AN INTRODUCTION TO SEMIALGEBRAIC GEOMETRY

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Octobre 2002

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Introduction

Semialgebraic geometry is the study of sets of *real* solutions of systems of polynomial equations and inequalities. These notes present the first results of semialgebraic geometry and related algorithmic issues. Their content is by no means original.

The first chapter explains algorithms for counting real roots (this is, in some sense, 0-dimensional semialgebraic geometry) and the Tarski-Seidenberg theorem.

In the next two chapters we study semialgebraic subsets of $\mathbb{R}^n$, which are defined by boolean combinations of polynomial equations and inequalities. The main tool for this study is the cylindrical algebraic decomposition, which is introduced in Chapter 2. The principal result of Chapter 3 is the triangulation theorem. This theorem shows that semialgebraic sets have a simple topology, which can be effectively computed from their definitions.

Chapter 4 contains examples of finiteness results and uniform bounds for semialgebraic families. In particular, we give an explicit bound for the number of connected components of a real algebraic set as a function of the degree of the equations and the dimension of the ambient space.

Many results of semialgebraic geometry also hold true for o-minimal structures (including for instance classes of sets definable with the exponential functions). See [D] (or the lecture notes [Co]) for this theory. The algorithmic aspects are specific to the semialgebraic case.

The bibliography is reduced to a minimum, containing mainly books and recent surveys. Further references (in particular, references to original papers) may be found there.

These notes have served as a basis for courses in Rennes, Paris and Pisa, for mini-courses at MSRI (Berkeley), at a summer school in Laredo (Cantabria) and at a CIMPA school in Niamey.

Chapter 1

Counting the real roots of a polynomial

1.1 Sturm's theorem

In this section, $P \in \mathbb{R}[X]$ is a nonconstant polynomial in one variable.

1.1.1 P without multiple root

We assume that P has no multiple root, i.e., $\gcd(P, P') = 1$. We construct a sequence of polynomials in the following way: $P_0 = P$, $P_1 = P'$ and, for $i > 0$, P_{i+1} is the negative of the remainder of the euclidean division of P_{i-1} by P_i ($P_{i-1} = P_i Q_i - P_{i+1}$, with $\deg P_{i+1} < \deg P_i$). We stop just before we get 0. The last polynomial P_K is then a nonzero constant (up to signs, this is just Euclide's algorithm for computing the gcd). The sequence $P_0, \ldots, P_K$ is called the *Sturm sequence of P and P'*. Let $a \in \mathbb{R}$, not a root of P. Denote by $v_P(a)$ the number of sign changes in the sequence $P_0(a), P_1(a), \ldots, P_K(a)$. For instance, if $P = X^3 - 3X + 1$, the Sturm sequence of P and P' is

$$(X^3 - 3X + 1,\ 3X^2 - 3,\ 2X - 1,\ \frac{9}{4})\ ,$$

which gives $(-1, 0, 1, \frac{9}{4})$ by evaluation at $a = 1$. Here $v_P(1) = 1$. We drop the zeroes which occur when counting the sign changes. The result is the following:

Theorem 1.1 (Sturm) *Let $a < b$ in $\mathbb{R}$, neither a nor b being a root of P. The number of roots of P in the interval (a, b) is equal to $v_P(a) - v_P(b)$.*

Proof. We consider how $v_P(x)$ changes when x passes through a root c of a polynomial of the Sturm sequence.

- If c is a root of P, the signs of P_0 and P_1 behave as follows:

$$\text{either} \quad \begin{array}{c|ccc} x & & c & \\ \hline P_0 & - & 0 & + \\ \hline P_1 & + & + & + \end{array} \qquad \text{or} \quad \begin{array}{c|ccc} x & & c & \\ \hline P_0 & + & 0 & - \\ \hline P_1 & - & - & - \end{array} \; .$$

  In both cases, the contribution to $v_P(x)$ decreases by 1.

- If c is a root of P_i, $0 < i < K$, we have $P_{i-1}(c) = -P_{i+1}(c) \neq 0$. Hence the contribution of the subsequence $P_{i-1}(x), P_i(x), P_{i+1}(x)$ to $v_P(x)$ does not change and remains equal to 1.

The theorem follows from the preceding remarks. □

1.1.2 P with multiple roots

The proof of Theorem 1.1 relies on the following properties of the sequence $P_0, \ldots, P_K$:

1. $P = P_0$, and P_K is a nonzero constant.
2. If c is a root of P_0, the product $P_0 P_1$ is negative on some interval $(c-\varepsilon, c)$ and positive on some interval $(c, c+\varepsilon)$.
3. If c is a root of P_i, $0 < i < K$, then $P_{i-1}(c)P_{i+1}(c) < 0$.

Assume now that P has multiple roots. We construct, as above, the sequence $P_0 = P, P_1 = P', \ldots, P_K$. Now P_K is no longer a constant, but the gcd of P and P'. Consider the sequence

$$P_0/P_K,\ P_1/P_K, \ldots,\ P_{K-1}/P_K,\ 1\ .$$

This sequence satisfies properties 1-2-3 above for the polynomial P_0/P_K, which has the same roots as P (not counting multiplicities). Moreover, if a is not a root of P, the number $v_P(a)$ of sign changes in the sequence $P_0(a), \ldots, P_K(a)$ is obviously the same as the number of sign changes in the sequence

$$P_0(a)/P_K(a), \ldots,\ P_{K-1}(a)/P_K(a),\ 1\ .$$

It follows:

Theorem 1.2 *Sturm's theorem still holds if P has multiple roots. The difference $v_P(a) - v_P(b)$ is equal to the number of* distinct *roots of P in the interval (a, b).*

1.1.3 A bound for the roots

Proposition 1.3 *Let $P = a_0X^d + \cdots + a_{d-1}X + a_d$, where $a_0 \neq 0$. If $c \in \mathbb{C}$ is a root of P, then*

$$|c| \leq \max_{i=1,\ldots,d} \left(d \left|\frac{a_i}{a_0}\right|\right)^{1/i} .$$

Proof. Set $M = \max_{i=1,\ldots,d} \left(d\ |a_i/a_0|\right)^{1/i}$ and let $z \in \mathbb{C}$ be such that $|z| > M$. Then $|a_i| < |a_0|\,|z|^i/d$, for $i = 1, \ldots, d$. Hence,

$$|a_1 z^{d-1} + \cdots + a_d| \leq |a_1|\,|z|^{d-1} + \cdots + |a_d| < |a_0 z^d|$$

and $P(z) \neq 0$. □

Set

$$M = \max_{i=1,\ldots,d} \left(d \left|\frac{a_i}{a_0}\right|\right)^{1/i} .$$

Then $v_P(x)$ is constant on $(-\infty, -M)$ (resp. $(M, +\infty)$) and equal to $v_P(-\infty)$ (resp. $v_P(+\infty)$), which is the number of sign changes in the sequence of leading coefficients of $P_0(-X)$, $P_1(-X)$, …, $P_K(-X)$ (resp. $P_0(X)$, …, $P_K(X)$).

Proposition 1.4 *The total number of distinct real roots of P is*

$$v_P(-\infty) - v_P(+\infty).$$

1.2 Real roots satisfying inequalities

In this section, $P \in \mathbb{R}[X]$ is a non constant polynomial in one variable, and Q, Q_1, …, Q_ℓ are polynomials in $\mathbb{R}[X]$.

1.2.1 One inequality

We want to count the number of real roots c of P such that $Q(c) > 0$. We modify the construction of the Sturm sequence by taking $P_0 = P$, $P_1 = P'Q$ and, as before, $P_{i+1} =$ the negative of the remainder of the euclidean division of P_{i-1} by P_i, for $i > 0$. We stop just before we obtain 0, i.e. we stop with P_K which is the gcd of P and $P'Q$. The sequence of polynomials we obtain in this way is called the Sturm sequence of P and $P'Q$. If the real number a is not a root of P, we denote by $v_{P,Q}(a)$ the number of sign changes in the sequence $P_0(a), P_1(a), \ldots, P_K(a)$.

Theorem 1.5 *Let $a < b$ be real numbers which are not roots of P. Then $v_{P,Q}(a) - v_{P,Q}(b)$ is equal to the number of distinct roots c of P in (a,b) such that $Q(c) > 0$ minus the number of those such that $Q(c) < 0$.*

Proof. First consider the case where P and $P'Q$ are relatively prime (P_K is a nonzero constant). This means that P has no multiple root, and no common root with Q. The property 2 of Section 1.1.2 is replaced with:

2' If c is a root of P_0, the product P_0P_1Q is negative on some interval $(c-\varepsilon, c)$ and positive on some interval $(c, c+\varepsilon)$.

The theorem follows from properties 1-2'-3.

If P_K is not a constant, the sequence $P_0/P_K, P_1/P_K, \ldots, P_{K-1}/P_K, 1$ satisfies properties 1-2'-3 for P_0/P_K. Hence, the difference between the numbers of sign changes in this sequence evaluated at a and b, respectively, is equal to the number we want to calculate, and it coincides with $v_{P,Q}(a) - v_{P,Q}(b)$. □

Remark. $v_{P,Q^2}(a) - v_{P,Q^2}(b)$ counts the number of distinct roots of P in (a,b) which are not real roots of Q. Therefore the number of distinct roots c of P in (a,b) such that $Q(c) > 0$ is equal to

$$\frac{1}{2}\left(v_{P,Q}(a) + v_{P,Q^2}(a) - v_{P,Q}(b) - v_{P,Q^2}(b)\right) .$$

We can replace v_{P,Q^2} with v_P if P and Q are relatively prime.

Exercise 1.6 The *Cauchy index* of a rational fraction $F \in \mathbb{R}(X)$ between a and b is the number of poles c, with $a < c < b$, such that $\lim_{x\to c_-} F(x) = -\infty$ and $\lim_{x\to c_+} F(x) = +\infty$, minus the number of those such that $\lim_{x\to c_-} F(x) = +\infty$ and $\lim_{x\to c_+} F(x) = -\infty$.
Given two polynomials P and Q in $\mathbb{R}[X]$, define the Sturm sequence of P and Q by taking $P_0 = P$, $P_1 = Q$ and the rest constructed as above. If a and b are not roots of P, show that the Cauchy index of Q/P between a and b is equal to the difference $v(a) - v(b)$ between the numbers of sign changes in the Sturm sequence evaluated at a and b, respectively. Recover Theorems 1.1 and 1.5 from this result.

Exercise 1.7 Show that we get the same result as Theorem 1.5 if we replace $P'Q$ with the remainder of the euclidean division of $P'Q$ by P in the construction of the Sturm sequence.

1.2.2 Several inequalities

We want to count the number of real roots of a system $P = 0,\ Q_1 > 0, ...,\ Q_\ell > 0$. First assume that P is relatively prime with all Q_i. Let $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_\ell) \in \{0,1\}^\ell$ and $Q^\varepsilon = Q_1^{\varepsilon_1} \cdots Q_\ell^{\varepsilon_\ell}$. By Theorem 1.5, $s_\varepsilon = v_{P,Q^\varepsilon}(-\infty) - v_{P,Q^\varepsilon}(+\infty)$ is equal to the number of distinct real roots c of P such that $Q^\varepsilon(c) > 0$ minus the number of those such that $Q^\varepsilon(c) < 0$. If $\varphi = (\varphi_1, \ldots, \varphi_\ell) \in \{0,1\}^\ell$, we denote by c_φ the number of distinct real roots c of P such that the sign of $Q_i(c)$ is $(-1)^{\varphi_i}$, for $i = 1, \ldots, \ell$. Let s (resp. c) be the vector whose coordinates are all s_ε (resp. c_φ).

Lemma 1.8 *There is an invertible $2^\ell \times 2^\ell$ matrix A_ℓ, depending only on ℓ, such that $s = A_\ell.c$.*

Proof. We proceed by induction on ℓ. For $\ell = 0$, we have trivially $s_\emptyset = c_\emptyset$. For $\ell = 1$, Sturm's theorem and Theorem 1.5 imply

$$\begin{pmatrix} s_0 \\ s_1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \end{pmatrix}.$$

The induction step from ℓ to $\ell + 1$ is as follows:

$$\begin{pmatrix} \vdots \\ s_{\varepsilon,0} \\ \vdots \\ s_{\varepsilon,1} \\ \vdots \end{pmatrix} = \begin{pmatrix} A_\ell & A_\ell \\ A_\ell & -A_\ell \end{pmatrix} \begin{pmatrix} \vdots \\ c_{\varphi,0} \\ \vdots \\ c_{\varphi,1} \\ \vdots \end{pmatrix}.$$

(Exercise: check this equality). The matrix $A_{\ell+1} = \begin{pmatrix} A_\ell & A_\ell \\ A_\ell & -A_\ell \end{pmatrix}$ has inverse

$$\frac{1}{2}\begin{pmatrix} A_\ell^{-1} & A_\ell^{-1} \\ A_\ell^{-1} & -A_\ell^{-1} \end{pmatrix}.$$

□

We obtain from the lemma $c = A_\ell^{-1}.s$. Since we can compute s, we get c and, in particular, the number of solutions of $P = 0, Q_1 > 0, ..., Q_\ell > 0$. In the general case ($P$ can have multiple roots or common roots with Q_i), we replace Q^ε with $\left(\prod_{i=1}^{\ell} Q_i^2\right)/Q^\varepsilon$, in order to get rid of the roots of P which are also roots of some Q_i.

Remark. The number of Sturm sequences to be computed is 2^ℓ, which grows exponentially with ℓ. Actually, it is possible to avoid this exponential growth: the idea for this (due to Ben-Or, Kozen and Reif) is that, since the total number of real roots of P is at most equal to $d = \deg(P)$, at most d among the c_φ are nonzero (see [R]).

1.2.3 Deciding the existence of a solution of a system of polynomial equations and inequalities

The preceding result allows us to decide the existence of a solution of a system $P = 0$, $Q_1 > 0$, ..., $Q_\ell > 0$, where P is nonconstant. If the system contains several equations $P_1 = 0$, ..., $P_m = 0$, we can replace them with one equation $P_1^2 + \cdots + P_m^2 = 0$. We can replace a nonstrict inequality $Q \geq 0$ with the disjunction $Q > 0$ or $Q = 0$. It remains the case where the system consists only of strict inequalities: $Q_1 > 0, \ldots, Q_\ell > 0$. In this case:

- the system is satisfied on some unbounded interval of the form $(a, +\infty)$ (resp. $(-\infty, a)$) if and only if the leading coefficients of Q_1, ..., Q_ℓ (resp. $Q_1(-X)$, ..., $Q_\ell(-X)$) are all positive;

- the system is satisfied on an interval (a, b), where a and b are real roots of the product $Q = \prod_{i=1}^{\ell} Q_j$, if and only if the system $Q' = 0$, $Q_1 > 0$, ..., $Q_\ell > 0$ has a real solution. Note that this case happens only if $\deg(Q) \geq 2$, and then the derivative Q' is nonconstant.

1.3 Systems of polynomial equations and inequalities with parameters

We consider a system of polynomial equations and inequalities

$$\mathcal{S}(T, X) \ : \quad \begin{cases} S_1(T, X) \rhd_1 0 \\ S_2(T, X) \rhd_2 0 \\ \cdots \\ S_\ell(T, X) \rhd_\ell 0 \end{cases} ,$$

where the S_i are real polynomials in $T = (T_1, \ldots, T_p)$ and X, and $\rhd_i$ are either $=$ or $\neq$ or $>$ or $\geq$. We consider X as the variable and T as parameters. In this section we explain how to discuss the existence of a real solution of this system, depending on the real parameters T.

1.3.1 Tarski-Seidenberg

We shall prove in this section the following result.

Theorem 1.9 (Tarski-Seidenberg – first form) *There exists an algorithm which, given a system of polynomial equations and inequalities in the variables* $T = (T_1, \ldots, T_p)$ *and* X *with coefficients in* $\mathbb{R}$

$$\mathcal{S}(T,X) \ : \quad \begin{cases} S_1(T,X) \rhd_1 0 \\ S_2(T,X) \rhd_2 0 \\ \cdots \\ S_\ell(T,X) \rhd_\ell 0 \end{cases}$$

(where the $\rhd_i$ *are either* $=$ *or* $\neq$ *or* $>$ *or* $\geq$*), produces a finite list* $\mathcal{C}_1(T), \ldots, \mathcal{C}_k(T)$ *of systems of polynomial equations and inequalities in* T *with coefficients in* $\mathbb{R}$ *such that, for every* $t \in \mathbb{R}^p$*, the system* $\mathcal{S}(t,X)$ *has a real solution if and only if one of the* $\mathcal{C}_j(t)$ *is satisfied.*

In other words, the formula "$\exists X \ \mathcal{S}(T,X)$" is equivalent to the disjunction "$\mathcal{C}_1(X)$ or $\ldots$ or $\mathcal{C}_k(X)$". The Tarski-Seidenberg theorem means that there is an algorithm for eliminating the real variable X. A well known example of elimination of a real variable is

$$\begin{array}{l} \exists X \ AX^2 + BX + C = 0 \quad \Leftrightarrow \\ (A \neq 0 \text{ and } B^2 - 4AC \geq 0) \text{ or } (A = 0 \text{ and } B \neq 0) \text{ or } (A = B = C = 0)\ . \end{array}$$

This example shows that, in order to discuss a system depending on parameters, it is convenient to fix the degrees (with respect to X) of the polynomials in the system. If $S \in \mathbb{R}[T,X]$, denote by $\mathrm{lc}(S)$ the leading coefficient (in $\mathbb{R}[T]$) of S, considered as a polynomial in X. We shall call *system with fixed degrees* a system of the form $(\mathcal{S}(T,X), \mathcal{D}(T))$, such that $\mathcal{D}(T)$ contains, or implies, the inequations $\mathrm{lc}(S_i) \neq 0$ for all polynomials appearing in $\mathcal{S}(T,X)$. Observe that any system is equivalent to a finite disjunction of systems with fixed degrees. For instance, the inequality $TX^3 + (U-1)X^2 + V > 0$ is equivalent to

$$\begin{array}{l} (T \neq 0 \ \text{ and } \ TX^3 + (U-1)X^2 + V > 0) \ \text{ or} \\ \qquad (T = 0 \ \text{ and } \ U - 1 \neq 0 \ \text{ and } \ (U-1)X^2 + V > 0) \ \text{ or} \\ \qquad\qquad (T = 0 \ \text{ and } \ U - 1 = 0 \ \text{ and } \ V > 0)\ . \end{array}$$

Hence, it is sufficient to discuss systems with fixed degrees. Observe also that we can consider only equations $= 0$ and strict inequalities > 0, since the other cases $\neq 0$ and ≥ 0 are equivalent to disjunctions of the preceding ones.

Remark. The proof of the Tarski-Seidenberg theorem will show the following important fact: if all polynomials in $\mathcal{S}(T, X)$ have coefficients in $\mathbb{Q}$, the algorithm produces systems $\mathcal{C}_1(T), \ldots, \mathcal{C}_k(T)$ where all polynomials have coefficients in $\mathbb{Q}$.

1.3.2 Systems with one equation

First we consider the particular case of a system with fixed degrees containing one equation of positive degree with respect to X. It is convenient to introduce the function $\operatorname{sign} : \mathbb{R} \to \{-1, 0, 1\}$ defined by

$$\operatorname{sign}(r) = \begin{cases} 1 & \text{if } r > 0 \ , \\ 0 & \text{if } r = 0 \ , \\ -1 & \text{if } r < 0 \ . \end{cases}$$

Let $P, Q_1, \ldots, Q_\ell$ be real polynomials in $T = (T_1, \ldots, T_p)$ and X, of positive degrees with respect to X. Let $\mathcal{D}(T)$ be the system $\operatorname{lc}(P) \neq 0$ and $\operatorname{lc}(Q_i) \neq 0$, $i = 1, \ldots, \ell$.

Lemma 1.10 *There is an algorithm which, given $(P, Q_1, \ldots, Q_\ell)$, produces a finite list $R_1, \ldots, R_k$ of polynomials in T and a function $c : \{-1, 0, 1\}^k \to \mathbb{N}$ such that, for every $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_k) \in \{-1, 0, 1\}^k$ and every $t \in \mathbb{R}^p$ which satisfies*

$$\mathcal{D}(t) \ \text{ and } \ \operatorname{sign}(R_1(t)) = \varepsilon_1 \ \text{ and } \ \ldots \ \text{ and } \ \operatorname{sign}(R_k(t)) = \varepsilon_k \ ,$$

the system

$$P(t, X) = 0 \ \text{ and } \ Q_1(t, X) > 0 \ \text{ and } \ \ldots \ \text{ and } \ Q_\ell(t, X) > 0$$

has exactly $c(\varepsilon)$ solutions.

Proof. We perform the computations of subsection 1.2, i.e. we compute Sturm sequences. For every new polynomial obtained in a Sturm sequence, we test whether its leading coefficient is zero or nonzero. In the case where the leading coefficient is zero, we replace the polynomial with its truncation. We do not test the leading coefficients of polynomials $P, Q_1, \ldots, Q_\ell$, since we assume they are all nonzero; this ensures, in particular, that all Sturm sequences start with nonconstant polynomials.

In this way we obtain a tree of computation of Sturm sequences; the branching tests are polynomial equations ($= 0$) and inequations ($\neq 0$) in the parameters T. Every branch of the computation tree gives a system of polynomial equations and inequations in T and the Sturm sequence corresponding to all parameters t satisfying this system. The signs (> 0 or < 0) of the leading coefficients of the polynomials in this Sturm sequence determine the difference $v(-\infty) - v(+\infty)$ between the numbers of sign changes.

The leading coefficients are rational fractions $A(T)/B(T)$ in T, where B is assumed to be nonzero in the branch. Note that the sign of $A(t)/B(t)$ is the same as the sign of $A(t)B(t)$. We take for $R_1, \ldots, R_k$ the $A(T)B(T)$, for all leading coefficients $A(T)/B(T)$ of polynomials appearing in all branches of trees of computation of Sturm sequences. If we fix the sign (-1, 0 or 1) of each $R_1(t), \ldots, R_k(t)$ and assume $\mathcal{D}(t)$ holds, the results of 1.2 give us the number of real solutions of the system

$$P(t, X) = 0 \ \text{ and } \ Q_1(t, X) > 0 \ \text{ and } \ \ldots \ \text{ and } \ Q_\ell(t, X) > 0 \, .$$

□

1.3.3 Example: polynomial of degree 4

In order to make the preceding lemma clearer, we treat the following example: the equation $X^4 + aX^2 + bX + c = 0$, where a, b, c are parameters. The leading coefficient of the equation does not vanish. Since there is no inequality, we have only one Sturm sequence to compute. The figure 1.1 shows the computation tree of this Sturm sequence. In this tree the following polynomial expressions in the parameters appear:

$$\begin{aligned}
\Gamma &= 2a^3 - 8ac + 9b^2, \\
\Delta &= 16a^4c - 4a^3b^2 - 128a^2c^2 + 144ab^2c - 27b^4 + 256c^3, \\
\Sigma &= b(a^2 + 12c), \\
\Lambda &= -27b^4 + 256c^3.
\end{aligned}$$

We have cleared the denominators in the presentation of the results of the computation. Observe that the properties of Sturm sequences are not altered if we multiply a polynomial in this sequence by a positive quantity (for instance, the square of a nonzero quantity). Remark that we find cases where $s = -2$ or -1; of course, these cases can never happen, since the number of roots has to be nonnegative.

We can draw the list of cases where $s = 0$. This gives a necessary and sufficient condition for the polynomial to have no real root. We can group some of the cases by noting that Δ equals Λ if $a = 0$, and $256c^3$ (which

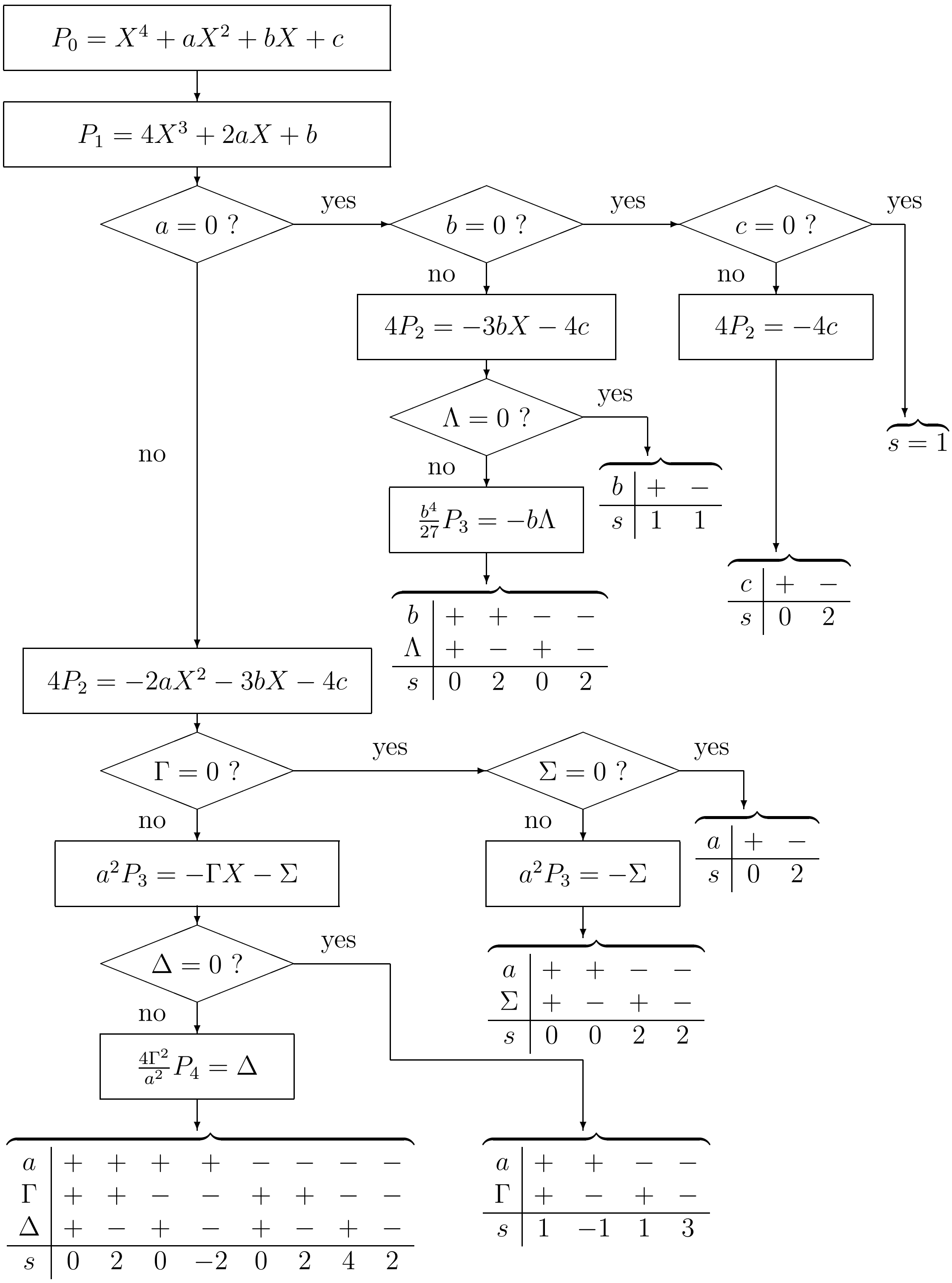


Figure 1.1: The computation tree of the Sturm sequence

has the same sign as c) if $a = b = 0$. Finally, we find that the polynomial $X^4 + aX^2 + bX + c$ has no real root if and only if

$$(a \geq 0 \ \text{ and } \ \Delta > 0) \ \text{ or } \ (a > 0 \ \text{ and } \ \Gamma = 0) \\ \text{ or } \ (a < 0 \ \text{ and } \ \Gamma > 0 \ \text{ and } \ \Delta > 0)\ .$$

The situation becomes clearer when we consider the picture in the space of parameters (a, b, c) (see Figure 1.2). The polynomial P has a multiple root if and only if $\Delta = 0$ (Δ is the *discriminant* of P). If P has a multiple root, we denote by $\alpha, \alpha, \beta, -(2\alpha + \beta)$ its four roots. It follows

$$\begin{aligned} a &= -3\alpha^2 - 2\alpha\beta - \beta^2\ , \\ b &= 2\alpha(\alpha^2 + 2\alpha\beta + \beta^2) = 2\alpha(-2\alpha^2 - a)\ , \\ c &= -\alpha^2(\beta^2 + 2\alpha\beta) = -\alpha^2(-3\alpha^2 - a)\ . \end{aligned}$$

We obtain in this way parametrizations of the curves $\Delta = 0$ in the planes $a =$ constant. Note that, for $a > 0$, the two imaginary roots of $\alpha^2 = -a/2$ give an isolated real point of the curve $\Delta = 0$ in the corresponding plane (in this case, P has two conjugate complex double roots). These points form the half-branch of parabola that we see on Figure 1.2; this parabola is part of the surface $\Delta = 0$.

When one moves continuously from one point of the space of parameters (a, b, c) to another without meeting the surface $\Delta = 0$, the implicit function theorem implies that the number of real roots does not change. Hence, the number of real roots is constant in each connected component of the complement of $\Delta = 0$.

Exercise 1.11 Identify on Figure 1.2 the set where $s = 0$ (no real root), and compare with the conditions we have found.

Exercise 1.12 Draw the semialgebraic subset

$$S = \{(a, c) \in \mathbb{R}^2\ ;\ X^4 + aX^2 + c = 0 \text{ has no real root}\}\ .$$

We want to show that S cannot be described by a conjunction of polynomial inequalities

$$P_1(a, c) > 0 \ \text{ and } \ \ldots \ \text{ and } \ P_s(a, c) > 0\ .$$

Suppose that S admits such a description. Show that one of the P_i vanishes on the parabola $c = a^2/4$ and can be written $P_i =$

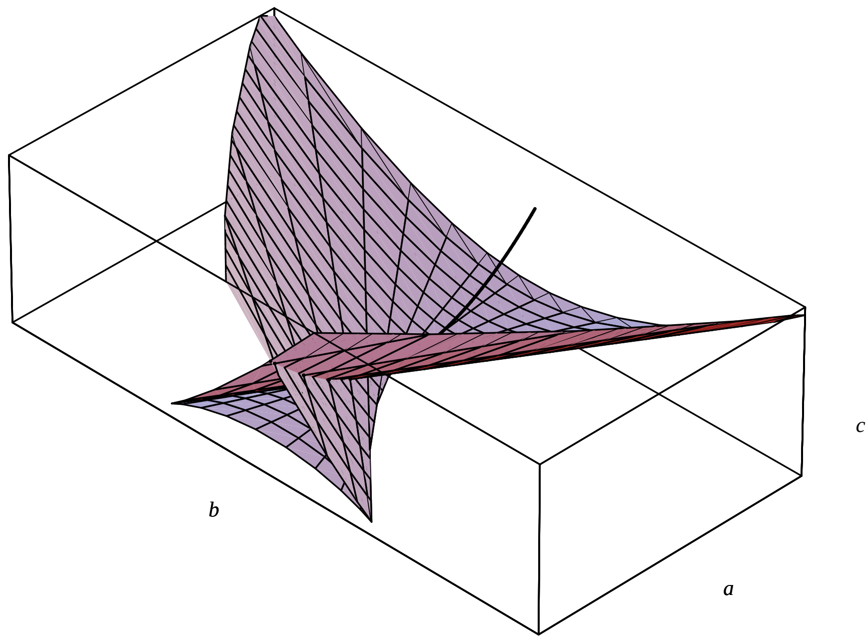


Figure 1.2: The discriminant of the polynomial of degree four

$(a^2 - 4c)^{2m+1}Q$, where $a^2 - 4c$ does not divide Q. Deduce that P_i should be negative on a part of S (hint: the sign of P_i changes along the parabola). Conclude.
Show that the semialgebraic set

$$\{(a, b, c) \in \mathbb{R}^3 \ ; \ X^4 + aX^2 + bX + c = 0 \text{ has no real root}\}$$

cannot be described by a conjunction of polynomial inequalities.

1.3.4 General systems

We now discuss general systems of equations and inequalities with fixed degrees. We repeat the arguments of 1.2.3

Firt consider the case of a system with several equations (of positive degree with respect to the variable X). We can replace the equations $P_1 = \ldots = P_k = 0$ with one equation $P_1^2 + \cdots + P_k^2 = 0$ and we can proceed as in susbsection 1.3.2.

Now consider the case where there is no equation of positive degree with respect to the variable X. There are only inequalities $Q_1 > 0, \dots, Q_\ell > 0$ with Q_j of positive degree with respect to X. It can be decided, by looking at the signs of the leading coefficients of the Q_j, whether the system is satisfied on an unbounded interval. The existence of an interval, whose endpoints are roots of $Q = \prod_{j=1}^{\ell} Q_j$, in which the system holds can be decided by discussing the system obtained by the adjunction of the equation $Q' = 0$. We then proceed as in susbsection 1.3.2.

1.4 Another method for counting real roots

1.4.1 Hermite's method

Let $P \in \mathbb{R}[X]$ be a polynomial of degree d, $\alpha_1, \dots, \alpha_d$ its roots in $\mathbb{C}$ (counted with multiplicities). If i is a nonnegative integer, set $N_i = \sum_{i=1}^{d} \alpha_j^i$. The N_i are called *the Newton sums of the roots of* P. They are symmetric polynomials of the roots of P, with integer coefficients. Hence, if $P = a_0X^d + a_1X^{d-1} + \cdots + a_d$, the Newton sums can be expressed as polynomials in $a_1/a_0, \dots, a_d/a_0$ with coeficients in $\mathbb{Z}$.

Exercise 1.13 Show that the expansion of the logarithmic derivative P'/P as a series in $1/X$ is $\sum_{i=0}^{\infty} N_i (1/X)^{1+i}$. Deduce from this fact the following relations:

$$\begin{array}{rcl} N_0 & = & d \\ a_0 N_1 & = & -a_1 \\ a_0 N_2 & = & -(N_1 a_1 + 2a_2) \\ & & \dots \\ a_0 N_i & = & -(N_{i-1}a_1 + N_{i-2}a_2 + \cdots + N_1 a_{i-1} + i a_i) \quad \text{if } i \le d \\ & & \dots \\ a_0 N_i & = & -(N_{i-1}a_1 + N_{i-2}a_2 + \cdots + N_{i-d} a_d) \quad \text{if } i > d \end{array}$$

Let us consider the quadratic form whose matrix is

$$\mathcal{H}(P) = \begin{pmatrix} N_0 & N_1 & \dots & N_{d-1} \\ N_1 & N_2 & \dots & N_d \\ \vdots & \vdots & \vdots & \vdots \\ N_{d-1} & N_d & \dots & N_{2d-2} \end{pmatrix}$$

Theorem 1.14 *The signature of the quadratic form with matrix $\mathcal{H}(P)$ is equal to the number of distinct real roots of P. Its rank is equal to the number of distinct roots of P (real and complex).*

Recall that the signature of a quadratic form $Q(U)$ in variables $U = (U_1, \ldots, U_d)$ with coefficients in $\mathbb{R}$ is computed in the following way: we decompose $Q(U)$ as

$$Q(U) = \sum_{i=1}^{p} L_i(U)^2 - \sum_{i=p+1}^{p+q} L_i(U)^2 ,$$

where the L_i are independant linear forms with coefficients in $\mathbb{R}$, and the signature of $Q(U)$ is the difference $p - q$.

Proof. The quadratic form with matrix $\mathcal{H}(P)$ in d variables $U_1, \ldots, U_d$ can be decomposed over $\mathbb{C}$ as the sum

$$(U_1 + \alpha_1 U_2 + \cdots + \alpha_1^{d-1} U_d)^2 + \cdots + (U_1 + \alpha_d U_2 + \cdots + \alpha_d^{d-1} U_d)^2 .$$

Indeed, if we denote by V the Vandermonde matrix

$$V = \begin{pmatrix} 1 & 1 & \ldots & 1 \\ \alpha_1 & \alpha_2 & \ldots & \alpha_d \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_1^{d-1} & \alpha_2^{d-1} & \ldots & \alpha_d^{d-1} \end{pmatrix} ,$$

we have $\mathcal{H}(P) = V\,({}^tV)$. Remark that the linear forms $L_\alpha(U) = U_1 + \alpha U_2 + \cdots + \alpha^{d-1} U_d$, for all *distinct* roots α of P, are linearly independent. Hence, the rank of $Q(U)$ is equal to the number of distinct complex roots of P. If α and $\overline{\alpha}$ are conjugate nonreal roots of P, we have

$$\begin{aligned} &(L_\alpha)^2 + (L_{\overline{\alpha}})^2 = \\ &2\left((U_1 + \Re(\alpha)U_2 + \cdots + \Re(\alpha^{d-1})U_d)^2 - (\Im(\alpha)U_2 + \cdots + \Im(\alpha^{d-1})U_d)^2\right) . \end{aligned}$$

The linear forms on the right-hand side of this equality have coefficients in $\mathbb{R}$. Hence, the pairs of conjugate nonreal roots contribute for 0 to the signature. Each distinct real root contributes for 1 to the signature. It follows that the signature is equal to the number of distinct real roots of P. $\square$

Exercise 1.15 Let Q be a polynomial in $\mathbb{R}[X]$. Replace all N_i with $N_i' = \sum_j Q(\alpha_j)\alpha_j^i$ in $\mathcal{H}(P)$. Show that the signature of the matrix obtained in this way is equal to the number of distinct real roots c of P such that $Q(c) > 0$ minus the number of those such that $Q(c) < 0$. Explain how to compute the N_i' by considering the expansion of $P'Q/P$ as a series of $1/X$.

1.4.2 Determination of the signature

Consider the *principal minors* of the matrix $\mathcal{H}(P)$, i.e. the determinants δ_i consisting of the first i rows and columns of the matrix, for $i = 1, \ldots, d$. Assume that none of the δ_i is zero. The signature of $\mathcal{H}(P)$ is equal to $d -$ twice the number of sign changes in the sequence $1, \delta_1, \ldots, \delta_d$. This result is known as Jacobi's theorem. We give indications on its proof in the following exercise.

Exercise 1.16 Consider a symmetric bilinear form b with matrix M in the basis $(e_1, \ldots, e_d)$. Assume that none of the principal minors $\delta_1, \ldots, \delta_d$ of M is zero. We construct an orthogonal basis $(\varepsilon_1, \ldots, \varepsilon_d)$ for b, of the form

$$\varepsilon_1 = e_1,\ \varepsilon_i = \lambda_{i,1} e_1 + \cdots + \lambda_{i,i-1} e_{i-1} + e_i \quad \text{for } 2 \leq i \leq d.$$

Show that, for $i \geq 2$, the determinant of the linear system $b(e_1, \varepsilon_i) = 0, \ldots, b(e_{i-1}, \varepsilon_i) = 0$ with unknowns $\lambda_{i,j}$, $j = 1, \ldots, i-1$, is equal to δ_{i-1}. Deduce from this fact that the orthogonal basis $(\varepsilon_1, \ldots, \varepsilon_d)$ is uniquely determined. Show that $b(\varepsilon_i, \varepsilon_i) = \delta_i/\delta_{i-1}$ for $i \geq 2$ (and, of course, $b(\varepsilon_1, \varepsilon_1) = \delta_1$). Prove Jacobi's theorem.

When one or several principal minors vanish, the situation is more complicated. For a general quadratic form, the principal minors are not sufficient to determine the signature. But the matrix $\mathcal{H}(P)$ has a special feature: it is a *Hankel matrix*, which means that all coefficients a_{ij} with $i + j =$ constant are equal. In this case, there is a method, due to Frobenius, to determine the signature by using only the principal minors. For more details concerning the theorem of Frobenius, see [G], Chapter 10, Theorem 24. Using this result and the fact that the rank of $\mathcal{H}(P)$ is equal to r if and only if $\delta_r \neq 0$ and $\delta_i = 0$ for $r < i \leq d$ (cf. 1.21), we obtain the following rule (see also [R]):

Proposition 1.17 *Let $P \in \mathbb{R}[X]$ be a polynomial of degree d, and let $\delta_1, \ldots, \delta_d$ be the principal minors of the matrix $\mathcal{H}(P)$. Let r be such that $\delta_r \neq 0$ and $\delta_{r+1} = \ldots = \delta_d = 0$ (note that $r \geq 1$ since $\delta_1 = d$). For $1 \leq i \leq r$, we define the "conventional signs" $\widetilde{\operatorname{sign}}(\delta_i)$ as follows:*

1. *If $\delta_i \neq 0$, $\widetilde{\operatorname{sign}}(\delta_i) = \operatorname{sign}(\delta_i)$.*

2. *If $\delta_i = \delta_{i-1} = \ldots = \delta_{i-j+1} = 0$ and $\delta_{i-j} \neq 0$, then*

$$\widetilde{\operatorname{sign}}(\delta_i) = (-1)^{j(j-1)/2} \operatorname{sign}(\delta_{i-j})\ .$$

Then the number of distinct real roots of P is equal to r minus twice the number of changes in the sequence $1, \widetilde{\operatorname{sign}}(\delta_1), \ldots, \widetilde{\operatorname{sign}}(\delta_r)$.

Continuing the exercise 1.15, we can obtain a similar result for counting the difference between the number of distinct real roots c of P such that $Q(c) > 0$ and the number of those such that $Q(c) < 0$.

Let us return to the example of a polynomial $P = X^4 + aX^2 + bX + c$ of degree 4. The Newton sums N_i are easily obtained by using the formulas of exercise 1.13:

$$\begin{aligned} N_0 &= 4 \\ N_1 &= 0 \\ N_2 &= -2a \\ N_3 &= -3b \\ N_4 &= 2a^2 - 4c \\ N_5 &= 5ab \\ N_6 &= -2a^3 + 6ac + 3b^2 \ . \end{aligned}$$

From this we obtain the principal minors of the matrix $\mathcal{H}(P)$:

$$\begin{aligned} \delta_1 &= 4 \\ \delta_2 &= -8a \\ \delta_3 &= -4(2a^3 - 8ac + 9b^2) = -4\Gamma \\ \delta_4 &= 16a^4c - 4a^3b^2 - 128a^2c^2 + 144acb^2 - 27b^4 + 256c^3 = \Delta \ , \end{aligned}$$

where we use the notation of Section 1.3.3.

Exercise 1.18 Using the rule of Proposition 1.17, recover a necessary and sufficient condition for the polynomial P to have no real root.

The example of the polynomial of degree 4 shows a great advantage of Hermite's method with respect to Sturm's method, in the case where there are parameters: the computations in Hermite's method are made *without branching*. In other words, the specialization to specific values of the parameters causes no trouble (if the degrees of the given polynomials are fixed). Actually the specialization problems in the computation of the Sturm sequence can also be avoided by using *subresultant polynomials*. These specialization problems are treated extensively in [R, GRRT]; see also [Loo] for a survey on subresultant polynomials.

Another nice feature of Hermite's method is that it can be generalized for systems of polynomial equations in several variables having finitely many solutions. The number of distinct real solutions can be computed as the signature of a quadratic form over the quotient of the ring of polynomials by the ideal generated by the equations. See for instance [R].

Hermite's and Sturm's method are actually strongly related. This is obvious in the example, if we compare the principal minors with the leading coefficients of the polynomials in the main branch of computation of the Sturm sequence (cf. 1.3.3). This is not mere coincidence, and it can be explained by using the theory of subresultant polynomials. We refer to [R, GRRT] for this explanation. We shall only consider principal subresultant coefficients in the next section. These are the leading coefficients of the subresultant polynomials, except that we take them to be zero if the subresultant polynomial has a degree smaller than its expected degree.

1.4.3 Principal subresultant coefficients

Consider two polynomials $P = a_0X^d + \cdots + a_d$ of degree d and $Q = b_0X^e + \cdots + b_e$ of degree e. The *resultant* of P and Q is the determinant of the *Sylvester matrix* of P and Q, which is the square matrix of size $d+e$ whose rows are the coordinates of $X^{e-1}P, \ldots, XP, P, Q, XQ, \ldots, X^{d-1}Q$, respectively, in the monomial basis $X^{d+e-1}, \ldots,$ X, 1. We draw this matrix in the case where $e = d-1$:

$$\begin{pmatrix}
a_0 & a_1 & a_2 & \ldots & . & . & . & a_d & 0 & . & \ldots & 0 \\
0 & a_0 & a_1 & \ldots & . & . & . & a_{d-1} & a_d & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & . & . & \ldots & 0 & a_0 & a_1 & a_2 & . & . & \ldots & a_d \\
0 & . & . & \ldots & . & 0 & b_0 & b_1 & . & . & \ldots & b_e \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & b_0 & . & \ldots & . & . & b_{e-1} & b_e & 0 & . & \ldots & 0 \\
b_0 & b_1 & . & \ldots & . & . & b_e & 0 & . & . & \ldots & 0
\end{pmatrix}.$$

Note that the Sylvester matrix is usually presented with another disposition of rows. The nonclassical presentation given here is borrowed from [GRRT] and has better properties with respect to the signs of the principal subresultant coefficients.

The resultant is zero if and only if P and Q have a common factor (i.e., their gcd is nonconstant). This is a well-known result (see for instance [L] chapter V, section 10). We shall prove a generalization of this theorem.

For $0 \leq j < \min(d, e)$, we call *principal subresultant coefficient* of order j of P and Q (denoted by $\mathrm{PSRC}_j(P, Q)$) the determinant of the square matrix of size $d + e - 2j$ which is obtained from the Sylvester matrix of P and Q by deleting the first j rows, the last j rows, the first j columns and the last j columns. The resultant of P and Q is $\mathrm{PSRC}_0(P, Q)$

Proposition 1.19 *Let ℓ be an integer, $0 \leq \ell < \min(d, e)$. The gcd of P et Q has degree $> \ell$ if and only if* $\mathrm{PSRC}_0(P, Q) = \ldots = \mathrm{PSRC}_\ell(P, Q) = 0$.

Proof. Consider the following problem:

(*) Do there exist nonzero polynomials U and V, with $\deg(U) < e - \ell$ and $\deg(V) < d - \ell$, such that $\deg(UP + VQ) < \ell$?

The last inequality can be translated to a homogeneous linear system of $d+e-2\ell$ equations in $d + e - 2\ell$ unknowns (the coefficients of U and V). The determinant of this system is $\pm\mathrm{PSRC}_\ell(P, Q)$. Hence, problem (*) has an affirmative answer if and only if $\mathrm{PSRC}_\ell(P, Q) = 0$.

Note that P and Q have a gcd of degree $> \ell$ if and only if there exist nonzero polynomials U and V as above, such that $UP + VQ = 0$. The case $\ell = 0$ of the proposition follows from the remarks already made. Let $\ell > 0$ and assume that we have proved that P and Q have a gcd with degree $\geq \ell$ if and only if $\mathrm{PSRC}_0(P, Q) = \ldots = \mathrm{PSRC}_{\ell-1}(P, Q) = 0$. If the gcd of P and Q has degree $> \ell$, the problem (*) has an affirmative answer and, therefore, $\mathrm{PSRC}_\ell(P, Q) = 0$. Conversely, if $\mathrm{PSRC}_0(P, Q) = \ldots = \mathrm{PSRC}_\ell(P, Q) = 0$, then (*) has an affirmative answer and there are nonzero polynomials U and V, with $\deg(U) < e - \ell$, $\deg(V) < d - \ell$ and $\deg(UP + VQ) < \ell$. By the inductive assumption, the gcd of P and Q has degree $\geq \ell$. Since this gcd divides $UP + VQ$, we have $UP + VQ = 0$. Hence, the gcd of P and Q has degree $> \ell$. □

The preceding proposition will be used in the next chapter. We shall now relate the principal minors of the Hankel matrix $\mathcal{H}(P)$ with certain principal subresultant coefficients.

Proposition 1.20 *Let δ_j be the principal minor constructed from the first j rows and columns of $\mathcal{H}(P)$. Then*

$$a_0^{2j-1}\delta_j = \mathrm{PSRC}_{d-j}(P, P') \ .$$

Proof. The expansion of P'/P as a series in $1/X$ (cf. exercise 1.13) allows one to check that the matrix of size $2j-1$ whose determinant is $\mathrm{PSRC}_{d-j}(P,P')$ is the product of the matrix

$$\begin{pmatrix} 1 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & N_0 & \dots & N_{j-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & N_0 & \dots & N_{j-3} & N_{j-2} & \dots & N_{2j-3} \\ N_0 & N_1 & \dots & N_{j-2} & N_{j-1} & \dots & N_{2j-2} \end{pmatrix}$$

with the matrix

$$\begin{pmatrix} a_0 & a_1 & \dots & a_{2j-2} \\ 0 & a_0 & \dots & a_{2j-1} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & . & \dots & a_0 \end{pmatrix},$$

where $a_\ell = 0$ if $\ell > d$. The determinant of the former matrix is δ_j, and the determinant of the latter is a_0^{2j-1}. □

Corollary 1.21 *The following properties are equivalent:*

1. *P has r distinct complex roots,*
2. *The matrix $\mathcal{H}(P)$ has rank r,*
3. *$\delta_r \neq 0$ and $\delta_j = 0$ for $r < j \leq d$,*
4. *$\mathrm{PSRC}_{d-r}(P,P') \neq 0$ and $\mathrm{PSRC}_\ell(P,P') = 0$ for $0 \leq \ell < d-r$.*

Corollary 1.22 *The number of distinct real roots of P depends only on the signs (> 0, < 0 or $= 0$) of the principal subresultant coefficients of P and P'.*

The rule to compute the number of distinct real roots follows from Proposition 1.17. Let us recall that the method of principal subresultant coefficients behaves better than Sturm's method when there are parameters: the computation of principal subresultant coefficients is uniform with respect to the parameters (as long as the degree of P is fixed), whereas the computation of the Sturm sequence is not uniform. Nevertheless, the best way to compute the principal subresultant coefficients is by using a variant of the Sturm sequence.

Chapter 2

Semialgebraic sets

2.1 Stability properties of the class of semialgebraic sets

2.1.1 Definition and first examples

A *semialgebraic subset* of $\mathbb{R}^n$ is the subset of $(x_1, \ldots, x_n)$ in $\mathbb{R}^n$ satisfying a boolean combination of polynomial equations and inequalities with real coefficients. In other words, the semialgebraic subsets of $\mathbb{R}^n$ form the smallest class $\mathcal{SA}_n$ of subsets of $\mathbb{R}^n$ such that:

1. If $P \in \mathbb{R}[X_1, \ldots, X_n]$, then $\{x \in \mathbb{R}^n \; ; \; P(x) = 0\} \in \mathcal{SA}_n$ and $\{x \in \mathbb{R}^n \; ; \; P(x) > 0\} \in \mathcal{SA}_n$.
2. If $A \in \mathcal{SA}_n$ and $B \in \mathcal{SA}_n$, then $A \cup B$, $A \cap B$ and $\mathbb{R}^n \setminus A$ are in $\mathcal{SA}_n$.

The fact that a subset of $\mathbb{R}^n$ is semialgebraic does not depend on the choice of affine coordinates.

Proposition 2.1 *Every semialgebraic subset of* $\mathbb{R}^n$ *is the union of finitely many semialgebraic subsets of the form*

$$\{x \in \mathbb{R}^n \; ; \; P(x) = 0 \;\text{ and }\; Q_1(x) > 0 \;\text{ and }\; \ldots \;\text{ and }\; Q_\ell(x) > 0\} \, ,$$

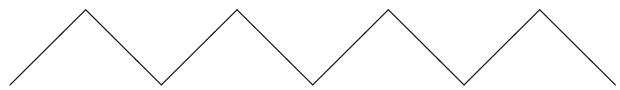
where $\ell \in \mathbb{N}$ *and* $P, Q_1, \ldots, Q_\ell \in \mathbb{R}[X_1, \ldots, X_n]$.

Proof. Check that the class of finite unions of such subsets satisfies the above properties 1 and 2. □

We give now some examples of semialgebraic sets.

- The semialgebraic subsets of $\mathbb{R}$ are the unions of finitely many points and open intervals.
- An algebraic subset of $\mathbb{R}^n$ (defined by polynomial equations) is semialgebraic.
- Let $F : \mathbb{R}^m \to \mathbb{R}^n$ be a polynomial mapping: $F = (F_1, \ldots, F_n)$, where $F_i \in \mathbb{R}[X_1, \ldots, X_n]$. Let A be a semialgebraic subset of $\mathbb{R}^n$. Then $F^{-1}(A)$ is a semialgebraic subset of $\mathbb{R}^m$.
- If A is a semialgebraic subset of $\mathbb{R}^n$ and $L \subset \mathbb{R}^n$ a line, then $L \cap A$ is the union of finitely many points and open intervals.
- If $A \subset \mathbb{R}^m$ and $B \subset \mathbb{R}^n$ are semialgebraic, $A \times B$ is a semialgebraic subset of $\mathbb{R}^m \times \mathbb{R}^n$.

Exercise 2.2 Show that the infinite zigzag



is not semialgebraic. Show that, for every compact semialgebraic subset K of $\mathbb{R}^2$, the intersection of K with the zigzag is semialgebraic.

2.1.2 Consequences of Tarski-Seidenberg principle

We have seen that the class of all semialgebraic subsets is closed under finite unions and intersections, taking complement, inverse image by a polynomial mapping, cartesian product. It is also closed under projection.

Theorem 2.3 (Tarski-Seidenberg – second form) *Let A be a semialgebraic subset of $\mathbb{R}^{n+1}$ and $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$, the projection on the first n coordinates. Then $\pi(A)$ is a semialgebraic subset of $\mathbb{R}^n$.*

Proof. Since A is the union of finitely many subsets of the form

$$\{x = (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \ ; \ P(x) = 0,\ Q_1(x) > 0, \ldots,\ Q_k(x) > 0\} \ ,$$

we may assume that A itself is of this form. It follows from the Tarski-Seidenberg theorem (first form, 1.9) that there is a boolean combination

$\mathcal{C}(X_1, \dots, X_n)$ of polynomial equations and inequalities in $X_1, \dots, X_n$ such that

$$\pi(A) = \{(x_1, \dots, x_n) \in \mathbb{R}^n \ ; \ \exists x_{n+1} \in \mathbb{R} \ (x_1, \dots, x_n, x_{n+1}) \in A\}$$

is the set of $(x_1, \dots, x_n)$ which satisfy $\mathcal{C}(x_1, \dots, x_n)$. This means that $\pi(A)$ is semialgebraic. $\square$

We now show some consequences of the Tarski-Seidenberg theorem.

Corollary 2.4 1. *If A is a semialgebraic subset of $\mathbb{R}^{n+k}$, its image by the projection on the space of the first n coordinates is a semialgebraic subset of $\mathbb{R}^n$.*

2. *If A is a semialgebraic subset of $\mathbb{R}^m$ and $F : \mathbb{R}^m \to \mathbb{R}^n$, a polynomial mapping, then the direct image $F(A)$ is a semialgebraic subset of $\mathbb{R}^n$.*

Proof. The first statement is easily obtained by induction on k. For the second statement, note that

$$\{(x, y) \in \mathbb{R}^m \times \mathbb{R}^n \ ; \ x \in A \ \text{ and } \ y = F(x)\}$$

is a semialgebraic subset of $\mathbb{R}^m \times \mathbb{R}^n$ and that $F(A)$ is its projection onto $\mathbb{R}^n$. $\square$

Corollary 2.5 *If A is a semialgebraic subset of $\mathbb{R}^n$, its closure in $\mathbb{R}^n$ is again semialgebraic.*

Proof. The closure of A is

$$\operatorname{clos}(A) = \left\{x \in \mathbb{R}^n \ ; \ \forall \varepsilon \in \mathbb{R}, \ \varepsilon > 0 \Rightarrow \exists y \in \mathbb{R}^n, \ y \in A \ \text{ and } \ \|x - y\|^2 < \varepsilon^2\right\}$$

and can be written as

$$\operatorname{clos}(A) = \mathbb{R}^n \setminus \left(\pi_1 \left(\{(x, \varepsilon) \in \mathbb{R}^n \times \mathbb{R} \ ; \ \varepsilon > 0\} \setminus \pi_2(B)\right)\right) \ ,$$

where

$$B = \left\{(x, \varepsilon, y) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \ ; \ y \in A \ \text{ and } \ \sum_{i=1}^{n} (x_i - y_i)^2 < \varepsilon^2)\right\} \ ,$$

$\pi_1(x, \varepsilon) = x$ and $\pi_2(x, \varepsilon, y) = (x, \varepsilon)$. Then observe that B is semialgebraic. $\square$

The example above shows that it is usually boring to write down projections in order to show that a subset is semialgebraic. We are more used to write down formulas. Let us make precise what is meant by a *first-order formula* (of the language of ordered fields with parameters in $\mathbb{R}$). A first-order formula is obtained by the following rules.

1. If $P \in \mathbb{R}[X_1, \ldots, X_n]$, then $P = 0$ and $P > 0$ are first-order formulas.
2. If Φ and Ψ are first-order formulas, then "Φ and Ψ", "Φ or Ψ", "not Φ" (often denoted by $\Phi \wedge \Psi$, $\Phi \vee \Psi$ and $\neg\Phi$, respectively) are first order formulas.
3. If Φ is a formula and X, a variable ranging over $\mathbb{R}$, then $\exists X \Phi$ and $\forall X \Phi$ are first-order formulas.

The formulas obtained by using only rules 1 and 2 are called *quantifier-free formulas.* By definition, a subset $A \subset \mathbb{R}^n$ is semialgebraic if and only if there is a quantifier-free formula $\Phi(X_1, \ldots, X_n)$ such that

$$(x_1, \ldots, x_n) \in A \Longleftrightarrow \Phi(x_1, \ldots, x_n).$$

The Tarski-Seidenberg theorem has the following useful formulation.

Theorem 2.6 (Tarski-Seidenberg – third form) *If $\Phi(X_1, \ldots, X_n)$ is a first-order formula, the set of $(x_1, \ldots, x_n) \in \mathbb{R}^n$ which satisfy $\Phi(x_1, \ldots, x_n)$ is semialgebraic.*

Proof. By induction on the construction of formulas. Rule 1 produces only semialgebraic sets. Rule 2 produces only semialgebrais sets from semialgebraic sets. For rule 3, if

$$\{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \ ; \ \Phi(x_1, \ldots, x_{n+1})\}$$

is semialgebraic, then

$$\{(x_1, \ldots, x_n) \in \mathbb{R}^n \ ; \ \exists x_{n+1} \ \Phi(x_1, \ldots, x_{n+1})\}$$

is its projection onto $\mathbb{R}^n$ and, hence, it is also semialgebraic. The case of $\forall X \Phi$ follows by observing that $\forall X \Phi$ is equivalent to $\neg\exists X \neg\Phi$. □

The preceding theorem can be formulated as follows.

Every first-order formula is equivalent to a quantifier-free formula, or, in other words,

> $\mathbb{R}$ *admits the elimination of quantifiers in the language of ordered fields.*

The reader who wants to learn more about model theory and its application to real algebraic geometry is invited to read the lecture notes [Pr].

Remark. One should pay attention to the fact that the quantified variables (or n-tuples of variables) have to range over $\mathbb{R}$, or $\mathbb{R}^n$, or possibly over a *semialgebraic* subset of $\mathbb{R}^n$. For instance,

$$\{(x,y) \in \mathbb{R}^2 \ ; \ \exists n \in \mathbb{N} \ y = nx\}$$

is not semialgebraic (why ?).

2.2 Semialgebraic functions

2.2.1 Definition and first properties

Let $A \subset \mathbb{R}^m$ and $B \subset \mathbb{R}^n$ be semialgebraic sets. A mapping $f : A \to B$ is said to be *semialgebraic* if its graph

$$\Gamma_f = \{(x,y) \in A \times B \ ; \ y = f(x)\}$$

is a semialgebraic subset of $\mathbb{R}^m \times \mathbb{R}^n$.

For instance:

- If $f : A \to B$ is a polynomial mapping (all its coordinates are polynomial), it is semialgebraic.
- If $f : A \to B$ is a regular rational mapping (all its coordinates are rational fractions whose denominators do not vanish on A), it is semialgebraic.
- If $f : A \to \mathbb{R}$ is a semialgebraic function, then $|f|$ is semialgebraic.
- If $f : A \to \mathbb{R}$ is semialgebraic and $f \geq 0$ on A, then $\sqrt{f}$ is a semialgebraic function.

Exercise 2.7 Let $A \subset \mathbb{R}^n$, $A \neq \emptyset$, be a semialgebraic set. Then the function

$$\begin{array}{rcl} \mathbb{R}^n & \longrightarrow & \mathbb{R} \\ x & \longmapsto & \operatorname{dist}(x,A) = \inf\{\|x-y\| \ ; \ y \in A\} \end{array}$$

is continuous semialgebraic.

Exercise 2.8 Show that if $f : (0, 1] \to \mathbb{R}$ is a semialgebraic function such that $f(x)$ is not bounded from above as $x \to 0$, then $\lim_{x\to 0} f(x) = +\infty$.

Important properties of semialgebraic mappings follow from the Tarski-Seidenberg theorem.

Corollary 2.9 *1. The direct image and the inverse image of a semialgebraic set by a semialgebraic mapping are semialgebraic. For instance, if $P(X_1, \ldots, X_n)$ is a polynomial and f, a semialgebraic mapping from $A \subset \mathbb{R}^m$ to $B \subset \mathbb{R}^n$, the set $\{y \in B \ ; \ P(f(y)) > 0\}$ is semialgebraic.*

2. The composition of two semialgebraic mappings is semialgebraic.

3. The semialgebraic functions from A to $\mathbb{R}$ form a ring.

Exercise 2.10 Let U be a semialgebraic open subset of $\mathbb{R}^n$, $f : U \to \mathbb{R}$ a semialgebraic function. Show that, if f admits a partial derivative $\partial f/\partial x_i$ on U, then this derivative is semialgebraic.

2.2.2 The Łojasiewicz inequality

The Łojasiewicz inequality gives information concerning the relative rate of growth of two continuous semialgebraic functions. First, we shall estimate the rate of growth of a semialgebraic function of one variable.

Proposition 2.11 *Let $f : (A, +\infty) \to \mathbb{R}$ be a semialgebraic (not necessarily continuous) function. There exist $B \geq A$ and an integer $N \in \mathbb{N}$ such that $|f(x)| \leq x^N$ for all $x \in (B, +\infty)$.*

Proof. Let Γ be the graph of f. It is a semialgebraic subset of $\mathbb{R}^2$. By 2.1, $\Gamma = G_1 \cup \ldots \cup G_p$, where each G_i is a nonempty subset of the form

$$G_i = \{(x, y) \in \mathbb{R}^2 \ ; \ P_i(x, y) = 0, \ Q_{i,1}(x, y) > 0, \ldots, \ Q_{i,k_i}(x, y) = 0\} \ .$$

All polynomials P_i have degree > 0 with respect to y: otherwise, if $(x_0, y_0) \in G_i$, Γ should contain a nonempty open interval of the vertical line $\{x_0\} \times \mathbb{R}$, which is impossible since Γ is a graph. Let

$$P(x, y) = a_0(x)y^d + a_1(x)y^{d-1} + \cdots + a_d(x)$$

be the product of all $P_i(x, y)$, where $d > 0$ and $a_0 \neq 0$. Choose $C \geq A$ big enough so that $a_0(x)$ does not vanish on $(C, +\infty)$. By Proposition 1.3, we obtain

$$|f(x)| \leq \max_{i=1,\ldots,d} \left(d \left| \frac{a_i(x)}{a_0(x)} \right| \right)^{1/i} \ .$$

As x tends to $+\infty$, the right-hand side of the equality is equivalent to λx^α, where $\lambda > 0$ and $\alpha \in \mathbb{Q}$. Taking N to be a nonnegative integer $> \alpha$, we obtain $B \geq C$, such that $|f(x)| \leq x^N$ for all $x > B$. $\square$

Theorem 2.12 (Łojasiewicz inequality) *Let $K \subset \mathbb{R}^n$ be a compact semialgebraic set, and let $f, g : K \to \mathbb{R}$ be continuous semialgebraic functions , such that*

$$\forall x \in K \quad (f(x) = 0 \Rightarrow g(x) = 0) \ .$$

Then there exist an integer $N \in \mathbb{N}$ and a constant $C \geq 0$, such that

$$\forall x \in K \quad |g(x)|^N \leq C|f(x)| \ .$$

Proof. For $t > 0$, set $F_t = \{x \in K \ ; \ t|g(x)| = 1\}$. Since F_t is closed in K, it is compact. Assume $F_t \neq \emptyset$; then f does not vanish on F_t and the continuous function $x \mapsto 1/|f(x)|$ has a maximum on F_t, which we denote by $\theta(t)$. If $F_t = \emptyset$, we set $\theta(t) = 0$. The function $\theta : (0, +\infty) \to \mathbb{R}$ is semialgebraic (check this fact by writing a formula which describes its graph). By Proposition 2.11, there exist $B > 0$ and $N \in \mathbb{N}$ such that

$$\forall t > B \quad |\theta(t)| \leq t^N \ .$$

This is equivalent to

$$\forall x \in K \quad \left(0 < |g(x)| < \frac{1}{B} \Rightarrow \frac{1}{|f(x)|} \leq \frac{1}{|g(x)|^N}\right) \ .$$

Let D be the maximum of the continuous function $|g(x)|^N/|f(x)|$ on the compact set

$$\{x \in K \ ; \ |g(x)| \geq 1/B\}$$

(observe that f does not vanish on this set), and let $C = \max(1, D)$. We obtain $|g(x)|^N \leq C|f(x)|$ for all $x \in K$. $\square$

Exercise 2.13 Let S be a closed semialgebraic subset of the plane which contains the graph of the exponential function $y = e^x$. Show that S contains some interval $(-\infty, A)$ of the x-axis. Hint: use the function "distance to S".

2.3 Decomposition of a semialgebraic set

We have seen that a semialgebraic subset of $\mathbb{R}$ can be decomposed as the union of finitely many points and open intervals. We shall see that every semialgebraic set can be decomposed as the disjoint union of finitely many pieces which are semialgebraically homeomorphic to open hypercubes $(0,1)^d$ of different dimensions. A semialgebraic homeomorphism $h : S \to T$ is a bijective continuous semialgebraic mapping from S onto T, such that $h^{-1} : T \to S$ is continuous.

Exercise 2.14 Check that h^{-1} is also semialgebraic.

The method of decomposition by using successive codimension 1 projections is the main tool for studying semialgebraic sets, and it is used in the foundational paper of S. Łojasiewicz (1964). We now explain the cylindrical algebraic decomposition of Collins [Cl], which makes precise the algorithmic content of this method.

2.3.1 Cylindrical algebraic decomposition

A *cylindrical algebraic decomposition* (abbreviated to *c.a.d.*) of $\mathbb{R}^n$ is a sequence $\mathcal{C}_1, \ldots, \mathcal{C}_n$, where, for $1 \le k \le n$, $\mathcal{C}_k$ is a finite partition of $\mathbb{R}^k$ into semialgebraic subsets (which are called *cells*), satisfying the following properties:

a) Each cell $C \in \mathcal{C}_1$ is either a point, or an open interval.

b) For every k, $1 \le k < n$, and for every $C \in \mathcal{C}_k$, there are finitely many continuous semialgebraic functions

$$\xi_{C,1} < \ldots < \xi_{C,\ell_C} : C \longrightarrow \mathbb{R} ,$$

and the cylinder $C \times \mathbb{R} \subset \mathbb{R}^{k+1}$ is the disjoint union of cells of $\mathcal{C}_{k+1}$ which are:

- either the *graph* of one of the functions $\xi_{C,j}$, for $j = 1, \ldots, \ell_C$:

$$A_{C,j} = \{(x', x_{k+1}) \in C \times \mathbb{R} \; ; \; x_{k+1} = \xi_{C,j}(x')\} ,$$

- or a *band* of the cylinder bounded from below and from above by the graphs of functions $\xi_{C,j}$ and $\xi_{C,j+1}$, for $j = 0, \ldots, \ell_C$, where we take $\xi_{C,0} = -\infty$ and $\xi_{i,\ell_C+1} = +\infty$:

$$B_{C,j} = \{(x', x_{k+1}) \in C \times \mathbb{R} \; ; \; \xi_{C,j}(x') < x_{k+1} < \xi_{C,j+1}(x')\} .$$

Proposition 2.15 *Every cell of a c.a.d. is semialgebraically homeomorphic to an open hypercube* $(0,1)^d$ *(by convention,* $(0,1)^0$ *is a point).*

Proof. We prove the property of the proposition for cells of $\mathcal{C}_k$, by induction on k. The key point is to observe that, using the notation above, every graph $A_{C,j}$ is semialgebraically homeomorphic to C and every band $B_{C,j}$ is semialgebraically homeomorphic to $C \times (0,1)$. In the case of $A_{C,j}$, the homeomorphism is simply

$$C \ni x' \longmapsto (x', \xi_{C,j}(x')) \in A_{C,j} \ .$$

For $B_{C,j}$ we take

$$\begin{array}{rcl} C \times (0,1) \ni (x',t) & \mapsto & (x', (1-t)\xi_{C,j}(x') + t\xi_{C,j+1}(x')) \text{ if } 0 < j < \ell_C \ , \\ & & \left(x', \dfrac{t-1}{t} + \xi_{C,1}(x')\right) \text{ if } j = 0,\ \ell_C \neq 0 \ , \\ & & \left(x', -\dfrac{1}{t} + \dfrac{1}{1-t}\right) \text{ if } j = \ell_C = 0 \ , \\ & & \left(x', \dfrac{t}{1-t} + \xi_{C,\ell_C}(x')\right) \text{ if } j = \ell_C \neq 0 \ . \end{array}$$

□

It is time to explain what we want to do with a c.a.d.. We shall use the following terminology: given a finite family $P_1, \ldots, P_r$ of polynomials in $\mathbb{R}[X_1, \ldots, X_n]$, we say that a subset C of $\mathbb{R}^n$ is $(P_1, \ldots, P_r)$-*invariant* if every polynomial P_i has a constant sign (> 0, < 0, or $= 0$) on C. We want to construct, from a finite family $P_1, \ldots, P_r$ of polynomials in $\mathbb{R}[X_1, \ldots, X_n]$, a c.a.d. of $\mathbb{R}^n$ such that:

c) Each cell $C \in \mathcal{C}_n$ is $(P_1, \ldots, P_r)$-invariant.

A c.a.d. of $\mathbb{R}^n$ satisfying this property will be called *adapted to* $(P_1, \ldots, P_r)$.

What is a c.a.d. adapted to $(P_1, \ldots, P_r)$ good for? First, the condition c) shows that every semialgebraic subset of $\mathbb{R}^n$ which is described by a boolean combination of equations $P_i = 0$ and inequalities $P_j > 0$ or $P_j < 0$, where P_i and P_j are among $P_1, \ldots, P_r$, is the union of some cells of $\mathcal{C}_n$. It follows that every semialgebraic set can be decomposed as the disjoint union of finitely many pieces, each semialgebraically homeomorphic to an open hypercube $(0,1)^d$. Moreover, the cylindrical arrangement of cells (property b) allows one to see that every semialgebraic subset of $\mathbb{R}^k$ described by a formula $Q_{k+1}x_{k+1} \ldots Q_n x_n \Phi$, where $Q_{k+1}, \ldots, Q_n$ are existential or universal quantifiers and Φ, a boolean combination of equations $P_i = 0$ and inequalities $P_j > 0$ or $P_j < 0$, is the union of some cells of $\mathcal{C}_k$. This can be useful, for instance, to decide whether such a formula is true or false.

2.3.2 Construction of an adapted c.a.d.

The definition of a c.a.d. shows the importance of the functions $\xi_{C,j}$ whose graphs cut the cylinders $C\times\mathbb{R}$. Since we want the bands of a cylinder contained in $\mathbb{R}^n$ to be $(P_1,\ldots,P_r)$-invariant, the $\xi_{C,j}$ have to describe the roots of the polynomials P_i, as functions of $(x_1,\ldots,x_{n-1})\in C$. We give now a first result in this direction.

Proposition 2.16 *Let $P(X_1,\ldots X_n)$ be a polynomial in $\mathbb{R}[X_1,\ldots X_n]$. Let $C\subset\mathbb{R}^{n-1}$ be a connected semialgebraic subset and $k\le d$ in $\mathbb{N}$ such that, for every point $x'=(x_1,\ldots,x_{n-1})\in C$, the polynomial $P(x',X_n)$ has degree d and exactly k distinct roots in $\mathbb{C}$. Then there are $\ell\le k$ continuous semialgebraic functions $\xi_1<\ldots<\xi_\ell : C\to\mathbb{R}$ such that, for every $x'\in C$, the set of real roots of $P(x',X_n)$ is exactly $\{\xi_1(x'),\ldots,\xi_\ell(x')\}$. Moreover, for $i=1,\ldots,\ell$, the multiplicity of the root $\xi_i(x')$ is constant for $x'\in C$.*

Proof. The argument relies on the "continuity of roots", in the following form (a proof is proposed in the next exercise):

CR Choose $a'\in C$ and let $z_1,\ldots,z_k$ be the distinct roots of $P(a',X_n)$, with multiplicities $m_1,\ldots,m_k$, respectively. Choose $\varepsilon>0$ so small that the open disks $D(z_i,\varepsilon)\subset\mathbb{C}$ with centers z_i and radius ε are disjoint. If $b'\in C$ is sufficiently close to a', the polynomial $P(b',X_n)$ has exactly m_i roots, counted with multiplicities, in the disk $D(z_i,\varepsilon)$, for $i=1,\ldots,k$.

Since $P(b',X_n)$ has k distinct roots and $d=m_1+\cdots+m_k$ roots counted with multiplicities, it follows that each $D(z_i,\varepsilon)$ contains exactly one root ζ_i of multiplicity k of $P(b',X_n)$. If z_i is real, ζ_i is real (otherwise, its conjugate $\overline{\zeta_i}$ would be another root of $P(b',X_n)$ in $D(z_i,\varepsilon)$). If z_i is nonreal, ζ_i is nonreal, since $D(z_i,\varepsilon)$ is disjoint from its image by conjugation. Hence, if $b'\in C$ is sufficiently close to a', $P(b',X_n)$ has the same number of distinct real roots as $P(a',X_n)$. Since C is connected, the number of distint real roots of $P(x',X_n)$ is constant for $x'\in C$. Let ℓ be this number. For $1\le i\le\ell$, denote by $\xi_i : C\to\mathbb{R}$ the function which sends $x'\in C$ to the i-th real root (in the increasing order) of $P(x',X_n)$. The argument above, with ε as small as we want, shows, moreover, that the functions ξ_i are continuous. It follows from the connectedness of C that each $\xi_i(x')$ has constant multiplicity. If C is described by the formula $\Theta(x')$, the graph of ξ_i is described by the formula

$$\begin{aligned}&\Theta(x')\ \text{ and }\ \exists y_1\ldots\exists y_\ell(y_1<\ldots<y_\ell\ \text{ and }\\ &\qquad P(x',y_1)=0\ \text{ and }\ \ldots\ \text{ and }\ P(x',y_\ell)=0\ \text{ and }\ x_n=y_i)\ ,\end{aligned}$$

which shows that ξ_i is semialgebraic. $\square$

Exercise 2.17 We identify monic polynomials $X^d + a_1 X^{d-1} + \cdots + a_d \in \mathbb{C}[X]$ of degree d with points $(a_1, \ldots, a_d) \in \mathbb{C}^d$. With this identification, let

$$\begin{array}{rcl} \mu : \mathbb{C}^e \times \mathbb{C}^{d-e} & \longrightarrow & \mathbb{C}^d \\ (R, S) & \longmapsto & RS \end{array}$$

be the mapping defined by the multiplication of monic polynomials.
1) Fix $R^0 \in \mathbb{C}^e$ and $S^0 \in \mathbb{C}^{d-e}$. Show that the jacobian determinant of μ at (R^0, S^0) is equal to $\pm$ the resultant of R^0 and S^0.
2) Let $Q^0 \in \mathbb{C}^d$ and assume that $Q^0 = R^0 S^0$, where R^0 and S^0 are relatively prime monic polynomials of degrees e and $d-e$, respectively. Show that for every Q sufficiently close to Q^0, there is a unique factorization $Q = RS$ with R close to R^0 and S close to S^0.
3) Assume $Q^0 = (X - z_1)^{m_1} \cdots (X - z_k)^{m_k}$, where $z_1, \ldots, z_k$ are the distinct roots of Q^0. Show that, for every Q close to Q^0, there is a unique factorization $Q = R_1 \cdots R_k$, where the R_i are monic polynomials close to $(X - z_i)^{m_i}$.
4) Fix $\varepsilon > 0$. Show that every monic polynomial sufficently close to X^m has its roots in $D(0, \varepsilon)$ (use Proposition 1.3). Deduce that every monic polynomial sufficiently close to $(X - z)^m$ has its roots in $D(z, \varepsilon)$.
5) Let Q^0 be a monic polynomial with distinct roots $z_1, \ldots, z_k$ of multiplicities $m_1, \ldots, m_k$, respectively. Choose $\varepsilon > 0$ such that all disks $D(z_i, \varepsilon)$ are disjoint. Show that every monic polynomial close to Q^0 has exactly m_i roots counted with multiplicities in $D(z_i, \varepsilon)$, for $i = 1, \ldots, k$.
6) Prove property CR above (if $P = a_0(x')X_n + \cdots + a_d(x')$, set $Q = P/a_0(x')$).

If we have several polynomials P_i, we have also to take care that the roots of the different P_i do not get mixed.

Proposition 2.18 *Let P and Q be polynomials of $\mathbb{R}[X_1, \ldots X_n]$. Let C be a connected semialgebraic subset of $\mathbb{R}^{n-1}$. Assume that the degree and the number of distinct roots of $P(x', X_n)$ (resp. $Q(x', X_n)$) and the degree of the gcd of $P(x', X_n)$ and $Q(x', X_n)$ are constant for all $x' \in C$. Let $\xi, \zeta : C \to \mathbb{R}$ be continuous semialgebraic functions such that $P(x', \xi(x')) = 0$ and $Q(x', \zeta(x')) = 0$ for every $x' \in C$. If there is $a' \in C$ such that $\xi(a') = \zeta(a')$, then $\xi(x') = \zeta(x')$ for every $x' \in C$.*

Proof. We use the same method of proof as in the preceding proposition. Let $z_1 = \xi(a') = \zeta(a'), \ldots, z_k$ be the distinct roots in $\mathbb{C}$ of the product

$P(a', X_n)Q(a', X_n)$. Let m_i (resp. p_i) be the multiplicity of z_i as a root of $P(a', X_n)$ (resp. $Q(a', X_n)$), where multiplicity zero means "not a root". The degree of $\gcd(P(a', X_n), Q(a', X_n))$ is $\sum_{i=1}^{k} \min(m_i, p_i)$, and each z_i has multiplicity $\min(m_i, p_i)$ as a root of this gcd. Choose $\varepsilon > 0$ such that all disks $D(z_i, \varepsilon)$ are disjoint. For every $x' \in C$ sufficiently close to a', each disk $D(z_i, \varepsilon)$ contains a root of multiplicity m_i of $P(x', X_n)$ and a root of multiplicity p_i of $Q(x', X_n)$. Since the degree of $\gcd(P(x', X_n), Q(x', X_n))$ is equal to $\sum_{i=1}^{k} \min(m_i, p_i)$, this gcd must have one root of multiplicity $\min(m_i, p_i)$ in each disk $D(z_i, \varepsilon)$ such that $\min(m_i, p_i) > 0$. In particular, it follows that $\xi(x') = \zeta(x')$. Since C is connected, this equality holds for every $x' \in C$. □

We have seen in Chapter 1 that the number of distinct complex roots of P and the degree of the gcd of P and Q, can be computed from the fact that the principal subresultant coefficients $\mathrm{PSRC}_i(P, P')$ and $\mathrm{PSRC}_i(P, Q)$ are zero or nonzero, as long as the degrees (with respect to X_n) of P and Q are fixed (cf. Corollary 1.21 and Proposition 1.19). For the values of the parameters (here, $X_1, \ldots, X_{n-1}$) such that some leading coefficients vanish, we have to use the principal subresultant coefficients for the truncated polynomials. This leads us to the following definition.

If P is a polynomial in $\mathbb{R}[X_1, \ldots, X_n]$, we consider it as a polynomial in the variable X_n with coefficients in $\mathbb{R}[X_1, \ldots, X_{n-1}]$. We denote by $\mathrm{lc}(P)$ its leading coefficient and by $\mathrm{trunc}(P)$ the truncated polynomial obtained by deleting its leading term. Let $P_1, \ldots, P_r$ be a family of polynomials in $\mathbb{R}[X_1, \ldots, X_n]$. We define $\mathrm{PROJ}(P_1, \ldots, P_r)$ to be the smallest family of polynomials in $\mathbb{R}[X_1, \ldots, X_{n-1}]$ satisfying the following rules:

- If $\deg_{X_n} P_i = d \geq 2$, $\mathrm{PROJ}(P_1, \ldots, P_i, \ldots, P_r)$ contains all nonconstant polynomials among $\mathrm{PSRC}_j(P_i, \partial P_i/\partial X_n)$ for $j = 0, \ldots, d-1$.
- If $1 \leq d = \min(\deg_{X_n}(P_i), \deg_{X_n}(P_k))$, $\mathrm{PROJ}(P_1, \ldots, P_i, \ldots, P_k, \ldots, P_r)$ contains all nonconstant $\mathrm{PSRC}_j(P_i, P_k)$ for $j = 0, \ldots, d$.
- If $\deg_{X_n} P_i \geq 1$ and $\mathrm{lc}(P_i)$ is not constant, $\mathrm{PROJ}(P_1, \ldots, P_i, \ldots, P_r)$ contains $\mathrm{lc}(P_i)$ and $\mathrm{PROJ}(P_1, \ldots, \mathrm{trunc}(P_i), \ldots, P_r)$.
- If $\deg_{X_n} P_i = 0$ and P_i is not constant, $\mathrm{PROJ}(P_1, \ldots, P_i, \ldots, P_r)$ contains P_i.

The following theorem is a consequence of the results previously proved in this section.

Theorem 2.19 *Let $(P_1, \ldots, P_r)$ be a family of polynomials in $\mathbb{R}[X_1, \ldots, X_n]$, and let C be a connected,* $\mathrm{PROJ}(P_1, \ldots, P_r)$*-invariant, semialgebraic subset of $\mathbb{R}^{n-1}$. Then there are continuous semialgebraic functions $\xi_1 < \ldots < \xi_\ell : C \to \mathbb{R}$, such that, for every $x' \in C$, the set $\{\xi_1(x'), \ldots, \xi_\ell(x')\}$ is the set of real roots of all nonzero polynomials $P_1(x', X_n), \ldots, P_r(x', X_n)$. The graph of each ξ_i, and each band of the cylinder $C \times \mathbb{R}$ bounded by these graphs, are connected semialgebraic sets, semialgebraically homeomorphic to C or $C \times (0,1)$, respectively, and $(P_1, \ldots, P_r)$-invariant.*

If we have constructed a c.a.d. of $\mathbb{R}^{n-1}$ adapted to $\mathrm{PROJ}(P_1, \ldots, P_r)$, the preceding theorem can be used to extend this c.a.d. to a c.a.d. of $\mathbb{R}^n$ adapted to $(P_1, \ldots, P_r)$. On the other hand, by iterating $(n-1)$ times the operation PROJ, we arrive to a finite family of polynomials in one variable X_1. It is easy to construct a c.a.d. of $\mathbb{R}$ adapted to this family: the real roots of the polynomials in the family cut the line in finitely many points and open intervals. Finally, we obtain:

Theorem 2.20 *For every finite family $P_1, \ldots, P_r$ in $\mathbb{R}[X_1, \ldots, X_n]$, there is an adapted c.a.d. of $\mathbb{R}^n$.*

We illustrate this result by constructing a c.a.d. of $\mathbb{R}^3$ adapted to the polynomial $P = X^2 + Y^2 + Z^2 - 1$. The Sylvester matrix of P and $\partial P / \partial Z$ is

$$\begin{pmatrix} 1 & 0 & X^2 + Y^2 - 1 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{pmatrix}.$$

Hence, $\mathrm{PSRC}_0(P, \partial P/\partial Z) = -4(X^2 + Y^2 - 1)$ and $\mathrm{PSRC}_1(P, \partial P/\partial Z) = 2$. Getting rid of irrelevant constant factors, we obtain $\mathrm{PROJ}(P) = (X^2 + Y^2 - 1)$, then $\mathrm{PROJ}(\mathrm{PROJ}(P)) = (X^2 - 1)$. The c.a.d. obtained is represented on Figure 2.1.

Exercise 2.21 How many cells of $\mathbb{R}^3$ are there in this c.a.d.? Is it possible to have a c.a.d. of $\mathbb{R}^3$, such that the sphere is the union of cells, with less cells?

Exercise 2.22 Let $f : A \to \mathbb{R}$ be a semialgebraic function, which is not supposed to be continuous. Show that there exists a finite semialgebraic partition $A = \bigcup_{i=1}^{s} C_i$ of A such that, for every i, the restriction of f to C_i is continuous. Hint: use a c.a.d. adapted to the graph of f.

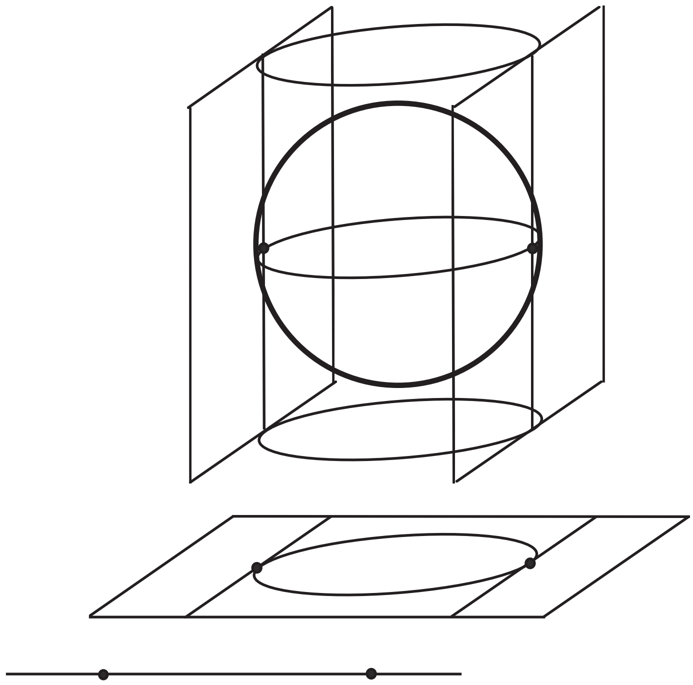


Figure 2.1: A c.a.d. adapted to the sphere

Remark. We return to the proof of Proposition 2.16. For every $x' \in C$, $\xi_i(x')$ is a root of $P(x', X_n)$, with constant multiplicity m_i. Hence, $\xi_i(x')$ is a simple root of the $(m_i - 1)^{\text{th}}$ derivative with respect to X_n. Therefore, if C is a $\mathcal{C}^\infty$ submanifold of $\mathbb{R}^{n-1}$, the function ξ_i is $\mathcal{C}^\infty$ on C. The graphs and the bands of the cylinder $C \times \mathbb{R}$ are also $\mathcal{C}^\infty$ submanifolds, diffeomorphic to C or $C \times (0,1)$, respectively (cf. the formulas in the proof of Proposition 2.15). By induction on n, one proves in this way that every semialgebraic set is the disjoint union of finitely many semialgebraic subsets C_i, which are $\mathcal{C}^\infty$ submanifolds each semialgebraically diffeomorphic to an open hypercube $(0,1)^{d_i}$. The semialgebraic $\mathcal{C}^\infty$ submanifolds are called *Nash manifolds*.

2.3.3 The c.a.d. algorithm

We shall now make more precise the algorithmic aspect of the construction of the c.a.d.. The c.a.d. algorithm receives as input a finite list of polynomials $P_1, \ldots, P_r$ in $\mathbb{Q}[X_1, \ldots, X_n]$ and produces as output the list of the cells of a c.a.d. adapted to $P_1, \ldots, P_r$ (with information on their cylindrical arrangement), together with a "test point" in each cell, whose coordinates are rational or real algebraic numbers. The algorithm works in the following way:

- Given a list of polynomials in one variable X_1, it counts and isolates in intervals with rational endpoints all real roots of these polynomials (this can be done by using Sturm's method). The cells of $\mathcal{C}_1$ are the roots and the intervals between the roots. The roots are characterized by the polynomial equation they satisfy and the interval with rational endpoints which isolates them. These endpoints may be taken as test points for the intervals between the roots (but there may be more convenient choices).

- Given a list $(P_1, \ldots, P_r)$ of polynomials in $\mathbb{Q}[X_1, \ldots, X_n]$, where $n > 1$, it computes $\mathrm{PROJ}(P_1, \ldots, P_r)$ and calls the c.a.d. algorithm for this list of polynomials in $\mathbb{Q}[X_1, \ldots, X_{n-1}]$. One obtains $\mathcal{C}_{n-1}$, which is a partition of $\mathbb{R}^{n-1}$ in $\mathrm{PROJ}(P_1, \ldots, P_r)$-invariant cells, and a test point a'_C for each cell $C \in \mathcal{C}_{n-1}$. For such a cell C, one can apply Theorem 2.19 to cut the cylinder $C \times \mathbb{R}$ in $(P_1, \ldots, P_r)$-invariant cells. In order to know how many cells there are in this cylinder and to produce a test point for each cell, one computes the real roots of $P_1(a'_C, X_n), \ldots, P_r(a'_C, X_n)$. Sturm's method can be used once again, but the coefficients of the polynomials may be real algebraic numbers.

We encounter here the problem of coding real algebraic numbers and computing with them. One possible method is to give the coordinates of the test point a'_C as polynomials in the primitive element of the extension of $\mathbb{Q}$ that they generate; this primitive element is given by its minimal polynomial over $\mathbb{Q}$ and an isolating interval with rational endpoints.

The c.a.d. algorithm can be used to solve the following problem: decide wether a formula without free variables is true or false (*decision problem*). More generally, given a first order formula, the c.a.d. algorithm allows one to decide whether the semialgebraic set S defined by this formula is empty or not, and in case not, produces a point in (every connected component of) S.

The complexity of the algorithm is *doubly exponential* in the number of variables (free variables or quantified ones) of the formula. This double exponential feature can be explained by the following remark. The resultant (with respect to X_n) of two polynomials of total degree d is, in general, of total degree d^2. Hence, taking the PROJ of a family of polynomials in n variables of maximum degree d will give a family of polynomials in $n-1$ variables of maximum degree d^2. Iterating the PROJ operation $n-1$ times will give a family of polynomials in one variable of maximum degree $d^{2^{n-1}}$.

The double exponential explains why the practical applications of the c.a.d. algorithm are very limited. In order to reduce the complexity, it is necessary to avoid the method of successive codimension 1 projections. We shall see in Chapter 4 an alternative method: the *critical point method*.

2.3.4 Connected components of semialgebraic sets

The c.a.d. shows that every semialgebraic set $S \subset \mathbb{R}^n$ is the disjoint union of semialgebraic subsets $C_1, \ldots, C_p$, such that each C_i is semialgebraically homeomorphic to an open hypercube $(0,1)^{d_i}$ (with $(0,1)^0 =$ a point). Each C_i is obviously connected.

Theorem 2.23 *Every semialgebraic set has finitely many connected components which are semialgebraic. Every semialgebraic set is locally connected.*

Proof. Using the notation above, we shall say that C_i is adjacent to C_j if $C_i \cap \mathrm{clos}(C_j) \neq \emptyset$. Let $\sim$ be the equivalence relation generated by the relation "is adjacent to": $C_i \sim C_j$ if there is a sequence $C_i = C_{i_0}, C_{i_1}, \ldots, C_{i_q} = C_j$ such that C_{i_j} is adjacent to $C_{i_{j+1}}$ or $C_{i_{j+1}}$ is adjacent to C_{i_j}. We obtain a partition of S into finitely many semialgebraic subsets $S_1, \ldots, S_r$, where each S_i is the union of all C_j in the same equivalence class for $\sim$. Each S_i is closed in S. Indeed, if $C_i \cap \mathrm{clos}(S_j) \neq \emptyset$, then C_j is adjacent to some $C_k \subset S_i$ and, hence, $C_j \subset S_i$. Since there are finitely many S_i, each one is also open in S. We now show that each S_i is connected. If $S_i = F_1 \cup F_2$, where F_1 and F_2 are disjoint closed subsets of S_i, we have:

- every $C_j \subset S_i$ is contained in F_1 or in F_2, since C_j is connected;
- if C_j and C_k are contained in S_i and C_j is adjacent to C_k, then C_j and C_k are both contained in F_1 or both in F_2.

According to the definition of S_i, it follows that $S_i = F_1$ or $S_i = F_2$. The first part of the theorem is proved.

The semialgebraic set $S \subset \mathbb{R}^n$ is locally connected if, for every $x \in S$, every open ball B with center x contains a connected neighborhood of x in S. Since $B \cap S$ is semialgebraic, it has a finite number of connected components. The connected component of $B \cap S$ containing x is a connected neighborhood of x in S. $\square$

Chapter 3

Triangulation of semialgebraic sets

In the preceding chapter, we have decomposed semialgebraic sets into simple pieces (the cells, which are semialgebraically homeomorphic to open hypercubes). We have also explained an algorithm which produces this decomposition. But the result obtained is not quite satisfactory, for the following reasons:

- We do not have a description of cells of the c.a.d. by a boolean combination of polynomial equations and inequalities. In particular, the c.a.d. algorithm described in Chapter 2 does not suffice to eliminate quantifiers.

- We have no information concerning which cells of a c.a.d. are adjacent to others, except for the cells in a cylinder. We do not know, in general, what happens when we pass from a cylinder to another. In the case of the c.a.d. adapted to the sphere, it is not difficult to determine the topology from the cell decomposition. The two functions on the disk $x^2 + y^2 < 1$, whose graphs are the two open hemispheres, have an obvious extension by continuity on the closed disk. We show an example where this is not so.

  Take $P = XYZ - X^2 - Y^2$. We have

  $$\begin{aligned} \mathrm{PROJ}(P) &= (XY, -X^2 - Y^2)\ , \\ \mathrm{PROJ}(\mathrm{PROJ}(P)) &= (X^4, 4X^2, X)\ . \end{aligned}$$

  The c.a.d. of $\mathbb{R}^2$ consists of 9 cells determined by the signs of X and Y. The cylinders over each open quadrant have three cells; the sign of P in

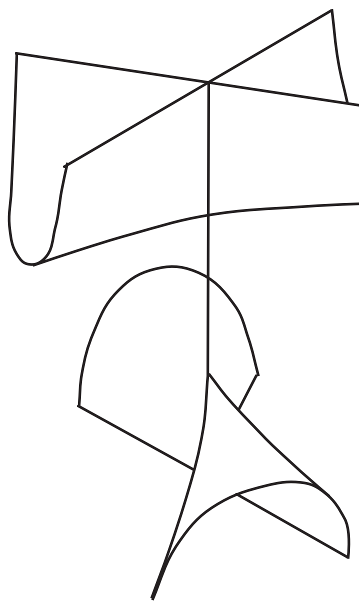


Figure 3.1: The surface $XYZ - X^2 - Y^2 = 0$.

these cells is

$$\begin{pmatrix} + \\ 0 \\ - \end{pmatrix} \quad \text{where } XY > 0, \qquad \begin{pmatrix} - \\ 0 \\ + \end{pmatrix} \quad \text{where } XY < 0\ .$$

The cylinders over each open half-axis have one cell, on which P is negative. The cylinder over the origin has one cell, on which $P = 0$. This information is not sufficient to determine the topology of the surface $P = 0$.

The main difference between the example of the sphere and the example above is the fact the polynomial P in the latter example is not monic as a polynomial in Z: its leading coefficent XY vanishes, and the polynomial is even identically zero for $X = Y = 0$. The functions describing the zeros of P on each open quadrant have no extension by continuity on the closed quadrants.

Exercise 3.1 Return to the last example $P = XYZ - X^2 - Y^2$, taking the variables in the order (Z, X, Y). Compute a c.a.d. adapted to P. Is it possible to recover the topology of $P = 0$ from this c.a.d.?

We shall see in this chapter how to modify the c.a.d. algorithm in order to solve the two problems listed above.

3.1 Thom's lemma

3.1.1 For one variable

We introduce a notation, which we shall use for relaxing inequalities. If $\varepsilon \in \{-1, 0, 1\}$ is a sign, we denote

$$\overline{\varepsilon} = \begin{cases} \{0\} & \text{if } \varepsilon = 0, \\ \{0, 1\} & \text{if } \varepsilon = 1, \\ \{0, -1\} & \text{if } \varepsilon = -1. \end{cases}$$

Proposition 3.2 (Thom's lemma) *Let $P_1, \ldots, P_s \in \mathbb{R}[X]$ be a finite family of nonzero polynomials, which is closed under derivation (i.e., if the derivative P_i' is nonzero, there is j such that $P_i' = P_j$). For $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_s) \in \{-1, 0, 1\}^s$, let $A_\varepsilon \subset \mathbb{R}$ be defined by*

$$A_\varepsilon = \{x \in \mathbb{R} \ ; \ \mathrm{sign}(P_i(x)) = \varepsilon_i \ \ \textit{for} \ \ i = 1, \ldots, s\} \ .$$

Then

- *either $A_\varepsilon = \emptyset$,*
- *or A_ε is a point (necessarily, at least one of the ε_i is 0),*
- *or A_ε is a nonempty open interval (necessarily, all ε_i are ± 1).*

Let

$$A_{\overline{\varepsilon}} = \{x \in \mathbb{R} \ ; \ \mathrm{sign}(P_1(x)) \in \overline{\varepsilon}_1 \ \text{ and } \ \ldots \ \text{ and } \ \mathrm{sign}(P_s(x)) \in \overline{\varepsilon}_s\} \ ,$$

which is obtained by relaxing the strict inequalities. Then $A_{\overline{\varepsilon}}$ is either empty, or a point, or a closed interval different from a point (and the interior of this interval is A_ε).

Proof. The proof is by induction on the number s of polynomials in the list. If $s = 1$, the only polynomial must be a nonzero constant, and in this case the statement holds trivially ($A_\varepsilon = \emptyset$ or $\mathbb{R}$). We prove the induction step from s to $s + 1$. We can assume that P_{s+1} has maximal degree in the list $(P_1, \ldots, P_{s+1})$. Then the list $(P_1, \ldots, P_s)$ is also closed under derivation. By the inductive assumption, if $\varepsilon \in \{-1, 0, 1\}^s$, the subset $A_\varepsilon \subset \mathbb{R}$ is either empty, or a nonempty open interval, or a point. Moreover, if A_ε is a nonempty open interval, P_{s+1} is monotone on A_ε, since P_{s+1}' has constant sign on A_ε. It follows that, for every $\varepsilon_{s+1} \in \{-1, 0, 1\}$, the set

$$A_{\varepsilon, \varepsilon_{s+1}} = A_\varepsilon \cap \{x \in \mathbb{R} \ ; \ \mathrm{sign}(P_{s+1}(x)) = \varepsilon_{s+1}\}$$

satisfies the properties of the proposition. $\square$

Exercise 3.3 Let a and b be distinct real roots of $P \in \mathbb{R}[X]$. Show that there exists a derivative $P^{(i)}$ such that $P^{(i)}(a)P^{(i)}(b) < 0$.

Thom's lemma allows one to answer to the first problem concerning c.a.d.: in order to obtain a description of each cell by a boolean combination of polynomial equations or inequalities, it is enough to add the derivatives. More precisely:

Corollary 3.4 *Let $(P_{i,j})$, where $1 \leq i \leq n$ and $1 \leq j \leq s_i$, be a family of nonzero polynomials such that:*

- *For fixed i, $P_{i,1}, \ldots, P_{i,s_i}$ is a family of polynomials in $\mathbb{R}[X_1, \ldots, X_i]$ which is closed under derivation with respect to X_i.*
- *For $i < n$, the family of polynomials $(P_{i,1}, \ldots, P_{i,s_i})$ contains the family $\mathrm{PROJ}(P_{i+1,1}, \ldots, P_{i+1,s_{i+1}})$.*

Then the families $\mathcal{C}_k$, for $k = 1, \ldots, n$, consisting of all nonempty semialgebraic susbsets of $\mathbb{R}^k$ of the form

$$\{x \in \mathbb{R}^k \ ; \ \mathrm{sign}(P_{i,j}(x)) = \varepsilon_{i,j} \text{ for } i = 1, \ldots, k \text{ and } j = 1, \ldots, s_i\} ,$$

where $\varepsilon_{i,j} \in \{-1, 0, 1\}$, constitute a c.a.d. of $\mathbb{R}^n$.

3.1.2 For several variables

In the situation of Thom's lemma, the different "pieces" (points and open intervals) are described by sign conditions on polynomials, and their closures are obtained by relaxing the strict inequalities. We shall extend these nice properties to the case of several variables. We return to the c.a.d., in order to see that, in "good situations", we can control what happens when we pass from a cylinder $C \times \mathbb{R}$ to another $C' \times \mathbb{R}$ such that $C' \subset \mathrm{clos}(C)$.

We shall say that a nonzero polynomial $P \in \mathbb{R}[X_1, \ldots, X_n]$ is *quasi-monic* with respect to X_n if its leading coefficient is a constant (we consider P as a polynomial in X_n with coefficients in $\mathbb{R}[X_1, \ldots, X_{n-1}]$).

Consider the following situation:

- $P_1, \ldots, P_s$ is a list of polynomials in $\mathbb{R}[X_1, \ldots, X_n]$, all quasi-monic with respect to X_n, closed under derivation with respect to X_n.
- C and C' are both connected, $\mathrm{PROJ}(P_1, \ldots, P_s)$-invariant, semialgebraic subsets of $\mathbb{R}^{n-1}$, and C' is contained in the closure of C.

It follows from Theorem 2.19 that there are continuous semialgebraic functions $\xi_1 < \ldots < \xi_\ell : C \to \mathbb{R}$ and $\xi'_1 < \ldots < \xi'_{\ell'} : C' \to \mathbb{R}$, which describe, as functions of $x = (x_1, \ldots, x_{n-1})$, the real roots of the polynomials $P_1(x, X_n), \ldots, P_s(x, X_n)$. Denote by A_j and A'_j the graphs of ξ_j and ξ'_j, respectively. Denote by B_j and B'_j the bands of the cylinders $C \times \mathbb{R}$ and $C' \times \mathbb{R}$, respectively, which are cut by these graphs.

Lemma 3.5 *In the situation above:*

1. *Every function ξ_j can be continuously extended to C', and this extension coincides with one of the functions $\xi'_{j'}$.*
2. *For every function $\xi'_{j'}$, there is a function ξ_j whose extension by continuity to C' is $\xi'_{j'}$.*
3. *For every $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_s) \in \{-1, 0, 1\}^s$, the set*

$$E_\varepsilon = \{(x, x_n) \in C \times \mathbb{R} \; ; \; \operatorname{sign}(P_i(x, x_n)) = \varepsilon_i \;\; \textit{for} \;\; i = 1, \ldots, s\}$$

is either empty, or one of the A_j, or one of the B_j. Let $E_{\overline{\varepsilon}}$ be the subset of $C \times \mathbb{R}$ obtained by relaxing the strict inequalities:

$$E_{\overline{\varepsilon}} = \{(x, x_n) \in C \times \mathbb{R} \; ; \; \operatorname{sign}(P_i(x, x_n)) \in \overline{\varepsilon}_i \;\; \textit{for} \;\; i = 1, \ldots, s\} \;,$$

and let

$$E'_{\overline{\varepsilon}} = \{(x, x_n) \in C' \times \mathbb{R} \; ; \; \operatorname{sign}(P_i(x, x_n)) \in \overline{\varepsilon}_i \;\; \textit{for} \;\; i = 1, \ldots, s\} \;.$$

If $E_\varepsilon \neq \emptyset$, we have $\operatorname{clos}(E_\varepsilon) \cap (C \times \mathbb{R}) = E_{\overline{\varepsilon}}$ and $\operatorname{clos}(E_\varepsilon) \cap (C' \times \mathbb{R}) = E'_{\overline{\varepsilon}}$. Moreover, $E'_{\overline{\varepsilon}}$ is either a graph $A'_{j'}$, or the closure of one of the bands $B'_{j'}$ in $C' \times \mathbb{R}$.

Proof. Let $x' \in C'$. Choose a function ξ_j. There is a polynomial P_μ of the family such that, for every $x \in C$, $\xi_j(x)$ is a simple root of

$$P_\mu(x, X_n) = a_0 X_n^d + a_1(x) X_n^{d-1} + \cdots + a_d(x) \;,$$

where a_0 is a nonzero constant. Set

$$M(x') = \max_{i=1,\ldots,d} \left(d \left| \frac{a_i(x')}{a_0} \right| \right)^{1/i} \;.$$

By Proposition 1.3, there is a neighborhood U of x' in $\mathbb{R}^{n-1}$ such that, for every $x \in U \cap C$, we have $\xi_j(x) \in [-M(x') - 1, M(x') + 1]$. Choose a sequence

(x^ν) in C, such that $\lim_{\nu\to\infty} x^\nu = x'$. The sequence $\xi_j(x^\nu)$ is bounded and has, therefore, a $\limsup y' \in [-M(x') - 1, M(x') + 1]$. The point (x', y') belongs to the closure of the graph of ξ_j. Let $\varphi_1 = \operatorname{sign}(P'_\mu(x, \xi_j(x)), \ldots, \varphi_d = \operatorname{sign}(P^{(d)}_\mu(x, \xi_j(x))$, for $x \in C$ (observe that these signs are constant for $x \in C$). Every point (x', x'_n) in the closure of the graph of ξ_j must satisfy

$$P_\mu(x', x'_n) = 0,\ \operatorname{sign}(P'_\mu(x', x'_n)) \in \overline{\varphi}_1, \ldots,\ \operatorname{sign}(P^{(d)}_\mu)(x', x'_n) \in \overline{\varphi}_d\ .$$

By Thom's lemma, there is at most one x'_n satisfying these inequalities. It follows that ξ_j extends continuously at x'. Hence, it extends continuously to C', and this extension coincides with one of the functions $\xi'_{j'}$. This proves 1.

We now prove 2. Choose a function $\xi'_{j'}$. Since $\xi'_{j'}$ is a simple root of some polynomial P_ν in the family, it follows from the implicit function theorem that there is a function ξ_j, also a root of P_ν, whose extension by continuity to C' is $\xi'_{j'}$.

We now turn to 3. The properties of E_ε and $E_{\overline{\varepsilon}}$ are straightforward consequences of Thom's lemma, since $P_1, \ldots, P_s$ have constant signs on each graph A_j and each band B_j, and the closure of B_j in $C \times \mathbb{R}$ is $A_j \cup B_j \cup A_{j+1}$ (as usual, $A_0 = \emptyset = A_{\ell+1}$). It is obvious that $\operatorname{clos}(E_\varepsilon) \cap (C' \times \mathbb{R}) \subset E'_{\overline{\varepsilon}}$. It follows from 1 and 2 that $\operatorname{clos}(E_\varepsilon) \cap (C' \times \mathbb{R})$ is either a graph A'_j or the closure of one of the bands $B'_{j'}$ in $C' \times \mathbb{R}$. By Thom's lemma, this is also the case for $E'_{\overline{\varepsilon}}$. It remains to check that the equality holds if $E'_{\overline{\varepsilon}}$ is the closure of a band $B'_{j'}$. In this case, all ε_i must be ± 1, and the sign of P_i is ε_i on every sufficiently small neighborhood V of a point x' of $B'_{j'}$. This implies $V \cap (C \times \mathbb{R}) \subset E_\varepsilon$ and, hence, $x' \in \operatorname{clos}(E_\varepsilon)$. This shows that $\operatorname{clos}(E_\varepsilon) \cap (C' \times \mathbb{R})$ is also the closure of $B'_{j'}$. □

The following theorem gives an answer to the problem of determining which cells of a c.a.d. are adjacent to another.

Theorem 3.6 *Let $(P_{i,j})$ be a family of polynomials with real coefficients, $1 \leq i \leq n$, $1 \leq j \leq s_i$, such that:*

- *for fixed i, $(P_{i,1}, \ldots, P_{i,s_i})$ is a family of polynomials in $\mathbb{R}[X_1, \ldots, X_i]$, all quasi-monic with respect to X_i, closed under derivation with respect to X_i,*
- *for $i < n$, the family of polynomials $(P_{i,1}, \ldots, P_{i,s_i})$ contains the family $\mathrm{PROJ}(P_{i+1,1}, \ldots, P_{i+1,s_{i+1}})$.*

For $0 < k \leq n$, given a family $\varepsilon = (\varepsilon_{i,j})$ of signs in $\{-1,0,1\}$ indexed by $i = 1,\ldots,k$ and $j = 1,\ldots,s_i$, set

$$\begin{aligned} C_\varepsilon &= \{x \in \mathbb{R}^k \ ; \ \operatorname{sign}(P_{i,j}(x)) = \varepsilon_{i,j} \text{ for } i = 1,\ldots,k \ \text{ and } \ j = 1,\ldots,s_i\} \ , \\ C_{\overline{\varepsilon}} &= \{x \in \mathbb{R}^n \ ; \ \operatorname{sign}(P_{i,j}(x)) \in \overline{\varepsilon_{i,j}} \text{ for } i = 1,\ldots,n \ \text{ and } \ j = 1,\ldots,s_i\} \ . \end{aligned}$$

Then the non empty C_ε are the cells of a c.a.d. of $\mathbb{R}^n$, and the closure of a nonempty cell C_ε is $C_{\overline{\varepsilon}}$, which is a union of cells.

The proof of the theorem is by induction on n, using the preceding lemma for the induction step and Thom's lemma for $n = 1$. This theorem may be seen as a *generalized Thom lemma*. Observe that the cells C_ε are actually Nash submanifolds, semialgebraically diffeomorphic to open hypercubes. The theorem above holds for a family of polynomials with special properties. Nevertheless, any finite family of polynomials in $\mathbb{R}[X_1,\ldots,X_n]$ can be, up to a linear change of variables, completed to a family satisfying these properties.

Proposition 3.7 *Let $P_1,\ldots,P_\ell \in \mathbb{R}[X_1,\ldots,X_n]$. There is a linear automorphism $u : \mathbb{R}^n \to \mathbb{R}^n$ and a family of polynomials $(P_{i,j})$ satisfying the conditions of Theorem 3.6, such that $P_{n,j}(X) = P_j(u(X))$ for $j = 1,\ldots,\ell$ (where $X = (X_1,\ldots,X_n)$).*

Proof. First, there is a linear change of variables

$$v(X_1,\ldots,X_n) = (X_1 + a_1X_n,\ X_2 + a_2X_n,\ldots,\ X_{n-1} + a_{n-1}X_n,\ X_n)$$

such that all polynomials $P_1(v(X)),\ldots,P_\ell(v(X))$ are quasi-monic with respect to X_n. Indeed, if $P_i(X) = \Pi_i(X) + \cdots$, where Π_i is the homogeneous part of highest degree (say d_i) of P_i, then $P_i(v(X)) = X_n^{d_i}\Pi_i(a_1,\ldots,a_{n-1},1)+$ terms of lower degree with respect to X_n. It suffices to choose $a_1,\ldots,a_{n-1}$ such that none of the $\Pi_i(a_1,\ldots,a_{n-1},1)$ is zero. Then we add to the list of polynomials $P_1(v(X)),\ldots,\ P_\ell(v(X))$ all their nonzero derivatives of every order with respect to X_n, say $P_{\ell+1},\ldots,\ P_{s_1}$. Now compute $(Q_1,\ldots,Q_t) = \mathrm{PROJ}(P_1(v(X)),\ldots,P_\ell(v(X)),P_{\ell+1},\ldots,P_{s_1})$. Using induction, there is a linear automorphism $u' : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ and a family $(P_{i,j})_{1\leq i\leq n-1\ ,\ 1\leq j\leq s_i}$ of polynomials satisfying the conditions of the theorem and such that $P_{n-1,j}(X') = Q_j(u'(X'))$, for $j = 1,\ldots,t$, where $X' = (X_1,\ldots,X_{n-1})$. Finally, set $u = (u' \times \mathrm{Id}) \circ v$ (where $(u' \times \mathrm{Id}\ )(X',X_n) = (u'(X'),X_n)$), $P_{n,j}(X) = P_j(u(X))$ for $1 \leq j \leq \ell$ and $P_{n,j}(X) = P_j(u'(X'),X_n)$ for $\ell + 1 \leq j \leq s_1$. $\square$

Corollary 3.8 *Let $S \subset \mathbb{R}^n$ be a semialgebraic set and $T_1, \ldots, T_q$, finitely many semialgebraic subsets of S. Then S can be decomposed as a disjoint finite union $S = \bigcup_{i=1}^{p} C_i$, where*

- *every C_i is semialgebraically homeomorphic (and even diffeomorphic) to an open hypercube $(0,1)^{d_i}$,*
- *the closure of C_i in S is the union of C_i and some C_j's, $j \neq i$, with $d_j < d_i$,*
- *every T_k is the union of some S_i.*

Proof. We start with a list of polynomials $(P_1, \ldots, P_\ell)$ such that S and all T_k are described by boolean combinations of sign conditions on polynomials of this list. We use Proposition 3.7 to be in the conditions of Theorem 3.6. Then S and all T_k are the unions of cells C_ε of this theorem. If $C_\varepsilon \neq \emptyset$, then $C_{\overline{\varepsilon}}$ is the union of A_ε and some $A_{\varepsilon'}$, $\varepsilon' \neq \varepsilon$. We can check by induction on n that $d_{\varepsilon'} < d_\varepsilon$. □

A decomposition $S = \bigcup_i C_i$ as in the above corollary is called a *stratification* of S, and the C_i are called *strata* of this stratification.

Exercise 3.9 We use the notation of Theorem 3.6. Assume that C_ε is nonempty and bounded. Show that the semialgebraic diffeomorphism $(0,1)^d \to C_\varepsilon$ induced by the c.a.d. extends to a surjective continuous mapping $[0,1]^d \to \mathrm{clos}(C_\varepsilon)$.

3.1.3 The finiteness theorem

The following theorem can be obtained as a consequence of the generalized Thom lemma.

Theorem 3.10 (Finiteness Theorem) *Let $S \subset \mathbb{R}^n$ be a semialgebraic set and U, a semialgebraic subset of S. Then U can be written as a finite union of open semialgebraic subsets of S of the form*

$$\{x \in S \ ; \ P_1(x) > 0 \ \text{ and } \ \ldots \ \text{ and } \ P_s(x) > 0\} \ ,$$

where $P_1, \ldots, P_s \in \mathbb{R}[X_1, \ldots, X_n]$.

Proof. Up to a linear change of variables, we can assume that S and U are defined by boolean combinations of sign conditions on polynomials of a list

$(P_{i,j})$ satisfying the conditions of the generalized Thom lemma 3.6. It follows that S and U are finite unions of cells C_ε (with the notation of Theorem 3.6). Define $C_{\underline{\varepsilon}}$ to be the set of $x \in \mathbb{R}^n$ such that $\mathrm{sign}(P_{i,j}(x)) = \varepsilon_{i,j}$ for every i, j such that $\varepsilon_{i,j} = \pm 1$ (i.e. we drop the equations and keep only the strict inequalities). Obviously, $C_{\underline{\varepsilon}}$ is an open semialgebraic set containing C_ε, and $C_{\underline{\varepsilon}}$ is the union of C_ε and the $C_{\varepsilon'}$ for all ε' such that $\varepsilon_{i,j} = \pm 1 \Rightarrow \varepsilon_{i,j} = \varepsilon'_{i,j}$. The open subset $S \cap A_{\underline{\varepsilon}}$ of S is defined by a conjunction of strict polynomial inequalities. Hence, it suffices to prove that

$$(*) \qquad U = \bigcup_{C_\varepsilon \subset U} (C_{\underline{\varepsilon}} \cap S) \ .$$

The inclusion of the left-hand side into the right-hand side is clear, since $C_\varepsilon \subset U$ implies $C_\varepsilon \subset C_{\underline{\varepsilon}} \cap S$. Let $C_\varepsilon \subset U$ and $C_{\varepsilon'} \subset C_{\underline{\varepsilon}} \cap S$. Then

$$C_{\overline{\varepsilon'}} = \mathrm{clos}(C_{\varepsilon'}) \supset C_\varepsilon \ ,$$

since $\varepsilon'_{i,j} = \pm 1$ implies $\varepsilon_{i,j} \in \overline{\varepsilon'_{i,j}}$. Therefore $\mathrm{clos}(C_{\varepsilon'}) \cap U \neq \emptyset$ and, since U is open in S, $C_{\varepsilon'} \cap U \neq \emptyset$. It follows that $C_{\varepsilon'} \subset U$. Hence, the equality $(*)$ is proved. $\square$

Exercise 3.11 The semialgebraic set

$$\{(x,y) \in \mathbb{R}^2 \ ; \ (y \neq 0 \ \text{ and } \ x^2 + y^2 < 1) \ \text{ or } \ (y = 0 \ \text{ and } \ 0 < x < 1)\}$$

is open in $\mathbb{R}^2$. Can you write it in the form described in the Finiteness Theorem?

Remark. Since U is open in S, it can be written as a union of open balls intersected with S, and each open ball is described by a strict polynomial inequality. The main point in Theorem 3.10 is that U is a *finite* union of semialgebraic subsets described by conjunctions of strict polynomial inequalities.

3.2 Triangulation

3.2.1 Simplicial complexes

We first recall some definitions concerning simplicial complexes that we shall need. Let $a_0, \ldots, a_d$ be points of $\mathbb{R}^n$ which are affine independent (i.e. not

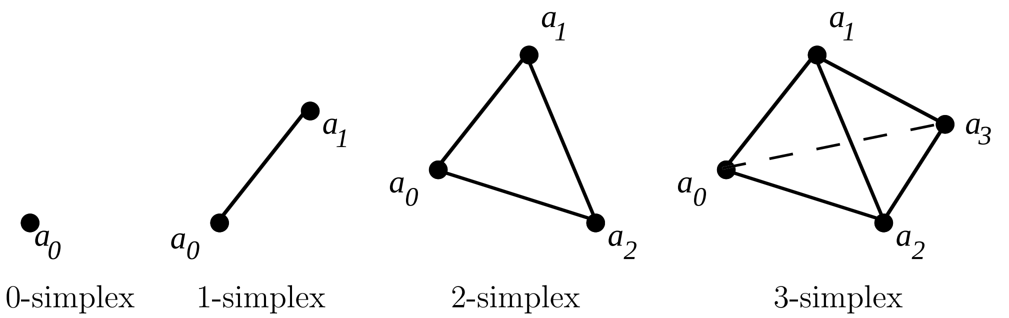


Figure 3.2: Simplices

contained in an affine subspace of dimension $d-1$). The *d-simplex* with vertices $a_0, \dots, a_d$ is

$$[a_0, \dots, a_d] = \{x \in \mathbb{R}^n \ ; \ \exists \lambda_0, \dots, \lambda_d \in [0,1] \ \sum_{i=0}^{d} \lambda_i = 1 \ \text{ and } \ x = \sum_{i=0}^{d} \lambda_i a_i\}$$

The corresponding *open simplex* is

$$(a_0, \dots, a_d) = \{x \in \mathbb{R}^n \ ; \ \exists \lambda_0, \dots, \lambda_d \in (0,1] \ \sum_{i=0}^{d} \lambda_i = 1 \ \text{ and } \ x = \sum_{i=0}^{d} \lambda_i a_i\}$$

We shall denote by $\mathring{\sigma}$ the open simplex corresponding to the simplex σ. A *face* of the simplex $\sigma = [a_0, \dots, a_d]$ is a simplex $\tau = [b_0, \dots, b_e]$ such that

$$\{b_0, \dots, b_e\} \subset \{a_0, \dots, a_d\} \ .$$

A *finite simplicial complex* in $\mathbb{R}^n$ is a finite collection $K = \{\sigma_1, \dots, \sigma_p\}$ of simplices $\sigma_i \subset \mathbb{R}^n$ such that, for every $\sigma_i, \sigma_j \in K$, the intersection $\sigma_i \cap \sigma_j$ is a common face of σ_i and σ_j (see Figure 3.3).

Figure 3.3: Simplicial complex

We set $|K| = \bigcup_{\sigma_i \in K} \sigma_i$; this is a semialgebraic subset of $\mathbb{R}^n$. A *polyhedron* in $\mathbb{R}^n$ is a subset P of $\mathbb{R}^n$, such that there exists a finite simplicial complex

K in $\mathbb{R}^n$ with $P = |K|$. Such a K will be called a *simplicial decomposition* of P. In the following, it will be convenient to agree that if a simplex σ belongs to a finite simplicial complex K, then all faces of σ also belong to K. With this convention, $|K|$ is the disjoint union of all $\mathring{\sigma}$ for $\sigma \in K$. Let K be a finite simplicial complex and, for $\sigma \in K$, let $\widehat{\sigma}$ be the barycenter of σ. The barycentric subdivision of K, denoted by K', is the finite simplicial complex whose simplices are all $[\widehat{\sigma}_0, \widehat{\sigma}_1, \ldots, \widehat{\sigma}_d]$, such that σ_i is a simplex of K, for $i = 0, \ldots, d$, and σ_i is a proper face of σ_{i+1}, for $i = 0, \ldots, d-1$. This is indeed a finite simplicial complex. The figure 3.4 shows examples of barycentric subdivisions for complexes which are reduced to a simplex with its faces.

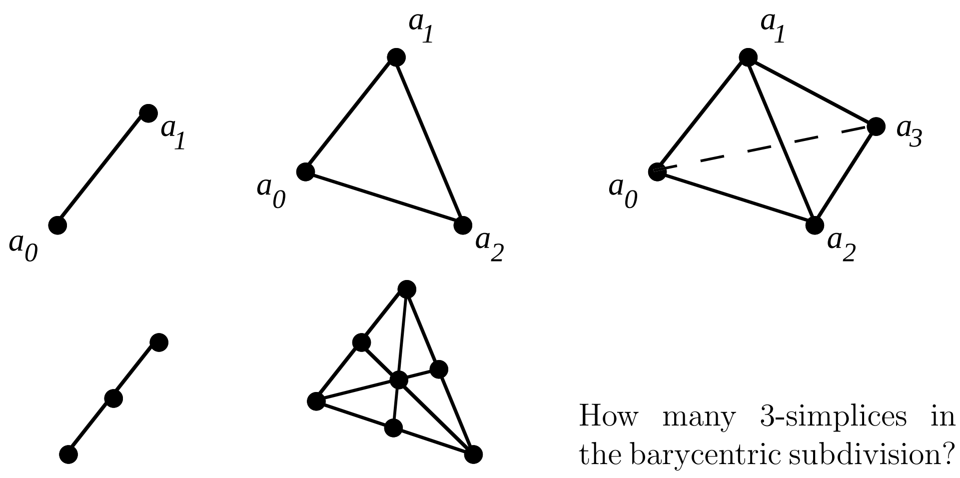


Figure 3.4: Barycentric subdivision

3.2.2 Triangulation of a compact semialgebraic set

Theorem 3.12 *Let $S \subset \mathbb{R}^n$ be a compact semialgebraic set, and $S_1, \ldots, S_p$, semialgebraic subsets of S. Then there exists a finite simplicial complex K in $\mathbb{R}^n$ and a semialgebraic homeomorphism $h : |K| \to S$, such that each S_k is the image by h of a union of open simplices of K.*

Proof. The proof is by induction on n. For $n = 1$, we can take $|K| = S$, and the only open simplices are points and bounded open intervals. We now prove the theorem for $n > 1$, assuming that it holds true for $n - 1$. Up to a linear change of variables (cf. Proposition 3.7), we can assume that S and all S_k are unions of cells of a c.a.d. satisfying the properties of the generalized Thom lemma 3.6, associated to a family of polynomials $(P_{i,j})$. In particular, we have a partition of $\mathbb{R}^{n-1}$ into finitely many connected semialgebraic sets C_i and continuous semialgebraic functions $\xi_{i,1} < \ldots < \xi_{i,\ell_i} : C_i \to \mathbb{R}$, which describe the real roots of the polynomials $P_{n,1}(x, X_n), \ldots, P_{n,s_n}(x, X_n)$ as functions of $x \in C_i$. We know that S and the S_k are unions of graphs of $\xi_{i,j}$ and bands of the cylinders $C_i \times \mathbb{R}$ bounded by these graphs. Denote by $\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ the projection onto the space of the first $n-1$ coordinates. The set $\pi(S)$ is compact, semialgebraic, and it is the union of some C_i; similarly, each $\pi(S_K)$ is the union of some C_i. By the inductive assumption, there is a triangulation $g : |L| \to \pi(S)$, where L is a finite simplicial complex in $\mathbb{R}^{n-1}$ and g a semialgebraic homeomorphism, such that each $C_i \subset \pi(S)$ is the union of images by g of open simplices of L. We can actually subdivide the $C_i \subset \pi(S)$ and assume that these C_i are of the form $g(\mathring{\tau}_i)$, where τ_i is a simplex of L.

We shall now construct, for each τ of L, a finite simplicial complex K_τ in $\mathbb{R}^n$ and a semialgebraic homeomorphism

$$h_\tau : |K_\tau| \longrightarrow \mathrm{clos}(S \cap (g(\mathring{\tau}) \times \mathbb{R})) \ .$$

We fix τ in L, say $\tau = [b_0, \ldots, b_d]$. Let $\xi : g(\mathring{\tau}) \to \mathbb{R}$ be one of the functions of the c.a.d., whose graph is contained in S. By Lemma 3.5, ξ has a continuous extension $\overline{\xi}$ defined on $\mathrm{clos}(g(\mathring{\tau})) = g(\tau)$. Set $a_i = (b_i, \overline{\xi}(g(b_i))) \in \mathbb{R}^n$, for $i = 0, \ldots, d$. Let σ_ξ be the simplex $[a_0, \ldots, a_d] \subset \mathbb{R}^n$. This σ_ξ will be one of the simplices of K_τ, and we define h_τ on σ_ξ by (see Figure 3.5)

$$h_\tau(\lambda_0 a_0 + \cdots + \lambda_d a_d) = (y, \overline{\xi}(y)) \ , \quad \text{where } y = g(\lambda_0 b_0 + \cdots + \lambda_d b_d) \ .$$

If $\xi' = g(\mathring{\tau}) \to \mathbb{R}$ is another function of the c.a.d. whose graph is contained in S, we define $\sigma_{\xi'} = [a'_0, \ldots, a'_d]$ in the same way. We do not want $\sigma_{\xi'}$ and σ_ξ to coincide: at least one of the a'_i must be different from a_i. Also, if the restrictions of $\overline{\xi}$ and $\overline{\xi'}$ to a face ρ of τ are different, the values of $\overline{\xi}$ and $\overline{\xi'}$ must differ for at least one of the vertices b_i of ρ (in order that the corresponding a_i and a'_i are distinct). So we should have the following property:

† for every simplex τ of L, if ξ and ξ' are distinct functions $g(\mathring{\tau}) \to \mathbb{R}$ of the c.a.d., there is at least one vertex b of τ such that $\overline{\xi}(g(b)) \neq \overline{\xi'}(g(b))$.

Observe that if τ_1 is a simplex of the barycentric subdivision L' of L, with $\mathring{\tau}_1 \subset \mathring{\tau}$, then the barycenter $\widehat{\tau}$ of τ is a vertex of τ_1, and $\xi \neq \xi'$ implies $\xi(g(\widehat{\tau})) \neq \xi'(g(\widehat{\tau}))$. Hence, replacing L with its barycentric subdivision L', we can assume that property † holds.

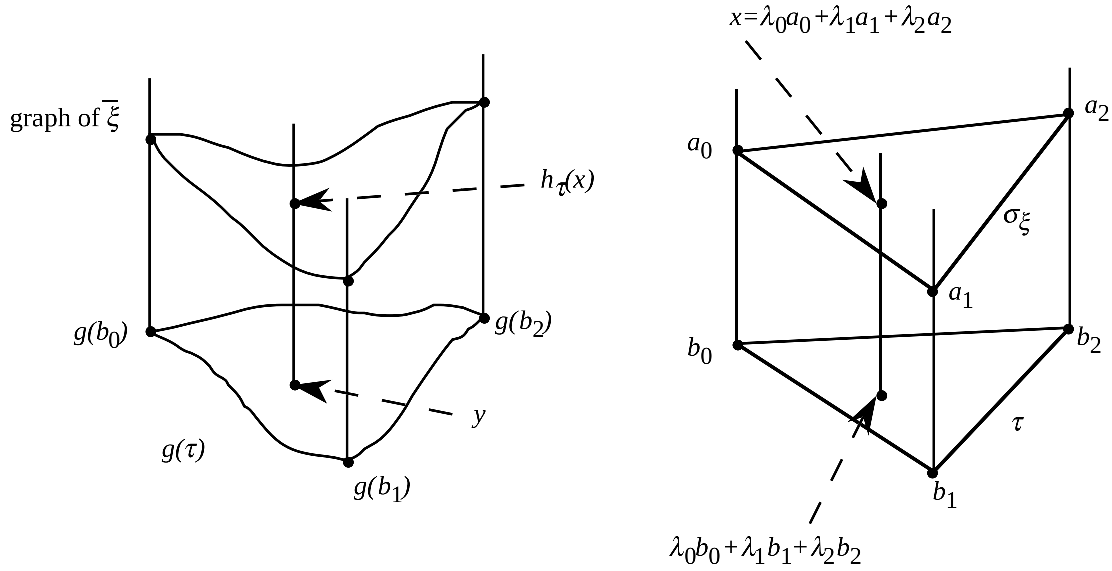


Figure 3.5: Construction of K_τ and h_τ: the case of a graph

We now consider the case of a band B contained in S (see Figure 3.6). Since S is compact, the band B is bounded from below and from above by the graphs of two consecutive functions

$$\xi < \xi' : g(\overset{\circ}{\tau}) \longrightarrow \mathbb{R} .$$

Let P be the polyhedron which is the part of the cylinder $\tau \times \mathbb{R}$ delimited by the simplices σ_ξ and $\sigma_{\xi'}$. The polyhedron P has the simplicial decomposition

$$P = \bigcup_{\substack{0 \le i \le d \\ a_i \ne a'_i}} [a'_0, \ldots, a'_i, a_i, \ldots, a_d] \ , .$$

These simplices and their faces will belong to K_τ, and we define h_τ on P so that it sends linearly the segment $[\lambda_0 a_0 + \cdots + \lambda_d a_d, \lambda_0 a'_0 + \cdots + \lambda_d a'_d]$ onto the segment $[(y, \overline{\xi}(y)), (y, \overline{\xi'}(y))]$, where $y = g(\lambda_0 b_0 + \cdots + \lambda_d b_d)$. By property †, the first segment is reduced to a point if and only if it is the case for the second.

We have constructed K_τ and h_τ for every simplex τ de L. We shall now check that the K_τ and h_τ can be glued together to give K and the triangulation $h : |K| \to S$. It is sufficient to check the gluing property for a simplex τ and one of its faces ρ. First observe that if σ_η is a simplex of K_ρ which meets $|K_\tau|$ and is sent by h_ρ onto the closure of the graph of the function $\eta : g(\overset{\circ}{\rho}) \to \mathbb{R}$ of

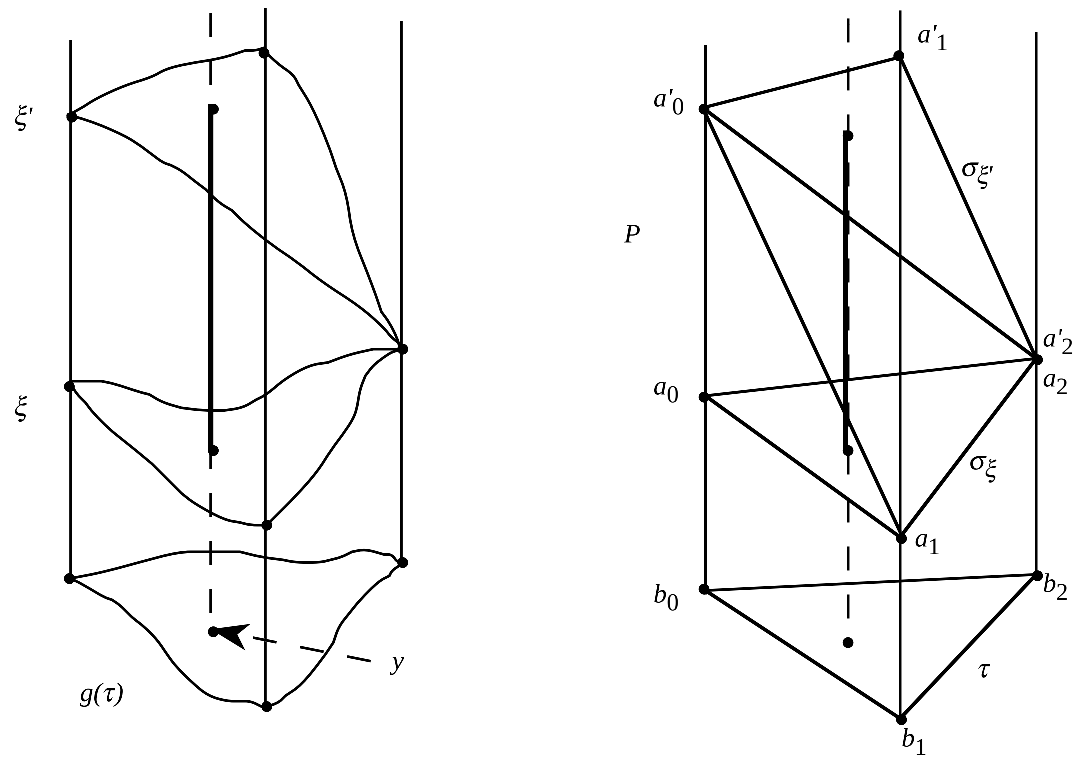


Figure 3.6: Construction of K_τ and h_τ: the case of a band

the c.a.d., then σ_η is a simplex of K_τ. Indeed, by property 2 of Lemma 3.5, η coincides with $\overline{\xi}$ on $g(\mathring{\rho})$, for some function $\xi : g(\mathring{\tau}) \to \mathbb{R}$ of the c.a.d. such that σ_ξ is a simplex of K_τ; then σ_η is a face of σ_ξ. It follows also that h_τ and h_ρ coincide on $|K_\tau| \cap |K_\rho|$. It remains to check that the simplicial decomposition of the polyhedron P in $\tau \times \mathbb{R}$ (cf. above) induces the simplicial decomposition of the polyhedron $P \cap (\rho \times \mathbb{R})$. This is the case if we have chosen a total ordering of the set of vertices of L, and if we make the simplicial decomposition

$$P = \bigcup_i [a'_0, \ldots, a'_i, a_i, \ldots, a_d] \ ,$$

where $b_0 < \ldots < b_i < \ldots < b_d$ for the chosen ordering. □

3.2.3 The curve selection lemma

The triangulation theorem allows one to give a short proof of the following.

Theorem 3.13 (Curve selection lemma) *Let $S \subset \mathbb{R}^n$ be a semialgebraic set. Let $x \in \operatorname{clos}(S)$, $x \notin S$. Then there exists a continuous semialgebraic mapping $\gamma : [0,1] \to \mathbb{R}^n$ such that $\gamma(0) = x$ and $\gamma((0,1]) \subset S$.*

Proof. Replacing S with its intersection with a ball with center x and radius 1, we can assume S bounded. Then $\operatorname{clos}(S)$ is a compact semialgebraic set. By the triangulation theorem, there is a finite simplicial complex K and a semialgebraic homeomorphism $h : |K| \to \operatorname{clos}(S)$, such that $x = h(a)$ for a vertex a of K and S is the union of some open simplices of K. In particular, since x is in the closure of S and not in S, there is a simplex σ of K whose a is a vertex, and such that $h(\mathring{\sigma}) \subset S$. Taking a linear parametrization of the segment joining a to the barycenter of σ, we obtain $\delta : [0,1] \to \sigma$ such that $\delta(0) = a$ and $\delta((0,1]) \subset \mathring{\sigma}$. Then $\gamma = h \circ \delta$ satisfies the property of the theorem. $\square$

Exercise 3.14 Show that a connected semialgebraic set A is *semialgebraically arcwise connected*: for every points a, b in A, there exists a continuous semialgebraic mapping $\gamma : [0,1] \to A$ such that $\gamma(0) = a$ and $\gamma(1) = b$.

3.3 Dimension

3.3.1 Dimension via c.a.d.

The c.a.d. allows one to decompose a semialgebraic set S as a finite union of cells semialgebraically homeomorphic (and even diffeomorphic) to open hypercubes $(0,1)^d$. This leads to the definition of the dimension of S as the maximum of these d.

Proposition 3.15 *Let $S \subset \mathbb{R}^n$ be a semialgebraic set, and let $S = \bigcup_{i=1}^p C_i$ be a decomposition of S into a disjoint union of semialgebraic subsets C_i, each semialgebraically diffeomorphic to $(0,1)^{d_i}$. The* dimension *of S is, by definition, $d = \max\{d_i \; ; \; i = 1, \ldots, p\}$. This dimension is independent of the decomposition.*

Proof. Assume that $S = \bigcup_{i=1}^p C_i$ and $S = \bigcup_{j=1}^q D_j$ are both decompositions of S into a disjoint union of semialgebraic subsets, each semialgebraically diffeomorphic to an open hypercube. By corollary 3.8, there is a semialgebraic stratification $S = \bigcup_{k=1}^r \Sigma_k$, which is a common refinement of the two decompositions. Each C_i and each D_j is a finite union of strata Σ_k. Let us compare the dimensions of C_i, D_j and Σ_k, as submanifolds of $\mathbb{R}^n$. If Σ_k is contained in C_i, the dimension of Σ_k is not greater than the dimension of C_i. On the other hand, if Σ_k is of maximal dimension among the strata contained in C_i, then Σ_k is open in C_i and, hence, has the same dimension as C_i. Indeed, the closure of $C_i \setminus \Sigma_k$ contains only $C_i \setminus \Sigma_k$ and strata of smaller dimension than Σ_k; therefore, it is disjoint from Σ_k. It follows that the maximum of the dimensions of the C_i is equal to the maximum of the dimensions of the Σ_k, which is, for the same reason, equal to the maximum of the dimensions of the D_j. □

The following properties of the dimension of a semialgebraic set are obvious: the dimension of the union of finitely many semialgebraic sets is the maximum of the dimensions of these semialgebraic sets; the dimension of the cartesian product of semialgebraic sets is the sum of their dimensions. The dimension of a semialgebraic set behaves well with respect to the topological closure.

Proposition 3.16 *Let $A \subset \mathbb{R}^n$ be a semialgebraic set. Then*

1. $\dim(\mathrm{clos}(A)) = \dim A$,
2. $\dim(\mathrm{clos}(A) \setminus A) < \dim A$.

Proof. Both properties follow from the definition of dimension and the fact that the closure of a stratum is the union of this stratum and strata of smaller dimensions (cf. 3.8). □

The dimension is invariant by a semialgebraic homeomorphism.

Lemma 3.17 *Let $A \subset \mathbb{R}^{n+k}$ be a semialgebraic set, $\pi : \mathbb{R}^{n+k} \to \mathbb{R}^n$ the projection on the space of the first n coordinates. Then* $\dim(\pi(A)) \leq \dim(A)$. *Moreover, if the restriction of π to A is one-to-one, then* $\dim(\pi(A)) = \dim A$.

Proof. If $k = 1$ and A is either the graph of a function or a band of a c.a.d. of $\mathbb{R}^{n+1}$, the lemma is obvious. If A is any semialgebraic subset of $\mathbb{R}^{n+1}$, it is the union of cells of a c.a.d.; hence, the lemma still holds true in this case. We prove the case of $k > 1$ by an easy induction. □

Theorem 3.18 *Let S be a semialgebraic subset of $\mathbb{R}^n$, and $f : S \to \mathbb{R}^k$ a semialgebraic mapping (not necessarily continuous). Then* $\dim f(S) \leq \dim S$. *If f is one-to-one, then* $\dim f(S) = \dim S$.

Proof. Let $A \subset \mathbb{R}^{n+k}$ be the graph of f. From the preceding lemma, it follows that $\dim(S) = \dim(A)$ and $\dim(f(S)) \leq \dim(A)$, with moreover $\dim(f(S)) = \dim(A)$ if f is one-to-one. $\square$

Exercise 3.19 Let $S \subset \mathbb{R}^n$ be a semialgebraic set, $x \in S$. Show that there exist a neighborhood V of x in $\mathbb{R}^n$ and a nonnegative integer d such that, for every semialgebraic neighborhood $W \subset V$ of x in $\mathbb{R}^n$, $\dim(W \cap S) = d$. The integer d is called the *dimension of S at x* and denoted by $\dim_x S$. Show that

$$\dim S = \max\{\dim_x S \ ; \ x \in S\} .$$

Show that $\{x \in S \ ; \ \dim_x S = \dim S\}$ is a closed semialgebraic subset of S.

3.3.2 Dimension of algebraic sets

We shall compare the dimension of semialgebraic sets, defined via decomposition, with the dimension of algebraic sets. We recall the results concerning the dimension of algebraic sets that we shall need (cf. [S], Chapter 1).

For $A \subset \mathbb{R}[X_1, \ldots, X_n]$, we denote by $\mathcal{Z}(A) \subset \mathbb{R}^n$ the common zeroset of all polynomials of A:

$$\mathcal{Z}(A) = \{x \in \mathbb{R}^n \ ; \ \forall P \in A \ P(x) = 0\} .$$

An algebraic subset of $\mathbb{R}^n$ is a subset of the form $\mathcal{Z}(A)$, for some subset A of $\mathbb{R}[X_1, \ldots, X_n]$. If I is the ideal of $\mathbb{R}[X_1, \ldots, X_n]$ generated by A, then $\mathcal{Z}(I) = \mathcal{Z}(A)$. Since every ideal of $\mathbb{R}[X_1, \ldots, X_n]$ is generated by finitely many polynomials, an algebraic set is the common zeroset of finitely many polynomials. Since, for $x \in \mathbb{R}^n$, $P_1(x) = \ldots = P_s(x) = 0$ is equivalent to $(P_1^2 + \cdots + P_s^2)(x) = 0$, an algebraic subset of $\mathbb{R}^n$ can always be described by *one* equation (this is not the case for complex algebraic sets).

For $S \subset \mathbb{R}^n$, we denote by $\mathcal{I}(S) \subset \mathbb{R}[X_1, \ldots, X_n]$ the subset of polynomials which vanish on S:

$$\mathcal{I}(S) = \{P \in \mathbb{R}[X_1, \ldots, X_n] \ ; \ \forall x \in S \ P(x) = 0\} .$$

$\mathcal{I}(S)$ is an ideal of $\mathbb{R}[X_1, \ldots, X_n]$. A subset $V \subset \mathbb{R}^n$ is algebraic if and only if $V = \mathcal{Z}(\mathcal{I}(V))$. The quotient ring $\mathcal{P}(S) = \mathbb{R}[X_1, \ldots, X_n]/\mathcal{I}(S)$ is called *the ring of polynomial functions on S*. Indeed, it can be identified with the ring of functions $S \to \mathbb{R}$ which are the restriction of a polynomial. A nonempty algebraic set V is said to be *irreducible* if it cannot be written as the union of two algebraic sets strictly contained in V. If V is irreducible, $\mathcal{P}(V)$ is an integral domain, and we shall denote by $\mathcal{K}(V)$ its field of fractions (the field of rational fractions on V).

The *dimension* of an algebraic set V is, by definition, the Krull dimension of the ring $\mathcal{P}(V)$, i.e. the maximal length of chains of prime ideals in $\mathcal{P}(V)$: $\dim(\mathcal{P}(V))$ is the maximum of d such that there exist prime ideals $\mathfrak{p}_0, \mathfrak{p}_1, \ldots, \mathfrak{p}_d$ of $\mathcal{P}(V)$, with $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \ldots \subsetneq \mathfrak{p}_d$. If V is irreducible, this dimension is equal to the transcendence degree of the field $\mathcal{K}(V)$ over $\mathbb{R}$ (i.e. $\mathcal{K}(V)$ is an algebraic extension of the field of rational fractions $\mathbb{R}(T_1, \ldots, T_d)$, where $d = \dim(V)$). An algebraic set V has a unique decomposition as a union of finitely many irreducible algebraic subsets $V_1, \ldots, V_p$, where $V_i \not\subset V_j$ for $i \neq j$. The V_i are called the *irreducible components* of V. If $W_1, \ldots, W_k$ are algebraic sets, so is $W_1 \cup \ldots \cup W_k$, and $\dim(W_1 \cup \ldots \cup W_k) = \max(\dim(W_i))$. In particular, the dimension of an algebraic set is the maximum of the dimensions of its irreducible components.

If $S \subset \mathbb{R}^n$ is any subset, $\mathcal{Z}(\mathcal{I}(S))$ is the smallest algebraic subset of $\mathbb{R}^n$ containing S. It is called the *Zariski closure of* S, and it will be denoted by $\overline{S}^Z$. The algebraic subsets of $\mathbb{R}^n$ are the closed sets of a topology on $\mathbb{R}^n$, which is called the Zariski topology, and $\overline{S}^Z$ is the closure of S for this topology. The Zariski topology is coarser than the usual topology, and it is not separated.

Theorem 3.20 *Let $S \subset \mathbb{R}^n$ be a semialgebraic set. Its dimension as a semialgebraic set (cf. 3.15) is equal to the dimension, as an algebraic set, of its Zariski closure $\overline{S}^Z$. In particular, if $V \subset \mathbb{R}^n$ is an algebraic set, its dimension as a semialgebraic set is equal to its dimension as an algebraic set (i.e. the Krull dimension of $\mathcal{P}(V)$).*

Proof. If $S = \bigcup_{i=1}^p C_i$, then $\overline{S}^Z = \bigcup_{i=1}^p \overline{C_i}^Z$. Hence, it is sufficient to prove the theorem for a cell $C \subset \mathbb{R}^n$ of a c.a.d.. The proof is by induction on n.

If $n = 1$, either C is a point and $\overline{C}^Z$ is equal to this point, or C is a nonempty open interval and $\overline{C}^Z = \mathbb{R}$. We have algebraic dimension 0 in the first case and 1 in the second case. Let $n > 1$, and assume the theorem proved for $n-1$. Let $\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ be the projection on the space of the first $n-1$ coordinates. Then $\pi(C) = D$ is a cell of the c.a.d., semialgebraically homeomorphic to $(0,1)^d$. By the inductive assumption, $\dim \overline{D}^Z = d$ (dimension as algebraic set). We have to consider two cases.

1. C is the graph of a function $\xi : D \to \mathbb{R}$ of the c.a.d., and, hence, C is semialgebraically diffeomorphic to $(0,1)^d$. There is a polynomial $P \in$

$\mathbb{R}[X_1, \ldots, X_n]$ such that, for every $x \in D$, $P(x, X_n)$ is not identically zero and $P(x, \xi(x)) = 0$. Let $Z = \mathcal{Z}(P) \subset \mathbb{R}^n$. Let $\overline{D}^Z = V_1 \cup \ldots \cup V_p$ be the decomposition into irreducible components (actually, it can be shown that $\overline{D}^Z$ is irreducible). We have

$$\overline{C}^Z \subset Z \cap ((V_1 \times \mathbb{R}) \cup (V_2 \times \mathbb{R}) \cup \ldots \cup (V_p \times \mathbb{R})),$$

and $Z \cap (V_i \times \mathbb{R}) \subsetneq V_i \times \mathbb{R}$, for $i = 1, \ldots, p$. We use the following facts concerning algebraic sets and their dimensions:

- $V_i \times \mathbb{R}$ is irreducible, since both V_i and $\mathbb{R}$ are irreducible.
- $\dim(V_i \times \mathbb{R}) = \dim V_i + 1$ (algebraic dimension).
- If W is an irreducible algebraic set and $V \subsetneq W$ a proper algebraic subset, then $\dim V < \dim W$.

It follows that

$$\dim Z \cap (V_i \times \mathbb{R}) < \dim V_i \times \mathbb{R} = \dim V_i + 1 \ ,$$

therefore $\dim \overline{C}^Z \leq \dim \overline{D}^Z = d$. It remains to prove the reverse inequality. Let W be an irreducible component of $\overline{C}^Z$. Then $Y = \overline{\pi(W)}^Z$ is irreducible. Indeed, if $Y = F_1 \cup F_2$, where F_1 and F_2 are algebraic sets, we have $W \subset \pi^{-1}(F_1)$ or $W \subset \pi^{-1}(F_2)$ since W is irreducible; hence, $Y \subset F_1$ or $Y \subset F_2$. The projection π induces an injective homomorphism from $\mathcal{P}(Y) = \mathcal{P}(\pi(W))$ into $\mathcal{P}(W)$ and, hence, a field homomorphism $\mathcal{K}(Y) \to \mathcal{K}(W)$. It follows that $\dim(Y) \leq \dim(W)$. Since $\overline{D}^Z$ is the union of these Y, we have $\dim \overline{D}^Z \leq \dim \overline{C}^Z$.

2. C is a band of the cylinder $D \times \mathbb{R}$. Then C is semialgebraically diffeomorphic to $(0,1)^{d+1}$. In this case, $\overline{C}^Z \supset D \times \mathbb{R}$, therefore $\dim \overline{C}^Z = \dim \overline{D}^Z + 1 = d + 1$.

The proof is completed. ☐

Chapter 4

Families of semialgebraic sets. Uniform bounds

4.1 Semialgebraic triviality of families

4.1.1 Hardt's theorem

Let $A \subset \mathbb{R}^n$ be a semialgebraic set, defined by a boolean combination of sign conditions on polynomials $P_1, \ldots, P_q$. Construct a c.a.d. of $\mathbb{R}^n$ adapted to $P_1, \ldots, P_q$. The set A is a union of graphs and bands in cylinders $C_i \times \mathbb{R}$, where $\mathbb{R}^{n-1} = C_1 \cup \ldots \cup C_r$ is a finite semialgebraic partition. Each $A \cap (C_i \times \mathbb{R})$ is semialgebraically homeomorphic to a product $C_i \times F_i$, where F_i is a semialgebraic subset of $\mathbb{R}$: one can take for instance $F_i = p^{-1}(b_i)$, where $p : A \to \mathbb{R}^{n-1}$ is the restriction of the projection onto the space of the $n-1$ first coordinates, and b_i, a point chosen in C_i. Hence, we have decomposed the target space $\mathbb{R}^{n-1}$ as the disjoint union of finitely many semialgebraic subsets C_i, such that p is semialgebraically trivial over each C_i in the following sense.

A continuous semi-algebraic mapping $p : A \to \mathbb{R}^k$ is said to be *semialgebraically trivial over a semialgebraic subset* $C \subset \mathbb{R}^k$ is there is a semialgebraic set F and a semialgebraic homeomorphism $h : p^{-1}(C) \to C \times F$, such that the composition of h with the projection $C \times F \to C$ is equal to the restriction of

p to $p^{-1}(C)$.

$$
\begin{tikzcd}
A \supset p^{-1}(C) \arrow[rr, "h"] \arrow[dr, "p"'] & & C \times F \arrow[dl, "\text{projection}"] \\
& \mathbb{R}^k \supset C &
\end{tikzcd}
$$

The homeomorphism h is called a semi-algebraic trivialization of p over C. We say that the trivialization h is compatible with a semialgebraic subset $B \subset A$ if there is a semialgebraic subset $G \subset F$ such that $h(B \cap p^{-1}(C)) = C \times G$.

The above property of the projection mapping $p : A \to \mathbb{R}^{n-1}$ holds actually for every continuous semialgebraic mapping.

Theorem 4.1 (Hardt's semialgebraic triviality) *Let $A \subset \mathbb{R}^n$ be a semi-algebraic set and $p : A \to \mathbb{R}^k$, a continuous semi-algebraic mapping. There is a finite semialgebraic partition of $\mathbb{R}^k$ into $C_1, \ldots, C_m$ such that p is semialgebraically trivial over each C_i. Moreover, if $B_1, \ldots, B_q$ are finitely many semialgebraic subsets of A, we can ask that each trivialization $h_i : p^{-1}(C_i) \to C_i \times F_i$ is compatible with all B_j.*

In particular, if b and b' are in the same C_i, then $p^{-1}(b)$ and $p^{-1}(b')$ are semialgebraically homeomorphic, since they are both semialgebraically homeomorphic to F_i. Actually we can take for F_i a fiber $p^{-1}(b_i)$, where b_i is a chosen point in C_i, and we ask in this case that $h_i(x) = (x, b_i)$ for all $x \in p^{-1}(b_i)$.

For the proof of Hardt's theorem, we refer to [BR] or [BCR]. Hardt's theorem for o-minimal structures is proved in [D] and in the lecture notes [Co] in the same collection.

We can easily derive from Hardt's theorem a useful information about the dimensions of the fibers of a continuous semialgebraic mapping. We keep the notation of the theorem. For every $b \in C_i$:

$$\dim p^{-1}(b) = \dim F_i = \dim p^{-1}(C_i) - \dim C_i \leq \dim A - \dim C_i \ .$$

From this observation follows:

Corollary 4.2 *Let $A \subset \mathbb{R}^n$ be a semialgebraic set and $f : A \to \mathbb{R}^k$, a continuous semialgebraic mapping. For $d \in \mathbb{N}$, the set*

$$\{b \in \mathbb{R}^k \ ; \ \dim(p^{-1}(b)) = d\}$$

is a semialgebraic subset of $\mathbb{R}^k$ of dimension not greater than $\dim A - d$.

Exercise 4.3 Let $p : \mathbb{R}^{n+1} \to \mathbb{R}^n$ be the projection on the space of the first n coordinates. Let A be a semialgebraic subset of $\mathbb{R}^{n+1}$, of dimension n. Show that there is an integer $N \geq 0$ such that

$$\{t \in \mathbb{R}^n \ ; \ p^{-1}(t) \cap A \text{ has exactly } N \text{ elements } \}$$

is a semialgebraic set of dimension n and

$$\{t \in \mathbb{R}^n \ ; \ p^{-1}(t) \cap A \text{ has } > N \text{ elements } \}$$

is a semialgebraic set of dimension $< n$ or empty.

4.1.2 Local conic structure of semialgebraic sets

Let A be a semialgebraic subset of $\mathbb{R}^n$ and a, a nonisolated point of A: for every $\varepsilon > 0$ there is $x \in A$, $x \neq a$, such that $\|x - a\| < \varepsilon$. Let $\overline{B}(a, \varepsilon)$ (resp. $S(a, \varepsilon)$) be the closed ball (resp. the sphere) with center a and radius ε. We denote by $a * (S(a, \varepsilon) \cap A)$ the cone with vertex a and basis $S(a, \varepsilon) \cap A$, i.e. the set of points in $\mathbb{R}^n$ of the form $\lambda a + (1 - \lambda)x$, where $\lambda \in [0, 1]$ and $x \in S(a, \varepsilon) \cap A$.

Theorem 4.4 *For $\varepsilon > 0$ sufficiently small, there is a semialgebraic homeomorphism $h : \overline{B}(a, \varepsilon) \cap A \to a * (S(a, \varepsilon) \cap A)$ such that $\|h(x) - a\| = \|x - a\|$ and $h_{|S(a,\varepsilon)\cap A} = \mathrm{Id}$.*

Proof. We apply Hardt's theorem 4.1 to the mapping $p : A \to \mathbb{R}$ defined by $p(x) = \|x - a\|$. We obtain semialgebraic trivializations of p over a finite semialgebraic partition of $\mathbb{R}$. We can assume that this partition has as member an interval $(0, \varepsilon_0)$. Choose ε such that $0 < \varepsilon < \varepsilon_0$. Since $p^{-1}(\varepsilon) = (A \cap S(a, \varepsilon))$, we have a semialgebraic homeomorphism

$$g : p^{-1}((0, \varepsilon_0)) \to (0, \varepsilon_0) \times (A \cap S(a, \varepsilon))$$

such that $g(x) = (\|x - a\|, g_1(x))$, where the restriction of g_1 to $S(a, \varepsilon) \cap A$ is the identity. Now define $h : \overline{B}(a, \varepsilon) \cap A \to C_\varepsilon$ by

$$\begin{cases} h(x) &= \left(1 - \dfrac{\|x - a\|}{\varepsilon}\right) a + \dfrac{\|x - a\|}{\varepsilon}\, g_1(x) \quad \text{if } x \neq a \ , \\ h(a) &= a \ . \end{cases}$$

We can check that h has the properties of the theorem. The inverse mapping of h is defined by

$$\begin{cases} h^{-1}(\lambda a + (1 - \lambda)x) &= g^{-1}((1 - \lambda)\varepsilon, x) \text{ for } \lambda \in [0, 1),\ x \in S(a, \varepsilon) \cap A \ , \\ h^{-1}(a) &= a \ . \end{cases}$$

□

Exercise 4.5 Let $A \subset \mathbb{R}^n$ be a semialgebraic set. For $r > 0$, we denote by $B(r)$ (resp. $\overline{B}(r)$, $S(r)$) the open ball (resp. the closed ball, the sphere) with center 0 and radius r in $\mathbb{R}^n$.
1) Show that there exist $r > 0$ and a continuous semialgebraic mapping

$$h_1 : A \cap (\mathbb{R}^n \setminus B(r)) \longrightarrow (A \cap S(r))$$

such that

$$\forall x \in A \cap S(r), \quad h_1(x) = x$$

and

$$\begin{array}{rcl} h : A \cap (\mathbb{R}^n \setminus B(r)) & \longrightarrow & (A \cap S(r)) \times [r, +\infty) \\ x & \longmapsto & (h_1(x), \|x\|) \end{array}$$

is a homeomorphism.
2) Using 1), construct a semialgebraic mapping

$$H : A \times [0,1] \longrightarrow A$$

such that

$$\forall x \in A, \quad H(x,0) = x \text{ and } H(x,1) \in A \cap \overline{B}(r) ,$$
$$\forall x \in A \cap \overline{B}(r),\ \forall t \in [0,1], \quad H(x,t) = x .$$

($A \cap \overline{B}(r)$ is a semialgebraic deformation retract of A.)

4.1.3 Finiteness of the number of topological types

Theorem 4.6 *For every positive integers d and n, there exist a positive integer $p = p(n,d)$ and algebraic subsets $V_1, \ldots, V_p \subset \mathbb{R}^n$, defined by polynomial equations of degrees at most d, such that, for every algebraic subset $W \subset \mathbb{R}^n$ defined by polynomial equations of degrees at most d, there exist $i \in \{1, \ldots, p\}$ and a semialgebraic homeomorphism $h : \mathbb{R}^n \to \mathbb{R}^n$ such that $h(W) = V_i$.*

In other words, there are finitely many semialgebraic topological types of inclusions $V \subset \mathbb{R}^n$, where V is an algebraic subset defined by equations of degrees at most d. By *(semialgebraic) topological type* of inclusion, we mean an equivalence class of subsets $V \subset \mathbb{R}^n$ for the equivalence relation $(V \subset \mathbb{R}^n) \sim (W \subset \mathbb{R}^n)$ if there exists a (semialgebraic) homeomorphism $h : \mathbb{R}^n \to \mathbb{R}^n$ such that $h(V) = W$.

Proof. An algebraic subset $V \subset \mathbb{R}^n$ defined by equations $P_1 = \ldots = P_q = 0$ of degrees $\leq d$ can always be regarded as defined by only one equation of degree $\leq 2d$, that is $P_1^2 + \ldots + P_q^2 = 0$. A polynomial of degree $2d$ in n variables has $\binom{2d+n}{n} = N(n,d)$ coefficients (check by induction on n). We identify the space of coefficients with $\mathbb{R}^N$, and we denote by $P_a \in \mathbb{R}[X_1, \ldots, X_n]$ the polynomial of degree $\leq 2d$ corresponding to $a \in \mathbb{R}^N$. The set

$$A = \{(a,x) \in \mathbb{R}^N \times \mathbb{R}^n \ ; \ P_a(x) = 0\}$$

is an algebraic subset of $\mathbb{R}^N \times \mathbb{R}^n$. Let $p : \mathbb{R}^N \times \mathbb{R}^n \to \mathbb{R}^N$ be the projection. Hardt's theorem implies that there is a finite semialgebraic partition $\mathbb{R}^N = C_1 \cup \ldots \cup C_q$ such that, for every $i = 1, \ldots, q$, there is a semialgebraic trivialization $h_i : C_i \times \mathbb{R}^n \to C_i \times \mathbb{R}^n$ of p over C_i, compatible with A. Choose a point a_i such that P_{a_i} is a sum of squares of polynomials in every C_i containing such points, say $C_1, \ldots, C_p$. For $i = 1, \ldots, p$, set

$$V_i = \{x \in \mathbb{R}^n \ ; \ P_{a_i}(x) = 0\} \ .$$

Any algebraic set W as in the statement of the theorem is of the form

$$W = \{x \in \mathbb{R}^n \ ; \ P_a(x) = 0\} \ ,$$

where $a \in C_i$ for some $i \in \{1, \ldots, p\}$. The semialgebraic trivialization h_i induces a semialgebraic homeomorphism $p^{-1}(a) \to p^{-1}(a_i)$ sending $A \cap p^{-1}(a)$ to $A \cap p^{-1}(a_i)$ which, in turn, gives a semialgebraic homeomorphism $h : \mathbb{R}^n \to \mathbb{R}^n$ such that $h(W) = V_i$. $\square$

4.2 Uniform bound on the number of connected components

It follows from Theorem 4.6 that there exists a positive integer $\varphi(n,d)$, such that every algebraic subset of $\mathbb{R}^n$ defined by equations of degrees $\leq d$ has at most $\varphi(n,d)$ connected components. Indeed, we can take for $\varphi(n,d)$ the maximum number of connected components of $V_1, \ldots, V_p$ (with the notation of Theorem 4.6). The aim of this section is to give an explicit upper bound for $\varphi(n,d)$. First note that we have the lower bound $d^n \leq \varphi(n,d)$. Indeed, the system of equations $(X_i - 1)(X_i - 2) \cdots (X_i - d)$, for $i = 1, \ldots, n$, has a set of solutions consisting of d^n real points. We assume everywhere in this section that d is a positive integer.

Theorem 4.7 *Let $V \subset \mathbb{R}^n$ be an algebraic subset defined by equations of degrees $\leq d$. The number of connected components of V is not greater than $d(2d-1)^{n-1}$.*

This result is related to the Thom-Milnor bound on the sum of the Betti numbers of a real algebraic set (the number of connected components is the Betti number b_0). For this result, see for instance [BR, BCR]. The proof of Theorem 4.7 is essentially the proof of the Thom-Milnor bound, without its homological part.

The proof proceeds by reducing to the case of a compact smooth hypersurface, and then looking for extremal points of a function on this hypersurface. This is an example of the "critical point method". This method can be applied to design algorithms (for deciding whether a semialgebraic set is empty, for instance) with better complexity than the c.a.d. algorithm. See [R] and the references cited there.

We state here a result which will be useful in this section. Let M (resp. N) be a smooth submanifold of $\mathbb{R}^n$ (resp. $\mathbb{R}^p$), and let $f : M \to N$ be a smooth map. For $x \in M$, denote by $df_x : T_xM \to T_{f(x)}N$ the tangent linear mapping from the tangent space to M at x to the tangent space to N at $f(x)$. We say that x is a *critical point* of f, and $f(x)$ a *critical value* of f, if df_x is not surjective.

Theorem 4.8 (Sard's theorem – semialgebraic version) *Let $M \subset \mathbb{R}^n$ and $N \subset \mathbb{R}^p$ be semialgebraic smooth submanifolds, and let $f : M \to N$ be a semialgebraic $\mathcal{C}^\infty$ mapping. Then the set of critical values of f is a semialgebraic subset of N, of dimension $< \dim N$.*

For a proof, we refer to [BCR] chap. 9 section 5, or [BR] 2.5.12. We propose a proof of a particular case in the following exercise.

Exercise 4.9 Let U be an open semialgebraic subset of $\mathbb{R}^n$. Let Q and f be polynomials on R^n. Assume that, for every $x \in U \cap Q^{-1}(0)$, $\overrightarrow{\mathrm{grad}}\, Q(x) \neq 0$ (hence, $M = U \cap Q^{-1}(0)$ is a smooth hypersurface). Show that the set of critical values of $f|_M$ is finite. Hints:
1) Show that $x \in M$ is a critical point of $f|_M$ if and only $\overrightarrow{\mathrm{grad}}\, f(x)$ is colinear to $\overrightarrow{\mathrm{grad}}\, Q(x)$. Deduce that the set Z of critical points of $f|_M$ is semialgebraic.
2) Show that f is constant along a smooth path in Z. Deduce that f is constant on each connected component of Z. Conclude.

4.2.1 Reduction to the case of a compact smooth hypersurface

Assume that

$$V = \{x \in \mathbb{R}^n \ ; \ P_1(x) = \ldots = P_q(x) = 0\} \ ,$$

where $\deg P_i \leq d$, and let $P = P_1^2 + \ldots + P_q^2$. We also assume that V has at least two connected components. Choose $R > 0$ larger than the maximum of the distances from the origin $0 \in \mathbb{R}^n$ to the connected components of V. The closed ball $\overline{B}$ with center 0 and radius R has a nonempty intersection with every connected component of V. Hence, the number of connected components of $\overline{B} \cap V$ is greater than or equal to the number of connected components of V.

Let $\mathcal{F}$ be the finite set of connected components of $\overline{B} \cap V$. For $C \in \mathcal{F}$, let K_C be the set of $x \in \overline{B}$ such that $\mathrm{dist}(x, C) = \mathrm{dist}(x, (\overline{B} \cap V) \setminus C)$. Let K be the union of the sets K_C, for all $C \in \mathcal{F}$. The set K is a closed semialgebraic subset of $\overline{B}$, disjoint from V.

If C_1 and C_2 are different connected components of $\overline{B} \cap V$, every continuous path in $\overline{B}$ joining a point of C_1 to a point of C_2 must intersect K_{C_1}. Hence, each connected component of $\overline{B} \setminus K$ contains at most one element of $\mathcal{F}$ (actually, exactly one).

For $0 < \varepsilon$, set

$$Q_\varepsilon(x) = P(x) + \varepsilon(\|x\|^2 - R^2) \ .$$

Note that Q_ε is a polynomial of degree $2d$ in x. Since the polynomial $P(x)$ is nonnegative on $\mathbb{R}^n$, the zeroset W_ε of Q_ε is contained in $\overline{B}$. Let $\varepsilon_0 > 0$ be the minimum of $P(x)/R^2$ for $x \in K$. Now assume $\varepsilon < \varepsilon_0$. It follows that W_ε is disjoint from K. Let A be a connected component of $\overline{B} \setminus K$ containing a $C \in \mathcal{F}$. Since Q_ε takes nonpositive values on $C \subset A$ and positive values on $K \cap \mathrm{clos}(A)$, its zeroset W_ε must have a nonempty intersection with A. This shows that the number of connected components of W_ε is greater than or equal to the number of connected components of $\overline{B} \cap V$, which is greater than or equal to the number of connected components of V.

Now we show that we can choose ε such that W_ε is a smooth hypersurface. This means that for every point $x \in W_\varepsilon$, the partial derivatives $\partial Q_\varepsilon / \partial x_i$ do not all vanish.

First, we claim that the set

$$X = \{(x, \varepsilon) \in \mathbb{R}^n \times \mathbb{R} \ ; \ \varepsilon > 0 \ \text{ and } \ Q_\varepsilon(x) = 0\} \ .$$

is a smooth hypersurface of $\mathbb{R}^{n+1}$. Since $\partial Q_\varepsilon / \partial \varepsilon = \|x\|^2 - R^2$, this partial derivative can vanish at $(x, \varepsilon) \in X$ only if $\|x\| = R$ and $P(x) = 0$. Since

$\partial P/\partial x_i$ vanishes when $P = 0$, we have moreover $(\partial Q_\varepsilon/\partial x_i)(x, \varepsilon) = 2\varepsilon x_i$. Hence, one of the partial derivatives $(\partial Q_\varepsilon/\partial x_i)$ is nonzero at (x, ε). This proves the claim.

Second, we claim that the function $f : X \to R$ defined by $f(x, \varepsilon) = \varepsilon$ has finitely many critical values. This follows from the semialgebraic version of Sard's theorem 4.8. Actually, it is sufficient to use here the result of Exercise 4.9. Hence, we can choose ε, with $0 < \varepsilon < \varepsilon_0$ and ε not a critical value of f. This implies that $\overrightarrow{\mathrm{grad}}\, Q_\varepsilon = (\partial Q_\varepsilon/\partial x_1, \ldots, \partial Q_\varepsilon/\partial x_n)$ is never 0 on W_ε.

Now it suffices to show that the number of connected components of W_ε is not greater than $d(2d-1)^{n-1}$. This will be done in the next lemma.

4.2.2 The case of a compact smooth hypersurface

Lemma 4.10 *Let $Q : \mathbb{R}^n \to \mathbb{R}$ be a polynomial of degree $2d$. Assume that $W = Q^{-1}(0)$ is compact and $\overrightarrow{\mathrm{grad}}\, Q$ has no zero on W (hence, W is a smooth compact hypersurface). Then the number of connected components of W is at most $d(2d-1)$.*

Proof. The connected components of W are compact smooth hypersurfacess. The coordinate function x_n has a maximum and a minimum on each of these components. At a point where x_n reaches its maximum or minimum, the tangent hyperplane to W is parallel to the hyperplane $x_n = 0$, which means that the first $n-1$ coordinates of $\overrightarrow{\mathrm{grad}}\, Q$ vanish. Hence, the points of W where x_n reaches an extremum are solutions of the system of equations

$$(S) \quad \left\{ \begin{array}{rcl} Q(x) & = & 0 \\ \dfrac{\partial Q}{\partial x_1}(x) & = & 0 \\ & \vdots & \\ \dfrac{\partial Q}{\partial x_{n-1}}(x) & = & 0 \end{array} \right. \qquad (n \text{ equations}, n \text{ variables}) .$$

Lemma 4.11 *We can choose the coordinates in $\mathbb{R}^n$ such that all real solutions of the above system (S) are nondegenerate, i.e. solutions where the jacobian* *determinant*

$$\det\begin{pmatrix} \dfrac{\partial Q}{\partial x_1} & \cdots & \dfrac{\partial Q}{\partial x_n} \\ \dfrac{\partial^2 Q}{\partial x_1 \partial x_1} & \cdots & \dfrac{\partial^2 Q}{\partial x_1 \partial x_n} \\ \vdots & \vdots & \vdots \\ \dfrac{\partial^2 Q}{\partial x_{n-1} \partial x_1} & \cdots & \dfrac{\partial^2 Q}{\partial x_{n-1} \partial x_n} \end{pmatrix}$$

does not vanish.

Assume for the moment that Lemma 4.11 is proved. Then, by Bezout's theorem (cf. for instance [BR], appendix B, or [BCR], chap. 9), the number of nondegenerate real solutions of the system (S) is $\leq 2d(2d-1)^{n-1}$, which is the product of the degrees of the equations. Since each connected component of W has at least two points where x_n has an extremum, W has at most $d(2d-1)^{n-1}$ connected components. This concludes the proof of Lemma 4.10 and also the proof of Theorem 4.7. □

Proof of Lemma 4.11: Apply the semialgebraic version of Sard's theorem 4.8 to the map

$$\varphi = \overrightarrow{\operatorname{grad}}\, Q / \| \overrightarrow{\operatorname{grad}}\, Q \| : W \longrightarrow S^{n-1} ,$$

where S^{n-1} is the unit sphere in $\mathbb{R}^n$. Since the set of critical values of φ is of dimension $< n-1$, we can find a pair of antipodal points b and $-b$ in S^{n-1}, which are not critical values. After rotating the coordinate axes, we can assume that $b = (0, \ldots, 0, 1)$. Observe that the real solutions a of the system (S) are exactly the points $a \in W$ such that $\varphi(a) = (0, \ldots, 0, \pm 1)$. For such an a, the tangent hyperplanes $T_a W$ and $T_{f(a)} S^{n-1}$ are both $x_n = 0$, and we have $\partial Q / \partial x_i(a) = 0$, for $i = 1, \ldots, n-1$. The matrix of $d\varphi_a$, in the coordinates $(x_1, \ldots, x_{n-1})$, is

$$\frac{1}{\| \overrightarrow{\operatorname{grad}}\, Q(a) \|} \left(\frac{\partial^2 Q}{\partial x_i \partial x_j}(a) \right)_{i=1,\ldots,n-1\ ;\ j=1,\ldots,n-1}$$

and, therefore,

$$\Delta = \det \left(\frac{\partial^2 Q}{\partial x_i \partial x_j}(a) \right)_{i=1,\ldots,n-1\ ;\ j=1,\ldots,n-1} \neq 0 .$$

Since the value of the jacobian determinant of the system (S) at a is equal to $\pm \Delta \dfrac{\partial Q}{\partial x_n}(a)$, it is nonzero. □

4.2.3 Bound for the number of connected components of a semialgebraic set

We have just seen that, for an algebraic set, the bound on the number of connected components depends only on the degree of the equations and not on the number of these equations. In the case of a semialgebraic set defined by polynomial inequalities, we have to take into account the number of these inequalities.

Exercise 4.12 Prove that any finite union of open intervals in $\mathbb{R}$ can be defined by a system of inequalities of degree 2.

Proposition 4.13 *Let (S) be a system of s polynomial equations and inequalities in k variables, of degrees at most $d \geq 2$. The number of connected components of the set of solutions of (S) in $\mathbb{R}^n$ is not greater than $d(2d-1)^{k+s-1}$.*

Proof. Assume $P > 0$ is a strict polynomial inequality in (S). Choose $\varepsilon > 0$ so small that there is a point x in each connected component of the set of solutions of (S), with $P(x) > \varepsilon$. Replacing $P > 0$ with $P - \varepsilon \geq 0$ in (S) can only increase the number of connected components of the set of solutions. Hence, we can assume that (S) contains only lax inequalities $Q_1 \geq 0, \ldots,$ $Q_t \geq 0$ and equations. Next, we replace every lax inequality $Q_i \geq 0$ with the equation $Q_i - T_i^2 = 0$, introducing a new variable T_i for each inequality. This replacement can only increase the number of connected components of the set of solutions. Finally, we have a system of s equations in $k + t \leq k + s$ variables of degrees $\leq d$. By Theorem 4.7, the number of connected components of the set of solutions is not greater than $d(2d-1)^{k+s-1}$. □

The bound of Proposition 4.13 is very coarse. Using more sophisticated arguments, one can obtain a bound with a polynomial dependence on the number of equations and inequalities, of the form $s^k O(d)^k$ (cf. [R]).

4.3 An application to lower bounds

This section is devoted to an application of the bound on the number of connected components of a semialgebraic set, due to Ben-Or (Proc. 15th ACM ann. Symp. on Theory of Comp., 80-86 (1983)). First, we describe a model of algorithm to decide whether an element $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ satisfies a boolean combination of sign conditions on polynomials in n variables (in other

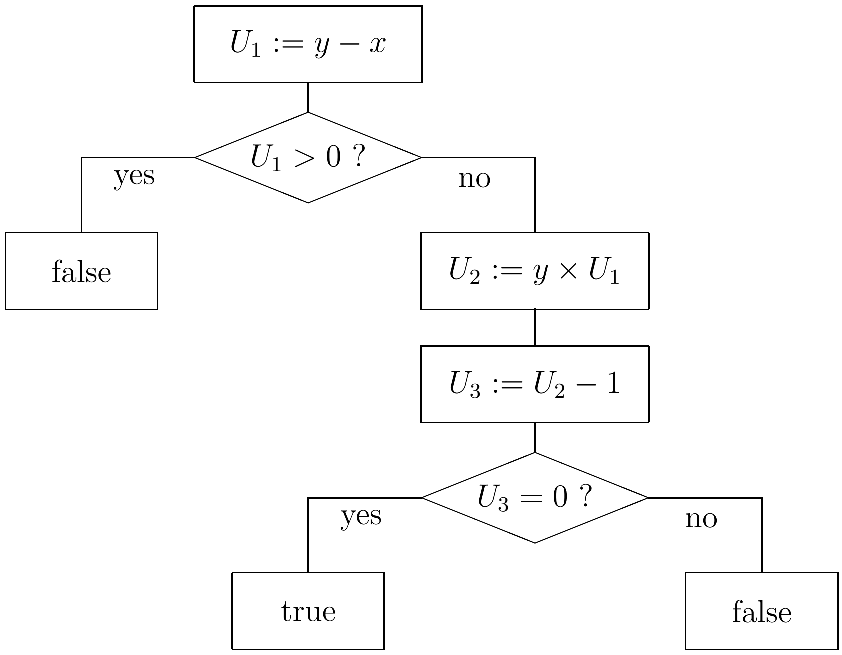


Figure 4.1: An algebraic computation tree deciding whether $y^2 - xy = 1$ and $y \leq x$.

words, to decide whether x belongs to a given semialgebraic subset $W \subset \mathbb{R}^n$). This model is an *algebraic computation tree.* Such a tree has one root and several leaves. The vertices different from the root have one father. The vertices different from the leaves have one or two sons. A vertex v with one son is labelled with a variable U_v and an instruction $U_v := a * b$, where $*$ is an arithmetic operation $(+, -, \times)$ and a and b are either x_i, $i \in \{1, \ldots, n\}$, or a real constant, or a variable $U_{v'}$ with v' ancestor of v. A vertex v with two sons is labeled with a test $a \,?\, 0$, where a is as above and ? is $=$, $>$, or $\geq$. A leaf v is labeled with a boolean constant b_v (true or false). The algorithm modelled by such a tree has as inputs n-tuples $x_1, \ldots, x_n$ of real numbers and goes down in the tree starting from the root. At a vertex v with one son, it computes the value of U_v following the instruction of the label. At a vertex with two sons, it chooses the left (resp. right) son if the answer to the test of the label is yes (resp. no). When the algorithm arrives to a leaf, it returns the boolean constant in the label of this leaf. The *cost* of an algorithm (in this model) is the maximal length of a path (from root to leaf) taken by an input $(x_1, \ldots, x_n) \in \mathbb{R}^n$.

Theorem 4.14 *If an algorithm with cost c decides whether $x \in W$, where W*

is a semialgebraic subset of $\mathbb{R}^n$*, then the number of connected components of* W *is not greater than* 2^{2n+5c}*.*

Proof. Let v be a leaf labelled with $b_v = \text{true}$. Denote by W_v the semialgebraic subset of W consisting of those inputs $x \in \mathbb{R}^n$ for which the algorithm arrives to the leaf v. Consider the system (S_v) obtained in the following way.

- For each ancestor v' of v with one son, take the equation $U_{v'} = a * b$ which is in the label of v'.

- For each ancestor v' of v with two sons, take the equation or inequality $a\,?\,0$ in the label of v' if v is a heir of v' on the left side, and its negation if v is a heir of v' on the right side.

The system (S_v) has s equations and inequalities in the variables

$$X_1, \ldots, X_n, U_{v_1}, \ldots, U_{v_m},$$

where $m \leq s$. Assume $W_v \neq \emptyset$. There are inputs for which the algorithm arrives to v. Hence $s \leq c$. Finally, $(x_1, \ldots, x_n) \in W_v$ if and only if there exists $(u_{v_1}, \ldots, u_{v_m}) \in \mathbb{R}^m$ such that $(x_1, \ldots, x_n, u_{v_1}, \ldots, u_{v_m})$ is a solution of (S_v). Hence, the number of connected components of W_v is not greater than the number of connected components of the set of solutions of (S_v). Since all equations and inequalities in (S_v) have degree ≤ 2, Proposition 4.13 implies that the number of connected components of W_v is not greater than $2 \times 3^{n+m+s-1} \leq 2 \times 3^{n+2c-1}$. The number of leaves v with $b_v = \text{true}$ and $W_v \neq \emptyset$ is at most 2^c, since the paths from the root to the leaves which are taken for some input have length c at most, and each vertex has at most two sons. Since W is the union of these W_v, the number of connected components of W is not greater than $2^c \times 2 \times 3^{n+2c-1} \leq 2^{2n+5c}$. □

Corollary 4.15 *The cost (for the algebraic computation tree model) of an algorithm deciding whether* n *real numbers* $(x_1, \ldots, x_n)$ *are all distinct is at least* $\Omega(n \log n)$ *(recall that* $f = \Omega(g)$ *means* $g = O(f)$*).*

Remark that there are algorithms solving this problem with cost $O(n \log n)$ (sorting by divide and conquer, for instance).

Proof. The set

$$W = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \ ; \ \forall i \neq j, x_i \neq x_j\}$$

has $n!$ connected components (there is a faithful and transitive action of the group of permutation of $\{1, \ldots, n\}$ on the set of connected components of W). The cost c of an algorithm deciding whether $x \in W$ must satisfy $n! \leq 2^{2n+5c}$. Taking logarithm and using $n \log n = O(\log(n!))$ we obtain $n \log n = O(2n + 5c)$. Hence $c = \Omega(n \log n)$. □

Exercise 4.16 (This exercise is taken from a paper by J.L. Montaña, L.M. Pardo and T. Recio in Effective Methods in Algebraic Geometry, Birkhäuser 1991).

1) We denote by $\mathcal{F}_d$ the family of all algebraic subsets of $\mathbb{R}^n$ defined by an equation $P = 0$, where P is a nonzero polynomial of degree $\leq d$. Let W be a semialgebraic subset of $\mathbb{R}^n$. Show that there exists $i \in \mathbb{N}$ such that, for every H in $\mathcal{F}_d$, the number of connected components of $H \cap W$ is $\leq i$. Let $I_d(W)$ be the smallest such integer.

2) We now assume W to be defined by a system of ℓ polynomial equations and inequalities in n variables, of degrees at most $d \geq 2$. Give an upper bound for $I_d(W)$.

We shall now find a lower bound for the cost of an algorithm deciding the following problem ("big hole" problem): given $n \geq 2$ real numbers $x_1, \ldots, x_n$, is there a closed interval of length 1 containing no x_i and contained in the convex hull of the x_i's in $\mathbb{R}$?

3) Let $W_n \subset \mathbb{R}^n$ be the subset of all $(x_1, \ldots, x_n)$ for which there is no big hole. Show that W_n is a connected semialgebraic set.

4) We assume that $(x_1, \ldots, x_n) \in W_n$ and $x_1 \leq x_2 \leq \ldots \leq x_n$. Show that

$$\sum_{i<j} (x_i - x_j)^2 \leq \sum_{k=1}^{n-1} k(n-k)^2 .$$

Show that the equality holds if and only if $x_{i+1} = x_i + 1$ for $i = 1, \ldots, n-1$. Deduce that the intersection of W_n with the algebraic set defined by the equation

$$\sum_{i<j} (x_i - x_j)^2 = \sum_{k=1}^{n-1} k(n-k)^2$$

is the union of $n!$ disjoint lines. Hence $I_2(W_n) \geq n!$.

5) Show that the cost (in the algebraic computation tree model) of an algorithm deciding the big hole problem is at least $\Omega(n \log n)$.

Bibliography

[BR] R. Benedetti, J-J. Risler: *Real Algebraic and Semi-algebraic Sets.* Hermann 1990

[BCR] J. Bochnak, M. Coste, M-F. Roy: *Real Algebraic Geometry.* Springer 1998

[B] F. Broglia, ed.: *Lectures on Real Geometry in memoriam of Mario Raimondo.* Expositions in Math. 23. De Gruyter 1996

[BCL] B. Buchberger, G.E. Collins, R. Loos, eds.: *Computer Algebra, Symbolic and Algebraic Computation.* Springer 1982

[CJ] B. Caviness, J. Johnson, eds.: *Quantifier Elimination and Cylindrical Algebraic Decomposition.* Texts and Monographs in Symbolic Computation. Springer 1998

[Ch] A. Cohen, ed.: *Some Tapas of Computer Algebra.* Springer 1998

[Cl] G.E. Collins: *Quantifier Elimination for Real Closed Fields by Cylindical Algebraic Decomposition.* Reprinted in [CJ]

[Co] M. Coste: *An Introduction to O-minimal Geometry.* Dottorato di Ricerca in Matematica, Dip. Mat. Univ. Pisa. Istituti Editoriali e Poligrafici Internazionali, 2000.

[D] L. van den Dries: *Tame Topology and O-minimal Structures.* London Math. Soc. Lecture Note 248. Cambridge Univ. Press 1998

[G] F.R. Gantmacher: *Théorie des matrices.* Dunod 1966

[GRRT] L. Gonzalez-Vega, F. Rouillier, M.-F. Roy, G. Trujillo: *Symbolic Recipes for Real Solutions.* In [Ch]

[L] S. Lang: *Algebra.* Addison-Wesley 1965

[Loo] R. Loos. *Generalized Polynomial Remainder Sequences*. Pp. 115-138 in [BCL]

[Pr] A. Prestel: *Model Theory for the Real Algebraic Geometer*. Dottorato di Ricerca in Matematica, Dip. Mat. Univ. Pisa. Istituti Editoriali e Poligrafici Internazionali 1998

[R] M-F. Roy: *Basic Algorithms in Real Algebraic Geometry: from Sturm Theorem to the Existential Theory of Reals*. Pp. 1-67 in [B]

[S] I.R. Shafarevich: *Basic Algebraic Geometry*. Springer 1974

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