let $U \subset M$. Then there exists a unique $A \in \mathcal{U}$ minimal with $U \subseteq A$. If U is finite and contains a non-zero element then $A^- \neq A$ and $A \cup U \nsubseteq A^-$.*

(c) [**c**] *Let $0 \neq m \in M$. Then there exists a unique jump (B, A) if $\mathcal{S}$ with $v \in A$ and $v \notin B$.*

Proof: (a) Suppose first that (B, A) is a jump. Then $B = A^-$. Let $C \in \mathcal{S}$ with $B \subseteq C \subseteq A$ Suppose $C \subset A$. Then $C \subseteq A^- = B$ and $C = B$.

Suppose next that $A = C$ or $B = C$ for all $C \in \mathcal{S}$ with $B \subseteq C \subseteq A$. Since $B \subseteq A$, $B \subseteq A^-$. Let $C \in \mathcal{S}$ with $C \subset A$. Since $\mathcal{S}$ is totally ordered, $C \subseteq B$ or $B \subseteq C$. In the latter case, $B \subseteq C \subset A$ and so by assumption $B = C$. So in any case $C \subseteq B$ and thus $A^- \subseteq B$. We conclude that $B = A^-$ and so (B, A) is a jump.

(b) Put $A = \bigcup\{S \in \mathcal{S} \mid U \subseteq S\}$. By $A \in \mathcal{S}$ and so clearly is minimal with respect to $U \subseteq A$ and is unique with respect to this property. Suppose now that U is finite and contains a non-zero element. Then $A \neq 0$. Suppose that $A = A^-$. Then for each $u \in U$ we can choose $B_u \in \mathcal{S}$ with $u \in B_u$ and $B_u \subset A$. Since U is finite $\{B_u, u \in U\}$ has a maximal elemeent B. Then $U \subseteq B \subset A$, contradicting the minimality of A

Thus $A \neq A^-$ and by minimality of A, $U \nsubseteq A$.

(c) Follows from (b) applied to $U = \{m\}$. □

Lemma 5.7.4 [**series and basis**] *Let R be a ring, M a free R-module with basis $\mathcal{B}$ and $\mathcal{S}$ be an R-series on M. Then the following four statements are equivalent. one of the follwing holds:*

(a) [**a**] *For each $A \in \mathcal{S}$, $A \cap \mathcal{B}$ spans A over R.*

(b) [**b**] *For each $B \in \mathcal{S}$, $(a + B \mid a \in \mathcal{B} \setminus B\}$ is R-linear independent in V/B. Then*

(c) [**c**] *For each jump (B, A) of $\mathcal{S}$, $(a+B \mid a \in \mathcal{B} \cap A \setminus B\}$ is R-linear independent in A/B.*

(d) [**d**] *For all $A, B \in \mathcal{S}$ with $B \subseteq A$, $(a + B \mid a \in \mathcal{B} \cap A \setminus B\}$ is an basis R-basis for A/B.*

Proof: (a)$\Longrightarrow$ (b): $(r_a) \in \bigoplus_{a \in \mathcal{B}\setminus A} R$ with $\sum_{a \in \mathcal{B}\setminus A} r_a a \in B$. Then by (a) applied to B there exists $(r_a) \in \bigoplus_{a \in \mathcal{B}\cap A}$ with

$$\sum_{a \in \mathcal{B}\setminus A} r_a a = \sum_{a \in \mathcal{B}\cap A} r_a a$$

Since $\mathcal{B}$ is linearly independent over R this implies $r_a = 0$ for all $a \in \mathcal{B}$ and so (b) holds.
(b)$\Longrightarrow$ (c): Obvious.

(c)$\Longrightarrow$ (a): Let $a \in A$. Since $\mathcal{B}$ spans M over R there exists afinite subset $\mathcal{C}$ of $\mathcal{B}$ and $(r_c) \in \bigoplus_{\mathcal{C}} R^\sharp$ with $a = \sum_{c \in \mathcal{C}} r_c c$. Let $D \in \mathcal{S}$ by minimal with $\mathcal{C} \subseteq D$. Then (D^-, D) is a jump and $\mathcal{C} \setminus D^- \neq \emptyset$. Suppose that $D \nsubseteq A$. Since $\mathcal{S}$ is totally ordered, $A \subseteq D^-$. Thus

$$0_{D/D^-} = a + D^- = \sum_{c \in \mathcal{C}} r_c c + D^- = \sum_{c \in \mathcal{C}\setminus D^-} r_c c + D^-$$

a contradiction to (c).
(a)$\Longrightarrow$ (d): (a) implies that $(a + B \mid a \in \mathcal{A}\}$ and so also $(a + B \mid a \in \mathcal{A}\}$ spans A/B. Since (a) implies (b), $(a + B \mid a \in \mathcal{B} \setminus B\}$ and so also $(a + B \mid a \in \mathcal{B} \cap A \setminus B\}$ is R-linear independent. So (d) holds.
(d)$\Longrightarrow$ (a): Just apply (d) with $B = 0$. □

5.8 The Branching Theorem

Definition 5.8.1 [**def:removable node**] *Let λ be partion of n*

(a) [**a**] *A node $d \in [\lambda]$ is called* removable *if $[\lambda] \setminus \{d\}$ is a Ferrers diagram.*

(b) [**b**] *$d_i = (r_i, c_i), 1 \leq i \leq k$ are the the removable nodes of $[\lambda]$ ordered such that $r_1 < r_2 < \ldots < r_k$. $\lambda^{(i)} = \lambda([\lambda] \setminus \{d_i\}$ and $\lambda \downarrow = \{\lambda^{(i)} \mid 1 \leq i \leq k\}$*

(c) [**c**] *$e \in \mathbb{Z}^+ \to \mathbb{Z}^+$ is called an* exterior node *of $[\lambda$ if $D \cup \{e\}$ is a Ferrers diagram . $\lambda \uparrow$ is the set of partions of n obtained by extending $[\lambda]$ by an exterior node.*

Lemma 5.8.2 [**basic removable**] *Let λ be a partition of n and $(i,j) \in D$. Then the following are equivalent*

(a) [**a**] *(i,j) is a removable node of $[\lambda]$.*

(b) [**b**] *$j = \lambda_i$ and $\lambda_i > \lambda_{i+1}$.*

(c) [**c**] *$i = \lambda'_j$ and $\lambda'_j > \lambda'_{j+1}$.*

(d) [**d**] *$j = \lambda_i$ and $i = \lambda'_j$.*

Proof: Obvious. $\square$

Definition 5.8.3 [**def:restrictable**] *Let λ be partition of n and t be a λ-tableau. We say that t is restrictable if $t^{-1}(n)$ is a removable node of $[\lambda]$. In this case $t\,|_{t^{-1}(I_{n-1})}$ is denoted by $t\downarrow$. $\overline{\underline{t}}$ is called restrictable if $\overline{\underline{t}}$ contains a restrictable tableau s. In this case we define $\overline{\underline{t}}\downarrow = \overline{\underline{s\downarrow}}$*

Lemma 5.8.4 [**basic restrictable**] *Let λ be a partion of t. If t is restricable then $t\downarrow$ is a tableau. If t is standard then t is restrictable and $t\downarrow$ is standard. Let $\pi \in \mathrm{Sym}(n-1)$. Then t is restrictable iff πt is restrictible. In this case $(\pi t)\downarrow = \pi(t\downarrow)$. $\overline{\underline{t}}$ is restrictable iff $\pi\overline{\underline{t}}$ is restrictable In this case $(\pi\overline{\underline{t}})\downarrow = \pi(\overline{\underline{t}}t\downarrow)$.*

Proof: Obvious.

Theorem 5.8.5 [**restricting specht**] *Let λ be a partition of n. For $0 \leq i \leq k$ let V_i be the F-submodule of S^λ spanned by all e_t where t is a restrictable λ-tableau with n in one of the rows $r_1, r_2, \ldots r_i$. Then*

$$0 = V_0 < V_1 \ldots < V_{k-1} < V_k = S^\lambda$$

as a series of $F\mathrm{Sym}(n-1)$-submodules with factors $V_i/V_{i-1} \cong S^{\lambda^{(i)}}$.

Proof: Clearly the the set of restrictable λ tableaux with n in row r_i is invariant under the action of $\mathrm{Sym}(n-1)$. Thus each V_i is an $FSym(n-1)$ submodule of S^λ. Also clearly $V_{i-1} \leq V_i$ and it remains to show that $V_i/V_{i-1} \cong S^{\lambda^{(i)}}$. For this define and F-linear map

$$\theta_i : M^\lambda \to M^{\lambda^{(i)}}, \quad \overline{\underline{t}} \to \begin{cases} \overline{\underline{t}}\downarrow & \text{if } n \text{ is in row } r_i \text{ of } t \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Clearly θ_i commutes with the action of $\mathrm{Sym}(n-1)$ and so θ_i is $F\mathrm{Sym}(n-1)$ linear. Let n be a restrictable tableau with n in row r_j. Then for all $\pi \in C_t$ n is in a row less or equal to r_i, with equality iff π fixes n, that is if $\pi \in C_{t\downarrow}$. Thus

(2) $$\theta_i(e_t) = \begin{cases} e_{t\downarrow} & \text{if } j = i \\ 0 & \text{if } j < i \end{cases}$$

If s is a $\lambda^{(i)}$-tableau, then $s = t\downarrow$ for a (unique) restrictable λ tableau t with n in row r_i. Hence

(3) $$V_{i-1} \leq V_i \cap \ker\theta_i \quad \text{and} \quad V_i/V_i \cap \ker\theta_i \cong \operatorname{Im}\theta_i = S^{\lambda^{(i)}}$$

Let $\mathcal{B}$ be the set of standard λ-polytabloids and $\mathcal{B}_i$ the e_t with t standard and n in row r_i. Then by (1) $\theta_i(\mathcal{B}_i)$ is the standard basis for $S^{\lambda^{(i)}}$ and so is linear independently. Thus also the image of $\mathcal{B}_i$ in $V_i/V_i \ker\theta_i$ is linearly independent. Consider the series of F-modules

$$0 = V_0 \leq V_1 \cap \ker\theta_1 \leq V_1 \leq V_2 \cap \ker\theta_2 < V_2 < \ldots < V_{k-1} \leq V_k \cap \ker\theta_k < V_k < S^\lambda$$

Each $e_t \in \mathcal{B}$ lies in a unique $\mathcal{B}_i$ and so in $V_i \setminus (V_i \cap \ker\pi_i)$. Thus $\mathcal{B} \cap V_i \cap \ker\theta_i \subseteq V_{i-1}$. So we can apply 5.7.4 to the series of F-modules and conlcude that $V_i \cap \ker\theta_i/V_{i-1}$ is as the emptyset as an R-basis. Hence $V_{i-1} = V_i \cap \ker\theta_i$. For the same reason $V_k = S^\lambda$ and theorem now follows from (3). □

Theorem 5.8.6 (Branching Theorem) [**branching theorem**] *Let F be a field with* $\operatorname{char} F = 0$ *and λ a partition of n.*

(a) [**a**]
$$S^\lambda \downarrow_{\mathrm{Sym}(n-1)} = \bigoplus_{\mu\in\lambda\downarrow} S^\mu$$

(b) [**b**]
$$S^\lambda \uparrow^{\mathrm{Sym}(n-1)} = \bigoplus_{\mu\in\lambda\uparrow} S^\mu$$

Proof: (a) Follows from 5.8.5 and Maschke's Theorem 2.3.2
(b) Follows from (a) and Frobenius Reprocity 2.7.4.

5.9 $S^{(n-2,2)}$

In this section we investigate the Specht modules $S^{(n)}$, $S^{(n-1,1)}$ and $S^{n-2,2}$.

Lemma 5.9.1 [**s(n)**] $M^{(n)} = S^{(n)} \cong D^{(n)} \cong F$.

Proof: There there a unique (n)-tabloid $\underline{\bar{t}}$ and $\pi\underline{\bar{t}} = \underline{\bar{t}}$ for all $\pi \in Sym(n)$. Moreover $e_t = \underline{\bar{t}}$ and so $S^{(n)} = M^{(n)}$. Also $S^{(n)\perp} = 0$ and the lemma is proved. □

Lemma 5.9.2 [**s(n-1)**] *Let x_i the unique $(n-1,1)$-tabloid with i in row 2. Let $z = \sum_{i=1}^n x_i$ be the sum of all λ-tabloids. Then*

(a) [**a**] $S^{(n-1,1)} = \{\sum_{i=1}^n f_i x_i \mid f_i \in F, \sum_{i=1}^n f_i = 0.$

(b) [**b**] $S^{(n-1,1)\perp} = Fz.$

(c) [**c**] $S^{(n-1,1)\perp} \cap S^{(n-1,1)} = \{fx \mid f \in F, nf = 0\}.$

Proof: (a) If t is tableau with $t(1,1) = i$ and $t(2,1) = j$, then $e_t = x_i - x_j$. This easily implies (a).

(b) $\sum_{f_i z_i} \perp x_i - x_j$ iff $f_i = f_j$.

(c) Follows from (a) and (b). □

Corollary 5.9.3 [**dim d(n-1)**] *Let F be a field and $p = \operatorname{char} \mathbb{F}$.*

(a) [**a**] *If $p \nmid n$, then $S^{(n-1,1)} \cong D^{(n-1,1)}$ has dimension $n-1$ over D.*

(b) [**b**] *If $p \mid n$, then $D^{(n-1,1)}$ has dimension $n-2$ over F.*

Proof: Follows immediately from 5.9.2. □

To analyze $S^(n-2,2)$ we introduce the follwing notation: Let $n \in \mathbb{N}$ with $n \geq 4$ and $\lambda = (n-2,2)$. Let $\mathcal{P}$ be the set for subsets of size two in I_n. For $P \in P_n$ let x_P be the λ-partition $(P, I_n \setminus P)$. Then $(x_P \mid P \in \mathcal{P})$ is an F-basis for M^λ. For a,b,c,d pairwise distinct elements in I_n put $e_{ab|cd} = x_{ac} + x_{bd} - x_{ad} - x_{bc}$. So $e_{ab|cd} = e_t$ for any λ tableau of the form $\dfrac{\overline{a\,c\ldots}}{\underline{b\,d}}$.

For $i \in I_n$ define $x_i := \sum_{i \in P \in \mathcal{P}} x_P$ and $y_i = \sum_{i \notin P \in \mathcal{P}} x_P$. Also let $z = \sum_{P \in \mathcal{P}} x_P$ and observe that $x_i + y_i = z$ for all $i \in I$.

Lemma 5.9.4 [**basis for s(n-2,2)perp**]

(a) [**a**] $x_1, x_2, \ldots x_{n-1}, y_n$ *is an F-basis for $S^{\lambda\perp}$.*

(b) [**b**] $x_1, x_2, \ldots x_{n-1}, z$ *is an F-basis for $S^{\lambda\perp}$.*

(c) [**c**] $y_1, y_2, \ldots y_{n-1}, z$ *is an F-basis for $S^{\lambda\perp}$.*

(d) [**d**] *If 2 is invertible in F then $x_1, x_2, \ldots x_n$ is an F-basis for $S^{\lambda\perp}$.*

(e) [**e**] *If $n-2$ is invertible in F, then $y_1, y_2, \ldots y_n$ is an F-basis for $S^{\lambda\perp}$.*

Proof: (a) We will first show that $x_i \perp e_{ab|cd}$ for all appropriate i, a, b, c, d. If $i \notin \{a, b, c, d\}$, x_i and $e_{ab|cd}$ have do not share a tabloid and so $(x_i \mid e_{ab|cd}) = 0$. So suppose $i = a$, then x_i and $e_{ab|cd}$ share x_{ac} and x_{ad} with opposite signs and so again $x_i \perp e_{ab|cd}$. Clearly $z \perp e_{ab|cd}$ and so also $y_i \perp e_{ab|cd}$. Thus x_i, y_i and z are all contained in $S^{\lambda\perp}$.

Now let $a = \sum_{P\in\mathcal{P}} r_P x_P \in S^{\lambda\perp}$. We need to show that a is a unique F-linear combination of $x_1, x_2, \ldots x_{n-1}, y_n$. For $n \neq i \in I_n$, x_i is the only one involving x_{in}. So replacing a by $a - \sum_{i=1}^{n-1} r_{in} x_i$ we assume that $r_{in} = 0$ for all $i \neq n$. And we need to show that a is scalar multiple of y_n. That is we need to show that $r_{ij} = r_{kl}$ whenever $\{i, j\}, \{k, l\} \in \mathcal{P}$ with $n \notin \{i, j, k, l\}$. Suppose first that $P \cap Q \neq \emptyset$ and say $i = k$ and withoutloss $j \neq l$. Since $a \in S^{\lambda\perp}$, $a \perp e_{in|jl}$. Thus $r_{ij} + r_{nl} - r_{il} - r_{nj} = 0$. By assumption $r_{nl} = r_{nj} = 0$ and so $r_{ij} = r_{il} = r_{kl}$. In the geneal case we conclude $r_{ij} = r_{ik} = r_{kl}$ and (a) is proved.

(b) Observe that $z = \sum_{i=1}^{n-1} x_i - y_n$. Thus (b) follows from (a).

(c) Since $y_i = z - x_i$ this follows from (b).

(d) Observe that $\sum_{i=1}^{n} x_i = 2z$ and so $x_n = -\sum_{i=1}^{n-1} x_i + 2z$. So (d) follows from (b).

(e) We have $\sum_{i=1}^{n} y_i = \sum_{i=1}^{n}(z - x_i) = nz - \sum_{i=1}^{n} x_i = (n-2)z$. So $y_n = -\sum_{i=1}^{n-1} y_i + (n-2)z$ and (e) follows from (c). □

It might be interesting to observe that $y_1, \ldots, y_{n-1}, x_n$ is only a basis if $n-2$ is invertible. Indeed $x_n = -\sum_{i=1}^{n-1} x_i + 2z = \sum_{i=1}^{n-1}(y_i - z) + 2z = \sum_{i=1} y_i + (n-2)z$.

We know proceed to compute $S^\lambda \cap S^{\lambda\perp}$ if F is a field.

Lemma 5.9.5 [**s(n-2) cap s(n-2)perp**] *Suppose F is field and put $p = \operatorname{char} F$.*

(a) [**a**] *Suppose $p = 0$ or p is odd and $n \not\equiv 1, 2 \mod p$ or $p = 2$ and $n \equiv 3 \mod 4$. Then n $S^\lambda \cap S^{\lambda\perp} = 0$.*

(b) [**b**] *Suppose p is odd and $n \equiv 1 \mod p$ or $p = 2$, $n \equiv 1 \mod 4$. Then $S^\lambda \cap S^{\lambda\perp} = Fz$.*

(c) [**c**] *Suppose p is odd and $n \equiv 2 \mod p$ or $p = 2$ and $n \equiv 2 \mod 4$, then $S^\lambda \cap S^{\lambda\perp} = \langle F y_i \mid 1 \leq i \leq n\rangle$ and $\sum_{i=1}^{n} y_i = 0$.*

(d) [**d**] *Suppose $p = 2$ and $n \equiv 0 \mod 4$. Then $S^\lambda \cap S^{\lambda\perp} = \langle F y_i y_j \mid 1 \leq i < j \leq n\rangle$ and $\sum_{i=1}^{n} y_n = 0$.*

Proof: Since F is a field and $(\cdot \mid \cdot)$ is non-degenerate, $S^{\lambda\perp\perp} = S^\lambda$ and so $S^\lambda \cap S^{\lambda\perp} = S^{\lambda\perp\perp} \cap S^{\lambda\perp}$ is the radical of the restriction of $(\cdot \mid \cdot)$ to S^λ.

By 5.9.4 $y_1, y_2 \ldots y_{n-1} z$ is basis for $S^{\lambda\perp}$. Let $a = r_0 z + \sum_{i=1}^{n-1} r_i y_i$. Then

Observe that

$$
\begin{aligned}
(y_i \mid y_i) &= \tbinom{n-1}{2} \\
(y_i \mid y_j) &= \tbinom{n-2}{2}\ i \neq j \\
(y_i \mid z) &= \tbinom{n-1}{2} \\
(z \mid z) &= \tbinom{n}{2}
\end{aligned}
$$

So $(a \mid y_j) = r_0\binom{n-1}{2} + r_j\binom{n-1}{2} + \sum_{i\neq j=1}^{n-1} r_i\binom{n-2}{2}$. Put $r = \sum_{i=1}^{n-1} r_i$. Since $\binom{n-1}{2} - \binom{n-2}{2} = \binom{n-2}{1} = n-1$ we conclude $a \in S^\lambda$ if and only if

(1) $$(a \mid y_j) = \binom{n-1}{2} r_0 + (n-2)r_j + \binom{n-2}{2} r = 0 \forall 1 \leq j < n$$

and

(2) $$(a \mid z) = r_0\binom{n}{2} + r\binom{n-1}{2} = 0$$

.

Sustracting (1) for two diffrent values of for j gives

(3) $$(n-2)r_j = (n-2)r_k \forall 1 \leq j < k \leq n-1$$

and so

(4) $$(n-2)r = (n-1)(n-2)r_j$$

Substracting (2) from (1) gives

(5) $$(n-1)r_0 + (n-2)r_j = (n-2)r$$

and using (4)

(6) $$(n-1)r_0 = (n-2)^2 r_j$$

Note also that (1) and (2) are equivalent to (2),(3) and (6).

Suppose first that $n-2 = 0$ in F. Then $\sum_{i=1}^n y_n = (n-2)z = 0$ and $\langle y_i \mid 1 \leq i \leq n\rangle_F = \langle y_i \mid 1 \leq i \leq n-1\rangle_F$ and

Also $n-1 \neq 0$. So (3) and (6) hold if and only if $r_0 = 0$. If $p \neq 2$ or $p = 2$ and $n \equiv 2 \mod 4$, then also $\binom{n-1}{2} = 0$ in F and so also (6) holds. Thus (c) holds in this case. If $p = 2$ and $n \equiv 0 \mod 4$, then $\binom{n-1}{2} = 1$ and so (6) holds if and only if $r = 0$. Observe also that $\sum_{i=1}^n y_i = 0$ and n even implies $\langle y_i + y_j \mid 1 \leq i < j \leq n\rangle_F = \langle y_i + y_j \mid 1 \leq i < j \leq n-1\rangle_F$ and so (d) holds.

Suppose next that $n-2 \neq 0$ in F. Then (3) just says $r_j = r_k$. Assume that $n-1 = 0$ in $\mathbb{F}$. Then (6) holds iff $r_j = 0$ for all j. Hence (2) says $r_0\binom{n}{2}r = 0$. If $p \neq 2$ or $p = 2$ and $n \equiv 1 \mod 4$, $\binom{n}{2} = 0$ and (b) holds. If $p = 2$ and $n \equiv 3 \pmod 4$, then $\binom{n}{2} = 1$. So $r_0 = 1$ and (a) holds.

Assume next that $n-1 \neq 0$ and so $p \neq 2$. Multipying (2) with $\frac{2}{n-1}$ gives $nr_0 = -(n-2)r$. Adding to (5) gives $r_0 = 0$. So also $0 = (n-2)r = (n-2)(n-1)r_j$ and $r_j = 0$. Thus (a) holds. $\square$

Corollary 5.9.6 [**dimension of d(n-2,2)**] *Suppose F is a field, then* $\dim_F S^{(n-2,2)} = \frac{n(n-3)}{2}$ *Moreover,*

(a) [**a**] *Suppose $p=0$ or p is odd and $n \not\equiv 1,2 \mod p$ or $p=2$ and $n \equiv 3 \mod 4$. Then* $\dim_F D^{(n-2,2)} = \frac{n(n-3)}{2}$.

(b) [**b**] *Suppose p is odd and $n \equiv 1 \mod p$ or $p=2$, $n \equiv 1 \mod 4$. Then* $\dim_F D^{(n-2,2)} = \frac{n(n-3)}{2} - 1$

(c) [**c**] *Suppose p is odd and $n \equiv 2 \mod p$ or $p=2$ and $n \equiv 2 \mod 4$. Then* $\dim_F D^{(n-2,2)} = \frac{(n-1)(n-4)}{2} - 1$.

(d) [**d**] *Suppose $p=2$ and $n \equiv 0 \mod 4$. Then* $\dim_F D^{(n-2,2)} = \frac{(n-1)(n-4)}{2}$.

Proof: Since $\dim D^\lambda = \dim S^\lambda - \dim(S^\lambda \cap S^{\lambda\perp})$, this follows from 5.9.5 and some simple calculations. □

Definition 5.9.7 [**def:shape**] *Let M be an R-module.*

(a) [**a**] *A shape of height n of M is inductively defined as follows:*

- *(i)* [**i**] *A shape of height 1 of M is any R-module isomorphic to M.*
- *(ii)* [**ii**] *A shape of height h of M is one of the following.*
  - *(a)* [**1**] *A triple $(A, \oplus, B)$ such that there exists R-submodules X, Y of M with $M = X \oplus Y$ such that A is a shape of height i of X, B is a shape of height j of Y and $k = i + j$.*
  - *(b)* [**2**] *A triple $(A, |, B)$ such that there exists R-submodules X of Y such that A is shape of height i of X, B is a shape of height j of M/X and $k = i + j$.*

(b) [**b**] *If $M \sim S$ means that S is a shape of M. A shape $(A, \oplus, B)$ as in (a:ii:a) is denoted by $A \oplus B$. A shape $(A, |, B)$ as in (a:ii:a) is denoted by $A \mid B$ or $\dfrac{A}{B}$.*

(c) [**c**] *A factor of a S shape of M is incuctively defined as follows: If S has height 1, then S itseld the only fcator of S. If $S = A \mid B$ or $S = A \oplus B$, then any factor of A or B is a factor of S.*

(d) [**d**] *A simple shape of M is a shape all of its factors are simple.*

Observe that if $M \sim A \mid (B \mid C$ then also $M \sim (A \mid B) \mid C$ and we just write $M \mid A \mid B \mid C$. Similar $M \sim (A \oplus B \oplus C)$ means $M \sim (A \oplus B) \oplus C$ and equally well $A \oplus B (\oplus C)$. We also have $M \sim A \oplus B$ iff $M \sim B \oplus A$. But $M \sim A \mid B$ does not imply $M \sim B \mid A$. We have $M \sim A \oplus (B \mid C)$ implies $M \mid (A \oplus B) \mid C$ and $M \sim B \mid (A \oplus C)$. But $M \sim (A \oplus B) \mid C$ does not imply $M \sim A \oplus (B \sim C)$.

For example if F is a field with char $F = p$ then by 5.9.2 $M^{(n-1,1)} \sim D^{(n)} \oplus D^{(n-1,1)}$ if $p \nmid n$ and $M^{(n-1,1)\sim D^{(n}} \mid D^{(n-1,1)} \mid D(n)$ if $p \mid n$.

If might also be worthwhile to define the following binary operation on classes of R-modules. If A, B are classes of R-modules, then $A \oplus B$ denotes the set of all R-modules M such that $M \cong X \oplus Y$ with $X \in A$ and $Y \in B$. $A \mid B$ is the class of all R-modules M such that M has an R-submodule X with $X \in A$ and $M/X \in B$. A shape of M then can be interpreted as a class of R-modules containing M obtained form the isomorphism classes of R modules and repeated application of the operations $\oplus$ and $|$.

To improve readabilty we write $D(a, b, c \dots)$ for $D^{(a,b,c,\dots)}$ in the next lemma.

Corollary 5.9.8 [**shape of m(n-2,2)**] *Suppose F is a field. Then $D^{(n-2,2)}$ has simply shapes as follows:*

(a) [**a**] *Suppose $p = 0$ or p is odd and $n \not\equiv 0, 1, 2 \mod p$ or $p = 2$ and $n \equiv 3 \mod 4$. Then*

$$M^{(n-2,2)} \sim D(n-2,2) \oplus D(n-1,1) \oplus D(n)$$

(b) [**b**] *Supose $p \neq 0, 2$ and $n \equiv 0 \mod p$. Then*

$$M^{(n-2,2)} \sim D(n-2,2) \quad \oplus \quad \begin{array}{c} D(n) \\ \hline D(n-1,1) \\ \hline D(n) \end{array}$$

(c) [**c**] *Suppose p is odd and $n \equiv 1 \mod p$ or $p = 2$, $n \equiv 1 \mod 4$. Then*

$$M^{(n-2,2)} \sim \begin{array}{c} D(n) \\ \hline D(n-2,2) \\ \hline D(n) \end{array} \quad \oplus \quad D(n-1,1)$$

(d) [**d**] *Suppose p is odd and $n \equiv 2 \mod p$. Then*

$$M^{(n-2,2)} \sim \begin{array}{c} D(n-1,1) \\ \hline D(n-2,2) \\ \hline D(n-1,1) \end{array} \quad \oplus \quad D(1)$$

(e) [**e**] *Suppose $p = 2$ and $n \equiv 2 \mod 4$. Then*

$$M^{(n-2,2)} \sim \begin{array}{c} D(n-1,1) \\ \hline D(n) \\ \hline D(n-2,2) \\ \hline D(n) \\ \hline D(n-1,1) \end{array} \quad \oplus \quad D(1)$$

(f) [**f**] *Suppose* $p = 2$ *and* $n \equiv 0 \mod 4$. *Then*

$$M^{(n-2,2)} \sim \frac{\dfrac{D(n-1,1) \oplus D(n)}{D(n-2,2)}}{D(n-1,1) \oplus D(n)}$$

Proof: This is straighforward from 5.9.5. As an example we consider the case $p = 2$ and $n \equiv 2 \pmod 4$. Observe that $(z \mid z) = \binom{n}{2} \neq 0$ and so $M^\lambda = \mathbb{F}z$. Thus $M^\lambda \sim D(n) \oplus z \perp$, and the restrition of $(\cdot \mid \cdot)$ to $z^\perp$ is a non-degenerate.

5.9.5 $B := S^\lambda \cap S^{\lambda\perp} = \langle y_i \mid 1 \leq 1 \leq n \rangle$. So B has the submodule, $A = \langle y_i y_j \mid 1 \leq u < j \leq n \rangle$. Since $\sum_{i=1}^n y_i = 0$, $B \cong D(n-1,1)$. Since n is even, $A/B \neq 1$ and $A/B \cong D(n)$. $S^\lambda/A = D^\lambda = D(n-2,2)$. Since $S^{\lambda\perp} = A + Fz$, $S^\lambda = z^\perp \cap A^\perp$. So $z^\perp \cap B^\perp / S^\lambda \cong (A/B)^* \cong D(n)^* \cong D(n)$. Moreover, $z^\perp / z^\perp \cap A^\perp \cong A^* \cong D(n-1,1)^* \cong D(n-1,1)$. Thus (e) holds. □

5.10 The dual of a Specht module

Definition 5.10.1 [**def:twisted module**] *Let* R *be a ring,* G *a group ,* M *an* RG*-module and* $\epsilon : G \to Z(R)^\sharp$ *a multiplicative homomoprhism. Then* M_ϵ *is the* RG*-module which is equal to* M *as an* R*-module and* $g \cdot_\epsilon m = \epsilon(g) gm$ *for all* $g \in G, m \in M$.

Note that this definition is consistent with our definition of the RG-module R_ϵ.

Proposition 5.10.2 [**slambdaprime**] *Let* λ *be a partion of* n. *Then*

$$S^{\lambda *} \cong M^\lambda / S^{\lambda\perp} \cong S^{\lambda'}_{\mathrm{sgn}}$$

as $F\mathrm{Sym}(n)$*-module.*

Proof: Fix a λ tableau s. Let $\pi \in R_s = C_G(\underline{s})$. Since $R_s = C_{s'}$, 5.3.4(e) gives $\pi e_{s'} = \mathrm{sgn}\pi e_{s'} = \pi \cdot_{\mathrm{sgn}} e_{s'}$. Hence there exists a unique $F\mathrm{Sym}(n)$-linear homorphism

$$\alpha_s : M^\lambda \to M^{\lambda'} \text{ with } \underline{s} \to e_{s'} \tag{1}$$

Let t be any λ-tabloids. Then the exists $\pi \in \mathrm{Sym} n$ with $\pi s = t$ (namely $\pi = ts^{-1}$) and so

$$\alpha_s(\underline{t}) \alpha_s(\underline{\pi s}) = \pi \cdot_{\mathrm{sgn}} e_{s'} = \mathrm{sgn}(\pi) e_{\pi s'} = \mathrm{sgn}(ts^{-1}) e_{t'}$$

that is

$$\alpha_s(\underline{t}) = \mathrm{sgn}(ts^{-1}) e_{t'} \tag{2}$$

Observe that (2) implies

$$\operatorname{Im}\alpha_s = S^{\lambda'} \tag{3}$$

Since $\lambda'' = \lambda$ we also obtain a unique $F\mathrm{Sym}(n-1)$ linear map

$$\alpha_{s'} : M^\lambda \to M^\lambda, \underline{\overline{t'}} \to \mathrm{sgn}(ts^{-1})e_t \tag{4}$$

Then

$$\operatorname{Im}\alpha_{s'} = S^\lambda \tag{5}$$

We claim that $\alpha_{s'}$ is the adjoint of α_s. That is

$$(\alpha_s(\underline{\overline{t}}) \mid \underline{\overline{r'}}) = (\underline{\overline{t}} \mid \alpha_{s'}(t))\underline{\overline{r}} \tag{6}$$

for all λ-tableaux t, r.

Indeed suppose that $\underline{\overline{r'}}$ is involved in involved in $\alpha_s(\underline{\overline{t}}) = \mathrm{sgn}ts^{-1}e_{t'}$. Then there exists $\beta \in C_{t'}$ with $\underline{\overline{r'}} = \underline{\overline{\beta t'}}$ and so there exists $\delta \in R_{r'}$ with $\delta r' = \beta t'$. Moreover

$$(\alpha_s(\underline{\overline{t}}) \mid \underline{\overline{r'}}) = \mathrm{sgn}(ts^{-1})\mathrm{sgn}\beta$$

Observe that $\delta \in C_r$ and $\beta \in R_t$. Thus $\underline{\overline{t}} = \underline{\overline{\beta t}} = \underline{\overline{\delta r}}$ and so $\underline{\overline{t}}$ is involved in e_r and

$$(\underline{\overline{t}} \mid \alpha_{s'}(\underline{\overline{r'}})) = \mathrm{sgn}(rs^{-1})\mathrm{sgn}\delta$$

$\delta r = \beta t$ implies $\delta rs^{-1} = \beta ts^{-1}$ and so

$$\mathrm{sgn}(rs^{-1})\mathrm{sgn}\delta = \mathrm{sgn}(ts^{-1})\mathrm{sgn}\beta$$

and so (6) holds.

Let $m \in M^\lambda$. $(\cdot \mid \cdot)$ is non-degenereate, (6) implies $\alpha_s(m) = 0$ iff $(\alpha_s(m) \mid m') = 0$ for all $m' \in M^{\lambda'}$ iff $(m \mid \alpha_{s'}(m')) = 0$ and iff $m \in (\operatorname{Im}\alpha_{s'})^\perp$. So by (5) $\ker\alpha_s = S^{\lambda^\perp}$ and so

$$M^\lambda / S^{\lambda\perp} \cong M^\lambda / \ker\alpha_s \cong \operatorname{Im}\alpha_s = S^\lambda$$

$\square$

Lemma 5.10.3 [**tensor and twist**] *Let R be a ring, G a group , M an RG-module and $\epsilon : G \to Z(R)^\sharp$ a multiplicative homomoprhism. Then*

$$M_\epsilon \cong R_\epsilon \otimes_R M$$

as an RG-module.

Proof: Observe first that there exists an R-isomorphism $\alpha : R_\epsilon \otimes_R M \to M$ with $r \otimes m \to rm$. Moreover, if $g \in G, r \in R$ and $m \in M$ then

$$\begin{aligned} \alpha(g(r \otimes m) = \alpha(g \cdot_\epsilon r \otimes gm) &= \alpha(\epsilon(g)r) \otimes gm \\ = \epsilon(g)rgm = & \epsilon(g)grm \\ = g \cdot_\epsilon rm &= g \cdot_\epsilon \alpha(r \otimes m) \end{aligned}$$

and so α is an RG-ismomorphism. □

Corollary 5.10.4 [**slambdaprime II**]

(a) [**a**] $S^{(1^n)} \cong F_{\text{sgn}}$.

(b) [**b**] *Let λ be a partition of n. Then $S^{\lambda *} \cong S(1^n) \otimes S^{\lambda'}$*

Proof: (a) By 5.9.1 $S^{(n)} \cong F$ and so by 5.10.2 $F \cong F^* \cong S^{(n)*} \cong S_{\text{sgn}}^{(n)'} = S_{\text{sgn}}^{(1^n)}$.
(b) $S^{\lambda *} \cong S_{\text{sgn}}^{\lambda'} \cong F_\epsilon \otimes S^{\lambda'} \cong S^{(1^n)} \otimes S^{\lambda'}$. □

Chapter 6

Brauer Characters

6.1 Brauer Characters

Let p be a fixed prime. Let $\mathbb{A}$ be the ring of algebraic integers in $\mathbb{C}$. Let I be an maximal ideal in $\mathbb{A}$ containing $p\mathbb{A}$ and put $\mathbb{F} = \mathbb{A}/I$. Then $\mathbb{F}$ is a field with with $\operatorname{char} \mathbb{F} = p$.

$$^* : \mathbb{A} \to \mathbb{F}, a \to a + I$$

be the correspoding ring homorphism.

Let $\tilde{\mathbb{A}}$ be the localization of $\mathbb{A}$ with respect to the maximal ideal I, that is $\tilde{\mathbb{A}} = \{\frac{a}{b} \mid a \in \mathbb{A}, b \in \mathbb{A} \setminus I$. Observe that * extends to a homomorphism

$$^* : \tilde{\mathbb{A}} \to \mathbb{F}, \frac{a}{b} \to a^*(b^*)^{-1}$$

In particular $\tilde{I} := \ker * = \{\frac{a}{b} \mid a \in I, b \in \mathbb{A} \setminus I\}$ is an maximal ideal in $\tilde{\mathbb{A}}$, $\tilde{\mathbb{A}}/\tilde{I} \cong \mathbb{F}$ and is the kernel of the homomorphism $\tilde{I} \cap \mathbb{A} = I$. Let U be the set of elements of finite p'-order in $\mathbb{A}^\sharp$.

Lemma 6.1.1 [**f=fpbar**]

(a) [**a**] *The restriction $U \to \mathbb{F}^\sharp, u \to u^*$ is an isomorphism of multiplicative groups.*

(b) [**b**] *$\mathbb{F}$ is an algebraic closure of its prime field $\mathbb{Z}^* \cong \mathbb{F}_p$.*

Proof: Let $u \in U$ and m the multiplicative order of u. Then

$$\sum_{i=0}^{m-1} x^i = \frac{x^m - 1}{x - 1} = \prod_{i=1}^{m-1} (x - u^i)$$

Substituting 1 for x we see that $1 - u$ divided m in $\mathbb{A}$. Thus $1 - u^*$ divides m^* in $\mathbb{F}$. Since $p \nmid 0$ and $\operatorname{char} F = p$, $m^* \neq 0$ and so also $1 - u^* \neq 0$. Thus * is 1-1 on U.

If $a \in \mathbb{A}$ then $f(a) = 0$ for some monic $f \in \mathbb{Z}[x]$. Then also $f^*(a) = 0$ and $f^* \neq 0$. So a^* is algebraic over $\mathbb{Z}^*$. Let $\mathbb{K}$ be an algebraic closure of $\mathbb{F}$ and so of $\mathbb{Z}^*$. Let $0 \neq k \in \mathbb{K}$. Then $k^m = 1$ where $m = |\mathbb{Z}^*[k]| - 1$ is coprime to p. Since U^* contains all m roots of $x^m - 1$ we get $k \in U^*$. Thus $\mathbb{K}^* \subseteq U^* \subseteq \mathbb{F}^* \subseteq \mathbb{K}^*$ and the lemma is proved. □

Definition 6.1.2 [**def:brauer character**] *Let G be a finite group and M an $\mathbb{F}G$-module. $\tilde{G}$ is the set of p-regular elements in G. Let $g \in \tilde{G}$ and choose $\xi_1, \dots \xi_n \in U$ such that $\eta_M(g) = \prod_{i=1}^n (x - \xi_i^*)$, where $\eta_M(g)$ is the characteristic polynomial of g on M. Put $\phi_M(g) = \sum_{i=1}^n \xi_i$. Then the function*

$$\phi_M : \tilde{G} \to \mathbb{A}, g \to \phi_M(g)$$

is called the Brauer character *of G with respect to M.*

Recall that if $H \subseteq G$ then we view RH as R an an R-submodule of RG. Also note that $\phi_M = \sum_{g \in \tilde{G}} \phi_M(g)g \in \mathbb{A}\tilde{G} \subseteq \mathbb{A}G$. Observe also that 1_{G° is the Brauer character of the trivial module $\mathbb{F}_G$.

Lemma 6.1.3 [**basic brauer**] *Let M be a G-module.*

(a) [**a**] *ϕ_M is a class function.*

(b) [**b**] *$\overline{\phi}_M(g) = \phi_M(g^{-1})$.*

(c) [**c**] *$\overline{\phi}_M = \phi_{M^*}$.*

(d) [**d**] *If $H \leq G$ then $\phi\mid_H = \phi_{M|_H}$.*

(e) [**e**] *$\mathcal{F}$ be the sets of factors of some $\mathbb{F}G$-series on M. Then*

$$\phi_M = \sum_{F \in \mathcal{F}} \phi_F$$

Proof: Readily verified. See 3.2.8. □

Definition 6.1.4 [**def tilde a**]

(a) [**a**] *For $g \in G$ let $g_p, g_{p'}$ be defined by $g_p, g_{p'} \in \langle g \rangle$, $g = g_p g_{p'}$, g_p is a p- and $g_{p'}$ is a p'-element.*

(b) [**b**] *For $a = \sum_{g \in G} a_g g \in \mathbb{C}G$, $\tilde{a} = a\mid_{\tilde{G}} = \sum_{g \in \tilde{G}} a_g g$.*

(c) [**c**] *For $a = \mathbb{C}\tilde{G}$ define $\check{a} \in \mathbb{C}G$ by $\check{a}(g) = a(g_{p'}$.*

Recall that $\chi_M(g) = \mathrm{tr}_M(g)$ is the trace of g on M.

Lemma 6.1.5 [**brauer and trace**] *Let M be a $\mathbb{F}G$-module. Then $(\check{\phi}_M)^* = \chi_M$.*

Proof: Let $W_i, 1 \leq i \leq n$ be the factors of an $\mathbb{F}\langle g\rangle$ composition series on M. Then since $\mathbb{F}$ is algebraically closed, W_i is 1-dimensionaly and g acts as a scalar μ_i on W_i. Since $\mathbb{F}$ contains no non-trivially p-root of unity g_p acts trivially on W_i and so also $g_{p'}$ acts as μ_i on W_i. Pick $\xi_i \in U$ with $\xi_i^* = \mu_i$. Then

$$\check{\phi}_M(g) = \phi_M(g_{p'}) = \sum_{i=1}^{n} \xi_i$$

and so

$$(\check{\phi}_M(g))^* = \sum_{i=1}^{n} \mu_i = \chi_M(g)$$

□

Let $\mathcal{S}_p$ be a set of representatives for the simple $\mathbb{F}G$-modules.

6.2 Algebraic integers

Definition 6.2.1 [**def:tracekf**] *Let $\mathbb{F} : \mathbb{K}$ be a finite separable field extension and $\mathbb{E}$ a splitting field of $\mathbb{F}$ over $\mathbb{K}$. Let Σ be set of $\mathbb{F}$-linear monomorphism from $\mathbb{F}$ to $\mathbb{K}$.*

$$\mathrm{tr} = \mathrm{tr}_{\mathbb{K}}^{\mathbb{F}} : \mathbb{F} \to \mathbb{K} \mid f \to \sum_{\sigma \in \Sigma} \sigma(f)$$

Lemma 6.2.2 [**basic tracekf**] *Let $\mathbb{F} : \mathbb{K}$ be a finite separable field extension. Then $s : \mathbb{F} \times \mathbb{F} \to \mathbb{K}, (a, b) \to \mathrm{tr}(ab)$ is a non-degenerate symmetric $\mathbb{K}$-bilinear form.*

Proof: Clearly s is$\mathbb{K}$-bilinear and symmetric. Suppose that $a \neq f \in \mathbb{F}^{\perp}$. Then $\mathrm{tr}(ab) = 0$ for all $b \in \mathbb{F}$ and since $a \neq o$, $\mathrm{tr}(f) = 0$ for all $f \in F$. Thus $\sum_{\sigma \in \Sigma} \sigma$, contradiction the linear idependence of filed monomorphism [Gr, III.2.4].

Corollary 6.2.3 [**trace dual basis**] *Let $\mathbb{F} : \mathbb{K}$ be a finite separable field extension and $\mathcal{B}$ a $\mathbb{K}$ basis for $\mathbb{F}$. Then $b \in \mathcal{B}$ there exists a unique $\tilde{b} \in \mathbb{F}$ with $\mathrm{tr}(a\tilde{b}) = \delta_{ab}$ for all $ab \in \mathbb{F}$.*

Proof: 6.2.2 and 4.1.8. □

Definition 6.2.4 [**def:integral**] *Let S be a commutative ring and R a subring.*

(a) [**a**] *$a \in R$ is called* integral *over S if there exists a monic $f \in S[x]$ with $f(a) = 0$.*

(b) [**b**] *$\overline{Int}_S(R)$ is the set of elements in S intgeral over R.*

(c) [**c**] *R is* integrally closed *in S if* $\mathrm{Int}_R(S)$*.*

(d) [**d**] *If Ris an integral domain, then R is called integrall closed if R is integraly closed in its field of fractions $\mathbb{F}_R$.*

Lemma 6.2.5 [**basic integral**] *Let S be a commutative ring, R a subring and $a \in S$. Then the following are equivalent:*

(a) [**a**] *a is integral over S.*

(b) [**b**] *$R[a]$ is finitely generated S-submodule of R.*

(c) [**c**] *There exists a faithful, finitely R-generated $R[a]$ module M*

Proof: (a)$\Longrightarrow$ (b): Let $f \in R[x]$ be monic with $f(a) = 0$. Then $a^n \in R\langle 1, \ldots, a^{n-1}\rangle$ and so $R[a] = R\langle 1, a, \ldots, a^{n-1}\rangle$ is finitely R-generated.

(a)$\Longrightarrow$ (b): Take $M = R[a]$.

(b)$\Longrightarrow$ (c): Let $\mathcal{B} \subseteq M$ be finite with $M = R\mathcal{B}$. Choose a matrix $D = (d_{ij}) \in \mathrm{M}_{\mathcal{B}}(R)$ with $ai = \sum_{i \in \mathcal{B}} d_{ij} j$ for all $i \in \mathcal{B}$. Let f be the characteristic polynomial of D. Then $f \in R[x]$ and f is monic. By Cayley-Hamilton [La, XV Theorem 8] $f(D) = 0$. Since $f(a)i = \sum_{j \in \mathcal{B}} f(D)_{ij} j$ for all $i \in I$ we get $f(a)M = 0$. Since $\mathrm{A}_R(M) = 0$ we have $f(a) = 0$. □

Lemma 6.2.6 [**integral closure**] *Let S be a commutative ring and R a subring of S.*

(a) [**a**] *Let $a \in S$. If a is integral over R, then also $R[a]$ is integral over R.*

(b) [**b**] *Let T be a subring of S with $R \subseteq T$. Then S is integral over R iff T is integral over R and S is integral over T.*

(c) [**c**] $\mathrm{Int}_S(R)$ *is a subring of R and* $\mathrm{Int}_R(S)$ *is integrally closed in S.*

Proof: (a) Let $b \in R[a]$. By 6.2.5(b), $R[a]$ is finitely R-generated. Since $R[a]$ is a faithful $R[b]$-module, 6.2.5(c) implies that b is integral over R.

(b) One direction is obvious. So suppose $S : T$ and $T : R$ are integral and let $a \in S$. Let $f = sum_{i=1}^n t_i x^i \in T[x]$ be monic with $f(a) = 0$. Put $R_0 = R$ and inductively $R_i = R_{i-1}[a_i]$. Then a_i is integral over R_{i-1}, R_i is finitely R_{i-1}-generated. Also $f \in R_n[x]$ and so $R_n[a]$ is finitely R_n-generated. It follows that $R_n[a]$ is finitely R-generated and so by 6.2.5(c), a is integral over R.

(c) Let $a, b \in \mathrm{Int}_S(R)$. By (a) $R[a] : R$ and $R[a,b] : R[a]$ are integral. So by (b) $R[a,b] : R$ is integral and so $R[a,b] \subseteq \mathrm{Int}_S(R)$ and $\mathrm{Int}_S(R)$ is a subring. Since both $\mathrm{Int}_S(\mathrm{Int}_S(R) : \mathrm{Int}_S(R)$ and $\mathrm{Int}_S(R)$ are integral, (b) implies that $\mathrm{Int}_S(R)$ is integrally closed in R. □

Lemma 6.2.7 [**f integral**] *Let R be a integral domain with field of fraction F and let K be a field extension of F. Let $a \in F$ be integral over R and f the minimal polynomial of a over $\mathbb{F}$.*

(a) [**a**] *All coefficents of f are integral over R.*

(b) [**b**] *If $\mathbb{K} : \mathbb{F}$ is finite seperable, then $\operatorname{tr}(a)$ is integral over R.*

Proof: (a) Let $\mathcal{A}$ be the set of roots of f in some splitting of f over $\mathbb{K}$. Alos let $g \in R[x]$ be monic with $f(a) = 0$. Then $f \mid g$ in $\mathbb{F}[x]$ and so $f(b) = 0$ for all $b \in \mathcal{A}$. Thus $\mathcal{A}$ is integral over R. Since $f \in R[\mathcal{A}][x]$, (a) holds.

(b) Let Σ be the set of monomorphism from $\mathbb{K}$ to the splitting field of $\mathbb{K}$ over $0\mathbb{F}$. Then each $\sigma(a), \sigma \in \Sigma$ is a root of f. Thus $\operatorname{tr}a = \prod_{\sigma\in\Sigma} \sigma(a) \in R[\mathcal{A}]$. □

Lemma 6.2.8 [**k=int/r**]*Suppose R is an integral domain with field of fraction $\mathbb{F}$. Let $\mathbb{K}$ be an algebraic field extension of $\mathbb{F}$. Then $\mathbb{K} = \{\frac{i}{r} \mid i \in \operatorname{Int}_{\mathbb{K}}(R), r \in R^\sharp\}$. In particular, $\mathbb{K}$ is the field of fraction of $\operatorname{Int}_R(S)$.*

Proof: Let $k \in \mathbb{K}$. Then ther exists a non-zero $f \in \mathbb{F}[x]$ with $f(k) = 0$. Multitiplying f with the product of the denominatos of its coeeficents we may assume that $f \in R[x]$. Let $f = \sum_{i=0}^n a_i x_i$ with $a_n \neq 0$. Put $g(x) = a_n^{n-1} f(\frac{x}{a_n}) = \sum_{i=0}^n a_i a^{n-1-i} x^i$. Then $g \in R[x]$, g is monic and $g(a_n k) = a_n^{n-1} f(k) = 0$. Thus $a_n k \in \operatorname{Int}_{\mathbb{K}}(R)$ and $k = \frac{a_n k}{k}$. □

Definition 6.2.9 [**def:lattice**] *Let R be a ring, S a subring of R, M an R-module and L an S-module of M. Then L is called a $R : S$-lattice for M provided that there exists an S-basis $\mathcal{B}$ for L such that $\mathcal{B}$ is also an R-basis for M.*

Lemma 6.2.10 [**intfr noetherian**] *Suppose R is an integral domain with field of fraction $\mathbb{F}$. Let $\mathbb{K}$ be a finite seperable extension of $\mathbb{F}$.*

(a) [**a**] *There exists an $\mathbb{F} : R$-lattice in $\mathbb{K}$ containing $\operatorname{Int}_{\mathbb{K}}(R)$.*

(b) [**b**] *If R is Noetherian, so is $\operatorname{Int}_{\mathbb{K}}(R)$.*

(c) [**c**] *If R is a PID, $\operatorname{Int}_{\mathbb{K}}(R)$ is an $\mathbb{F} : R$-lattice in $\mathbb{K}$.*

(a) Let $\mathcal{B}$ be a $\mathbb{F}$ basis for $\mathbb{K}$. For each $b \in \mathcal{B}$ there exisst $i_b \in \operatorname{Int}_{\mathbb{K}}(R)$ and $r_b \in R^\sharp$ with $b = \frac{i_B}{r_b}$. So replacing $\mathcal{B}$ by $b \prod_{d\in\mathcal{B}} r_b$ we may assume that $\mathcal{B} \subseteq \operatorname{Int}_{\mathbb{K}}(R)$. By 6.2.2 and 4.1.8 there exists $b^* \in\in \mathbb{K}$ with $\operatorname{tr}(b^* d) = \delta_{bd}$ for all $b, d \in \mathcal{B}$ and $(b^* \mid b \in \mathcal{B})$ is a $\mathbb{F}$-basis for $\mathbb{K}$. Thus $L = \operatorname{Int}_{\mathbb{K}}(R)\langle b^* \mid b \in \mathcal{B}\rangle$ is an $\operatorname{Int}_{\mathbb{K}}(R)$-lattice in $\mathbb{K}$. Let $i \in \operatorname{Int}_{\mathbb{K}}(R)$. Then $i = \sum_{b\in\mathcal{T}} \operatorname{tr}(bi) b^*$. Since $\operatorname{Int}_{\mathbb{K}}(R)$ is a subring $bi \in \operatorname{Int}_{\mathbb{K}}(R)$. So by 6.2.7(b) $\operatorname{tr}(bi) \in \operatorname{Int}_{\mathbb{K}}(R)$ and so $i \in L$.

(b) By (a) $\operatorname{Int}_{\mathbb{K}}(R)$ is contained in a finitely generated R-module. Since R is Noetherian we conclude that $\operatorname{Int}_{\mathbb{K}}(R)$ is a Noetherian R- and so also a Neotherian $\operatorname{Int}_{\mathbb{K}}(R)$-module.

(c) By (a) $\mathrm{Int}_{\mathbb{K}}(S)$ ia a finitely generated, torsion free R-module and so is free with R- basis say $\mathcal{D}$. It is easy to see that $\mathcal{D}$ is also linearly independent over $\mathbb{F}$. From 6.2.8, $\mathbb{K} = \mathbb{F}\mathrm{Int}_K(S)$ and so $\mathbb{F}\mathcal{D} = \mathbb{K}$ and $\mathcal{D}$ is also an $\mathbb{F}$ basis. □

Definition 6.2.11 [**def:algebraic number field**] *An* algebraic number field *is a finite field extension of* $\mathbb{Q}$.

Lemma 6.2.12 [**primes are maximal**] *Let* $\mathbb{K}$ *be an algebraic number field and* J *a non-zero prime ideal in* $R := \mathrm{Int}_{\mathbb{K}}(\mathbb{Z})$. R/J *is a finite field and in particular* J *is a maximal ideal in* R.

Proof: Let $0 \neq j \in J$ and let $f \in \mathbb{Z}[x]$ monic of minimal degree with $f(j)$. Let $f(x) = g(x)x + a$ with $a \in \mathbb{Z}$. Then $f(j) = 0$ gives $a = -g(j)j \in J$. By minimality of $\deg f$, $g(j) \neq 0$ and so also $a \neq 0$. Thus $J \cap \mathbb{Z} \neq 0$ and so $\mathbb{Z} + J/J$ is finite. By 6.2.10(a) R is a finite generate $\mathbb{Z}$-module. Thus R/J is a finitely generated $\mathbb{Z} + J/J$-module and so R/J is a finite. Since J is prime, R/J is an integral domain and so R/J is a finite field. □

Definition 6.2.13 [**def:dedekind domain**] *A* Dedekind domain *is an integrally closed Noetherian domain in which every which every non-zero prime ideal is maximal.*

Corollary 6.2.14 [**algebraic integers are dedekind**] *The set of algebriac integers in an algebraic number field form a Dedekind domain.*

Proof: Let $\mathbb{K}$ be an algebraic number field and $R := \mathrm{Int}_{\mathbb{K}}(\mathbb{Z})$. By 6.2.8 $\mathbb{K}$ is the field of fraction of R. So by 6.2.6(c) R is integrally closed. By 6.2.10 R is Noetherian and by 6.2.12 all prime ideals in R are maximal. □

Lemma 6.2.15 (Noetherian Induction) [**noetherian induction**] *R be a ring and M be an Noetherian R-module and $\mathcal{A}$ and $\mathcal{B}$ sets of R-submodules of M. Suppose that for all $A \in \mathcal{A}$ such that $D \in \mathcal{B}$ for all $A < D \in \mathcal{A}$, then $\mathcal{A} \subseteq \mathcal{B}$.*

Proof: Suppose not. Then $\mathcal{A} \setminus \mathcal{B}$ has a maximal element element A. But then $D \in \mathcal{B}$ for all $A < D \in \mathcal{A}$ and so by assumption $A \in \mathcal{B}$, a contradiction. □

Lemma 6.2.16 [**contains product of prime**] *Let R be a commutative Noetherian ring and J an ideal in R. Then there exist prime ideals $P_1, P_2 \ldots P_n \in R$ with $J \subseteq P_i$ and $\prod_{i=1}^n P_i \in J$.*

Proof: If J is is a prime ideal the lemma holds with $n = 1$ and $P_1 = J$. So suppose J is not a prime ideal. The there exists ideal $J < J_k < R$, $k = 1, 1$ with $J_1 J_2 \subseteq R$. By Notherian induction we may assume that there exists prime ideals $J_k \subseteq P_{ik}$ in R with $\prod_{i=1}^{n_k} P_{ik} \subseteq J_k$. Thus $\prod_{k=1}^2 \prod_{i=1}^{n_k} P_{ik} \leq J_1 J_2 \subseteq J$. □

Definition 6.2.17 [**def:division**] *Let M be an R module and $N \subseteq M$ and $J \subseteq R$. Then* $N \div_M J =: \{m \in M \mid Jm \subseteq N\}$.

For example $0 \div_M J = \mathrm{A}_M(J)$ and if N is an R-submodule of M, then $N \leq N \div_M J$ and $N \div_M J/N = \mathrm{A}_{M/N}(J)$. If R is an integral domain with field of fraction $\mathbb{K}$ and $a, b \in \mathbb{K}$ with $b \neq 0$, then $Ra \div_{\mathbb{K}} Rb = R\frac{a}{b}$.

Definition 6.2.18 [**def:fractional ideal**] *Let R be a integral domain with field of fraction $\mathbb{K}$. A* fractional ideal *of R is a non-zero R-submodule J of R such that $kJ \subseteq R$ for some $k \in K^\sharp$. $\mathcal{FI}(R)$ is the set of fractional ideals of R. Observe that $\mathcal{FI}(R)$ is an abelian monoid under multiplication with identity element R. A fractional ideal is called* invertible *if its invertible in the monoid $\mathcal{FI}(R)$. $\mathcal{FI}^*(R)$ is the group of invertible elements in $\mathcal{FI}(R)$.*

Lemma 6.2.19 [**basic monoid**] *Let H be a monoid.*

(a) [**a**] *Every h has at most one inverse.*

(b) [**b**] *Let $a, b \in H$. If H is abelian and ab is invertible, then a and b are invertible. invertible.*

Proof: (a) If $ah = 1$ and $hb = 1$, then $b = (ah)b = a(hb) = a$.

(b) Let h be an inverse of a. Then $1 = h(ab) = (ha)b$ and so since H is abelian, ha is an inverse of b. By symmetry hb is an inverse for a. □

Lemma 6.2.20 [**basic invertible**] *Let R be a integral domain with field of fraction $\mathbb{K}$ and let J be a fractional ideal of R.*

(a) [**a**] *If $T \neq 0$ is an R-submodule of J, then T is a fraction ideal of R and $R \div_{\mathbb{K}} J \subseteq R \div_{\mathbb{K}} T$.*

(b) [**b**] *$R \div_{\mathbb{K}} J$ is a fractional ideal of I.*

(c) [**c**] *J is invertible iff and only if $(R \div_{\mathbb{K}} J)J = R$. In this case its inverse is $(R \div_{\mathbb{K}} J)J$.*

Proof: By defintion of a fractiona ideal there exists $k \in \mathbb{K}\sharp$ with $kJ \subseteq R$.

(a) Note that $kT \subseteq R$ and so T is a fractional ideal. If $lK \subseteq R$ then also $lT \subseteq R$ and (a) is proved.

(b) Since $k \in R \div_{\mathbb{K}} J$, $R \div_{\mathbb{K}} J \neq 0$. Let $t \in J^\sharp$. Then by (a) applied to $T = Rt$,

$$R \div_{\mathbb{K}} J \subseteq R \div_{\mathbb{K}} Rrt = R\frac{1}{t}$$

and so $t(R \div_{\mathbb{K}} J) \subseteq R$ and $R \div_{\mathbb{K}} J$ is a fractional ideal.

(c) If $(R \div_{\mathbb{K}} J)J = R$, then $R \div_{\mathbb{K}} J$ is an inverse for J in $\mathcal{FI}(R)$. Suppose now that $T \in \mathcal{FI}(R)$ with $TJ = R$. Then clearly $T \subseteq R \div_{\mathbb{F}} J$. Thus

$$R = TJ \subseteq (R \div_{\mathbb{F}} J)J \subseteq R$$

Thus both T and $R \div_{\mathbb{K}} F$ are inverse of J and so $T = R \div_{\mathbb{K}} F$. □

Lemma 6.2.21 [**partial inverse**] *Let R be an Dedekind domain with field of fraction $\mathbb{K}$ and J proper ideal in R. Then $R < R \div_{\mathbb{K}} J$.*

Proof: Let P be a maximal ideal in R with $J \leq P$. Let $a \in J^{\sharp}$. By 6.2.16 there exists non-zero prime ideals $P_1, P_2, \dots P_n$ with $\prod_{i=1}^{n} P_i \leq Ra$. We also assume that n is minimal with with property. Since $Ra \leq P$ and P is a prime ideal we must have $P_i \leq P$ for some i. By definition of a Dekind domain, P_i is a maximal ideal and so $P_i = P$. Let $Q = \prod_{i \neq j=1}^{n} P_j$. Then $PQ \leq Ra$ and by minimality of n, $Q \not\leq Ra$. Thus $Ja^{-1}Q \leq PQa^{-1} \leq R$ and and $a^{-1}Q \not\leq R$. So $a^{-1}Q \leq R \div_{\mathbb{K}} J$ and hence $R \div_{\mathbb{K}} J \not\leq R$. Clearly $R \leq R \div_{\mathbb{K}} J$ and the lemma is proved.

Proposition 6.2.22 [**fi for dekind**] *et R be an Dedekind domain with field of fraction $\mathbb{K}$. Let P be a nonzero prime ideal in the Dedekind domain R and J a non-zero ideal with $J \subseteq P$. Then P invertible and $J < JP^{-1} \leq R$.*

Proof: Put $Q := R \div_{\mathbb{K}}$. Then $R \leq Q$ and $J \subseteq JQ \subseteq R$. Suppose that $J = JQ$. Since R is Noetherian, J is finitely R-generated. Since $\mathbb{K}$ is an integral domain and $J \neq 0$, J is a faithful Q-module. Thus 6.2.5(c) implies that Q is integral over R. By defintition of a Dekind domain, R is integrally closed in $\mathbb{K}$ and so $Q \leq R$. But this contradicts 6.2.21

Thus $J < JQ^{-1}$ and inparticular $P < PQ \leq R$. By definition of a Dekind Domain P is a maximal ideal in R and so $PQ = P$. Thus $Q = P^{-1}$ and the proposition is proved. □

Theorem 6.2.23 [**structure of dedekind**] *Let R be a Dedekind domain and let $\mathcal{P}$ be the set of non-zero prime ideals in R. Then the map*

$$\tau : \oplus_{\mathcal{P}} \mathbb{Z} \to \mathcal{FI}(R) \mid (z_P) \to \prod_{P \in \mathcal{P}} P^{z_P}$$

is an isomorphism of monoids. In particular, $\mathcal{FI}(R)$ is a group. Moreover $\tau(z) \leq R$ if and only if $z \in \oplus_{\mathcal{P}} \mathbb{N}$.

Proof: Clearly τ is an homomorphism. Suppose there exists $0 \neq z \in \ker \tau$. Let $X = \{P \in \mathcal{P} \mid z_P < 0$ and $Y = \{P \in \mathcal{P} \mid z_P > 00$. Then $X \cap Y = \emptyset$ and $X \cup Y \neq \emptyset$. Moreover, $\tau(z) = R$ implies

$$\prod_{P \in X} P^{-z_p} = \prod_{P \in Y} P^{z_P}$$

In particular both X and not empty. Let $Q \in X$. Then

$$\prod_{P \in Y} P^{z_P} \leq Q$$

a contrdiction since $P \not\leq Q$ for all $P \in Y$ and sinceR/Q is a prime ideal.

Thus τ is $1-1$.

Next let J be a proper ideal in R and P a maximal ideal in R with $J \leq P$. By 6.2.22 $J < JP^{-1} \leq R$. By Noetherian induction $JP^{-1} = P_1 \ldots P_n$ for some prime ideals $P_1, \ldots P_n$ and so $J = PP_1 \ldots P_n$, that is $J = \tau(z)$ for some $z \in \oplus_{\mathcal{P}}\mathbb{N}$.

Finally let J be an arbitray fraction ideal in $\mathbb{K}$. Then by definition ther exists $kJ \subseteq R$ for some $k \in \mathbb{K}^\sharp$. Then $k = \frac{r}{s}$ with $r, s \in R^\sharp$ and so $rJ = skJ \subseteq R$. Let $u, v \in \bigoplus_{\mathcal{P}} \mathbb{N}$ with $\tau(u) = Rr$ and $\tau(v) = rJ$. Then

$\tau(v - u) = (Rr)^{-1}(rJ) = Rr{-}1rJ = J$ and so τ is onto. □

The next proposition shows that Dedekind domains are not far away from being principal domains.

Proposition 6.2.24 [**nearly principal**] *Let R be a Dedekind domain.*

(a) [**a**] *Let A and B be a fractional ideals of R with $B \leq A$. Then A/B is a cyclic R-module.*

(b) [**b**] *Let A be a fractional ideal of R. Then there exists $a, b \in A$ with $A = Ra + Rb$.*

Proof: (a) Replacing A and B by kA and kB for a suitable $k \in R$ we may assume that $B \leq A \leq R$, Let $\mathcal{Q}$ be a finite set of prime ideals in R with $A = \prod_{P \in \mathcal{Q}} P^{a_P}$ and $B = \prod_{P \in \mathcal{Q}} P^{b_P}$ for some $a_p, b_P \in \mathbb{N}$. Choose $x_P \in P^{a_p} \setminus P^{a_p+1}$. Observe that $P^{a_p+1} + Q^{a_Q+1} = R$ for disctinct $P, Q \in \mathcal{Q}$. So by the Chinese Remainder Theorem 2.5.15(e) the exists $x \in R$ with $x + P^{a_p+1} = x_p + P^{a_p+1}$ for all $P \in \mathcal{Q}$. Thus $x \in \bigcap_{P \in \mathcal{Q}} P^{a_p} = A$ and $x \notin P^{a_P+1}$. Since $B \leq Rx + B$, $Rx + B = \prod_{P \in \mathcal{Q}} P^{c_P}$ for some $c_P \in \mathbb{N}$. Since $Rx + B \leq A$, $c_P \geq a_P$. Since $x \notin P^{a_P+1}$, $c_P \leq a_p$. Thus $a_P = c_P$ for all $P \in \mathcal{Q}$ and so $A = Rx + B$.

(b) Let $0 \neq b \in A$ and put $B = Ra$. By (a) $A/B = Ra + B/B$ for some $a \in A$. Thus $A = Ra + Rb$. □

6.3 The Jacobson Radical II

Lemma 6.3.1 (Nakayama) [**nakayama**] *Let R be a ring and M a non zero finitely generated R-module then* $\mathrm{J}(R)M \neq 0$.

Let $\mathcal{B} \subseteq M$ be minimal with $R\mathcal{B} = M$. Let $b \in \mathcal{B}$, then $M \neq R(\mathcal{B} \setminus \{b\}$ and repplacing M be $M/R(\mathcal{B} \setminus \{b\}$ we mau assume that $M = Rb$. Then $M \cong R/\mathrm{A}_R(b)$. Let J be maximal left ideal of R with $A_R(b) \leq J$. Then $\mathrm{J}(R) + A_R(b) \leq J < R$ and so also $\mathrm{J}(R) < M$. □

Lemma 6.3.2 [**jr and inverses**] *Let R be a ring and $x \in R$.*

(a) [**a**] *$x \in \mathrm{J}(R)$ iff $rx - 1$ has a left inverse for all $x \in R$.*

(b) [**b**] *x is left invertible in R iff $x + \mathrm{J}(R)$ is left invertible in $R/\mathrm{J}(R)$.*

(c) [**c**] *The $J(R)$ is equal to the right Jacobson radical* $\mathrm{J}(R^{\mathrm{op}}$.

(d) [**d**] *x is invertible in R iff $x + \mathrm{J}(R)$ is invertible in $R/\mathrm{J}(R)$.*

Proof: (a) Let $x \in R$ and let $\mathcal{M}$ be the set of maximal left ideals in R. The the follwing are equivalent

$$\begin{array}{cc}
x \notin \mathrm{J}(R) & \\
x \notin M & \text{for some} M \in \mathcal{M} \\
Rx + M = R & \text{for some} M \in \mathcal{M} \\
rx + m = 1 & \text{for some} M \in \mathcal{M}, m \in \mathcal{M}, r \in R \\
rx - 1 \in \mathcal{M} & \text{for some } r \in R, M \in \mathcal{M} \\
R(rx-1) \neq R & \text{for some} r \in R \\
(rx-1) \text{ is not left invertible} & \text{for some} r \in R
\end{array}$$

(b) If x is left invertible, then $x + \mathrm{J}(R)$ is left invertible. Suppose now that $x + \mathrm{J}(R)$ is left invertible. Then $1 - yx \in \mathrm{J}(R)$ for some $y \in R$. By (a) $yx = 1 - (1 - yx)$ has a left inverse. Hence also x as a left inverse.

As a step towards (c) and (d) we prove next:

1° [**1**] *If $x - 1 \in \mathrm{J}(R)$. Then x is invertible.*

By (b) there exists $k \in R$ with $kx = 1$. Thus $k - 1 = k - kx = k(1 - x) \in \mathrm{J}(R)$ and so by (b) again k has a left inverse l. So by 2.2.2 $x = l$ and k is an inverse of x.

(c) Let $j \in \mathrm{J}(R)$ and $r \in \mathrm{J}(R)$. Since $\mathrm{J}(R)$ is an ideal, $jr \in \mathrm{J}(R)$. Thus by (1°) $1 + jr$ is invertible. So by (a) applied to R^{op}, $j \in \mathrm{J}(R^{\mathrm{op}}$. Hence $\mathrm{J}(R) \leq \mathrm{J}(R^{\mathrm{op}}$. By symmetry $\mathrm{J}(R) \leq \mathrm{J}(R^{\mathrm{op}}$.

(d) Follows from (b) applied to R and R^{op}. □

Lemma 6.3.3 [**jr cap za**] *Let A be a ring, R a subring and suppose that A is finite generated as an R-module. Then $\mathrm{J}(R) \cap Z(A) \leq \mathrm{J}(A)$.*

Proof: Let M be a simple A-module. Then M is cylcic as an A-module and so finitely generated as an R-module. Thus by 6.3.1, $\mathrm{J}(R)M \neq M$. Hence also $(\mathrm{J}(R) \cap \mathrm{Z}(A))M < M$ and since $(\mathrm{J}(R) \cap \mathrm{Z}(A))M$ is an A-submodule we conclude that $\mathrm{J}(R) \cap \mathrm{Z}(A) \leq \mathrm{A}_A(M)$. Thus $\mathrm{J}(R) \cap \mathrm{Z}(A) \leq J(A)$. □

Proposition 6.3.4 [**jza**] *Let A be a ring.*

(a) [**a**] *If K is a nilpotent left ideal in A, then $K \leq \mathrm{J}(A)$*

(b) [**b**] *If A is artian, $\mathrm{J}(A)$ is the largest nilpotent ideal in A.*

(c) [**c**] *If A is artian and finitely $Z(A)$-generated then* $\mathrm{J}(A) \cap \mathrm{Z}(A) = \mathrm{J}(\mathrm{Z}(A))$.

Proof:

(a) Let $k \in K$. Then rk is nilpotent and so $1 + rk$ is invertible in in R. So by 6.3.2(a), $k \in \mathrm{J}(A)$.

(b) Since A is Artinian we can choose $n \in \mathbb{N}$ with $\mathrm{J}(A)^n$ minimal. Then $\mathrm{J}(A)\mathrm{J}(A)^n = \mathrm{J}(A)^n$. Suppose $\mathrm{J}(A)^n \neq 0$ and choose a left ideal K in A minimal with $\mathrm{J}(A)^n K \neq 0$. Let $k \in K$ with $\mathrm{J}(A)^n k \neq 0$. Then $\mathrm{J}(A)^n \mathrm{J}(A)k = J^(A)^n k \neq 0$ and so by mimimality of K, $K = \mathrm{J}(A)k$. Thus $k = jk$ for some $j \in \mathrm{J}(A)$. Thus $(1-j)k = 0$. By 6.3.2 $1-j$ is invertible and so $k = 0$, a contradiction.

(c) By (b) $\mathrm{J}(A) \cap \mathrm{Z}(A)$ is a nilpotent ideal in $\mathrm{Z}(A)$ and so by (a) $\mathrm{J}(A) \cap \mathrm{Z}(A) \leq \mathrm{Z}(\mathrm{J}(A))$. By 6.3.3 $\mathrm{J}(\mathrm{Z}(A)) \leq \mathrm{J}(A) \cap \mathrm{Z}(A)$ and (c) is proved. □

Lemma 6.3.5 [**invertible in ere**] *Let R be a ring, $S \leq \mathrm{Z}(R)$ and suppose that R is a finitely generated S-module. Let $e \in R$ be an idempotent and $x \in eRe$ with $x + \mathrm{J}(S)R = e + \mathrm{J}(S)R$. Then there exists a unique $y \in eRe$ with $xy = yx = e$.*

Proof: Since $(ere)(ete) = e(eter)e$, eRe is a ring with identity e. We need to show that x is invertible in eRe. If $R = ST$ for a finite subset T of R then also $eRe = eS(eTe)$ and so eRe is a finitely geneerated eS-module. Also $eS = eSe \leq \mathrm{Z}(eRe)$ and so by 6.3.3 $\mathrm{J}(eS) \leq \mathrm{J}(eRe)$. Since $e : S \to eS$ is an onto ring homomorphism, $eJ(S) \leq \mathrm{J}(eS) \leq \mathrm{J}(eRe)$. Since $x \in eRe$ and $x - e \in \mathrm{J}(S)R$

$$x - e = e(x-e)e \in e\mathrm{J}(S)Re = e\mathrm{J}(s)eRe \leq \mathrm{J}(eRe)eRe \leq \mathrm{J}(eRe)$$

Thus $x - e \in \mathrm{J}(eRe)$ and by 6.3.2 x has an inverse in eRe. □

6.4 A basis for $\mathbb{C}\tilde{G}$

Lemma 6.4.1 [**from oq to f**] *Let X be non-empty finite subset of $\overline{\mathbb{Q}}^\sharp$. Then there exists $b \in \mathbb{Q}(X)$ with $bX \subseteq \mathbb{A}$ and $bX \nsubseteq I$.*

Proof: By 6.2.22 applied with $\mathbb{K} = \mathbb{Q}(X)$ we have $I^{-1}I = \mathbb{A}$. So there exists $b \in I^{-1}$ with $bX \nsubseteq I$. □

Corollary 6.4.2 [**f linearly independent**] *Let V be an $\overline{\mathbb{Q}}$-space and $(v_i)_{i=1}^n \in V^n$. Let $W = \mathbb{A} < v_i \mid 1 \leq i \leq n$. and suppose that $(v_i + IW)_{i=1}^n$ is $\mathbb{F}$-linearly independent in W/IW. Then $(v_i)_{i=1}^n$ is linearly idenpendet over $\overline{\mathbb{Q}}$.*

Proof: Suppose there exists $a_i \in \overline{\mathbb{Q}}$ not all zero with $\sum_{i=1}^n a_i v_i = 0$. By 6.4.1 there exists $b \in \overline{\mathbb{Q}}$ with $ba_i \in \mathbb{A}$ anf $ba_j \notin I$ for some $1 \leq j \leq n$. Then $\sum_{i=1}^n (ba_i + I)(v_i + IW) = 0$ but $ba_j + I \neq I$, a contradcition. □

Lemma 6.4.3 [**linear independence of characters**]

(a) [**a**] $(\chi_M \mid M \in \mathcal{S}_p)$ *is* $\mathbb{F}$*-linear independent in* $\mathbb{F}G$*.*

(b) [**b**] $(\phi_M \mid M \in \mathcal{S}_p)$ *is* $\mathbb{C}$*-linearly independent in* $\mathbb{C}\tilde{G}$*.*

Proof: (a) Let $f_M \in \mathbb{F}$ with $\sum f_M\chi_M = 0$. Pick $e_M \in \mathrm{End}_{\mathbb{F}}(M)$ with $\mathrm{tr}_M(e_M) = 1$. 2.5.18 there exists $a_M \in \mathbb{F}G$ such that a_M acts as e_M on N and trivially on N for all $M \neq N \in \mathcal{S}_p$. Then

$$0 = \sum_{N \in \mathcal{S}_p} f_N\chi_N(e_M) = f_M$$

and so (a) holds.

(b) Since all coefficents of ϕ_M are in $\mathbb{A}$, $\phi_M \mid M \in \mathcal{S}_p)$ is $\mathbb{C}$-linearly independent iff $(\phi_M \mid M \in \mathcal{S}_p)$ is $\overline{\mathbb{Q}}$-linearly independent and iff $(\check{\phi}_M \mid M \in \mathcal{S}_p)$ is $\overline{\mathbb{Q}}$-linearly independent. By 6.1.5 $(\check{\phi}_M)^* = \chi_M$ and so by (a) $(\check{\phi}_M)^* \mid M \in \mathcal{S}_p)$ is $\mathbb{F}$-linearly independent. So (b) follows from 6.4.2. □

Lemma 6.4.4 [**existence of a lattice**] *Let* V *be an* $\rtimes Q$*-space and* W *a finitely generated* $\mathbb{A}_I$ *submodule of* V *with* $V = \mathbb{Q}W$*. Then* W *is an* $\mathbb{A}_I$*-lattice in* V*.*

Proof: Note that W/I_IW is a finite dimensional vector space over $\mathbb{A}_I/I_I = \mathbb{F}$ and so has a basis $u_i + I_IW, 1 \leq i \leq n$. By 6.4.2 $(u_i)_{i=1}^n$ is linearly independent over $\overline{\mathbb{Q}}$ and so also over $\mathbb{A}_I$. Let $U = \mathbb{A}_i\langle u_i\, od 1 \leq i \leq n$. Then $W = U + I_IW$. Since I_I is the unique maximal ideal in $\mathbb{A}_I$, $I_I = (\,\mathbb{A}_I)$. Thus by the Nakayama Lemma 6.3.1 applied to W/U gives $W = U$. Hence also $V = \overline{\mathbb{Q}}W = \overline{\mathbb{Q}V}\langle u_i \mid 1 \leq i \leq n\rangle$ □

Lemma 6.4.5 [**existence of oq lattice**] *Let* $\mathbb{E} : \mathbb{K}$ *be a field extension and* M *a simple* $\mathbb{K}G$*-module. If* $\mathbb{K}$ *is algebraicly closed then there exists an* G*-invariant* $\mathbb{K}$ *lattice* L *is* M*. For any such* L*,* L *is a simple* $\mathbb{K}G$*-module and* $M \cong \mathbb{E} \otimes_{\mathbb{K}} L$*.*

Proof: Since G is finite there exists a simple $\mathbb{K}G$-submodule L in M. Moreover there is a non-zero $\mathbb{E}G$-linear map $\alpha : \mathbb{E} \otimes_{\mathbb{K}} L \to M, e \otimes l \to el$. Since $\mathbb{K}$ is algebraicly closed, $\mathbb{E} \otimes_{\mathbb{K}} L$ is a simple $\mathbb{E}G$-module. The same is true for M and so α is an isomorphism. In particular, any $\mathbb{K}$ basis for L is also a $\mathbb{E}$-basis for M and so L is a K-lattice in M.

Now let L is any $\mathbb{K}$-lattice in G. If $) \neq N \leq L$ is a $\mathbb{K}G$-submodule then $\mathbb{E}N$ is a $\mathbb{E}G$-submodule of M. Thus $\mathbb{E}N = M$ and $\dim_{\mathbb{K}} N = \dim_{\mathbb{E}} \mathbb{E}N = \dim_{\mathbb{E}} M = \dim_{\mathbb{K}} L$ and so $N = L$ and L is a simple $\mathbb{K}G$-module. □

Lemma 6.4.6 [**existence of ai lattice**] *Let* M *be an* $\mathbb{C}G$*-module. Then there exists a* G*-invariant* $\mathbb{A}_I$*-lattice* L *in* M*.*

Proof: By 6.4.5 there exists a G-invariant $\overline{\mathbb{Q}}$-lattice V in M. Let X be a $\overline{\mathbb{Q}}$-basis for V and put $L = \mathbb{A}_I GX$. Since G and X are finite, L is finitely $\mathbb{A}_I$-generated. Thus by 6.4.4, L is an $\mathbb{A}_I$-lattice in V and so also in M. $\square$

Lemma 6.4.7 [**characters are brauer characters**] *Let M be an $\mathbb{C}G$-module and L a G-invariant $\mathbb{A}_I$-lattice in M. Let M° be the $\mathbb{F}G$-module, $L/I_I L$. Then $\chi_M^* = \chi_{M^\circ}$ and $\tilde{\chi}_M = \phi_{M^\circ}$*

Proof: Let $\mathcal{B}$ be an $\mathbb{A}_I$ basis for L, $g \in G$ and D the marix for g with respect to $\mathcal{B}$. Then D^* is the matrix for g with respect to the basis $(b + I_L L)_{b\in\mathcal{B}}$ for M°. Since $\eta_M(g) = \det(x\mathrm{I}\,d_n - D)$ we conclude that $\eta_M(g)^* = \eta_{M^\circ}(g)$. In particular $\chi_M(g)^* = \chi_{M^\circ}(g)$ and if $\eta_M(g) = \prod_{i=1}^n (x - \xi_i)$ then $\eta_{M^\circ}(g) = \prod_{i=1}^n (x - \xi_i^*)$. So if $g \in G^\circ$, then $\chi_M(g) = \phi_{M^\circ}(g)$. $\square$

Definition 6.4.8 [**def:Irr G**]

(a) [**a**] $\mathrm{Irr}(G) = \{\chi_M \mid M \in \mathcal{S}\}$ *is the set of simple characters of G.*

(b) [**b**] $\mathrm{IBr}(G) = \{\phi_M \mid M \in \mathcal{S}_p\}$ *is the set of simple Brauer characters of G.*

(c) [**c**] $Z\mathbb{C}\tilde{G} := \mathbb{C}\tilde{G} \cap Z(\mathbb{C}G)$ *is the set of complex valued class function on $\tilde{G}$.*

(d) [**d**] *If M be an $\mathbb{C}G$-module and L an G invariant $\mathbb{C} : \mathbb{A}_I$ lattice in M, then $M^\circ = L/I_I L$ is called a* reduction modulo p *of M.*

Theorem 6.4.9 [**ibr basis**]

(a) [**a**] $Z\mathbb{C}(\tilde{G})$ *is the $\mathbb{C}$-span of the Brauer characters.*

(b) [**b**] $\mathrm{IBr}(G)$ *is a $\mathbb{C}$-basis for*$Z\mathbb{C}(\tilde{G})$

(c) [**c**] $|\mathcal{S}|_p = |\mathrm{IBr}(G)$ *is the number of p'-conjugacy classes.*

Proof: (a) Observe that the map $\tilde{}: Z(\mathbb{C}G) \to Z\mathbb{C}(\tilde{G})$ is an orthogonal projection and so onto. On the otherhand since $Z(\mathbb{C}G)$ is an $\mathbb{C}$ -span of the G-characters we conclude from 6.4.7 that the image of $\tilde{}$ is conatained in $\mathbb{C}$-span of the Brauer characters. So (a) holds.

(b) By 6.1.3(e) every Brauer chacter is a sum of simple Brauet charcters. So by (a), $\mathrm{IBr}(G)$ spans $Z\mathbb{C}(\tilde{G})$ By 6.4.3(b) $\mathrm{IBr}(G)$ is linearly independent over $\mathbb{C}$ and so (b) holds.

(c) Both $\mathrm{IBr}(G)$ and $(a_C \mid C a p'$ conjugacy class$\}$ are bases for $Z\mathbb{C}(\tilde{G})$ $\square$

Definition 6.4.10 [**def:decomposition matrix**]

(a) [**a**] *$D = D(G) = (d_{phi\chi})$ is the matrix of $\tilde{}: Z\mathbb{C}G \to Z\mathbb{C}\tilde{G}$ with respect to* $\mathrm{Irr}(G)$ *and* $\mathrm{IBr}(G)$. *D is called the* decompositon matrix *of G.*

(b) [**b**] $C = C(G) = (c_{\phi\psi})$ *is the inverse of Gram matrix of* $(\cdot \mid \cdot)$ *with respect to* $\mathrm{IBr}(G)$. C *is called the* Cartan matrix *of* G.

(c) [**c**] *For* $\phi \in \mathrm{IBr}(G)$, $\Phi_\phi = \sum_{\chi \in \mathrm{Irr}(G)} d_{\phi\chi}\chi$ *is called the* projective indecomposable *character associated to* ϕ. *For* $M \in \mathcal{S}_p$ *put* $\Phi_M = \Phi_{\phi_M}$.

Lemma 6.4.11 [**basic decomposition**]

(a) [**a**] *Let* $\chi \in \mathrm{Irr}(G)$. *Then* $\tilde{\chi} = \sum_{\phi \in \mathrm{IBr}(G)} d_{\phi\chi}\phi$.

(b) [**z**] *Let* $M \in \mathcal{S}(G)$, M° *a p-reduction of* M, $N \in \mathcal{S}_p(G)$ *and* $\mathcal{F}$ *a* $\mathbb{F}G$*-composition series on* M. *Then* $d_{\phi_N\chi_M}$ *is the number of factors of* $|caF$ *isomorphic to* N.

(c) [**b**] *Let* $\phi, \psi \in \mathrm{IBr}(G)$. *Then* $\Phi_\phi \in Z\mathbb{C}\tilde{G}$ *and* $(\Phi_\phi \mid \psi) = \delta_{\phi\psi}$. *So* $(\Phi_\phi \mid \phi \in \mathrm{Irr}(G))$ *is the dual basis for* $Z\mathbb{C}\tilde{G}$.

(d) [**c**] $C^{-1} = ((\phi \mid \psi))_{\phi\psi}$

(e) [**d**] $C = ((\Phi_\phi \mid \Phi_\psi))$ *is Gram matrix of* $(\mathrm{cot} \mid \cdot)$ *with respect to* $(\Phi_\phi \mid \phi \in \mathrm{IBr}(G)$.

(f) [**e**] *Let* $\phi \in \Psi$. *Then* $\Phi_\phi = \tilde{\Phi}_\phi = \sum_{\psi \in \mathrm{IBr}(G)} c_{\phi\psi}\psi$.

(g) [**f**] $C = DD^{\mathrm{T}}$.

Proof: (a) Immediate from the definition of D.

(b) For $N \in \mathcal{S}_p(G)$ Let a_N be the number of compostion factors of G isomorphic to N. Then by 6.1.3(e), $\phi_{M^\circ} = \sum_{N \in \mathcal{S}_p(G)} a_N \phi_N$.

By 6.4.7 $\phi_{M^\circ} = \tilde{\chi}_M$. So (a) and the linearly independence of $\mathrm{IBr}(G)$ implies $d_{\phi_N\chi_M} = a_N$.

(c) Follows from 4.1.14

(d) Immediate from the definition of C.

(e) and (f) follows from 4.1.16

(g) From (d) and the definition of Φ_π:

$$c_{\phi\psi} = (\sum_{\chi \in \mathrm{Irr}(G)} d_{\phi\chi}\chi \mid \sum_{\chi \in \mathrm{Irr}(G)} d_{\psi\chi}\chi) = \sum_{\chi \in \mathrm{Irr}(G)} d_{\phi\chi}d_{\psi\chi}$$

and so (g) holds.

Corollary 6.4.12 [**dphichi not zero**] *For each* $\phi \in \mathrm{IBr}(G)$, *there exists* $\chi \in \mathrm{Irr}(G)$ *with* $d_{\phi\chi \neq 0}$. *In otherwords, for each* $M \in \mathcal{S}_p$ *there exists a* $\check{M} \in \mathcal{S}$ *such that* M *is isomorphic to a composition factor of nay p-reduction of* $\check{M}$.

Proof: Follows from the fact that $\tilde{}: Z(\mathbb{C}G) \to Z\mathbb{C}\tilde{G}$ is onto. $\square$

Corollary 6.4.13 [**projective is regular**] *Let $M \in \mathcal{S}_p$ and $P \in \mathrm{Syl}_p(M)$. Then $\dim \Phi_M$ is divisiple $|P|$. Moreover, Φ_M restricted to P is an integral multiple of the regular character for P.*

Proof: Since $\Phi_M = \tilde{\Phi}_M$ we have $\Phi_M(g) = 0$ for all $g \in P^\sharp$. Thus $(\Phi_M \mid_P \mid 1_P)_P = \frac{1}{|P|}\Phi_M(1)$ and so $|P|$ divides $\Phi_M(1)$. Therefore

$$\Phi_M(1) = \frac{\Phi_M(1)}{|P|}\chi^P_{\mathrm{reg}}$$

□

Theorem 6.4.14 [**pprime=0**] *Suppose G is a $p\prime$ group.*

(a) [**a**] $\mathrm{Irr}(G) = \mathrm{IBr}(G)$ *and* $D = (\delta_{\phi\psi})$.

(b) [**b**] *For $M \in \mathcal{S}$ let M° be a reduction modulo p. Then M° is a simple $\mathbb{F}G$-module and the map $\mathcal{S} \to \mathcal{S}_p, M \to M^\circ$ is bijection.*

Proof: By 3.1.3(c) $|G| = \sum_{\phi\in\mathrm{IBr}(G)} \phi(1)^2 = \sum_{\chi\in\mathrm{Irr}(G)} \chi(1)^2$ Thus

$$\begin{array}{rcccc}
|G| & = & \sum\limits_{\chi\in\mathrm{Irr}(G)} \chi(1)^2 & = & \sum\limits_{\chi\in\mathrm{Irr}(G)} \left(\sum\limits_{\phi\in\mathrm{IBr}(G)} d_{\phi\chi}\phi(1)\right)^2 \\
 & \geq & \sum\limits_{\chi\in\mathrm{Irr}(G)} \sum\limits_{\phi\in\mathrm{IBr}(G)} d_{\phi\chi})^2\phi(1)^2 & = & \sum\limits_{\phi\in\mathrm{IBr}(G)} \left(\sum\limits_{\chi\in\mathrm{Irr}(G)} d_{\phi\chi})^2\right)\phi(1)^2 \\
 & \geq & \sum\limits_{\phi\in\mathrm{IBr}(G)} \phi(1)^2 & = & |G|
\end{array}$$

Hence equality holds everythere. In particular $\sum_{\chi\in\mathrm{Irr}(G)} d_{\phi\chi})^2 = 1$ for all $\phi \in \mathrm{IBr}(G)$. So there exists a unique $\chi_\phi \in \mathrm{Irr}(G)$ with $d_{\phi\chi_\phi} \neq 0$. Moreover $d_{\phi\chi_\phi} = 1$.

Also $\left(\sum_{\phi\in\mathrm{IBr}(G)} d_{\phi\chi}\right)^2 = \sum_{\phi\in\mathrm{IBr}(G)}(d_{\phi\chi})^2$ and so for each $\chi \in \mathrm{IBr}(G)$ there exists unique $\phi_\chi \in \mathrm{IBr}(G)$ with $d_{\phi_\chi\chi} \neq 0$. Hence $\chi = \chi_{\phi_\chi}$, $d_{\phi_\chi\chi} = 1$, $\chi = \tilde{\chi} = \phi_\chi = \chi_\chi$ and (a) holds.

(b) follows from (a) and 6.4.11(b). □

Proposition 6.4.15 [**fong**] *Suppose that $p = 2$ and $\phi \in \mathrm{IBr}(G)$. If ϕ is real valued and $\phi(1)$ is odd, then $\phi = 1_{\tilde{G}}$.*

Proof: Let $M \in \mathcal{S}_p$ with $\phi = \phi_M$. Then $\phi_{M^*} = \overline{\phi}_M = \Phi_M$ and some $M \cong M^*$. Thus the proposition follows from 4.1.22 and 4.1.21. □

Lemma 6.4.16 [**opg trivial**] *Let $M \in \mathcal{S}_p$. Then $O_p(G) \leq C_G(M)$.*

Proof: Let W be a simple $\mathbb{F}O_p(G)$ submodule in M. The number of p' conjugacy classes of $O_p(G) = 1$. So up to isomorphism $O_p(G)$ has a unique simple module, namely $\mathbb{F}_{O_p(G)}$. Thus $0 \neq W \leq C_M(O_p(G))$. Since $C_M(O_p(G))$ is an $\mathbb{F}G$-submodule we conclude $M = C_M(O_p(G))$ and $O_p(G) \leq C_G(M)$. □

6.5 Blocks

Lemma 6.5.1 [**omegam**] *Let $\mathbb{K}$ be an algebraicly closed field and M a simple $\mathfrak{G}G$-moudle.*

(a) [**a**] $a \in Z(\mathbb{K}G)$ *there exists a unique* $\omega_M \in \mathbb{K}$ *with* $\rho_M(a) = \omega_M(a)\mathrm{id}_M$.

(b) [**b**] $\omega_M : Z(\mathbb{K}G) \to \mathbb{K}$ *is a ring homomorphism.*

(c) [**c**] $\chi_M(a) = \dim_{\mathbb{K}} M \cdot \omega_M(a) = \chi_M(1)\omega_M(a)$.

(d) [**d**] *If* $\mathbb{K} = \mathbb{C}$ *then and* $a \in Z(\mathbb{A}G)$, *then* $\omega_M(a) \in \mathbb{A}$.

Proof: (a) follows from Schurs Lemma 2.5.3.

(b) and (c) are obvious.

(d) By 3.2.13 $\omega_M(a_C) \in \mathcal{A}$ for all $C \in \mathcal{C}$. Since $(a_C \mid C \in \mathcal{C})$ is a $\mathbb{A}$-basis for $Z(\mathbb{A}G)$, (d) follows from (b). □

Definition 6.5.2 [**def:lambdaphi**]

(a) [**a**] *Let* $M \in \mathcal{S}$ *and* $\chi = \chi_M$. *Then* $\omega_\chi = \omega_M$.

(b) [**b**] *Let* $M \in \mathcal{S}$ *and* $\chi = \chi_M$. *Then* $\lambda_\chi : Z(\mathbb{F}G) \to \mathbb{F}$ *is define by* $\lambda_\chi(a^*) = \omega_\chi(a)^*$ *for all* $a \in Z(\mathbb{A}_I G)$.

(c) [**c**] *Let* $M \in \mathcal{S}_p$ *and* $\phi = \phi_M$. *Then* $\lambda_\phi = \omega_M$.

(d) [**d**] *Define the relation* $\sim_p$ *on* $\mathrm{Irr}(G) \cup \mathrm{IBr}(G)$ *by* $\alpha \sim_p \beta$ *if* $\lambda_\alpha = \lambda_\beta$. *A* block *(or p-block) of* G *is an equivalence class of* $\sim_p$.

(e) [**e**] $\mathrm{Bl}(G)$ *is the set of blocks of* G.

(f) [**f**] *If* B *is a block of* G *then* $\mathrm{Irr}(B) = B \cap \mathrm{Irr}(G)$ *and* $\mathrm{IBr}(B) = B \cap \mathrm{IBr}(G)$.

(g) [**g**] *For* $\mathcal{A} \subseteq \mathrm{Irr}(G)$, *put* $\mathcal{A}^\dagger = \{\phi \in \mathrm{IBr}(G) \mid d_{\phi\chi \neq 0}$ *for some* $\chi \in \mathcal{A}\}$.

(h) [**h**] *For* $\mathcal{B} \subseteq \mathrm{IBr}(G)$, *put* $\mathcal{B}^\dagger = \{\chi \in \mathrm{Irr}(G) \mid d_{\phi\chi \neq 0}$ *for some* $\phi \in \mathcal{B}\}$.

Proposition 6.5.3 [**d and lambda**] *(a)* [**a**] *Let* $\chi \in \mathrm{Irr}(G)$ *and* $\phi \in \mathrm{IBr}(G)$*. If* $d_{\phi\chi} \neq 0$ *then* $\lambda_\phi = \lambda_\chi$*.*

(b) [**b**] *Let* B *be a block of* G *then* $\mathrm{IBr}(B) = \mathrm{Irr}(B)^\dagger$ *and* $\mathrm{Irr}(B) = \mathrm{IBr}(B)^\dagger$*.*

Proof: (a) Let $M \in \mathcal{S}$ with $\chi = \chi_M$ and $N \in \mathcal{S}_p$ with $\phi = \phi_N$. Let L be an G-invariant A_I-lattice in M. Since $d_{\phi\chi\neq 0}$, N is isomorphic to $\mathbb{F}G$ composition factor of $M^\circ = L/I_I L$. Let $a \in Z(\mathbb{A}G)$. Then a acts as the scalar $\omega_\chi(a)$ on M and on L. Thus a acts as the scalar $\omega_\chi(a)^* = \lambda_\chi(a^*)$ on M° and on N. Thus $\lambda_\chi(a^*) = \lambda_\phi(a^*)$ and (a) holds.

(b) $\phi \in \mathrm{IBr}(G)$ with $d_{\phi\chi}$ for some $\chi \in \mathrm{Irr}(B)$ then by (a) $\phi \in B$. Thus $\mathrm{Irr}(B)^\dagger \subseteq \mathrm{IBr}(B)$. Conversely if *phi* $\in \mathrm{IBr}(B)$ we can choose (by 6.4.12) $\chi \in \mathrm{IBr}(G)$ with $d_{\phi\chi} \neq 0$. Then by (a) $\chi \in B$ and so $\mathrm{IBr}(B) \subseteq \mathrm{Irr}(B)^\dagger$. Thus $\mathrm{IBr}(B) = \mathrm{Irr}(B)^\dagger$. Similary $\mathrm{Irr}(B) = \mathrm{IBr}(B)^\dagger$. $\square$

Let $\chi \in \mathrm{Irr}(G)$ and $\phi \in \mathrm{IBr}(G)$. Then λ_χ is defined by **??**(**??**) and λ_ϕ by **??**(**??**). If $\lambda = \phi$ then 6.5.3(a) shows that $\lambda_\chi = \lambda_\phi$.

Definition 6.5.4 [**brauer graph**] *Let* $\chi, \psi \in \mathrm{Irr}(G)$*. We say that* ϕ *and* ψ *are* linked *if there exists* $\phi \in \mathrm{IBr}(G)$ *with* $d_{\phi\chi} \neq 0 \neq d_{\phi\psi}$*. The graph on* $\mathrm{IBr}(G)$ *with edges the linked pairs is called the* Brauer graph *of* G*. We say* χ *and* ψ *are connected if* ϕ *and* ψ *lie in the same connected component of the Brauer graph.*

Corollary 6.5.5 [**blocks and connected component**]

(a) [**a**] *Let* $\mathcal{A} \subseteq \mathrm{Irr}(G)$*. Then* $\mathcal{A}^{\dagger\dagger}$ *consist of all simple characters linked to some element of* $\mathcal{A}$*.*

(b) [**b**] *Let* $\mathcal{A} \subseteq \mathrm{Irr}(G)$*. Then* $\mathcal{A}$ *is union of connected components of the Brauer graph iff and only if* $\mathcal{A} = \mathcal{A}^{\dagger\dagger}$*.*

(c) [**c**] *If* B *is a block then* $\mathrm{Irr}(B)$ *is a union of connected components of the Brauer Graph.*

Proof: (a) Let $\psi \in \mathrm{Irr}(G)$. Then

$$\begin{gathered}\psi \text{ is linked to some element of } \mathcal{A}\\ \text{iff}\\ \text{there exists } \chi \in \mathcal{A} \text{ and } \phi \in \mathrm{IBr}(G) \text{ with } d_{\phi\chi} \neq 0 \neq d_{\phi\psi}\\ \text{iff}\\ \text{there exists } \phi \in \mathcal{A}^\dagger \text{ with } d_{\phi\psi} \neq 0\\ \text{iff}\\ \psi \in \mathcal{A}^{\dagger\dagger}\end{gathered}$$

So (a) holds.

(b) follows immediately from (a).

(c) By 6.5.3 $\mathrm{Irr}(B)^{\dagger\dagger} = \mathrm{IBr}(B)^\dagger = \mathrm{Irr}(B)$.

Proposition 6.5.6 [**osima**] *Let $\mathcal{A} \subseteq \mathrm{Irr}(G)$ with $\mathcal{A} = \mathcal{A}^{\dagger\dagger}$. Let $x \in \tilde{G}$ and $y \in G$. Then*

$$\sum_{\chi \in \mathcal{A}} \chi(x)\chi(y) = \sum_{\phi \in \mathcal{A}^\dagger} \phi(x)\Phi_\phi(y)$$

Proof: We compute

$$\begin{array}{lccc} & \displaystyle\sum_{\chi \in \mathcal{A}} \chi(x)\chi(y) & = & \displaystyle\sum_{\chi \in \mathcal{A}} \left(\sum_{\phi \in \mathrm{IBr}(G)} d_{\phi\chi}\phi(x) \right) \chi(y) \\ = & \displaystyle\sum_{\chi \in \mathcal{A}} \left(\sum_{\phi \in \mathcal{A}^\dagger} d_{\phi\chi}\phi(x) \right) \chi(y) & = & \displaystyle\sum_{\chi \in \mathcal{A}^\dagger} \left(\sum_{\phi \in \mathcal{A}} d_{\phi\chi}\chi(y) \right) \phi(x) \\ = & \displaystyle\sum_{\chi \in \mathcal{A}^\dagger} \left(\sum_{\phi \in \mathrm{Irr}(G)} d_{\phi\chi}\chi(y) \right) \phi(x) & = & \displaystyle\sum_{\chi \in \mathcal{A}^\dagger} \Phi_\phi(y)\phi(x) \end{array}$$

□

Corollary 6.5.7 (Weak Block Orthogonality) [**weak block orthogonality**] *Let B be block of G, $x \in \tilde{G}$ and $y \in G \setminus \tilde{G}$. Then*

$$\sum_{\chi \in \mathrm{Irr}(B)} \chi(x)\overline{\chi(y)} = 0$$

Since $\mathrm{Irr}(G)^{\dagger\dagger} = \mathrm{Irr}(G)$ we can apply 6.5.6:

$$\sum_{\chi \in \mathrm{Irr}(B)} \chi(x)\overline{\chi(y)} = \sum_{\chi \in \mathrm{Irr}(B)} \chi(x)\chi(y^{-1}) = \sum_{\phi \in \mathcal{A}^\dagger} \phi(x)\Phi_\phi(y^{-1})$$

Since y^{-1} $\not{\tilde{G}}$ 6.4.11(c) implies $\Phi_\phi(y^{-1} = 0$ and so the Corollary is proved. □

Definition 6.5.8 [**def:ea**]

(a) [**a**] *For $M \in \mathcal{S}$ and $\chi = \chi_M$ put $e_\chi = e_M$(see 3.1.3(d).*

(b) [**b**] *For $\mathcal{A} \subseteq \mathrm{Irr}(G)$, put $e_\mathcal{A} = \sum_{\chi \in \mathcal{A}} e_\chi$.*

Corollary 6.5.9 [**ea in ai(tilde g)**] *Let $\mathcal{A} \subseteq \mathrm{Irr}(G)$ with $\mathcal{A} = \mathcal{A}^{\dagger\dagger}$. Then $e_\mathcal{A} \in ZA_I\tilde{G}$.*

Proof: Let $\chi \in \mathcal{A}$ and $g \in G$. By 3.2.12(a), g coefficents of e_χ is $\frac{1}{|G|}\chi(1)\overline{\chi}(x)$ Let f_g be the g-coefficent of $e_\mathcal{A}$. Then by 6.5.6

$$f_g = \frac{1}{|G|} \sum_{\chi \in \mathcal{A}} \chi(1)\chi(x^{-1}) = \frac{1}{|G|} \sum_{\phi \in \mathcal{A}^\dagger} \phi(1)\Phi_\phi(g^{-1})$$

If $g \notin \tilde{G}$ we conclude that $f_g = 0$ and so

$$(*) \qquad e_{\mathcal{A}} \in \mathbb{C}\tilde{G}$$

Suppose now that $g \in \tilde{G}$. Then using 6.5.6 one more time:

$$f_g = \frac{1}{|G|}\sum_{\chi\in\mathcal{A}} \chi(g^{-1})\chi(1) = \frac{1}{|G|}\sum_{\phi\in\mathcal{A}^\dagger} \phi(g^{-1})\Phi_\phi(1) = \sum_{\phi\in\mathcal{A}^\dagger} \phi(g^{-1})\frac{\Phi_\phi(1)}{|G|}$$

By 6.4.13 $\frac{\Phi_\phi(1)}{|G|} \in \mathbb{A}_I$. Also $\phi(g^{-1} \in \mathbb{A} \in \mathbb{A}_I$ and so $f_g \in \mathbb{A}_i$. Thus $e_{\mathcal{A}} \in \mathbb{A}G$. Together with (*) and the fact that e_χ is class function we see that the Corollary holds. □

Lemma 6.5.10 [**unions of blocks**] *Let $\mathcal{A} \subseteq \mathrm{Irr}(G)$ with $e_{\mathcal{A}} \in Z(\mathbb{A}_I(G))$. Then $\mathcal{A} = \bigcup_{i=1}^k \mathrm{Irr}(B_i)$ for some blocks $B_1, \ldots B_k$.*

Proof: Let $\chi, \psi \in \mathrm{Irr}(G)$. Then $\omega_\chi(e_\psi) = \delta_{\chi\psi}$ and so $\omega_\chi(e_{\mathcal{A}}) = 1$ if $\chi \in \mathcal{A}$ and $\omega_\chi(e_{\mathcal{A}}) = 0$ otherwise. By assumption $e_{\mathcal{A}} \in Z(\mathbb{A}_I(G))$ and so $\lambda_\chi(e^*_{\mathcal{A}}) = \omega_\chi(e_{\mathcal{A}})$ and so

$$(*) \qquad \chi \in \mathcal{A} \text{ iff } \lambda_\chi(e^*_{\mathcal{A}}) = 1$$

Let B be the block containg χ and $\psi \in \mathrm{Irr}(B)$. Then $\lambda_\chi(e^*_{\mathcal{A}}) = \lambda_\psi(e^*_{\mathcal{A}})$ and so by (*), $\chi \in \mathcal{A}$ iff $\psi \in \mathcal{A}$. □

Theorem 6.5.11 [**block=connected components**] *If B is block, then $\mathrm{Irr}(B)$ is connected in the Brauer Graph. So the connected components of the Brauer graph are exactly the $\mathrm{Irr}(B)$, B a block.*

Proof: If B is a block then by 6.5.5(c), $\mathrm{Irr}(B)$ is the union of connected components. Connversely if $\mathcal{A}$ is a connected component then by 6.5.9 $e_A \in Z(A_I G)$ and so by 6.5.10 $\mathcal{A}$ is a union of blocks. □

Definition 6.5.12 [**def:fb**]

(a) [**a**] *Let B be a block. Then $e_B = e^*_{\mathrm{Irr}(B)}$ and $f_B = e_{\mathrm{Irr}(B)}$.*

(b) [**b**] *Let $\mathcal{A}$ be set of blocks. Then $e_{\mathcal{A}} = \sum_{B\in\mathcal{A}} e_B$ and $f_{\mathcal{A}} = \sum_{Bin\mathcal{B}} f_B$*

(c) [**c**] *Let B be block, then $\mathbb{F}B := \mathbb{F}Ge_B$.*

(d) [**d**] *If $\mathcal{A}$ is a set of blocks, then $\mathbb{F}\mathcal{A} = \mathbb{F}Ge_{\mathcal{A}}$.*

(e) [**e**] *Let B be a block then $\lambda_B = \lambda_\phi$ for any $\phi \in \mathrm{IBr}(G)$.*

(f) [**f**] *Let B be a block, then $\mathcal{S}_p(B) = \{M \in \mathcal{S}_p \mid \phi_M \in B\}$ and $\mathcal{S}(B) = \{M \in \mathcal{S} \mid \chi_M \in B\}$*

Lemma 6.5.13 [**omega chi fy**] *Let X, Y be blocks and $\chi \in X$. Then $\omega_\chi(f_Y) = \delta XY$ and $\lambda_X(e_Y) = \delta_{XY}$*

Proof: This follows from $\omega_\chi(e_\psi) = \delta_{\chi\psi}$ for all $\chi\psi \in \mathrm{Irr}(G)$. □

Theorem 6.5.14 [**structure of fg**]

(a) [**a**] $\sum_{B \in \mathrm{Bl}(G)} e_B = 1$.

(b) [**b**] $e_B \in Z(\mathbb{F}G)$ *for all blocks B*

(c) [**c**] $e_X e_Y = 0$ *for any distinct blocks X and Y.*

(d) [**d**] $e_B^2 = e_B$ *for all blocks b*

(e) [**e**] $\mathbb{F}G = \bigoplus_{B \in \mathcal{B}} \mathbb{F}B$.

(f) [**f**] $Z(\mathbb{F}G) = \bigoplus_{B \in \mathcal{B}} Z(\mathbb{F}B)$.

(g) [**g**] $\mathrm{J}(\mathbb{F}G) = \bigoplus_{B \in \mathcal{B}} \mathrm{J}(\mathbb{F}B)$.

(h) [**h**] *Let X, Y be blocks. Then $\lambda_X(e_Y) = \delta_{XY}$.*

(i) [**i**] *Let X and Y be distincts blocks. Then $\mathbb{F}X$ annihilates all $M \in \mathcal{S}_p(Y)$.*

(j) [**j**] *Let B be a block. Then $\S_p(B)$ is set of representativves for the isomorphism classes classes of simple $\mathbb{F}B$-modules.*

Proof: (a) $\sum_{\chi \in \mathrm{Irr}(G)} e_\chi = 1$ and so also $\sum_{B \in \mathrm{Bl}(G)} e_{\mathrm{Irr}(B)} = 1$. Applying * gives (a).

(b) Since $e_\chi \in \mathrm{Z}(\mathbb{C}G)$, $e_{\mathrm{Irr}G} \in Z(\mathbb{A}_I G)$ and so (b) holds.

(c) $e_\chi e_\psi = 0$ for distinct simple characters. So $e_{Irr(X)} e_{Irr(Y)} = 0$ and so (c) holds.

(d) follows from $e^2_{\mathrm{Irr}(B)} = e_{\mathrm{Irr}(B)}$.

(e) (a) implies $\mathbb{F}G = \sum_{B \in \mathrm{Bl}(G)} \mathbb{F}B$. Let $B \in \mathcal{B}$ and $\mathcal{B} = \mathrm{Bl}(G) \setminus \{B\}$. Then by (c) $\mathbb{F}B \cdot \mathbb{F}\mathcal{B} = 0$. Moreover if $x \in \mathbb{F}B$ then $e_B x = x$ and if $x \in \mathbb{F}\mathcal{B}$ then $e_B x = 0$. Thus $\mathbb{F}B \cap \mathbb{F}\mathcal{B} = 0$ and so (d) holds.

(f) follows from (d).

(g) follows from (d) and 2.5.16(e).

(h) Let $\chi \in \mathrm{Irr}(X)$. Then $\lambda_X(e_Y) = \lambda_X(e^*_{\mathrm{Irr}(Y)}) = \omega_X((e_{\mathrm{Irr}(Y)})^* = \delta^*_{XY} = \delta_{XY}$.

(i) Let $M \in \mathcal{S}_p(Y)$. Then e_X acts as the scalar $\lambda_\phi(e_X) = \lambda_Y(e_X)$ on M. So by (h) e_X annhilates M. Thus also $\mathbb{F}X = \mathbb{F}Ge_X$ annihilates M.

(j) Any simple $\mathbb{F}B$-module is also a simple $\mathbb{F}G$-module. So (j) follows from (i). □

Theorem 6.5.15 [**zfb is local**] *$Z(\mathbb{F}B)$ is a local ring with unique maximal ideal* $\mathrm{J}(\mathrm{Z}(\mathbb{F}B)) = \ker\lambda_B \cap Z(\mathbb{F}B)$.

Proof: Let $M \in \mathcal{S}_p(B)$ and $z \in \mathrm{Z}(\mathbb{F}(B))$. Then z acts as the scalar $\lambda_B(z)$ on M. So z annihilates M if and only if $z \in \ker\lambda_B$. Thus $\mathrm{Z}(\mathbb{F}(B)) \cap \mathrm{A}_{\mathbb{F}B}(M) = \mathrm{Z}(\mathbb{F}B) \cap \ker\lambda_B$ and so

$$\mathrm{J}(\mathrm{Z}(\mathbb{F}B)) \overset{6.3.4}{=} \mathrm{Z}(\mathbb{F}B)) \cap \mathrm{J}(\mathbb{F}(B)) \underset{6.5.14(j)}{\overset{2.4.7}{=}} \mathrm{Z}(\mathbb{F}(B)) \cap \bigcap_{M \in \mathcal{S}_p(B)} \mathrm{A}_{\mathbb{F}B}(M) = Z(\mathbb{F}B) \cap \ker\lambda_B$$

So $\mathrm{J}(\mathrm{Z}(\mathbb{F}B)) = \ker\lambda_B \cap Z(\mathbb{F}B)$. Since $\mathrm{Z}(\mathbb{F}B)/ker\lambda_B \cap Z(\mathbb{F}B) \cong \mathrm{Im}\,\lambda_B = \mathbb{F}$ we conclude that $\mathrm{J}(\mathrm{Z}(\mathbb{F}B))$ is a maximal ideal in $Z(\mathbb{F}(B))$. This clearly implies that $\mathrm{J}(\mathrm{Z}(\mathbb{F}B))$ is the unique maximal ideal in $\mathbb{F}(B)$. □

Corollary 6.5.16 [**blocks indecomposable**] *Let B be a block.*

(a) [**a**] *Then $\mathbb{F}B$ is indecompsable as a ring.*

(b) [**b**] *Let e be an idempotent in $\mathrm{Z}F(G)$ then e_T for some $T \subseteq \mathrm{Bl}(G)$.*

Proof: (a) Suppose $\mathbb{F}B = X \oplus Y$ for some proper ideals X and Y. Then both X and Y have an identity. Thus $Z(X) \neq 0$, $Z(Y) \neq 0$ and $Z(\mathbb{F}(B) = \mathrm{Z}(X) \oplus Z(Y)$, a contradiction to 6.5.15.

(b) Since $e = \sum_{B \in \mathrm{Bl}(B)} ee_B$ and each non-zero ee_B is an idempotent we may assume that $e = ee_B \in \mathbb{F}B$ for some block B. Then $\mathbb{F}B = e\mathbb{F}B \oplus (e - e_B)\mathbb{F}B$ and (a) implies $e - e_B = 0$ and so $e = e_B$. □

Lemma 6.5.17 [**phi fb**] *Let B be a block then*

$$\phi_{\mathbb{F}B} = \sum_{\chi \in \mathrm{Irr}(B)} \chi(1)\tilde{\chi} = \sum_{\phi \in IBr} \Phi_\phi(1)\phi$$

Proof: By 3.2.11(c) $\chi_{\mathbb{C}G} = \sum_{\chi \in \mathrm{Irr}(G)} \chi(1)\chi$. So by 6.4.7 applied to the $\mathbb{A}_I$-lattice $\mathbb{A}_I G$ in $\mathbb{C}G$,

$$(1) \qquad \phi_{\mathbb{F}G}G = \tilde{\chi}_{\mathbb{C}G} = \sum_{\chi \in \mathrm{Irr}(G)} \chi(1)\tilde{\chi} = \sum_{B \in \mathrm{Bl}(G)} \sum_{\chi \in B} \chi(1)\tilde{\chi}$$

Observe that

$$(2) \qquad \sum_{\chi \in B} \chi(1)\tilde{\chi} = \sum_{\chi \in \mathrm{Irr}(B)} \chi(1) \left(\sum_{\phi \in \mathrm{Irr}(B)} d_{\phi\chi}\phi \right) = \sum_{\phi \in \mathrm{IBr}(B)} \Phi_\phi(1)\phi$$

and so by (1)

$$\phi_{\mathbb{F}G} = \sum_{B\in \mathrm{Bl}(G)} \sum_{\phi\in \mathrm{IBr}(B)} \Phi_\phi(1)\phi \tag{3}$$

Now let B a block. If M is composition factor for $\mathbb{F}G$ of $\mathbb{F}B$ then e_B acts identity on M. So by 6.5.14 $\phi_M \in B$. It follows that

$$\phi_{\mathbb{F}B} = \sum_{\phi\in \mathrm{IBr}(G)} d_\phi \phi \tag{4}$$

for some $d_\phi \in \mathbb{N}$. Since $\mathbb{F}G = \sum_{B\in \mathrm{Bl}(G)} \mathbb{F}B$ we conclude

$$\phi_{\mathbb{F}G} = \sum_{B\in \mathrm{Bl}(G)} \sum_{\phi\in \mathrm{IBr}(B)} d_\phi \phi \tag{5}$$

From (3) and (5) and the linear independence of $\mathrm{IBr}(G)$ we get $d_\phi = \Phi_\phi(1)$ for all $\phi \in \mathrm{IBr}(G)$. The lemma now follows from (4) and (2). □

6.6 Brauer's Frist Main Theorem

Definition 6.6.1 [**def:defect group c**] *Let C be a conjugacy class of G.*

(a) [**z**] *A* defect group *of C is a Sylow p-subgroup of $C_G(x)$ for some $x \in C$.*

(b) [**a**] $\mathrm{Syl}(C)$ *is the set of all defect groups of G.*

(c) [**b**] *We fix $g_C \in C$ and $D_C \in \mathrm{Syl}_p(C_G(g_C))$.*

(d) [**d**] *Let $\mathcal{A}$ and $\mathcal{B}$ be set of subgroups of G. We write $\mathcal{A} \prec \mathcal{B}$ if for all $A \in \mathcal{A}$ there exists $B \in \mathcal{B}$ with $A \leq B$.*

(e) [**e**] *Let $\mathcal{A}$ be a set subgroups of G. Then $\mathcal{C}_\mathcal{A} = \{C \in \mathcal{C} \mid \mathrm{Syl}(C) \prec \mathcal{A}\}\}$ and $\mathrm{Z}_\mathcal{A}(\mathbb{F}G) = \mathbb{F}\langle a_C \mid C \in \mathcal{C}_\mathcal{A}\rangle$.*

(f) [**f**] *For $A \subseteq \mathrm{Z}(\mathbb{F}G)$ set $\mathcal{C}_A = \{C \in \mathcal{C}(G) \mid a(g_C) \neq 0$ for some $a \in A\}$.*

(g) [**g**] *For $A, B, C \in \mathcal{C}$ put $K_{ABC} = \{(a, b) \in A \times B \mid ab = g_C\}$.*

Lemma 6.6.2 [**trivial zdfg**] *Let $z \in \mathrm{Z}(\mathbb{F}G)$ and $\mathcal{D}$ a set of subgroups of G. Then $z \in \mathrm{Z}_\mathcal{D}(\mathbb{F}G)$ iff $a_C \in \mathrm{Z}_\mathcal{D}(\mathbb{F}G)$ for all $C \in \mathcal{C}_z$ and iff $\mathrm{Syl}(C) \prec \mathcal{D}$ for all $C \in \mathcal{C}_z$.*

Proof: Since $z = \sum_{C\in\mathcal{C}(G)} z(g_C)a_C$ and $(a_C \mid C \in \mathcal{C}(G))$ is linearly independent this follows immediately from the definition of $\mathrm{Z}_{\mathcal{D}}(\mathbb{F}G)$. $\square$

Lemma 6.6.3 [**syl c prec syl a**] *Let* $A, B, C \in \mathcal{C}$

(a) [**a**] $|K_{ABC}| \equiv |\{(a,b) \in \mathcal{A} \times \mathcal{B} \mid a, b \in C_G(D_C), ab = g_C\}| \pmod p$.

(b) [**b**] *If* $p \nmid |K_{ABC}|$ *then* $\mathrm{Syl}(C) \prec \mathrm{Syl}(A)$.

Proof: (a) Observe that $C_G(g_C)$ acts on K_{ABC} by coordinate wise conjugation. All non-trivial orbits of D_C on K_{ABC} have length divisble by p and so (a) holds.

(b) By (a) there exists $a \in \mathcal{A}$ with $D_C \in C_G(a)$ and so $D_C \leq D$ for some $D \in \mathrm{Syl}_p(C_G(a)$. Since G acts transitively on $\mathrm{Syl}(C)$, $\mathrm{Syl}(C) \prec \mathrm{Syl}(A)$. $\square$

Proposition 6.6.4 [**zdfg ideal**] *Let* $\mathcal{D}$ *be set of subgroups of* G*. Then* $\mathrm{Z}_{\mathcal{D}}(\mathbb{F}G)$ *is an ideal in* G*.*

Proof: Let $A, B \in \mathcal{C}$ with $\mathrm{Syl}(A) \prec \mathcal{D}$. Then in $\mathbb{F}G$:

$$a_A a_B = \sum_{C\in\mathcal{C}} |K_{ABC}| a_C = \sum_{C\in\mathcal{C}, \phi\nmid|K_{ABC}|} |K_{ABC} a_C$$

By 6.6.3 $\mathrm{Syl}(C) \prec \mathrm{Syl}(A) \prec \mathcal{D}$ whenever $p \nmid |K_{ABC}|$. Then $a_C \in \mathrm{Z}_{\mathcal{D}}(\mathbb{F}G)$ and so $a_A a_B \in \mathrm{Z}_{\mathcal{D}}(\mathbb{F}G)$. $\square$

Definition 6.6.5 [**def:fa**]

(a) [**a**] $\mathfrak{G}$ *be the set of sets of of subgroups of* G*.* $\mathfrak{G}_\circ$ *consist of all* $\mathcal{A} \in \mathfrak{G}$ *such that* $A, B \in \mathcal{A}$ *with* $A \subseteq B$ *implies* $A = B$*.*

(b) [**b**] *If* $\mathcal{A} \in \mathfrak{G}$*, then* $\max(\mathcal{A})$ *is the set maximal elements of* $\mathcal{A}$ *with respect to inclusion.*

(c) [**c**] *Let* $\mathcal{A}, \mathcal{B} \in \mathfrak{G}$*. Then* $\mathcal{A} \wedge \mathcal{B} := \max(\{A \cap B \mid A \in \mathcal{A}, B \in \mathcal{B}\})$*.*

(d) [**d**] *Let* $\mathcal{A}, \alpha B \in \mathfrak{G}$*. The* $\mathcal{A} \vee \mathcal{B} = \max(\mathcal{A} \cup \mathcal{B})$*.*

Lemma 6.6.6 [**basis fa**] *Let* $\mathcal{A}, \mathcal{B}, \mathcal{D} \in \mathfrak{G}$*.*

(a) [**a**] $\prec$ *is reflexive and transitive.*

(b) [**b**] $\mathcal{A} \prec \max \mathcal{A}$ *and* $\max \mathcal{A} \prec \mathcal{A}$*.*

(c) [**c**] $\max(A) \in \mathfrak{G}_\circ$ *and if* $\mathcal{A}$ *is* G*-invariant so is* $\max \mathcal{A}$*.*

(d) [**d**] $\mathcal{A} \prec \mathcal{B}$ *iff* $\max(\mathcal{A}) \prec \max(\mathcal{B})$*.*

(e) [**e**] *If all elements in $\mathcal{A}$ have the same size, $\mathcal{A} \in \mathfrak{G}_\circ$.*

(f) [**f**] *If $\mathcal{A}$ is conjugacy class of subgroups of G, then $\mathcal{A} \in \mathfrak{G}_\circ$.*

(g) [**g**] $\mathcal{C}_\mathcal{A} = \mathcal{C}_{\max(\mathcal{A})}$ *and* $\mathrm{Z}_\mathcal{A}(\mathbb{F}G) = \mathrm{Z}_{\max(\mathcal{A})}(\mathbb{F}G)$.

(h) [**h**] *Restricted to $\mathfrak{G}_\circ$, $\prec$ is a partial ordering.*

(i) [**i**] $(\mathcal{A} \vee \mathcal{B}) \prec \mathcal{D}$ *iff* $\mathcal{A} \prec \mathcal{D}$ *and* $\mathcal{B} \prec \mathcal{D}$.

(j) [**j**] $\mathcal{D} \prec (\mathcal{A} \wedge \mathcal{B})$ *iff* $\mathcal{D} \prec \mathcal{A}$ *and* $\mathcal{D} \prec \mathcal{B}$.

Proof:

(a) Obvious.

(b) Clearly $\max \mathcal{A} \prec \mathcal{A}$. Let $A \in \mathcal{A}$ since G is finite we can choose $B \in \mathcal{A}$ of maxial size with $A \subseteq B$. Then $B \in \max(\mathcal{A}0$ and so $\mathcal{A} \prec \max \mathcal{A}$.

(c) If $A, B \in \max(\mathcal{A})$ with $A \subseteq B$, then $A = B$ by maximalty of A.

(d) Follows from (a) and (b).

(e) is obvious.

(f) follows from (e).

(g) The first statement follows from (d) and the second from the first.

(h) Let $\mathcal{A}, \mathcal{B} \in \mathfrak{A}(G)$ with $\mathcal{A} \prec \mathcal{B}$. Let $A \in \mathcal{A}$ and choose $B \in \mathcal{B}$ with $A \leq B$. Then choose $D \in \mathcal{A}$ with $B \leq D$. Then $A \leq D$ and so $A = D$ and $A = B$. Thus $\mathcal{A} \subseteq \mathcal{B}$. By symmetry $\mathcal{B} \subseteq \mathcal{A}$. So $\mathcal{A} = \mathcal{B}$. (h) now follows from (a).

(i) By (d) $(\mathcal{A} \vee \mathcal{B}) \prec \mathcal{D}$ iff $(\mathcal{A} \cup \mathcal{B}) \prec \mathcal{D}$ and so iff $\mathcal{A} \prec \mathcal{D}$ and $\mathcal{B} \prec \mathcal{D}$.

(j) By (d) $\mathcal{D} \prec (\mathcal{A} \wedge \mathcal{B})$ iff $\mathcal{D} \prec \{A \cap B \mid A \in \mathcal{A}, B \in \mathcal{B}\}$ and so iff $\mathcal{D} \prec \mathcal{A}$ and $\mathcal{D} \prec \mathcal{B}$. $\square$

Lemma 6.6.7 [**basic zdfg**] *Let $\mathcal{D}, \mathcal{E} \in \mathfrak{D}_\circ$.*

(a) [**a**] *If $\mathcal{D} \prec \mathcal{E}$, then $\mathcal{C}_\mathcal{D} \subseteq \mathcal{C}_\mathcal{E}$ and* $\mathrm{Z}_\mathcal{D}(\mathbb{F}G) \leq \mathrm{Z}_\mathcal{E}(\mathbb{F}G)$.

(b) [**b**] $(\mathcal{D} \wedge \mathcal{E}) \prec \mathcal{D}$.

(c) [**c**] $\mathcal{C}_\mathcal{D} \cap \mathcal{C}_\mathcal{E} = \mathcal{C}_{\mathcal{D} \wedge \mathcal{E}}$ *and* $\mathrm{Z}_\mathcal{D}(\mathbb{F}G) \cap \mathrm{Z}_\mathcal{E}(\mathbb{F}G) = \mathrm{Z}_{\mathcal{D} \wedge \mathcal{E}}(\mathbb{F}G)$

(d) [**d**] *Let $A \subseteq \mathrm{Z}(\mathbb{F}(G))$. Let $\mathfrak{G}_\circ(A) := \{\mathcal{A} \in \mathfrak{G}_\circ \mid \mathrm{Z}_\mathcal{D}(\mathbb{F}G)$. Then there exists a unique $\mathcal{E} \in \mathfrak{G}_\circ(A)$ with $\mathcal{E} \prec \mathcal{D}$ for all $\mathcal{D} \in \mathfrak{G}_\circ(A)$. We denote this $\mathcal{E}$ by* $\mathrm{Syl}(A)$.

(e) [**e**] *If $A \subseteq B \subseteq \mathrm{Z}(\mathbb{F}(G))$, then* $\mathrm{Syl}(A) \prec \mathrm{Syl}(B)$.

(f) [**f**] *For all $C \in \mathcal{C}$,* $\mathrm{Syl}(a_C) = \mathrm{Syl}(C)$

(g) [**g**] $\mathrm{Syl}(\mathrm{Z}(\mathbb{F}G)) = \mathrm{Syl}(G)$

(h) [**h**] *For all $A \subseteq \mathrm{Z}(\mathbb{F}(G))$,* $\mathrm{Syl}(A) \prec \mathrm{Syl}(G)$, *that is* $\mathrm{Syl}(A)$ *is a set of p subgroups of G.*

(i) [**i**] *Let $A, B \subseteq \mathrm{Z}(\mathbb{F}G)$. Then* $\mathrm{Syl}(A \cup B) = \mathrm{Syl}(A) \vee \mathrm{Syl}(B)$.

(j) [**j**] *Let* $A \subset \mathrm{Z}(\mathbb{F}G)$ *then* $\mathrm{Syl}(A) = \mathrm{Syl}(\{a_C \mid C \in \mathcal{A}\}) = \bigvee_{C \in \mathcal{C}_A} \mathrm{Syl}(C)$.

Proof: (a) and (b) are obvious.

(c) Let $C \in \mathcal{C}$. Then $C \in \mathcal{C}_{\mathcal{D}} \cap \mathcal{C}_{\mathcal{E}}$ iff $\mathrm{Syl}(C) \prec \mathcal{D}$ and $\mathrm{Syl}(C) \prec \mathcal{E}$. Thus by **??** iff $\mathrm{Syl}(C) \prec \mathcal{D} \wedge \mathcal{E}$ and iff $C \in \mathcal{C}_{D \wedge \mathcal{E}}$. So the first statement in (b) holds.

Since $\{a_C \mid C \in \mathcal{C}\}$ is $\mathbb{F}$-linearly independent

$$\mathrm{Z}_{\mathcal{D}}(\mathbb{F}G) \cap \mathrm{Z}_{\mathcal{E}}(\mathbb{F}G) = \mathbb{F}\{a_C \mid C \in \mathcal{C}_{\mathcal{D}} \cap \mathcal{C}_{\mathcal{E}}\}$$

So the second statement in (c) follows from the first.

(d) Put $\mathcal{E} = \bigwedge_{\mathcal{D} \in \mathfrak{G}_\circ(A)} \mathcal{D}$. By (c), $A \leq \mathrm{Z}_{\mathcal{E}}(\mathbb{F}G)$ and by (b) $\mathcal{E} \prec \mathcal{D}$ for all $\mathcal{D} \in \mathfrak{A}$. Since $\prec$ is antisymmetric on $\mathfrak{G}_\circ$, $\mathcal{E}$ is unique.

(e) Observe that $\mathrm{Syl}(B) \in \mathfrak{G}_\circ$ and so (e) follows from (d).

(f) Since $\mathrm{Syl}(C) \prec \mathrm{Syl}(C)$, $C \in \mathcal{C}_{\mathrm{Syl} C}$ and so $a_C \in \mathrm{Z}_{\mathrm{Syl}(C)}(\mathbb{F}G)$. Since $a_C \in \mathrm{Z}_{\mathrm{Syl}(a_C)}(\mathbb{F}G)$ we conclude from 6.6.2 that $C \in \mathcal{C}_{\mathrm{Syl}(a_c)}$ and so $\mathrm{Syl}(C) \prec \mathrm{Syl}(a_C)$. Since $\prec$ is anti-symmetric (f) holds.

(g) Let $S \in \mathrm{Syl}(G)$, $1 \neq x \in Z(S)$ and $C = {}^G x$. Then clearly $\mathrm{Syl}(C) = \mathrm{Syl}(G)$ and so by (e) and (f), $\mathrm{Syl}(\mathrm{Z}(\mathbb{F}G)) \prec \mathrm{Syl}(G)$. Clearly $\mathrm{Syl}(C) \prec \mathrm{Syl}(G)$ for all $C \in \mathcal{C}$. So $\mathcal{C}_{\mathrm{Syl}(G)} = \mathcal{C}$ and $\mathrm{Z}_{\mathrm{Syl}(G)}(\mathbb{F}G) = \mathrm{Z}(\mathbb{F}G)$. (d) implies $\mathrm{Syl}(\mathrm{Z}(\mathbb{F}G)) \subseteq \mathrm{Syl}(G)$ and so (g) holds.

(h) follows from (e) and (g).

(i) We have $\mathrm{Z}_{\mathrm{Syl}(A) \vee \mathrm{Syl}(B)}(\mathbb{F}G) = \mathrm{Z}_{\mathrm{Syl}(A) \cup \mathrm{Syl}(B)}(\mathbb{F}G) = \mathrm{Z}_{\mathrm{Syl}(A)}(\mathbb{F}G) + \mathrm{Z}_{\mathrm{Syl}(B)}(\mathbb{F}G)$ and so $A \cup B \subseteq \mathrm{Z}_{\mathrm{Syl}(A) \vee \mathrm{Syl}(B)}(\mathbb{F}G)$. Thus $\mathrm{Syl}(A \cup B) \prec \mathrm{Syl}(A) \vee \mathrm{Syl}(B)$. Since $A \leq Z_{\mathrm{Syl}(A \cup B)}(\mathbb{F}G$, $\mathrm{Syl}(A) \prec \mathrm{Syl}(A \cup B)$ and by symmetry $\mathrm{Syl}(B) \prec \mathrm{Syl}(A \cup B)$. Thus $\mathrm{Syl}(A) \vee \mathrm{Syl}(B) \prec \mathrm{Syl}(A \cup B)$ and (i) holds.

(j) By 6.6.2 $\mathrm{Syl}(A) = \mathrm{Syl}(\{a_C \mid C \in \mathcal{C}_A\}$. By (i) and (f) $\mathrm{Syl}(\{a_C \mid C \in \mathcal{C}_A\} = \bigvee_{C \in \mathcal{C}_A} \mathrm{Syl}(a_C)$. □

Lemma 6.6.8 [**eb in sum k**] *Let* B *be a block and* $\mathcal{K}$ *a set of ideals in* $\mathrm{Z}(\mathbb{F}G)$ *with* $e_B \in \sum \mathcal{K}$. *Then* $Z(\mathbb{F}B) \leq K$ *for some* $K \in \mathcal{K}$.

Proof: Since $e_B = e_B^2 \in \sum_{K \in \mathcal{K}} e_B K$ there exists $K \in \mathcal{K}$ with $e_B K \not\leq \mathrm{J}(\mathrm{Z}(\mathbb{F}B))$. Since by 2.2.4 all elements in $\mathrm{Z}(\mathbb{F}B)) \setminus \mathrm{J}(\mathrm{Z}(\mathbb{F}B))$ are invertible, $\mathrm{Z}(\mathbb{F}B) = e_B K \leq K$. □

Definition 6.6.9 [**sylb**] *Let* B *be a block. Then* $\mathrm{Syl}(B) := \mathrm{Syl}(e_B)$. *The members of* $\mathrm{Syl}(B)$ *are called the* defect groups *of* B.

Proposition 6.6.10 [**sylow theorem for blocks**] *Let* B *be block of* G. *Then* G *acts transitively on* $\mathrm{Syl}(B)$.

Proof: Let $\mathfrak{D}$ be the set of orbits for G on $\mathrm{Syl}(B)$. Then clearly $\mathcal{C}_{\mathrm{Syl}(B)} = \bigcup_{\mathcal{D} \in \mathfrak{D}} C_{\mathcal{D}}$ and so

$$e_B \in \mathrm{Z}_{\mathrm{Syl}(B)}(\mathbb{F}G) = \sum_{\mathcal{D}\in\mathfrak{D}} \mathrm{Z}_{\mathcal{D}}(\mathbb{F}G)$$

So by 6.6.8 $e_B \in \mathrm{Z}_{\mathcal{D}}(\mathbb{F}G)$ for some $\mathcal{D} \in \mathfrak{D}$. Thus by 6.6.7(d) implies $\mathrm{Syl}(B) = \mathrm{Syl}(e_B) \prec \mathcal{D}$. Since $\mathcal{D} \subseteq \mathrm{Syl}(e_B)$ we get $\mathrm{Syl}(e_B) = \mathcal{D}$. □

Definition 6.6.11 [**def:defect class**] *Let B be a block and $C \in \mathcal{C}(G)$. Then C is called a* defect class *of B provided that $\lambda_B(a_C) \neq 0 \neq \epsilon_B(g_C)$.*

Lemma 6.6.12 [**existence of defect class**] *Every block has at least one defect class.*

Proof: We have $e_B = \sum_{C\in\mathcal{C}(G)} e_B(g_C)a_C$ and so

$$1 = \lambda_B(e_B) = \sum_{C\in\mathcal{C}(G)} e_B(g_C)\lambda(a_C).$$

Proposition 6.6.13 [**min-max**] *Let B be a block of G and C a conjuagacy class.*

(a) [**a**] *If $\lambda_B(a_C) \neq 0$, then* $\mathrm{Syl}(B) \prec \mathrm{Syl}(C)$.

(b) [**b**] *If $\epsilon_B(a_C) \neq 0$ then* $\mathrm{Syl}(C) \prec Syl(B)$

(c) [**c**] *If C is a defect class of B, then* $\mathrm{Syl}(C) = \mathrm{Syl}(B)$.

Proof: (a) Since $\lambda_B(a_C) \neq 0$ and $a_C \in Z_{\mathrm{Syl}(C)}(\mathbb{F}G)$ we have $Z_{\mathrm{Syl}(C)}(\mathbb{F}G) \nleq \ker\lambda_B$. Since λ_B has codimension 1 on $\mathrm{Z}(\mathbb{F}G)$ we conclude

$$\mathrm{Z}(\mathbb{F}G) = \ker\lambda_B + \mathrm{Z}_{\mathrm{Syl}(C)}(\mathbb{F}G)$$

Since $e_B \notin \ker\lambda_B$ 6.6.8 implies $e_B \in \mathrm{Z}_{\mathrm{Syl}(C)}(\mathbb{F}G)$. Thus by 6.6.7(d), $\mathrm{Syl}(B) \prec \mathrm{Syl}(C)$.
(b) This follows from 6.6.7(j).
(c) Follows from (a) and (b). □

Lemma 6.6.14 [**ac in jzfg**] *Let $C \in \mathcal{C}(G)$ with $C \cap C_G(O_p(G)) = 1$, then $a_C \in \mathrm{J}(\mathrm{Z}(\mathbb{F}(G))$ and so $\lambda_B(a_C) = 0$ for all blocks B.*

Proof: Let $M \in \mathcal{S}_p(G)$ and let P be an orbit for $O_p(G)$ on C and $g \in P$. By assumption $|P| \neq 1$ and so $p \mid |P|$. By 6.4.16 $\rho_M(O_p(G)) = 1$ and so $\rho_M({}^q g) = \rho_M(g)$ for all $g \in O_p(G)$. Thus $\rho_M(a_P) = |P|\rho_M(g) = 0$ and so also $\rho_M(a_C) = 0$. Thus $a_C \in \mathrm{J}(\mathbb{F}(G))$. 6.3.4 completes the proof. □

Lemma 6.6.15 [**defect classes**] *All defect class of G are contained in $C_G(O_p(G))$.*

Proof: Let C be a defect class of the block B. Then $\lambda_B(a_C) \neq 0$ and so $a_C \notin \mathrm{J}(\mathrm{Z}(\mathbb{F}B))$. Thus by 6.6.14 $C \cap C_G(O_p(G)) \neq \emptyset$. Since G is transitive on C, $C \subseteq C_G(O_p(G))$. □

Proposition 6.6.16 [**opg in defect group**]

(a) [**a**] *$O_p(G)$ is contained in any defect group of any block of G.*

(b) [**b**] *If P is a defect group of some block of G and $P \trianglelefteq G$ then $P = O_p(G)$*

(a)Let B be a block, C a defect class of B. By 6.6.15 $O_p(G) \leq C_G(g_C)$ and so $O_p(G) \leq D_C$.
(b) Follows immediateley from (a) □

Definition 6.6.17 [**def:brauer map**] *Let P be a p-subgroup. Then* $\mathrm{Br}_P : \mathrm{Z}(\mathbb{F}G) \to \mathrm{Z}(\mathbb{F}C_G(P)), a \to a \mid_{C_G(P)}$ *is called the* Brauer map *of P.*

Proposition 6.6.18 [**basic brauer map**]

(a) [**a**] *Let $K \subseteq G$. Then* $\mathrm{Br}_P(a_K) = a_{K \cap C_G(P)}$.

(b) [**b**] Br_P *is an algebra homomophism.*

(c) [**c**] *If $C_G(P) \leq H \leq N_G(P)$ then* $\operatorname{Im}\mathrm{Br}_P \leq \mathrm{Z}(\mathbb{F}H)$ *and so we obtain algebra homomorphism*

$$\mathrm{Br}_P^H : \mathrm{Z}(\mathbb{F}G) \to \mathrm{Z}(\mathbb{F}H), a \in \mathrm{Br}_P(H)$$

Proof: (a) is obvious.

(b) Let $A, B \in \mathcal{C}(G)$. We need to show that $\mathrm{Br}_P(a_A a_B) = \mathrm{Br}_P(a_A)\mathrm{Br}_P(a_B)$. Let $g \in C_G(P)$. Then the coeficient of g in $\mathrm{Br}_P(a_A a_B)$ is the order of the set

$$\{(a, b) \in A \times B \mid ab = g\}$$

The coefficient of g in $\mathrm{Br}_P(a_A a_B)$ is the order of

$$\{(a, b) \in A \times B \mid a \in C_G(P), b \in C_G(P), ab = g\}$$

Since P centralizes g, P acts on the first set and the second set consists of the fixedpoints of P. So the size of the two sets are equal modulo p and (b) holds.

(c) Let $\alpha : \mathbb{F}G \to \mathbb{F}C_G(P)$ be the restriction map. Since $C_G(P) \trianglelefteq H$, $\alpha(hah^{-1}) = \alpha(hah^{-1})$ for all $a \in G$ and all $h \in H$. Hence the same is true for all $a \in \mathbb{F}G$, $h \in H$. Thus $\operatorname{Im} Br_P = \alpha(\mathrm{Z}(\mathbb{F}G)) \leq Z(\mathbb{F}H)$. □

Lemma 6.6.19 [**kernel of brauer map**] *Let P be a p-subgroup of G.*

(a) [**a**] *Let* $C \in \mathcal{C}(G)$*. Then* $C \cap C_G(P) \neq \emptyset$ *iff* $P \prec \mathrm{Syl}(C)$*.*

(b) [**b**]

$$\ker \mathrm{Br}_P = \mathbb{F}\langle a_C \mid C \in \mathcal{C}(G), P \not\prec \mathrm{Syl}(C)\rangle$$

Proof: (a) $C \cap C_G(P) \neq \emptyset$ iff $P \leq C_G(g)$ for some $g \in C$ and so iff $P \leq D$ for some $D \in \mathrm{Syl}(C)$, that is iff $P \prec \mathrm{Syl}(C)$.

(b) Let $z = \sum_{g \in G} z(g)g = \sum_{C \in \mathcal{C}(G)} z(g_c)a_C \in \mathrm{Z}(\mathbb{F}(G))$. Then $\mathrm{Br}_P(z) = 0$ iff $z(g) = 0$ for all $g \in P$, iff $z(g_c) = 0$ for all $C \in \mathcal{C}$ with $C \cap P \neq \emptyset$ and iff $z \in \mathbb{F}\langle a_C \mid C \cap P = \emptyset\rangle$. So (a) implies (b). □

Proposition 6.6.20 [**defect and brauer map**] *Let* B *be a block of* G *and* P *be a* p*-subgroup of* G*.*

(a) [**a**] $\mathrm{Br}_P(e_B) \neq 0$ *iff* $P \prec \mathrm{Syl}(B)$*.*

(b) [**b**] $P \in \mathrm{Syl}(B)$ *iff* P *is* p*-subgroup maximal with respect to* $\mathrm{Br}_P(e_B) \neq 0$*.*

Proof: (a) By 6.6.19(b), $\mathrm{Br}_P(e_P) \neq 0$ iff $e_B \notin \mathbb{F}\langle a_C \mid C \in \mathcal{C}(G), P \not\prec \mathrm{Syl}(C)\rangle$ and so iff $P \prec \mathrm{Syl}(C)$ for some $C \in \mathcal{C}(G)$ with $e_B(g_C) \neq 0$.

If $P \prec \mathrm{Syl}(B)$, then by 6.6.13(c), $P \prec \mathrm{Syl}(C)$ for amy defect class C of B. Thus $\mathrm{Br}_P(e_B) \neq 0$.

Conversely suppose $\mathrm{Br}_P(e_P) \neq 0$ and let $C \in \mathcal{C}(G)$ with $e_B(g_C) \neq 0$ and $P \prec \mathrm{Syl}(C)$. By 6.6.13(b), $\mathrm{Syl}(C) \prec Syl(B)$ and so (a) is proved.

(b) follows immediately from (a). □

Definition 6.6.21 [**def:lbg**] *Let* $H \leq G$ *and* b *a block of* H*.*

(a) [**a**] $\lambda_b^G : \mathrm{Z}(\mathbb{F}G) \to \mathbb{F}, a \to \lambda_b(a\mid_H)$*.*

(b) [**b**] *If* λ_b^G *is an algebra homomorphsim, the* b^G *is the unique block of* G *with* $\lambda_{b^G} = \lambda_b^G$*.*

Lemma 6.6.22 [**syl(b) in syl(bg)**] *Let* b *be a block of* $H \leq G$*. If* b^G *is defined then* $\mathrm{Syl}(b) \prec \mathrm{Syl}(b^G)$*.*

Proof: Let C be a defect class of B. Then $0 \neq \lambda_{b^G}(a_C) = \lambda_b^G(a_C) = \lambda_b(a_{C \cap H})$. Ot follows that there exists $c \in \mathcal{C}(H)$ with $c \subseteq C$ and $\lambda_b(a_c) \neq 0$. Hence by 6.6.13(a), $\mathrm{Syl}(b) \prec \mathrm{Syl}(c)$. Clearly $\mathrm{Syl}(c) \prec \mathrm{Syl}(C) = \mathrm{Syl}(B)$ and the lemma is proved. □

Proposition 6.6.23 [**lbg=brplb**] *Suppose that* P *is a* p*-subgroup of* G *and* $PC_G(P) \leq H \leq N_G(P)$*.*

(a) [**a**] $\lambda_b^G = \lambda_b \circ \mathrm{Br}_P$ *for all blocks* b *of* H*.*

(b) [**b**] *b^G is defined for all blocks b of H.*

(c) [**c**] *Let B be a block if G and b a block of H. Then $B = b^G$ iff $\lambda_b(\mathrm{Br}_P(e_B)) = 1$.*

(d) [**d**] *Let B be a block. Then $\mathrm{Br}_P(e_B) = \sum\{e_b \mid b \in \mathrm{Bl}(H), b^G = B\}$.*

(e) [**e**] *Let B be a block of G. Then $B = b^G$ for some block b of H iff $P \prec \mathrm{Syl}(B)$.*

Proof: (a) Let $C \in (G)$ we have to show that

$$(*) \qquad \lambda_b(a_{C\cap H}) = \lambda_b(a_{C\cap C_G(P)})$$

Since H nomrmalizes $C \cap H$ and $C \cap C_G(P)$. $C \cap H \setminus C_G(P)$ is a union of conjugacy classes of H. Let $c \in \mathcal{C}(H)$ with $c \subseteq C$ and $c \cap C_G(P)\emptyset$. Since $P \leq O_p(H)$, $C_H(O_p(H)) \leq C_G(P)$ and thus $c \cap C_H(O_p(H)) = 1$. 6.6.14 implies $a_c \in \mathrm{J}(\mathrm{Z}(\mathbb{F}H))$ and so $\lambda_b(a_c) = 0$. This implies (*) and so (a) holds.

(b) Since both Br_P and λ_b are homomorphism this follows from (a).

(c) By (b) $\lambda_b(\mathrm{Br}_B(e_B) = \lambda_{b^G}(e_B) = \delta_{B,b^G}$.

(d) Since Br_P is a homomorphism, $\mathrm{Br}_P(e_B)$ is either zero or an idempotent in $\mathrm{Z}(\mathbb{F}H)$. Hence by 6.5.16(b) (applied to H $\mathrm{Br}(e_B) = e_T$ for some (possible empty) $T \subseteq \mathrm{Bl}(H)$. Let $b \in \mathrm{Bl}(H)$. The $\lambda_b(e_T) = 1$ if $b \in T$ and 0 otherwise. So by (c), $T = \{b \in \mathrm{Bl}(G) \mid B = b^G\}$.

(e) By (d) $\mathrm{Br}_P(e_B) \neq 0$ iff ther exists $b \in \mathrm{Bl}(G)$ with $B = b^G$. Thus (e) follows from 6.6.20(a). □

Definition 6.6.24 [**def:G—P**] *Let P be a p-sugbroups of G. Then $\mathcal{C}(G|P) = \{C \in \mathcal{C}(G) \mid P \in \mathrm{Syl}(C)\}$ and $\mathrm{Bl}(G|P) = \{B \in \mathrm{Bl}(G) mid P \in \mathrm{Syl}(G)\}$.*

Proposition 6.6.25 [**defect opg**] *Let B be a block of G with defect group $O_p(G)$. Then $\mathrm{Syl}(C) = \{O_p(G)\}$ for all $C \in \mathcal{C}(G)$ with $e_B(g_C) \neq 0$ and so $e_B \in \mathbb{C}\langle a_C \mid C \in \mathcal{C}(G|O_p(G))\rangle$*

Proof: Let $C \in \mathcal{C}(G)$ with $e_B(g_C) \neq 0$. Then by 6.6.13(b), $\mathrm{Syl}(C) \prec \mathrm{Syl}(B) = \{O_p(G)\}$. On the otherhand $b = B$ is the unique block of G with $B = b^G$ and so by 6.6.23(d), $\mathrm{Br}_{O_p(G)} = e_B$. It follows that $C \leq C_G(O_p(G))$ and so $O_p(G) \prec \mathrm{Syl}(C)$. □

Lemma 6.6.26 [**first for classes**] *Let P be a p-subgroup of G. Then the map*

$$\mathcal{C}(G|P) \to \mathcal{C}(N_G(P)|P), C \to C \cap C_G(P)$$

is a well defined bijection.

Proof: Let $C \in \mathcal{C}(G|P)$. To show that out map us well defined we have to show that $C \cap C_G(P)$ is a conjugacy class for $N_G(P)$. Since $N_G(P)$ normalizes C and $C_G(P)$ it normalizes $C \cap C_G(P)$. Note that Gacst on the set $\{(x,Q) \mid x \in C, Q \in \mathrm{Syl}_p(G) = \{(x,Q) \mid x \in C, Q \in\cong GP, [x,Q] = 1\}$. Let $x \in C$. Then $C_G(x)$ acts tranistively on $Syl_p(C_G(x))$ and so by 1.1.10 $N_G(P)$ is tranistive on $C \cap C_G(P)$. So $C \cap C_G(P)$ is a conjugacy class of $N_G(P)$.

Since distinct conjugacy clases are disjoint, our map is injective. Let $L \in \mathcal{C}(N_G(P)|P)$ and let C be the unique conjugacy class of G containing L. Let $x \in L$. Since $P \in \mathrm{Syl}(L)$ and $P \trianglelefteq N_G(P)$, $\mathrm{Syl}(L) = \{P\}$ and so $P \in \mathrm{Syl}_p(N_G(P) \cap C_G(x))$. Let $P \leq Q \in \mathrm{Syl}_p(C_G(x))$. Then $PleqN_Q(P) \in N_G(P) \cap C_G(x)$ and so $P = N_Q(P)$. 1.4.5(c) implies $P = Q$ and so $P \in \mathrm{Syl}(C)$ and $C \in \mathcal{C}(G \mid P)$. Since $C \cap C_G(P)$ is a conjugacy class of $N_G(P)$, $C \cap C_G(P) = L$ and so our map is onto. □

Theorem 6.6.27 (Brauer's First Main Theorem) [**first**] *Let P be a p-subgroup of G.*

(a) [**a**] *The map* $\mathrm{Bl}(N_G(P)|P) \to \mathrm{Bl}(G|P), b \to b^G$ *is well defined bijection.*

(b) [**b**] *Let* $B \in \mathrm{Bl}(G|P)$ *and* $b = \mathrm{Bl}(N_G(P)|P)$, *then* $B = b^G$ *iff* $\mathrm{Br}_P(e_B) = e_b$.

Proof: Let b be a block of $N_G(P)$ with defect group P. Since $P \trianglelefteq N_G(P)$, $\mathrm{Syl}(b) = \{P\}$. By 6.6.23 b^G is defined and $\lambda_{b^G} = \lambda_b^G = \lambda_b \circ \mathrm{Br}_P$.To show that our map is well defiend we need to show P is a defect group of b^G. Let L be a defect class of b. Then by 6.6.13(c), $\mathrm{Syl}(L) = \mathrm{Syl}(b) = \{P\}$ and thus $L \in \mathcal{C}(N_G(P)|P)$. Let C be the unique conjugacy class of G containin L. By 6.6.26 $P \in \mathrm{Syl}(C)$ and $C \cap C_G(P) = L$. Hence

$$\lambda(b^G)(a_C) = \lambda_(\mathrm{Br}_P(a_C)) = \lambda_b(a_{C \cap C_G(P)}) = \lambda_b(a_L) \neq 0$$

Thus by 6.6.13(a), $\mathrm{Syl}(b^G) \prec \mathrm{Syl}(C)$ and so P contains a defect group of $\mathrm{Syl}(b^G)$. By 6.6.22, $\{P\} = \mathrm{Syl}(b) \prec \mathrm{Syl}(b^G)$. Thus P is contained in a defect group of b^G. Hence P is a defect group of b^G.

To show that $b \to b^G$ is onto let $B \in Bl(G|P)$. Let T be the set of blocks of $N_G(P)$ with $B = b^G$. Then by By 6.6.23(d), $e_B = e_T$ and by 6.6.23(e), $T \neq 0$. Let $b \in T$. Since $P \leq O_p(N_G(P))$, 6.6.16 implies that P is contained in any defect group of b. By 6.6.22 any defect groups of b is contained in a defect group of $B = b^G$. Thus P is a defect group of b.

Finally assume that $b^G = d^G$ for some $b, d \in \mathrm{Bl}(N_G(P)|P)$. Then $\lambda_b \circ \mathrm{Br}_P = \lambda_{b^G} = \lambda_d \circ \mathrm{Br}_P$. Thus $\lambda_b(a_{C \cap C_G(P)}) = \lambda_d(a_{C \cap C_G(P)}$ for all $C \in \mathcal{C}(G)$. Hence by 6.6.26, $\lambda_b(a_L) = \lambda_d(a_L)$ for all $L \in \mathcal{C}(N_G(P) \mid P)$. Observe that by 6.6.16(b), $P = O_p(N_G(P))$ and so by 6.6.25 e_b is a $\mathbb{C}$-linear combination of the $a_L, L \in \mathcal{C}(N_G(P)|P$. Thus

$$1 = \lambda_b(e_b) = \lambda_d(e_b) = \delta_{bd}$$

and $b = d$. So our map is 1-1. □

Corollary 6.6.28 [**p=opng**] *Let P be the defect group of some block of G. Then $P = O_p(N_G(P))$.*

Proof: By 6.6.27 P is a defect group of some block of $N_G(P)$. So by 6.6.16(b), $P = O_p(N_G(P))$. $\square$

6.7 Brauer's Second Main Theorem

Lemma 6.7.1 [**x invertible in zag**] *Let B be block of G and $x \in \mathrm{Z}(\mathbb{A}_I G)$ with $\lambda_B(x^*) = 1$. Then there exists $y \in f_B \mathrm{Z}(\mathbb{A}_I G)$ with $yx = f_B$.*

Proof: Since $\lambda_B((f_B x)^*) = \lambda_B(e_B)\lambda_B(x) = 1$ we may replace x by $f_B x$ and assume that $x \in f_B \mathrm{Z}(\mathbb{A}_I G))$. Then $f_B x = x$, $e_B x^* = x^*$ and $x^* \in \mathbb{F}B$. Since $\lambda_B(x^*) = 1\lambda_B(e_B)$ and $\ker \lambda_B \cap \mathrm{Z}(\mathbb{F}B) = \mathrm{J}(\mathrm{Z}(\mathbb{F}B))$ we conclude for 6.7.1 that x^* is invertible in $\mathrm{Z}(\mathbb{F}B)) = e_B \mathrm{Z}(\mathbb{F}G) = (f_B \mathrm{Z}(\mathbb{A}_I G))^*$. So there exists $u \in f_B \mathrm{Z}(\mathbb{A}_I G))$ with $(ux)^* = e_B$. Observe that $\ker({}^*\colon \mathbb{A}_I H \to \mathbb{F}G) = I_I G = \mathrm{J}(A_I) \cdot \mathbb{A}_I G$ and $ux \in f_B \cdot \mathbb{A}_I G \cdot f_B$. Thus 6.3.5 shows that there exists a unique $v \in f_B \cdot \mathbb{A}_I G \cdot f_B$ with $vux = f_B$. Let $g \in G$. Then $t \cong gv \cdot ux = {}^g(vux) = {}^g f_B = f_B$ and so by uniqueness of v, ${}^g v = v$ and $v \in \mathrm{Z}(\mathbb{A}_I G)$. So the lemma holds with $y = vu$. $\square$

Lemma 6.7.2 [**fb on fbprime**] *Let $H \leq G$, b a block of H. Suppose that b^G is define and put $B = b^G$. Then there exists $w \in \mathbb{A}_I(G \setminus H)$ such that*

(a) [**a**] $f_b f_{B'} = w f_{B'}$.

(b) [**b**] $f_b w = w = w f_b$.

(c) [**c**] *H centralizes.*

Proof: Let $x = f_B \mid_H$ and $z = f_B \mid_{H \setminus H}$. Then $f_B = a + c$. By defintion of $B = B^G$, $\lambda_B = \lambda_b^G$ and so

$$1 = \lambda_B(e_B) = \lambda_n(e_B \mid H) = \lambda_B((f_B \mid_H)^*) = \lambda_B(x^*).$$

Hence by 6.7.1 applied to H in place of G there exists $y \in f_B \mathrm{Z}(\mathbb{A}_I H)$ with $yx = f_B$. Put $w = -yz$ and note that H centralizes w. Since $H \cdot (G \setminus H) \subseteq G \setminus H$, $w \in \mathbb{A}_I(G \setminus H)$. Since $f_b y = f_b$ also $f_b w = w$. It remains to prove (a).

$$y f_B = y(x + z) = yx + yz = f_B - w$$

Hence

$$(f_b - w) f_{B'} = y f_B f_{B'} = 0$$

This (a) holds.

Lemma 6.7.3 [**p partition**]

(a) [**a**] *Let $\langle h\rangle$ be a finite cyclic group acting on a set Ω. Suppose h_p acts fixed-point freely on Ω. Then there exists there exists an $< h >$-invariant partion of $(\Omega_i)_{i\in\mathbb{F}_p}$ of Ω with $h\Omega_i = \Omega_{i+1}$.*

(b) [**b**] *If $h \leq H \leq G$ with $C_H(h_p) \leq H$, S a ring and $w \in S[G \setminus H]$. If h centralizes w, then there exists $w_i \in S[G \setminus H], i \in F_p$ with $hw_ih^{-1} = w_{i+1}$ and $\sum_{i\in\mathbb{F}_p} w_i = w$.*

(a) Put $H = \langle h\rangle$ act transitively on Ω. Let Ω_0 be an orbit for H^p on Ω. Suppose that $\Omega_0 = \Omega$. Then by the Frattinargument, $H = H^pC_H(\omega)$ and so $H/C_H(\omega)$ is a p' group. Thus $h_p \in C_H(\omega)$ contrary to the assumptions. Thus $\Omega_0 \neq \Omega$ Since $H^p \trianglelefteq H$, $H/H^p \cong C_p$ acts tranistively on the set of orbits of H^p on Ω. So (a) holds with $\Omega_i = h^i\Omega_0$, for $i \in \mathbb{F}_p$.

(b) Since $C_G(h_p) \leq H$, h_p acts fixed-point freely on $G \setminus H$ via conjuagtion. Let Ω_i be as in (a) with $\Omega = G \setminus H$ and put $w_i = w \mid_{\Omega_i}$. Then clearly $w = \sum_{i\in\mathbb{F}_p} w_i$. Now

$$^hw_i = {}^h(w \mid \Omega_i) = {}^hw \mid_{^h\Omega_i} = w \mid_{\Omega_{i+1}} = w_{i+1}$$

and (b) is proved.

Lemma 6.7.4 [**eigenvector for h**] *Let $H \leq G$ and b a block for G. Suppose that $B = b^G$ us defined and that $h \in H$ with $C_G(h_p) \in H$.*

(a) [**a**] *Let $\omega \in \mathbb{C}$ with $\omega^p = 1$. If $f_{B'}f_b \neq 0$, then the exists a unit t in the ring $f_{B'}f_b \cdot \mathbb{A}_IG \cdot f_{B'}f_b$ with $^ht = \omega t$.*

(b) [**b**] *If $\chi \in \mathrm{Irr}(G)$ with $\chi \notin B$. Then $\chi(hf_b) = 0$.*

Proof: (a) Let w be a as in 6.7.2. By 6.7.3(b) theer exists $w_i \in \mathbb{A}_IG$ with $w = s\sum_{i\in\mathbb{F}_p} w_i$ and $^hw_i = w_{i+1}$. By 6.7.2(b), $w = f_bwf_b$ and so replacing w_i by $f_bw_if_b$ we may assume that $w_i \in f_b \cdot \mathbb{A}_IG \cdot f_b$. Put $s = \sum_{i\in\mathbb{F}_p} \omega^iw_i$. Then clearly $^hs = \omega s$ and $s \in f_b \cdot \mathbb{A}_IG \cdot f_b$. Put $t = f_{B'}s$. $f_{B'} \in \mathrm{Z}(\mathbb{A}_IG)$ is a central idempotent, $t \in f_{B'}f_b \cdot \mathbb{A}_IG \cdot f_{B'}f_b$ and $^ht = \omega t$. To complete the proof of (a) we need to show that t is unit in the ring $f_{B'}f_b \cdot \mathbb{A}_IG \cdot f_{B'}f_b$.

Since $\mathbb{F}$ has no element of multiplicative order p, $\omega^* = 1$ and so $s^* = \sum_{i\in\mathbb{F}_p} w_i^* = w^*$ and so by 6.7.2(a),

$$f_{B'}f_b)^* = (f_{B'}w)^* = (f_{B'}s)^* = t^*$$

So 6.3.5 applied with the idempotent $f = f_{B'}f_b$ yields that t is a unit in $f_{B'}f_b\cdot\mathbb{A}_IG\cdot f_{B'}f_b$.

(b) Let $M \in \mathcal{S}(G)$ with $\chi = \chi_M$. Put $V = f_bM$. Observe that V that $\mathbb{C}H$ submodule of M. Moreover, $M = \mathbb{A}_M(f_b) \oplus V$ and f_b acts as id_V on V. Thus $\chi_M(hf_b) = \chi_V(f_b)$. Since $\chi \notin B$, $f_BM = 0$ and so $f_{B'}$ act as identity on M and on V. So also $f_{\mathrm{B}'}f_b$ acts as indentity on V. The $V = f_{B'}f_bM$ is a module for the ring $f_{B'}f_b \cdot \mathbb{A}_IG \cdot f_{B'}f_b$

If $V = 0$ clearly (b) holds. So suppose $V \neq 0$ and so also $f_{B'}f_b \neq 0$.

For L be the set of eigenvalues for h on V and for $l \in L$ let V_l be the corresponding eigenspace. Then $V = \bigoplus_{l \in L} V_l$. Let ω be a primitive p-root of unity in U and choose t as in (a). Then t is invertible on V. Moreover, if $l \in L$ and $v \in V_l$, then $htv = hth^{-1}hv = \omega tlv = (\omega l)tv$. Thus $tV_l \leq V_{tl}$. In particular $t^pV_l = V_{t^pL} = V_l$ and since t^p is invertible, $t^pV_l = V_l$ and so also $tV_l = V_{tl}$. T Inparticular $< \omega\rangle$ acts an L be left multiplication and $\dim V_l = \dim V_{\omega l}$. Let L_0 be a set of representatoves for the orbits of $\langle \omega \rangle$ in L. Then

$$\begin{array}{ccccc} & \chi_V(h) & = & \sum_{l \in L} \chi_{V_l}(h) & = \sum_{l \in L} l \dim_{V_l} \\ = & \sum_{l \in L_0} \sum_{i=0}^{p-1} \omega^i l \dim V_{\omega^i l} & = & \sum_{l \in L_0} \left(\sum_{i=0}^{p-1} \omega^i \right) l \dim V_l & = \quad 0 \end{array}$$

□

Definition 6.7.5 [**def:p-section**] *Let $x \in G$ be a p-element. Then $\mathrm{S}_G(x) = \mathrm{S}(x) = \{y \in G \mid y_p \in {}^G x\}$ is called the p-*section *if x in G.*

Lemma 6.7.6 [**basic p-section**] *Let $x \in G$ be a p-elemenent and Y a set of representatives for the p'-conjugact classes in $C_G(x)$. Then $\{xy \mid y \in Y\}$ is a set of representaives for the conjugacy classes of G in $\mathrm{S}(x)$.*

Proof: Any $s \in \mathrm{S}(x)$ is uniquely determined by the pair $(s_p, s_{p'})$. So the lemma follows from 1.1.10 □

Definition 6.7.7 [**def:bx**] *Let $x \in G$ be a p-element and B a block p-block and $\theta \in \mathbb{C}G)$.*

(a) [**a**] *Let T a block or a set of blocks. Then $\theta_T : G \to \mathbb{C} \mid g \to \theta(f_T g)$.*

(b) [**b**] *$\theta^x : G \to \mathbb{C}$, $x \to \theta(xh)$.*

(c) [**c**] *$B^x = \{b \in \mathrm{Bl}(C_G(x))\} \mid b^G = B\}$.*

Lemma 6.7.8 [**fchi selfadjoint**] *Let $T \subseteq \mathrm{Irr}(G)$. Then*

(a) [**a**] *$f_{T^\circ} = \overline{f}_T$*

(b) [**b**] *$(af_T \mid b) = (a \mid bf_T)$ for all $a, b \in \mathbb{C}G$.*

Proof: By linearity we may assume $T = \{\chi\}$ for some $\chi \in \mathrm{Irr}(G)$.

(a) Since $\chi^\circ = \bar{c}hi$ and $f_\chi = \frac{\chi(1)}{|G|}\overline{\chi}$ we have $f_{\chi^\circ} = \overline{f}_\chi$.

(b) By (a) $\overline{f}_\chi^\circ = f_\chi$ and 3.4.2(c) implies $(af_\chi \mid b) = (a \mid bf_\chi)$.

Lemma 6.7.9 [**dual of a block**] *Let B be a block.*

(a) [**a**] *$\overline{B} = \{\psi \mid \psi \in B\}$ is a block.*

(b) [**b**] $\lambda_{\overline{B}}(a) = \lambda_B(a^\circ)$.

(c) [**c**] $f_{\overline{B}} = \overline{f}_B = f_B^\circ$.

(d) [**d**] $e_{\overline{B}} = e_B^\circ$.

Proof: (a) and (b): Let $\psi \in B$ and M the correspoding module. Then $\overline{\psi}$ corresponse to M^*. By the definition of the action of a group ring on the dual $\rho_{M^*}(a) = \rho_M(a^\circ)^{\text{dual}}$. It follows that $\lambda_{\overline{\psi}}(a) = \lambda_\psi(a^\circ)$. Thus $\lambda_\alpha = \lambda_\beta$ iff $\lambda_{\overline{\alpha}} = \lambda_{\overline{b}}$ and so (a) and (b) hold.

(c): Clearly $f_{\overline{B}} = \overline{f}_B$. By 6.7.8, $\overline{f}_B = f_T^\circ$ and so (c) holds.

(d): Apply * to (c). □

Lemma 6.7.10 [**theta b**] *Let T be a block or or a set of blocks and $\theta \in \mathbb{C}G$. Then $\theta_B = \theta f_{\overline{B}}$.*

Proof: Let $b \in G$. Then by 6.7.8

$$\theta_T(b) = \theta(f_B b) = |G|(\theta \mid \overline{f_T b}) = |G|(\theta\overline{f_T} \mid \overline{b}) = (\theta f_{\overline{B}})(b).$$

□

Lemma 6.7.11 [**theta fb**] *Let B be a block.*

(a) [**a**] $\mathrm{Irr}(B)$ *is a basis for* $\mathbb{C}\overline{B} := \mathbb{C}G f_B$.

(b) [**b**] *Both* $\mathrm{IBr}(G)$ *and* $(\Phi_\phi \mid \phi \in \mathrm{IBr}(G)$ *are a basis for* $\mathbb{C}\tilde{\overline{B}}$*, where* $\mathbb{C}\tilde{B} := \mathbb{C}\tilde{G} \cap \mathbb{C}B$.

(c) [**c**] *If* $\chi \in \mathrm{Irr}(B)$*, then* $\tilde{\chi} \in \mathbb{F}\overline{B}$.

(d) [**d**] *For all* $\theta \in \mathrm{Z}(\mathbb{C}G)$, $\widetilde{\theta f_B} = \tilde{\theta} f_B$ *and* $\tilde{\theta_B} = \tilde{\theta}_B$.

(e) [**e**] *Let* $\theta \in \mathrm{Z}(\mathbb{C}G)$ *and* B *a block of* G*. Then* $\theta f_B = \sum_{\chi \in \mathrm{Irr}(\overline{B})} (\theta \mid \chi)\chi$.

Proof: (a): Let $\chi \in \mathrm{Irr}(B)$. Then $\chi = \frac{|G|}{\phi(1)} f_{\overline{\chi}} \in \mathbb{C}G\overline{B}$ and so (a) holds.

(b) Let $\phi \in \mathrm{IBr}(B)$. Then by (a)

$$\Phi_\psi = \sum_{\chi \in \mathrm{Irr}(B)} d_{\phi\chi}\chi \in \mathbb{C}\overline{B}$$

and so $(\Phi_\phi \mid \phi \in \mathrm{IBr}(G)$ is a basis for $\mathbb{C}\tilde{\overline{B}}$. Moreover,

$$\phi = \sum_{\psi \in \mathrm{IBr}(B)} (\phi \mid \psi)\Phi_\psi \in \mathbb{C}\overline{B}$$

and so (b) holds.

(c) $\tilde{\chi} = \sum_{\phi \in \mathrm{IBr}(B)} d_{\phi\chi}\phi$. So (c) follows from (b).
(d) By linearity we may assume that $\theta \in \mathrm{Irr}(G)$. If $\theta \in \overline{B}$ then by (b) and (c)

$$\tilde{\theta} f_B = \tilde{\theta} = \widetilde{\theta f_B}$$

and if $\theta \notin \overline{B}$, then

$$\tilde{\theta} f_B = 0 = \tilde{0} = \widetilde{\theta f_B}$$

So the first statement holds. The second now follows from 6.7.10
(e) follows from $\theta = \sum_{\chi \in \mathrm{Irr}(G)} (\theta \mid \chi)$ and (a). □

Lemma 6.7.12 [**decomposing theta x**] *Let $x \in G$ be a p-element, B a block of G.*

(a) [**a**] *If $\chi \in \mathrm{Irr}(B)$, then $\widetilde{\chi^x} = \widetilde{\chi^x}_{B^x}$.*

(b) [**b**] *Let $\theta \in \mathrm{Z}(\mathbb{C}G)$, then $((\tilde{\theta_B})^x) = (\tilde{\theta^x})_{B^x}$.*

Proof: (a) Let $b \in \mathrm{Bl}(C_G(x)) \setminus B^x$ and $y \in \widetilde{C_G(x))}$. Then

$$\widetilde{\chi^x}_b(y) = \widetilde{\chi^x}(f_b y) \overset{6.7.11(d)}{=} \chi^x(f_b y) = \chi(f_b x y) \overset{6.7.4(b)}{=} 0$$

Thus $\widetilde{\chi^x}_b = 0$ and so $\widetilde{\chi^x} = \sum_{b \in \mathrm{IBr}(C_G(x))} \widetilde{\chi^x}_b = \sum_{b \in \mathrm{IBr}(B^x)} \widetilde{\chi^x}_b = \widetilde{\chi^x}_{B^x}$.
(b) By linearity we may assume $\theta \in Irr(G)$ and say $\theta \in A \in \mathrm{Bl}(G)$. So (b) follows from (a). □

□

Theorem 6.7.13 [**my second**] *Let $\mathcal{X}$ a set of representatives for the p-element classes. Define*

$$\mu : \mathrm{Z}(\mathbb{C}G) \to \bigoplus_{x \in X} \mathrm{Z}\mathbb{C}\widetilde{C_G(x)}, \theta \to (\tilde{\theta}^x)_x$$

and

$$\nu : \bigoplus_{x \in X} \mathrm{Z}\mathbb{C}\widetilde{C_G(x)} \to \mathrm{Z}(\mathbb{C}G), (\tau_x)_x \to \theta$$

where $\theta(g) = \tau_x(y)$ for $x \in \mathcal{X}$ and $y \in \widetilde{C_G(x)}$ with $xy \in {}^G x$.

(a) [**a**] *μ and ν are inverse to each other and so both are $\mathbb{C}$-isomorphism*

(b) [**b**] *$\mu(\mathrm{Z}\mathbb{C}\widetilde{C_G(x)}) = \mathrm{Z}\mathbb{C}\,\mathrm{S}(x)$.*

(c) [**c**] *μ and ν are isometries.*

(d) [**d**] *$\mathrm{Z}(\mathbb{C}G) = \bigoplus_{x \in \mathcal{X}} \mathrm{Z}\mathbb{C}\,\mathrm{S}(x)$.*

(e) [**e**] *For each block B of G,* $\Xi(\mathrm{Z}(\mathbb{C}B)) = \bigoplus_{x\in X} \mathrm{Z}\mathbb{C}\widetilde{B^x}$

(f) [**f**] $\mathrm{Z}(\mathbb{C}B) = \bigoplus_{x\in\mathcal{X}} \nu(\mathrm{Z}\mathbb{C}\widetilde{B^x}))$

Proof: Observe that by 6.7.6 ν is well defined. Also we view $\mathrm{Z}\mathbb{C}\widetilde{C_G(x)}$ has subring of $\bigoplus_{x\in X} \mathrm{Z}\mathbb{C}\widetilde{C_G(x)}$.

(a) and (b) are obvious.

(c) Let $r, x \in \mathcal{X}$, $s \in \widetilde{C_G(r)}$ and $y \in \widetilde{C_G(x)}$. Let $C \neq D \in \mathcal{C}(G)$, $E \in (C_G(x)$ and $F \in C_G(r)$ with $rs \in C, xy \in D$, $s \in E$ and $y \in F$. Then $\mu(a_C) = a_E$ and $\widetilde{\mu(a_D)} = F$. Since $C \neq D$ either $x \neq y$ or $E \neq F$ and in both cases $a_E \perp a_F$ in $\bigoplus_{x\in X} \mathrm{Z}\mathbb{C}\widetilde{C_G(x)}$. Note that also $a_C \perp a_D$ in $\mathrm{Z}(\mathbb{C}G)$. Moreover

$$(a_D \mid a_D)_G = \frac{|D|}{|G|} = \frac{1}{|C_G(xy)|} = \frac{1}{|C_{C_x}(y)|} = \frac{|F|}{|C_G(x)|} = (a_F \mid a_F)_{C_G(x)}$$

and so (c) holds.

(d) Follows since G is the disjoint union of the $opS(x), x \in \mathcal{X}$. Alternaively it folloes from (a) -(c).

(e) Follows from 6.7.12.

(f) follows from (e) and and (c). $\square$

Lemma 6.7.14 [**x decomposition**] *Let $x \in G$. Define the complex* $\mathrm{IBr}(C_G(x)) \times \mathrm{Irr}(G)$-*matrix $D^x = (d^x_{\phi\chi})$ by*

$$\tilde{\chi^x} = \sum_{\phi\in\mathrm{Irr}(\mathcal{G})} \delta^x_{\phi\chi}\phi$$

any $\chi \in \mathrm{Irr}(G)$ Then

$$d^x_{\phi\chi} = \sum_{\psi\in\mathrm{Irr}(C_G(x))} (\chi \mid_H \mid \psi)_H \frac{\psi(x)}{\psi(1)}\phi(y)$$

Proof:

Let $\chi = \chi_M$ with $M \in \mathcal{S}(G)$ an d$y \in \widetilde{C_G(x)}$. Then as an $C_G(x)$-module, $M \cong \sum_{N\in\mathcal{S}(H)} N^{d_N}$ for some $d_N \in \mathbb{N}$. Since $x \in Z(C_G(x))$, x acts as a scalar λ^x_N on N. Then $\chi_N(f_{\mathcal{B}}xy) = \lambda^x_N\chi_N(f_{\mathcal{B}}y)$. Moreover $f_{\mathcal{B}}$ annhilates N if $N \notin \mathcal{S}(\mathcal{B})$ and acts as identiity on N if $N \in \mathcal{S}(\mathcal{B})$. Hence

$$(*) \qquad \chi(f_{\mathcal{B}}xy) = \sum_{N\in\mathcal{S}(C_g(x))} d_N\lambda^x_N\chi_N(f_{\mathcal{B}}y) = \sum_{N\in\mathcal{S}(\mathcal{B})} \chi_N(y)$$

Observe that $\delta_N = (\chi \mid H \mid \chi_N)$, $\lambda^x_N = \frac{\chi_N(x)}{\chi_N(1)}$ and $\tilde{\chi}_N = \sum_{\phi\in\mathrm{IBr}(C_G(x))} d_{\phi\chi_N}\phi_N$. Substitution into (*) gives the lemma. $\square$

Theorem 6.7.15 (Brauer's Second Main Theorem) [**second**] *Let x be a p-element in G and $b \in \mathrm{Bl}(C_G(x))$. If $\chi \in \mathrm{Irr}(G)$ but $\chi \notin \mathrm{Irr}(b^G)$, then $d^x_{\phi\chi} = 0$ for all $\phi \in \mathrm{IBr}(G)$.*

Proof: Follows from 6.7.12(a).

Corollary 6.7.16 [**chixy**] *Let x be a p-element in G, $y \in C_G(x)$ a p'-element, B a block of B and $\chi \in \mathrm{Irr}(B)$. Then*

$$\chi(xy) = \sum \{ d^x_{\phi\chi} \mid b \in \mathrm{Bl}(C_G(x)), B = b^G \}$$

Proof: This just rephrases 6.7.12(a).

Corollary 6.7.17 [**gp in defect group**] *Let B be a block of G, $\chi \in \mathrm{Irr}(B)$ and $g \in G$. If $\chi(g) \neq 0$ then g_p is contained in a defect group of B,*

Proof: Let $x = g_p, y = g_{p'}$. Since $\chi(g) = \chi(xy) \neq 0$, 6.7.16 implies tat there exists $b \in IBr(G)$ with $B = b^G$. Since $x \in O_p(C_G(x)$ is contained in any defect group of b, 6.6.22 implies that x is contained a defect group of B. □

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