LECTURES ON APPLIED MATHEMATICS

Part 1: Linear Algebra

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PREFACE

To Part 1

It is common for Departments of Mathematics to offer a junior-senior level course on Linear Algebra. This book represents one possible course. It evolved from my teaching a junior level course at Texas A&M University during the several years I taught after I served as President. I am deeply grateful to the A&M Department of Mathematics for allowing this Mechanical Engineer to teach their students.

This book is influenced by my earlier textbook with C.-C Wang, Introductions to Vectors and Tensors, Linear and Multilinear Algebra. This book is more elementary and is more applied than the earlier book. However, my impression is that this book presents linear algebra in a form that is somewhat more advanced than one finds in contemporary undergraduate linear algebra courses. In any case, my classroom experience with this book is that it was well received by most students. As usual with the development of a textbook, the students that endured its evolution are due a statement of gratitude for their help.

As has been my practice with earlier books, this book is available for free download at the site http://www1.mengr.tamu.edu/rbowen/ or, equivalently, from the Texas A&M University Digital Library's faculty repository, http://repository.tamu.edu/handle/1969.1/2500. It is inevitable that the book will contain a variety of errors, typographical and otherwise. Emails to rbowen@tamu.edu that identify errors will always be welcome. For as long as mind and body will allow, this information will allow me to make corrections and post updated versions of the book.

College Station, Texas *R.M.B.*

Posted February, 2014

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Part 1 Linear Algebra

PART I

LINEAR ALGEBRA

Selected Reading for Part I

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Chapter 1

ELEMENTARY MATRIX THEORY

When we introduce the various types of structures essential to the study of linear algebra, it is convenient in many cases to illustrate these structures by examples involving matrices. Also, many of the most important practical applications of linear algebra are applications focused on matrix algebra. It is for this reason we are including a brief introduction to matrix theory here. We shall not make any effort toward rigor in this chapter. In later chapters, we shall return to the subject of matrices and augment, in a more careful fashion, the material presented here.

Section 1.1. Basic Matrix Operations

We first need some notations that are convenient as we discuss our subject. We shall use the symbol $\mathscr{R}$ to denote the set of *real* numbers, and the symbol $\mathscr{C}$ to denote the set of complex numbers. The sets $\mathscr{R}$ and $\mathscr{C}$ are examples of what is known in mathematics as a *field*. Each set is endowed with two operations, *addition* and *multiplication* such that

For Addition:

1. The numbers x_1 and x_2 obey (commutative)

$$x_1 + x_2 = x_2 + x_1$$

2. The numbers x_1, x_2, and x_3 obey (associative)

$$(x_1 + x_2) + x_3 = x_1 + (x_2 + x_3)$$

3. The real (or complex) number 0 is unique (identity) and obeys

$$x + 0 = 0 + x$$

4. The number x has a unique "inverse" $-x$ such that.

$$x + (-x) = 0$$

For Multiplication

5. The numbers x_1 and x_2 obey (commutative)

$$x_1 x_2 = x_2 x_1$$

6. The numbers x_1, x_2, and x_3 obey (associative)

$$(x_1x_2)x_3 = x_1(x_2x_3)$$

7. The real (complex) number 1 is unique (identity) and obeys

$$x(1) = (1)x = x$$

8. For every $x \neq 0$, there exists a number $\frac{1}{x}$ (inverse under multiplication) such that

$$x\left(\frac{1}{x}\right) = \left(\frac{1}{x}\right)x = 1$$

9. For every x_1, x_2, x_3, (distribution axioms)

$$x_1(x_2 \pm x_3) = x_1x_2 \pm x_1x_3$$

$$(x_1 \pm x_2)x_3 = x_1x_2 \pm x_1x_3$$

While it is not especially important to this work, it is appropriate to note that the concept of a field is not limited to the set of real numbers or complex numbers.

Given the notation $\mathscr{R}$ for the set of real numbers and a positive integer N, we shall use the notation $\mathscr{R}^N$ to denote the set whose elements are N-tuples of the form $\left(x_1,...,x_N\right)$ where each element is a real number. A convenient way to write this definition is

$$\mathscr{R}^N = \left\{\left(x_1,...,x_N\right) \middle| x_j \in \mathscr{R}\right\} \tag{1.1.1}$$

The notation in (1.1.1) should be read as saying "$\mathscr{R}^N$ equals the set of all N-tuples of real numbers." In a similar way, we define the N-tuple of complex numbers, $\mathscr{C}^N$, by the formula

$$\mathscr{C}^N = \left\{\left(z_1,...,z_N\right) \middle| z_j \in \mathscr{C}\right\} \tag{1.1.2}$$

An M by N *matrix* A is a rectangular array of real or complex numbers A_{ij} arranged in M rows and N columns. A matrix is usually written

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdot & \cdot & \cdot & A_{1N} \\ A_{21} & A_{22} & \cdot & \cdot & \cdot & A_{2N} \\ \cdot & & & & & \\ \cdot & & & & & \\ A_{M1} & A_{M2} & \cdot & \cdot & \cdot & A_{MN} \end{bmatrix} \tag{1.1.3}$$

and the numbers A_{ij} are called the *elements* or *components* of A. The matrix A is called a *real* matrix or a *complex* matrix according to whether the components of A are real numbers or complex numbers. Frequently these numbers are simply referred to as *scalars*.

A matrix of M rows and N columns is said to be of *order* M by N or $M \times N$. The location of the indices is sometimes modified to the forms A^{ij}, $A^{i}{}_{j}$, or $A_{i}{}^{j}$. Throughout this chapter the placement of the indices is unimportant and shall always be written as in (1.1.3). The elements $A_{i1}, A_{i2}, \ldots, A_{iN}$ are the elements of the i^{th} row of A, and the elements $A_{1k}, A_{2k}, \ldots, A_{Nk}$ are the elements of the k^{th} column. The convention is that the first index denotes the row and the second the column. It is customary to assign a symbol to the set of matrices of order $M \times N$. We shall assign this set the symbol $\mathscr{M}^{M\times N}$. More formally, we can write this definition as

$$\mathscr{M}^{M\times N} = \left\{ A \,\middle|\, A \text{ is an } M \times N \text{ matrix} \right\} \tag{1.1.4}$$

A *row matrix* is a $1 \times N$ matrix, e.g.,[1]

$$\begin{bmatrix} A_{11} & A_{12} & \cdot & \cdot & \cdot & A_{1N} \end{bmatrix}$$

while a *column matrix* is an $M \times 1$ matrix, e.g.,

$$\begin{bmatrix} A_{11} \\ A_{21} \\ \cdot \\ \cdot \\ \cdot \\ A_{M1} \end{bmatrix}$$

The matrix A is often written simply

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[1] A row matrix as defined by $\begin{bmatrix} A_{11} & A_{12} & \cdot & \cdot & \cdot & A_{1N} \end{bmatrix}$ is mathematically equivalent to an *N*-tuple that we have previously written $\left(A_{11}, A_{12}, \ldots, A_{1N} \right)$. For our purposes, we simply have two different notations for the same quantity.

$$A = \left[A_{ij} \right] \tag{1.1.5}$$

A *square* matrix is an $N \times N$ matrix. In a square matrix A, the elements $A_{11}, A_{22}, \ldots, A_{NN}$ are its *diagonal* elements. The sum of the diagonal elements of a square matrix A is called the *trace* and is written $\operatorname{tr} A$. In other words,

$$\operatorname{tr} A = A_{11} + A_{22} + \cdots + A_{NN} \tag{1.1.6}$$

Two matrices A and B are said to be *equal* if they are identical. That is, A and B have the same number of rows and the same number of columns and

$$A_{ij} = B_{ij}, \qquad i = 1, \ldots, N, \qquad j = 1, \ldots, M \tag{1.1.7}$$

A matrix, every element of which is zero, is called the *zero* matrix and is written simply 0.

If $A = \left[A_{ij} \right]$ and $B = \left[B_{ij} \right]$ are two $M \times N$ matrices, their *sum* (*difference*) is an $M \times N$ matrix $A + B$ $(A - B)$ whose elements are $A_{ij} + B_{ij}$ $(A_{ij} - B_{ij})$. Thus

$$A \pm B = \left[A_{ij} \pm B_{ij} \right] \tag{1.1.8}$$

Note that the symbol $\pm$ on the right side of (1.1.8) refers to addition and subtraction of the complex or real numbers A_{ij} and B_{ij}, while the symbol $\pm$ on the left side is an operation defined by (1.1.8). It is an operation defined on the set $\mathscr{M}^{M \times N}$. Two matrices of the same order are said to be *conformable* for addition and subtraction. Addition and subtraction are not defined for matrices which are not conformable.

If λ is a number and A is a matrix, then λA is a matrix given by

$$\lambda A = \left[\lambda A_{ij} \right] = A \lambda \tag{1.1.9}$$

Just as (1.1.8) defines addition and subtraction of matrices, equation (1.1.9) defines multiplication of a matrix by a real or complex number. It is a consequence of the definitions (1.1.8) and (1.1.9) that

$$-A = (-1)A = \left[-A_{ij} \right] \tag{1.1.10}$$

These definitions of addition and subtraction and, multiplication by a number imply that

$$A + B = B + A \tag{1.1.11}$$

$$A + (B + C) = (A + B) + C \tag{1.1.12}$$

$$A + 0 = A \tag{1.1.13}$$

$$A - A = 0 \tag{1.1.14}$$

$$\lambda(A+B) = \lambda A + \lambda B \tag{1.1.15}$$

$$(\lambda + \mu)A = \lambda A + \mu A \tag{1.1.16}$$

and

$$1A = A \tag{1.1.17}$$

where A, B and C are as assumed to be conformable.

The applications require a method of multiplying two matrices to produce a third. The formal definition of matrix multiplication is as follows: If A is an $M \times N$ matrix, i.e. an element of $\mathscr{M}^{M\times N}$, and B is an $N \times K$ matrix, i.e. an element of $\mathscr{M}^{N\times K}$, then the *product* of B by A is written AB and is an element of $\mathscr{M}^{M\times K}$ with components $\sum_{j=1}^{N} A_{ij}B_{js}$, $i = 1,...,M$, $s = 1,...,K$. For example, if

$$A = \underbrace{\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \\ A_{31} & A_{32} \end{bmatrix}}_{3\times 2} \qquad \text{and} \qquad B = \underbrace{\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}}_{2\times 2} \tag{1.1.18}$$

then AB is a 3×2 matrix given by

$$\begin{aligned} AB &= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \\ A_{31} & A_{32} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \\ &= \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \\ A_{31}B_{11} + A_{32}B_{21} & A_{32}B_{12} + A_{32}B_{22} \end{bmatrix} \end{aligned} \tag{1.1.19}$$

The product AB is defined only when the number of columns of A is equal to the number of rows of B. If this is the case, A is said to be *conformable* to B for *multiplication*. If A is conformable to B, then B is not necessarily conformable to A. Even if BA is defined, it is not necessarily equal to AB. The following example illustrates this general point for particular matrices A and B.

Example 1.1.1: If you are given matrices A and B defined by

$$A=\begin{bmatrix}3 & -2\\ 2 & 4\\ 1 & -3\end{bmatrix} \quad \text{and} \quad B=\begin{bmatrix}-2 & 1 & 3\\ 4 & 1 & 6\end{bmatrix} \tag{1.1.20}$$

The multiplications, AB and BA, yield

$$AB=\begin{bmatrix}-14 & 1 & -3\\ 12 & 6 & 30\\ -14 & -2 & -15\end{bmatrix} \tag{1.1.21}$$

and

$$BA=\begin{bmatrix}-1 & -1\\ 20 & -22\end{bmatrix} \tag{1.1.22}$$

On the assumption that A, B ,and C are conformable for the indicated sums and products, it is possible to show that

$$A(B+C)=AB+AC \tag{1.1.23}$$

$$(A+B)C=AC+BC \tag{1.1.24}$$

and

$$A(BC)=(AB)C \tag{1.1.25}$$

However, $AB \neq BA$ in general, $AB=0$ does not imply $A=0$ or $B=0$, and $AB=AC$ does not necessarily imply $B=C$.

If A is an $M \times N$ matrix and B is an $M \times N$ then the products AB and BA are defined but not equal. It is a property of matrix multiplication and the trace operation that

$$\operatorname{tr}\left(AB\right)=\operatorname{tr}\left(BA\right) \tag{1.1.26}$$

The square matrix I defined by

$$I = \begin{bmatrix} 1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 1 & \cdot & \cdot & \cdot & 0 \\ \cdot & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ 0 & 0 & \cdot & \cdot & \cdot & 1 \end{bmatrix} \tag{1.1.27}$$

is the *identity* matrix. The identity matrix is a special case of a *diagonal* matrix. In other words, a matrix which has all of its elements zero except the diagonal ones. It is often convenient to display the components of the identity matrix in the form

$$I = \left[\delta_{ij} \right] \tag{1.1.28}$$

where

$$\delta_{ij} = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases} \tag{1.1.29}$$

The symbol δ_{ij}, as defined by (1.1.29), is known as the *Kronecker delta*.[2]

A matrix A in $\mathscr{M}^{M\times N}$ whose elements satisfy $A_{ij} = 0$, $i > j$, is called *an upper triangular matrix* , i.e.,

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} & \cdot & \cdot & \cdot & A_{1N} \\ 0 & A_{22} & A_{23} & \cdot & \cdot & \cdot & A_{2N} \\ 0 & 0 & A_{33} & \cdot & \cdot & \cdot & \\ \cdot & & & & & & \\ \cdot & & & & & & \\ \cdot & & & & & & \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & A_{MN} \end{bmatrix} \tag{1.1.30}$$

A *lower triangular matrix* can be defined in a similar fashion. A *diagonal matrix* is a square matrix that is both an upper triangular matrix and a lower triangular matrix.

If A and B are square matrices of the same order such that $AB = BA = I$, then B is called the *inverse* of A and we write $B = A^{-1}$. Also, A is the inverse of B, i.e. $A = B^{-1}$.

Example 1.1.2: If you are given a 2×2 matrix

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[2] The Kronecker is named after the German mathematician Leopold Kronecker. Information about Leopold Kronecker can be found, for example, at http://en.wikipedia.org/wiki/Leopold_Kronecker.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \tag{1.1.31}$$

then it is a simple exercise to show that the matrix B defined by

$$B = -\frac{1}{2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} \tag{1.1.32}$$

obeys

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\left(-\frac{1}{2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}\right) = -\frac{1}{2}\begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{1.1.33}$$

and

$$\left(-\frac{1}{2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}\right)\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{1.1.34}$$

Therefore, $B = A^{-1}$ and $A = B^{-1}$.

Example 1.1.3: Not all square matrices have an inverse. A matrix that does not have an inverse is

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \tag{1.1.35}$$

If A has an inverse it is said to be *nonsingular*. If A has an inverse, then it is possible to prove that it is unique. If A and B are square matrices of the same order with inverses A^{-1} and B^{-1} respectively, we shall show that

$$(AB)^{-1} = B^{-1}A^{-1} \tag{1.1.36}$$

In order to prove (1.1.36), the definition of an inverse requires that we establish that $(B^{-1}A^{-1})AB = I$ and $(AB)B^{-1}A^{-1} = I$. If we form, for example, the product $(B^{-1}A^{-1})AB$, it follows that

$$(B^{-1}A^{-1})AB = B^{-1}A^{-1}AB = B^{-1}\left(A^{-1}A\right)B = B^{-1}\left(I\right)B = B^{-1}B = I \tag{1.1.37}$$

Likewise,

$$(AB)B^{-1}A^{-1} = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I \tag{1.1.38}$$

Equations (1.1.37) and (1.1.38) confirm our assertion (1.1.36).

Exercises

1.1.1 Add the matrices

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} -5 \\ 6 \end{bmatrix}$$

1.1.2 Add the matrices

$$\begin{bmatrix} 2i & 3 & 7+2i \\ 5 & 4+3i & i \end{bmatrix} + \begin{bmatrix} 2 & 5i & -4 \\ -4i & 3+3i & -i \end{bmatrix}$$

1.1.3 Add

$$\begin{bmatrix} 1 \\ 2i \end{bmatrix} + 3\begin{bmatrix} -5i \\ 6 \end{bmatrix}$$

1.1.4 Multiply

$$\begin{bmatrix} 2i & 3 & 7+2i \\ 5 & 4+3i & i \end{bmatrix} \begin{bmatrix} 2i & 8 \\ 1 & 6i \\ 3i & 2 \end{bmatrix}$$

1.1.5 Multiply

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 3 & 5 \\ 2 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 1 \\ 5 & 4 & 1 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 1 \\ 5 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 3 & 3 & 5 \\ 2 & 4 & 1 \end{bmatrix}$$

1.1.6 Show that the product of two upper (lower) triangular matrices is an upper lower triangular matrix. Further, if

$$A = \left[A_{ij} \right], \qquad B = \left[B_{ij} \right]$$

are upper (lower) triangular matrices of order $N \times N$, then

$$(AB)_{ii} = (BA)_{ii} = A_{ii} B_{ii}$$

for all $i = 1, \ldots, N$. The off diagonal elements $(AB)_{ij}$ and $(BA)_{ij}$, $i \neq j$, generally are not equal, however.

1.1.7 If you are given a square matrix

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

with the property that $A_{11}A_{22} - A_{12}A_{21} \neq 0$, show that the matrix $\dfrac{1}{A_{11}A_{22} - A_{12}A_{21}} \begin{bmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{bmatrix}$ is the inverse of A. Use this formula to show that the inverse of

$$A = \begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix}$$

is the matrix

$$A^{-1} = \begin{bmatrix} -\dfrac{1}{10} & \dfrac{2}{5} \\ \dfrac{3}{10} & -\dfrac{1}{5} \end{bmatrix}$$

1.1.8 Confirm the identity (1.1.26) in the special case where A and B are given by (1.1.20).

Section 1.2. Systems of Linear Equations

Matrix algebra methods have many applications. Probably the most useful application arises in the study of systems of M linear algebraic equations in N unknowns of the form

$$
\begin{array}{l}
A_{11}x_1 + A_{12}x_2 + A_{13}x_3 + \cdots + A_{1N}x_N = b_1 \\
A_{21}x_1 + A_{22}x_2 + A_{23}x_3 + \cdots + A_{2N}x_N = b_2 \\
\cdot \\
\cdot \\
\cdot \\
A_{M1}x_1 + A_{M2}x_2 + A_{M3}x_3 + \cdots + A_{MN}x_N = b_M
\end{array}
\tag{1.2.1}
$$

The system of equations (1.2.1) is *overdetermined* if there are more equations than unknowns, i.e., $M > N$. Likewise, the system of equations (1.2.1) is *underdetermined* if there are more unknowns than equations, i.e., $N > M$.

In matrix notation, this system can be written

$$
\begin{bmatrix}
A_{11} & A_{12} & \cdot & \cdot & \cdot & A_{1N} \\
A_{21} & A_{22} & & & & A_{2N} \\
\cdot & & & & & \\
\cdot & & & & & \\
\cdot & & & & & \\
A_{M1} & A_{M2} & \cdot & \cdot & \cdot & A_{MN}
\end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_N \end{bmatrix}
=
\begin{bmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ \cdot \\ b_M \end{bmatrix}
\tag{1.2.2}
$$

The above matrix equation can now be written in the compact notation

$$A\mathbf{x} = \mathbf{b} \tag{1.2.3}$$

where $\mathbf{x}$ is the $N \times 1$ column matrix

$$
\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_N \end{bmatrix}
\tag{1.2.4}
$$

and $\mathbf{b}$ is the $M \times 1$ column matrix

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ \cdot \\ b_M \end{bmatrix} \tag{1.2.5}$$

A *solution* to the $M \times N$ system is a $N \times 1$ column matrix $\mathbf{x}$ that obeys (1.2.3). It is often the case that overdetermined systems do not have a solution. Likewise, undetermined solutions usually do not have a unique solutions. If there are an equal number of unknowns as equations, i.e., $M = N$, he system may or may not have a solution. If it has a solution, it may not be unique.

In the special case where A is a square matrix that is also nonsingular, the solution of (1.2.3)is formally

$$\mathbf{x} = A^{-1}\mathbf{b} \tag{1.2.6}$$

Unfortunately, the case where A is square and also has an inverse is but one of many cases one must understand in order to fully understand how to characterize the solutions of (1.2.3).

Example 1.2.1: For $M = N = 2$, the system

$$\begin{aligned} x_1 + 2x_2 &= 5 \\ 2x_1 + 3x_2 &= 8 \end{aligned} \tag{1.2.7}$$

can be written

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \end{bmatrix} \tag{1.2.8}$$

By substitution into (1.2.8), one can easily confirm that

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \tag{1.2.9}$$

is the solution. In this case, the solution can be written in the form (1.2.6) with

$$A^{-1} = \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix} \tag{1.2.10}$$

In the case where $M = N = 2$ and $M = N = 3$ the system (1.2.2) can be view as defining the common point of intersection of straight lines in the case $M = N = 2$ and planes in the case $M = N = 3$. For example the two straight lines defined by (1.2.7) produce the plot

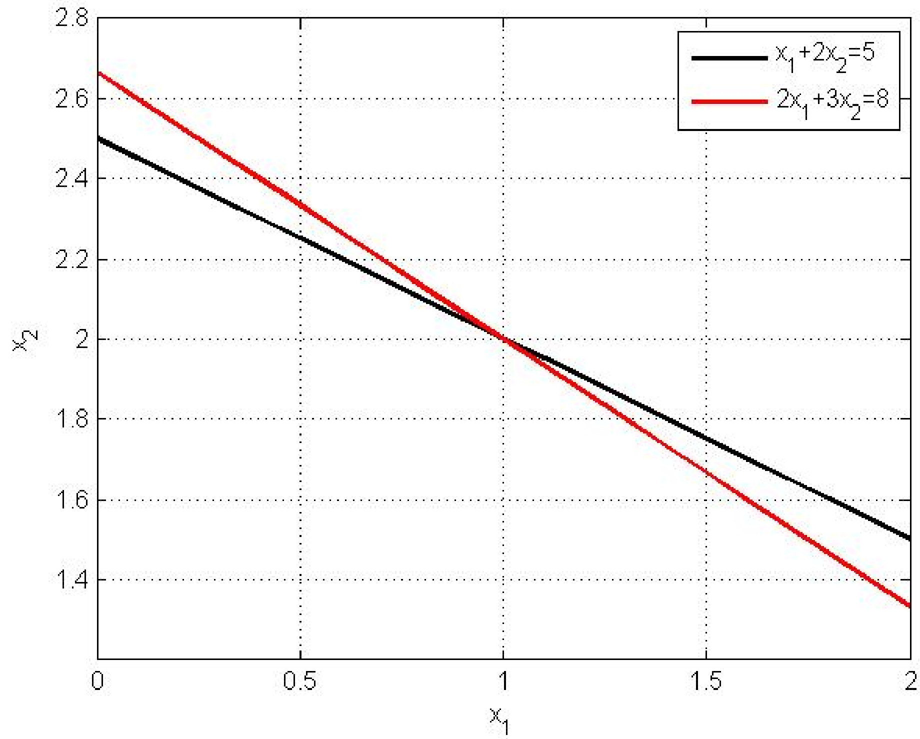


Figure 1. Solution of (1.2.8)

which displays the solution (1.2.9). One can easily imagine a system with $M = N = 2$ where the resulting two lines are parallel and, as a consequence, there is no solution.

Example 1.2.2: For $M = N = 3$, the system

$$\begin{aligned} 2x_1 - 6x_2 - x_3 &= -38 \\ -3x_1 - x_2 + 7x_3 &= -34 \\ -8x_1 + x_2 - 2x_3 &= -20 \end{aligned} \tag{1.2.11}$$

defines three planes. If this system has a unique solution then the three planes will intersect in a point. As one can confirm by direct substation, the system (1.2.11) does have a unique solution given by

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ 2 \end{bmatrix} \tag{1.2.12}$$

The point of intersection (1.2.12) is displayed by plotting the three planes (1.2.11) on a common axis. The result is illustrated by the following figure.

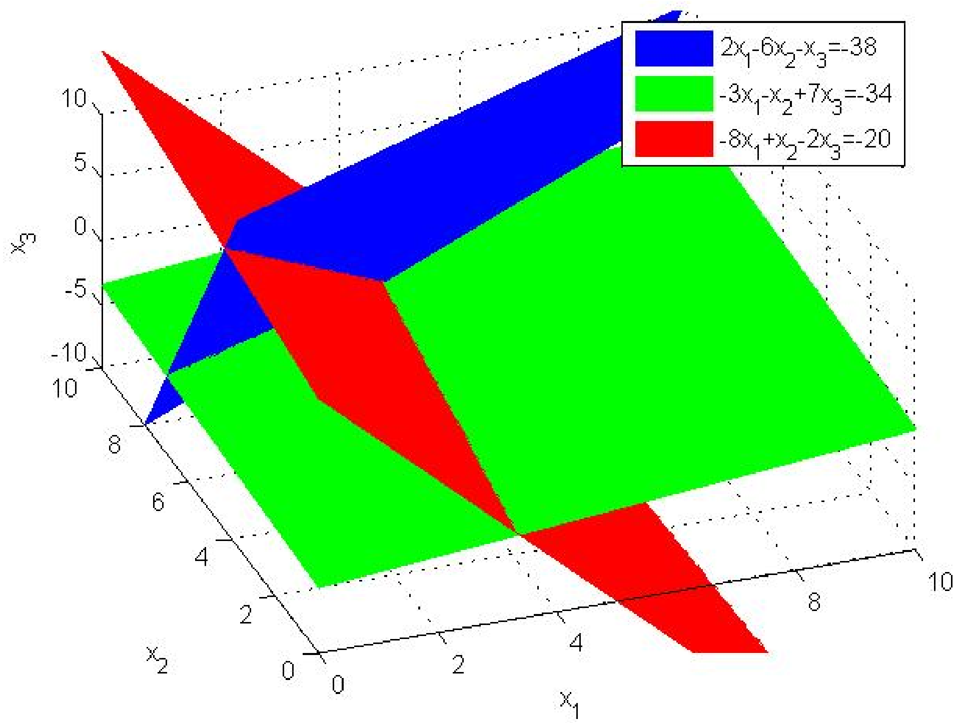


Figure 2. Solution of (1.2.11)

It is perhaps evident that planes associated with three linear algebraic equations can intersect in a point, as with (1.2.11), or as a line or, perhaps, they will not intersect. This geometric observation reveals the fact that systems of linear equations can have unique solutions, solutions that are not unique and no solution. An example where there is not a unique solution is provided by the following:

Example 1.2.3:

$$\begin{aligned} 2x_1 + 3x_2 + x_3 &= 1 \\ x_1 + x_2 + x_3 &= 3 \\ 3x_1 + 4x_2 + 2x_3 &= 4 \end{aligned} \tag{1.2.13}$$

By direct substitution into (1.2.13) one can establish that

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 - 2x_3 \\ x_3 - 5 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ -5 \\ 0 \end{bmatrix} + \begin{bmatrix} -2 \\ -5 \\ 1 \end{bmatrix} x_3 \tag{1.2.14}$$

obeys (1.2.13) for all values of x_3. Thus, there are an infinite number of solutions of (1.2.13). Basically, the system (1.2.13) is one where the planes intersect in a line, the line defined by $(1.2.14)_3$. The following figure displays this fact.

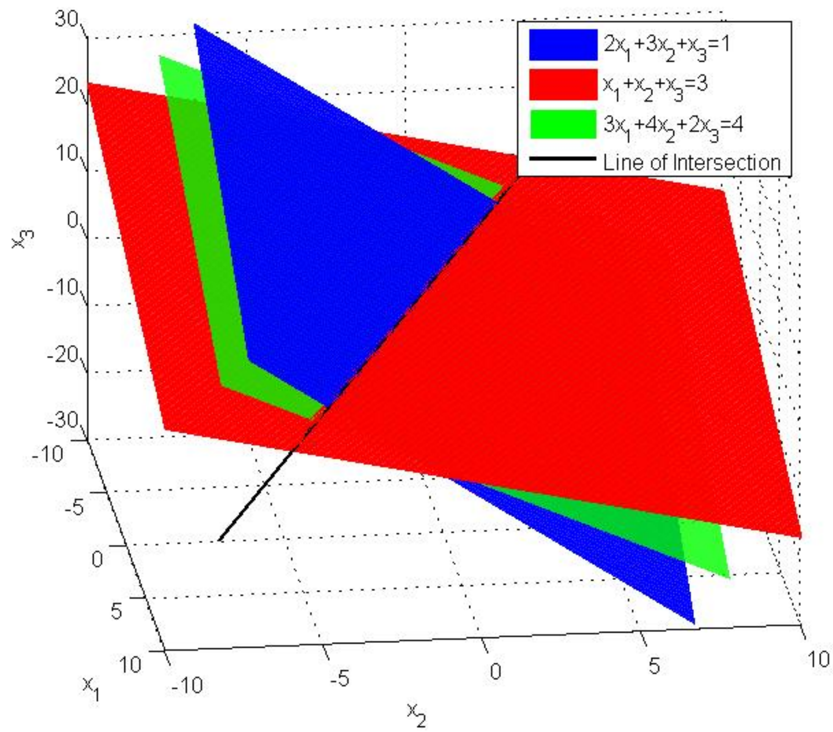


Figure 3. Solution of (1.2.13)

An example for which there is no solution is provided by

Example 1.2.4:

$$\begin{aligned} 2x_1 + 3x_2 + x_3 &= 1 \\ 3x_1 + 4x_2 + 2x_3 &= -80 \\ x_1 + x_2 + x_3 &= 10 \end{aligned} \tag{1.2.15}$$

The plot of these three equations yields

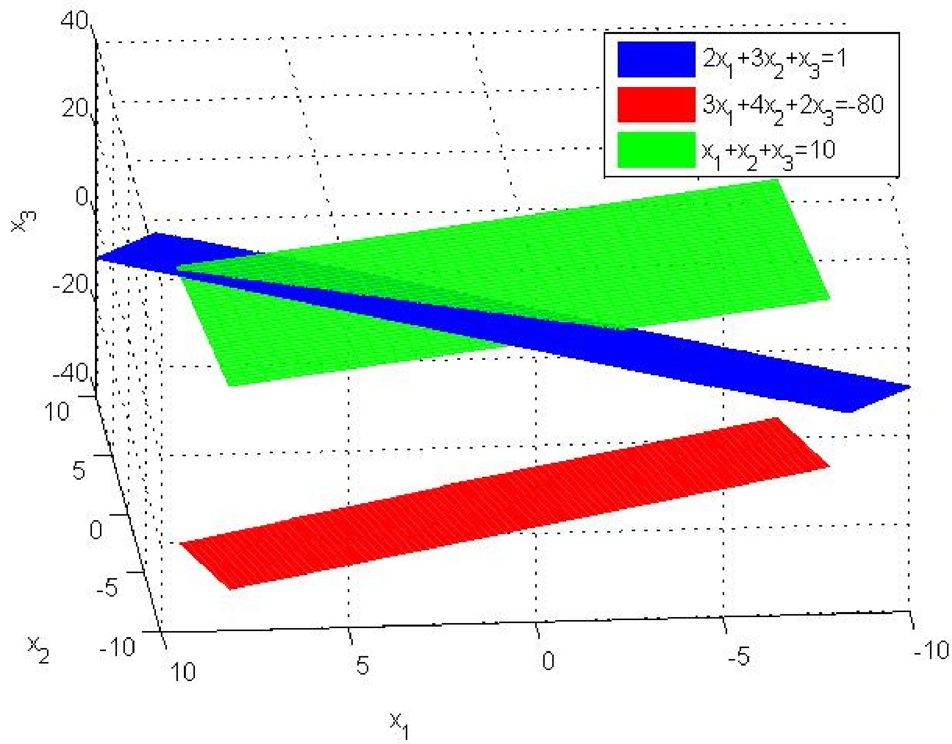


Figure 4. Plot of (1.2.15)

A solution does not exist in this case because the three planes do not intersect.

Example 1.2.5: Consider the undetermined system

$$\begin{aligned} x_1 - x_2 + x_3 &= 2 \\ 2x_1 + x_2 - x_3 &= 4 \end{aligned} \tag{1.2.16}$$

By direct substitution into (1.2.16) one can establish that

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ x_3 \\ x_3 \end{bmatrix} \tag{1.2.17}$$

is a solution for all values x_3. Thus, there are an infinite number of solutions of (1.2.16).

Example 1.2.6: Consider the overdetermined system

$$\begin{aligned} x_1 + x_2 &= 2 \\ x_1 - x_2 &= 1 \\ x_1 &= 4 \end{aligned} \tag{1.2.18}$$

If $(1.2.18)_3$ is substituted into $(1.2.18)_1$ and $(1.2.18)_2$ the inconsistent results $x_2 = -2$ and $x_2 = 3$ are obtained. Thus, this overdetermined system does not have a solution.

The above six examples illustrate the range of possibilities for the solution of (1.2.3) for various choices of M and N. The graphical arguments used for Examples 1.2.1, 1.2.2, 1.2.3 and 1.2.4 are especially useful when trying to understand the range of possible solutions. Unfortunately, for larger systems, i.e., for systems where $M = N > 3$, we cannot utilize graphical representations to illustrate the range of solutions. We need solution procedures that will yield numerical values for the solution developed within a theoretical framework that allows one to characterize the solution properties in advance of the attempted solution. Our goal, in this introductory phase of this linear algebra course is to develop components of this theoretical framework and to illustrate it with various numerical examples.

Section 1.3. Systems of Linear Equations: Gaussian Elimination

Elimination methods, which represent methods learned in high school algebra, form the basis for the most powerful methods of solving systems of linear algebraic equations. We begin this discussion by introducing the idea of an *equivalent system* to the given system (1.2.1). An equivalent system to (1.2.1) is a system of M linear algebraic equations in N unknowns obtained from (1.2.1) by

a) switching two rows,
b) multiplying one of the rows by a nonzero constant
c) multiply one row by a nonzero constant and adding it to another row, or
d) combinations of a),b) and c).

Equivalent systems have the same solution as the original system. The point that is embedded in this concept is that given the problem of solving (1.2.1), one can convert it to an equivalent system which will be easier to solve. Virtually all of the solution techniques utilized for large systems involve this kind of approach.

Given the system of M linear algebraic equations in N unknowns (1.2.1), repeated,

$$\begin{aligned}
&A_{11}x_1 + A_{12}x_2 + A_{13}x_3 + \cdots + A_{1N}x_N = b_1 \\
&A_{21}x_1 + A_{22}x_2 + A_{23}x_3 + \cdots + A_{2N}x_N = b_2 \\
&. \\
&. \\
&. \\
&A_{M1}x_1 + A_{M2}x_2 + A_{M3}x_3 + \cdots + A_{MN}x_N = b_M
\end{aligned} \tag{1.3.1}$$

the *elimination method* consists of the following steps:

- Solve the first equation for one of the unknowns, say, x_1 if $A_{11} \neq 0$
- Substitute the result into the remaining $M-1$ equations to obtain $M-1$ equations in $N-1$ unknowns, $x_2, x_3, ..., x_N$.
- Repeat the process with these $M-1$ equations to obtain an equation for one of the unknowns.
- This solution is then *back substituted* into the previous equations to obtain the answers for the other two variables.

If the original system of equations *does not* have a solution, the elimination process will yield an *inconsistency* which will not allow you to proceed. This elimination method described by the above steps is called *Gauss Elimination* or *Gaussian Elimination*. The following example illustrates how this elimination can be implemented.

Example 1.3.1: Given

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 1 \\ 2x_1 - x_2 + x_3 &= 3 \\ -x_1 + 2x_2 + 3x_3 &= 7 \end{aligned} \tag{1.3.2}$$

Step 1: The object is to use the first equation to eliminate x_1 from the second. This can be achieved if we multiple the first equation by 2 and subtract it from the second. The result is

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 1 \\ -5x_2 + 3x_3 &= 1 \\ -x_1 + 2x_2 + 3x_3 &= 7 \end{aligned} \tag{1.3.3}$$

Step 2: This step eliminates x_1 from $(1.3.3)_3$ by adding $(1.3.3)_1$ to $(1.3.3)_3$. The result is

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 1 \\ -5x_2 + 3x_3 &= 1 \\ 4x_2 + 2x_3 &= 8 \end{aligned} \tag{1.3.4}$$

Step 3: The second and third equations in (1.3.4) involve the unknowns x_2 and x_3. The elimination method utilizes these two equations to eliminate x_2. This elimination is achieved if we multiply $(1.3.4)_2$ by $\frac{4}{5}$ and add it to $(1.3.4)_3$. The result is

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 1 \\ -5x_2 + 3x_3 &= 1 \\ \frac{22}{5}x_3 &= \frac{44}{5} \end{aligned} \tag{1.3.5}$$

Step 4: The next step starts a *back substitution* process. First, we recognize that $(1.3.5)_3$ yields

$$x_3 = 2 \tag{1.3.6}$$

This result is substituted into $(1.3.5)_2$ to yield

$$-5x_2 + 6 = 1 \tag{1.3.7}$$

and, as a result,

$$x_2 = 1 \tag{1.3.8}$$

Step 5: We continue the back substitution process and use (1.3.6) and (1.3.8) to derive from $(1.3.5)_1$

$$x_1 = 1 \tag{1.3.9}$$

Therefore, the solution is

$$\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \tag{1.3.10}$$

It should be evident that the above steps are not unique. We could have reached the same endpoint with a different sequence of rearrangements. Also, it should be evident that one could generalize the above process to very large systems.

Example 1.3.2:

$$\begin{aligned} x_1 + 3x_2 + x_3 &= 1 \\ 2x_1 + x_2 + x_3 &= 5 \\ -2x_1 + 2x_2 - x_3 &= -8 \end{aligned} \tag{1.3.11}$$

We shall use Gaussian elimination to show that

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} \tag{1.3.12}$$

Unlike the last example, we shall *not* directly manipulate the actual equations (1.3.11). We shall simply do matrix manipulations on the coefficients. This is done by first writing the system (1.3.11) as a matrix equation. The result is

$$\begin{bmatrix} 1 & 3 & 1 \\ 2 & 1 & 1 \\ -2 & 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ -8 \end{bmatrix} \tag{1.3.13}$$

The next formal step is to form what is called the *augmented matrix*. It is simply the matrix $A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 1 & 1 \\ -2 & 2 & -1 \end{bmatrix}$ augmented by the column matrix $\begin{bmatrix} 1 \\ 5 \\ -8 \end{bmatrix}$. It is customarily given the notation $\left(A \middle| \mathbf{b}\right)$. In our example, the augmented matrix is

$$\left(A\middle|\mathbf{b}\right)\equiv\left[\begin{array}{ccc|c}1 & 3 & 1 & 1\\ 2 & 1 & 1 & 5\\ -2 & 2 & -1 & -8\end{array}\right] \tag{1.3.14}$$

Next, we shall do the Gaussian elimination procedure *directly* on the augmented matrix.

Step 1: Multiply the first row by 2 (the A_{21} element), divide it by 1 (the A_{11} element) and subtract the first row from the second. The result is

$$\left[\begin{array}{ccc|c}1 & 3 & 1 & 1\\ 2 & 1 & 1 & 5\\ -2 & 2 & -1 & -8\end{array}\right]\xrightarrow[\substack{2\times\text{row 1}\\ \text{subtracted}\\ \text{from row 2}}]{}\left[\begin{array}{ccc|c}1 & 3 & 1 & 1\\ 0 & -5 & -1 & 3\\ -2 & 2 & -1 & -8\end{array}\right] \tag{1.3.15}$$

Repeating this process, which is called *pivoting*,

$$\begin{aligned}\left[\begin{array}{ccc|c}1 & 3 & 1 & 1\\ 2 & 1 & 1 & 5\\ -2 & 2 & -1 & -8\end{array}\right]&\xrightarrow[\substack{2\times\text{row 1}\\ \text{subtracted}\\ \text{from row 2}}]{}\left[\begin{array}{ccc|c}1 & 3 & 1 & 1\\ 0 & -5 & -1 & 3\\ -2 & 2 & -1 & -8\end{array}\right]\\ &\xrightarrow[\substack{2\times\text{row 1 added}\\ \text{to row 3}}]{}\left[\begin{array}{ccc|c}1 & 3 & 1 & 1\\ 0 & -5 & -1 & 3\\ 0 & 8 & 1 & -6\end{array}\right]\\ &\xrightarrow[\substack{\frac{8}{5}\times\text{row 2}\\ \text{added}\\ \text{to row 3}}]{}\left[\begin{array}{ccc|c}1 & 3 & 1 & 1\\ 0 & -5 & -1 & 3\\ 0 & 0 & -\frac{3}{5} & -\frac{6}{5}\end{array}\right]\end{aligned} \tag{1.3.16}$$

The last augmented matrix coincides with the system

$$\begin{aligned} x_1+3x_2+x_3&=1\\ -5x_2-x_3&=3\\ -\frac{3}{5}x_3&=-\frac{6}{5}\end{aligned} \tag{1.3.17}$$

The next step starts the back substitution part of the process. Equation $(1.3.17)_3$ yields

$$x_3=2 \tag{1.3.18}$$

Therefore, from equations $(1.3.17)_2$ and $(1.3.17)_1$,

$$\begin{aligned} -5x_2 - x_3 = 3 \Rightarrow x_2 = \frac{-x_3 - 3}{5} = -1 \\ x_1 + 3x_2 + x_3 = 1 \Rightarrow x_1 = 1 - x_3 - 3x_2 = 2 \end{aligned} \tag{1.3.19}$$

Therefore, we have found the result (1.3.12)

The above steps can be generalized without difficulty. For simplicity, we shall give the generalization for the case where $M = N$. The other cases will eventually be discussed but the details can get too involved if we allow those cases at this point in our discussions. For a system of N equations and N unknowns, we have the equivalence between the system of equations (1.3.1) and its representation by the augmented matrix as follows:

$$\left.\begin{aligned} & A_{11}x_1 + A_{12}x_2 + A_{13}x_3 + \cdots + A_{1N}x_N = b_1 \\ & A_{21}x_1 + A_{22}x_2 + A_{23}x_3 + \cdots + A_{2N}x_N = b_2 \\ & \cdot \\ & \cdot \\ & \cdot \\ & A_{N1}x_1 + A_{N2}x_2 + A_{N3}x_3 + \cdots + A_{NN}x_N = b_N \end{aligned}\right\} \Leftrightarrow \left(A|\mathbf{b}\right) \equiv \underbrace{\left[\begin{array}{ccccccc|c} A_{11} & A_{12} & A_{13} & \cdot & \cdot & \cdot & A_{1N} & b_1 \\ A_{21} & A_{22} & A_{23} & \cdot & \cdot & \cdot & A_{2N} & b_2 \\ A_{31} & A_{32} & A_{33} & \cdot & \cdot & \cdot & & b_3 \\ \cdot & & & & & & & \cdot \\ \cdot & & & & & & & \cdot \\ \cdot & & & & & & & \cdot \\ A_{N1} & A_{N2} & A_{N3} & \cdot & \cdot & \cdot & A_{NN} & b_N \end{array}\right]}_{\text{Augmented Matrix}} \tag{1.3.20}$$

We then, as the above example illustrate, can perform the operations on the rows of the augmented matrix, rather than on the equations themselves.

Note: In matrix algebra, we are using what is known as *row operations* when we manipulate the augmented matrix.

Step 1: Forward Elimination of Unknowns:

If $A_{11} \neq 0$, we first multiply the row of the augmented matrix equation by $\dfrac{A_{21}}{A_{11}}$ and subtract the result from the second row. The result is the augmented matrix

$$\left[\begin{array}{cccccc|c} A_{11} & A_{12} & A_{13} & \cdot & \cdot & \cdot & A_{1N} & b_1 \\ 0 & A_{22}-\dfrac{A_{21}}{A_{11}}A_{12} & A_{23}-\dfrac{A_{21}}{A_{11}}A_{13} & \cdot & \cdot & \cdot & A_{2N}-\dfrac{A_{21}}{A_{11}}A_{1N} & b_2-\dfrac{A_{21}}{A_{11}}b_1 \\ A_{31} & A_{32} & A_{33} & \cdot & \cdot & \cdot & & b_3 \\ \cdot & & & & & & & \cdot \\ \cdot & & & & & & & \cdot \\ \cdot & & & & & & & \cdot \\ A_{N1} & A_{N2} & A_{N3} & \cdot & \cdot & \cdot & A_{NN} & b_N \end{array}\right] \tag{1.3.21}$$

In order to keep the notation from becoming unwieldy, we shall assign different symbols to the second row and write (1.3.21) as

$$\left[\begin{array}{ccccccc|c} A_{11} & A_{12} & A_{13} & \cdot & \cdot & \cdot & A_{1N} & b_1 \\ 0 & A_{22}^{(1)} & A_{23}^{(1)} & \cdot & \cdot & \cdot & A_{2N}^{(1)} & b_2^{(1)} \\ A_{31} & A_{32} & A_{33} & \cdot & \cdot & \cdot & & b_3 \\ \cdot & & & & & & & \cdot \\ \cdot & & & & & & & \cdot \\ \cdot & & & & & & & \cdot \\ A_{N1} & A_{N2} & A_{N3} & \cdot & \cdot & \cdot & A_{NN} & b_N \end{array}\right] \tag{1.3.22}$$

Next, we repeat the last procedure for the third row by multiply the first row by $\dfrac{A_{31}}{A_{11}}$ and subtract the result from the third equation. The result can be written

$$\left[\begin{array}{ccccccc|c} A_{11} & A_{12} & A_{13} & \cdot & \cdot & \cdot & A_{1N} & b_1 \\ 0 & A_{22}^{(1)} & A_{23}^{(1)} & \cdot & \cdot & \cdot & A_{2N}^{(1)} & b_2^{(1)} \\ 0 & A_{32}^{(1)} & A_{33}^{(1)} & \cdot & \cdot & \cdot & & b_3^{(1)} \\ \cdot & & & & & & & \cdot \\ \cdot & & & & & & & \cdot \\ \cdot & & & & & & & \cdot \\ A_{N1} & A_{N2} & A_{N3} & \cdot & \cdot & \cdot & A_{NN} & b_N \end{array}\right] \tag{1.3.23}$$

This process is continued until the all of the entries below A_{11} are zero in the augmented matrix. The result is the augmented matrix

$$\left[\begin{array}{ccccccc|c} A_{11} & A_{12} & A_{13} & \cdot & \cdot & \cdot & A_{1N} & b_1 \\ 0 & A_{22}^{(1)} & A_{23}^{(1)} & \cdot & \cdot & \cdot & A_{2N}^{(1)} & b_2^{(1)} \\ 0 & A_{32}^{(1)} & A_{33}^{(1)} & \cdot & \cdot & \cdot & & b_3^{(1)} \\ \cdot & & & & & & & \cdot \\ \cdot & & & & & & & \cdot \\ \cdot & & & & & & & \cdot \\ 0 & A_{N2}^{(1)} & A_{N3}^{(1)} & \cdot & \cdot & \cdot & A_{NN}^{(1)} & b_N^{(1)} \end{array}\right] \tag{1.3.24}$$

The augmented result (1.3.24) corresponds to the system of equations
The result is that the original N equations are replaced by

$$\begin{aligned} A_{11}x_1 + A_{12}x_2 + A_{13}x_3 + \cdots + A_{1N}x_N &= b_1 \\ A_{22}^{(1)}x_2 + A_{23}^{(1)}x_3 + \cdots + A_{2N}^{(1)}x_N &= b_2^{(1)} \\ A_{32}^{(1)}x_2 + A_{33}^{(1)}x_3 + \cdots + A_{3N}^{(1)}x_N &= b_3^{(1)} \\ &\cdot \\ &\cdot \\ &\cdot \\ A_{N2}^{(1)}x_2 + A_{N3}^{(1)}x_3 + \cdots + A_{NN}^{(1)}x_N &= b_N^{(1)} \end{aligned} \tag{1.3.25}$$

Note: In the above sequence, the first row is the *pivot row* and its coefficient A_{11} is called the *pivot coefficient* or *pivot element*.

The next step is to apply the same process to the set of $N-1$ equations with $N-1$ unknowns

$$\begin{aligned} A_{22}^{(1)}x_2 + A_{23}^{(1)}x_3 + \cdots + A_{2N}^{(1)}x_N &= b_2^{(1)} \\ A_{32}^{(1)}x_2 + A_{33}^{(1)}x_3 + \cdots + A_{3N}^{(1)}x_N &= b_3^{(1)} \\ &\cdot \\ &\cdot \\ &\cdot \\ A_{N2}^{(1)}x_2 + A_{N3}^{(1)}x_3 + \cdots + A_{NN}^{(1)}x_N &= b_N^{(1)} \end{aligned} \tag{1.3.26}$$

to eliminate the second unknown, x_2. This process begins by multiplying the second row of (1.3.24) by $\dfrac{A_{32}^{(1)}}{A_{22}^{(1)}}$ (this step assumes $A_{22}^{(1)} \neq 0$) and subtracting the result from the third row of (1.3.24). This step is repeated for the remaining rows until the augmented matrix is transformed into the following:

$$\left[\begin{array}{cccccc|c}
A_{11} & A_{12} & A_{13} & \cdot & \cdot & \cdot & A_{1N} & b_1 \\
0 & A_{22}^{(1)} & A_{23}^{(1)} & \cdot & \cdot & \cdot & A_{2N}^{(1)} & b_2^{(1)} \\
0 & 0 & A_{33}^{(2)} & \cdot & \cdot & \cdot & A_{2N}^{(2)} & b_3^{(2)} \\
\cdot & & \cdot & & & & & \cdot \\
\cdot & & \cdot & & & & & \cdot \\
\cdot & & \cdot & & & & & \cdot \\
0 & 0 & A_{N3}^{(2)} & \cdot & \cdot & \cdot & A_{NN}^{(2)} & b_N^{(2)}
\end{array}\right] \tag{1.3.27}$$

You should now have the idea. You continue this process until the augmented matrix $(1.3.20)_2$ is replaced by the upper triangular form

$$\left[\begin{array}{ccccccc|c}
A_{11} & A_{12} & A_{13} & \cdot & \cdot & \cdot & A_{1N} & b_1 \\
0 & A_{22}^{(1)} & A_{23}^{(1)} & \cdot & \cdot & \cdot & A_{2N}^{(1)} & b_2^{(1)} \\
0 & 0 & A_{33}^{(2)} & \cdot & \cdot & \cdot & A_{2N}^{(2)} & b_3^{(2)} \\
\cdot & & \cdot & \cdot & & & & \cdot \\
\cdot & & \cdot & & \cdot & & & \cdot \\
\cdot & & \cdot & & & A_{N-1,N-1}^{(N-2)} & A_{N-1,N}^{(N-2)} & b_{N-1}^{(N-2)} \\
0 & 0 & 0 & 0 & \cdot & \cdot & A_{NN}^{(N-1)} & b_N^{(N-1)}
\end{array}\right] \tag{1.3.28}$$

Each step in the process leading to (1.3.28) has assumed we have not encountered the situation where the lead coefficient in the pivot row was zero. The augmented matrix (1.3.28) corresponds to the system of equations

$$\begin{aligned}
A_{11}x_1 + A_{12}x_2 + A_{13}x_3 + \cdots + A_{1N}x_N &= b_1 \\
A'_{22}x_2 + A'_{23}x_3 + \cdots + A'_{2N}x_N &= b'_2 \\
A''_{33}x_3 + \cdots + A''_{3N}x_N &= b''_3 \\
&\cdot \\
&\cdot \\
&\cdot \\
A_{NN}^{(N-1)}x_N &= b_N^{(N-1)}
\end{aligned} \tag{1.3.29}$$

Step 2: Back Substitution

If $A_{NN}^{(N-1)} \neq 0$, the last equation can be solved as follows:

$$x_N = \frac{b_N^{(N-1)}}{A_{NN}^{(N-1)}} \tag{1.3.30}$$

This answer can then be back substituted into the previous equations to solve for $x_{N-1}, x_{N-2}, ..., x_1$. The formula for these unknowns, should it ever prove useful, is

$$x_i = \frac{b_i^{(i-1)} - \sum_{j=i+1}^{N} A_{ij}^{(i-1)} x_j}{A_{ii}^{(i-1)}} \qquad \text{for} \quad i = N-1, N-2, ..., 1 \tag{1.3.31}$$

The process just described make repeated use of the assumption that certain coefficients were nonzero in order for the process to proceed. If one cannot find a coefficient with this property, then the system is degenerate in some way and may not have a unique solution or any solution. Frequently one avoids this problem by utilizing a procedure by what is called *partial pivoting*. The following example illustrates this procedure.

Example 1.3.3: Consider the system of equations

$$\begin{aligned} 2x_2 + 3x_3 &= 8 \\ 4x_1 + 6x_2 + 7x_3 &= -3 \\ 2x_1 - 3x_2 + 6x_3 &= 5 \end{aligned} \tag{1.3.32}$$

The procedure we described above would first create the auxiliary matrix representation of this system. The result is

$$\left(A \middle| \mathbf{b}\right) = \left[\begin{array}{ccc|c} 0 & 2 & 3 & 8 \\ 4 & 6 & 7 & -3 \\ 2 & -3 & 6 & 5 \end{array}\right] \tag{1.3.33}$$

Because $A_{11} = 0$, we immediately encounter a problem with our method. The partial pivoting procedure simply reorders the equations such that the new $A_{11} \neq 0$. For example, we can begin the elimination process with the auxiliary matrix

$$\left(A \middle| \mathbf{b}\right) = \left[\begin{array}{ccc|c} 4 & 6 & 7 & -3 \\ 2 & -3 & 6 & 5 \\ 0 & 2 & 3 & 8 \end{array}\right] \tag{1.3.34}$$

The usual practice is to switch the order of the equations so as to make the A_{11} the largest, in absolute value, of the elements in the first column.

Example 1.3.4: In Section 1.2 we discussed Example 1.2.3 which was the system

$$\begin{aligned} 2x_1 + 3x_2 + x_3 &= 1 \\ x_1 + x_2 + x_3 &= 3 \\ 3x_1 + 4x_2 + 2x_3 &= 4 \end{aligned} \tag{1.3.35}$$

This system has the solution (1.2.14), repeated,

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 - 2x_3 \\ x_3 - 5 \\ x_3 \end{bmatrix} \tag{1.3.36}$$

It is helpful to utilize the Gaussian Elimination procedure to see this solution. The first step is to form the augmented matrix

$$\left(A|\mathbf{b}\right) = \left[\begin{array}{ccc|c} 2 & 3 & 1 & 1 \\ 1 & 1 & 1 & 3 \\ 3 & 4 & 2 & 4 \end{array}\right] \tag{1.3.37}$$

The sequence of steps described above, applied to this example, is

$$\left[\begin{array}{ccc|c} 2 & 3 & 1 & 1 \\ 1 & 1 & 1 & 3 \\ 3 & 4 & 2 & 4 \end{array}\right] \xrightarrow[\substack{\frac{1}{2}\times\text{row 1} \\ \text{subtracted} \\ \text{from row 2}}]{} \left[\begin{array}{ccc|c} 2 & 3 & 1 & 1 \\ 0 & -\frac{1}{2} & \frac{1}{2} & \frac{5}{2} \\ 3 & 4 & 2 & 4 \end{array}\right] \tag{1.3.38}$$

Repeating this process,

$$\left[\begin{array}{ccc|c} 2 & 3 & 1 & 1 \\ 1 & 1 & 1 & 3 \\ 3 & 4 & 2 & 4 \end{array}\right] \xrightarrow[\substack{\frac{1}{2}\times\text{row 1} \\ \text{subtracted} \\ \text{from row 2}}]{} \left[\begin{array}{ccc|c} 2 & 3 & 1 & 1 \\ 0 & -\frac{1}{2} & \frac{1}{2} & \frac{5}{2} \\ 3 & 4 & 2 & 4 \end{array}\right]$$

$$\xrightarrow[\substack{\frac{3}{2}\times\text{row 1 subtracted} \\ \text{from row 3}}]{} \left[\begin{array}{ccc|c} 2 & 3 & 1 & 1 \\ 0 & -\frac{1}{2} & \frac{1}{2} & \frac{5}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} & \frac{5}{2} \end{array}\right] \tag{1.3.39}$$

$$\xrightarrow[\substack{\text{subtract row 2} \\ \text{from row 3}}]{} \left[\begin{array}{ccc|c} 2 & 3 & 1 & 1 \\ 0 & -\frac{1}{2} & \frac{1}{2} & \frac{5}{2} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The occurrence of the zero in the 33 position of the last matrix means that we cannot proceed with the back substitution process as it was described above. The modified back substitution process proceeds as follows: The last augmented matrix coincides with the system

$$\begin{aligned} 2x_1 + 3x_2 + x_3 &= 1 \\ -\frac{1}{2}x_2 + \frac{1}{2}x_3 &= \frac{5}{2} \end{aligned} \tag{1.3.40}$$

The occurrence of the row of zeros in the third row, results in only two equations for the three unknowns x_1, x_2 and x_3. The next step starts the back substitution part of the process. Equation $(1.3.40)_2$ yields

$$x_2 = x_3 - 5 \tag{1.3.41}$$

Therefore, from equation $(1.3.40)_1$,

$$x_1 = \frac{1}{2}\left(1 - 3x_2 - x_3\right) = 8 - 2x_3 \tag{1.3.42}$$

Therefore, we have found the result (1.3.36)

Example 1.3.5: In Section 1.2 we discussed Example 1.2.4 which was the system

$$
\begin{aligned}
2x_1 + 3x_2 + x_3 &= 1 \\
3x_1 + 4x_2 + 2x_3 &= -80 \\
x_1 + x_2 + x_3 &= 10
\end{aligned}
\tag{1.3.43}
$$

It was explained in Section 1.2 that this system does not have a solution. This conclusion arises from the Gaussian Elimination procedure by the following steps. As usual, the first step is to form the augmented matrix

$$
\left(A|\mathbf{b}\right) = \left[\begin{array}{ccc|c} 2 & 3 & 1 & 1 \\ 3 & 4 & 2 & -80 \\ 1 & 1 & 1 & 10 \end{array}\right] \tag{1.3.44}
$$

The sequence of steps described above, applied to this example, is

$$
\left[\begin{array}{ccc|c} 2 & 3 & 1 & 1 \\ 3 & 4 & 2 & -80 \\ 1 & 1 & 1 & 10 \end{array}\right] \xrightarrow[\substack{\frac{3}{2}\times\text{row 1} \\ \text{subtracted} \\ \text{from row 2}}]{} \left[\begin{array}{ccc|c} 2 & 3 & 1 & 1 \\ 0 & -\frac{1}{2} & \frac{1}{2} & -\frac{163}{2} \\ 1 & 1 & 1 & 10 \end{array}\right] \tag{1.3.45}
$$

Repeating this process,

$$
\begin{aligned}
&\left[\begin{array}{ccc|c} 2 & 3 & 1 & 1 \\ 3 & 4 & 2 & -80 \\ 1 & 1 & 1 & 10 \end{array}\right] \xrightarrow[\substack{\frac{3}{2}\times\text{row 1} \\ \text{subtracted} \\ \text{from row 2}}]{} \left[\begin{array}{ccc|c} 2 & 3 & 1 & 1 \\ 0 & -\frac{1}{2} & \frac{1}{2} & -\frac{163}{2} \\ 1 & 1 & 1 & 10 \end{array}\right] \\
&\xrightarrow[\substack{\frac{1}{2}\times\text{row 1 subtracted} \\ \text{from row 3}}]{} \left[\begin{array}{ccc|c} 2 & 3 & 1 & 1 \\ 0 & -\frac{1}{2} & \frac{1}{2} & -\frac{163}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} & \frac{19}{2} \end{array}\right] \\
&\xrightarrow[\substack{\text{subtract row 2} \\ \text{from row 3}}]{} \left[\begin{array}{ccc|c} 2 & 3 & 1 & 1 \\ 0 & -\frac{1}{2} & \frac{1}{2} & -\frac{163}{2} \\ 0 & 0 & 0 & 91 \end{array}\right]
\end{aligned}
\tag{1.3.46}
$$

The last augmented matrix coincides with the system

$$\begin{aligned} 2x_1 + 3x_2 + x_3 &= 1 \\ -\frac{1}{2}x_2 + \frac{1}{2}x_3 &= -\frac{163}{2} \\ 0x_3 &= 91 \end{aligned} \tag{1.3.47}$$

Of course, the last equation is inconsistent. The only conclusion is that there is no solution to the system (1.3.43). This is the analytical conclusion that is reflective of the graphical solution attempted with Figure 4 of Section 1.2.

Example 1.3.6: All of examples in this section are examples where $M = N = 3$. The assumption $M = N$ was made when we went through the detailed development of the Gaussian Elimination process. The method also works for cases where the number of equations and the number of unknowns are not the same. The following undetermined system is an illustration of this case.

$$\begin{aligned} x_1 + 2x_2 - 4x_3 + 3x_4 + 9x_5 &= 1 \\ 4x_1 + 5x_2 - 10x_3 + 6x_4 + 18x_5 &= 4 \\ 7x_1 + 8x_2 - 16x_3 &= 7 \end{aligned} \tag{1.3.48}$$

As usual, the first step is to form the augmented matrix

$$(A|\mathbf{b}) = \left[\begin{array}{ccccc|c} 1 & 2 & -4 & 3 & 9 & 1 \\ 4 & 5 & -10 & 6 & 18 & 4 \\ 7 & 8 & -16 & 0 & 0 & 7 \end{array}\right] \tag{1.3.49}$$

The sequence of steps that implement the Gaussian Elimination is

$$\begin{aligned} &\left[\begin{array}{ccccc|c} 1 & 2 & -4 & 3 & 9 & 1 \\ 4 & 5 & -10 & 6 & 18 & 4 \\ 7 & 8 & -16 & 0 & 0 & 7 \end{array}\right] \xrightarrow[\text{from row2}]{\text{Subtract } 4\times\text{row1}} \left[\begin{array}{ccccc|c} 1 & 2 & -4 & 3 & 9 & 1 \\ 0 & -3 & 6 & -6 & -18 & 0 \\ 7 & 8 & -16 & 0 & 0 & 7 \end{array}\right] \\ &\xrightarrow[\text{from row3}]{\text{Subtract } 7\times\text{row1}} \left[\begin{array}{ccccc|c} 1 & 2 & -4 & 3 & 9 & 1 \\ 0 & -3 & 6 & -6 & -18 & 0 \\ 0 & -6 & 12 & -21 & -63 & 0 \end{array}\right] \xrightarrow[\text{from row3}]{\text{Subtract } 2\times\text{row2}} \left[\begin{array}{ccccc|c} 1 & 2 & -4 & 3 & 9 & 1 \\ 0 & -3 & 6 & -6 & -18 & 0 \\ 0 & 0 & 0 & -9 & -27 & 0 \end{array}\right] \\ &\xrightarrow[\text{and row 3 by -9}]{\text{Divide row 2 by -3}} \left[\begin{array}{ccccc|c} 1 & 2 & -4 & 3 & 9 & 1 \\ 0 & 1 & -2 & 2 & 6 & 0 \\ 0 & 0 & 0 & 1 & 3 & 0 \end{array}\right] \end{aligned} \tag{1.3.50}$$

The last augmented matrix coincides with the system

$$\begin{aligned} x_1+2x_2-4x_3+3x_4+9x_5&=1\\ x_2-2x_3+2x_4+6x_5&=0\\ x_4+3x_5&=0 \end{aligned} \tag{1.3.51}$$

The back substitution process takes the third equation of the set (1.3.51) and eliminates x_4 or x_5 from the first two. In this case, the result turns out to be

$$\begin{aligned} &x_1=1\\ &x_2-2x_3=0\\ &x_4+3x_5=0 \end{aligned} \tag{1.3.52}$$

The Gaussian Elimination process applied to the augmented matrix produces attempts to produce a triangular form as illustrated with (1.3.28). Example 1.3.2, which involved a system (1.3.11) that had a unique solution produced a final augmented matrix of the form (see equation (1.3.16))

$$\left[\begin{array}{ccc|c} 1 & 3 & 1 & 1\\ 0 & -5 & -1 & 3\\ 0 & 0 & -\frac{3}{5} & -\frac{6}{5} \end{array}\right] \tag{1.3.53}$$

Example 1.3.4, which involved a system (1.3.35) that did not have a unique solution produced a final augmented matrix of the form (see equation (1.3.39))

$$\left[\begin{array}{ccc|c} 2 & 3 & 1 & 1\\ 0 & -\frac{1}{2} & \frac{1}{2} & \frac{5}{2}\\ 0 & 0 & 0 & 0 \end{array}\right] \tag{1.3.54}$$

Example 1.3.5, which involved a system (1.3.43) that did not have a solution produced a final augmented matrix of the form (see equation (1.3.46))

$$\left[\begin{array}{ccc|c} 2 & 3 & 1 & 1\\ 0 & -\frac{1}{2} & \frac{1}{2} & -\frac{163}{2}\\ 0 & 0 & 0 & 91 \end{array}\right] \tag{1.3.55}$$

Our last example, Example 1.3.6, which involved an undetermined system (1.3.51) produced a final augmented matrix of the form (see equation (1.3.50))

$$\left[\begin{array}{ccccc|c} 1 & 2 & -4 & 3 & 9 & 1 \\ 0 & 1 & -2 & 2 & 6 & 0 \\ 0 & 0 & 0 & 1 & 3 & 0 \end{array}\right] \tag{1.3.56}$$

These examples illustrate that we might not reach the triangular form if the equations are inconsistent or if the solution is not unique. The final step in the Gaussian elimination process, regardless of where it ends, is known as the *row echelon form*. This upper triangular matrix can be given a more formal definition as follows:

Definition: A $M \times N$ matrix A is in is in *row echelon form* if

1) Rows with at least one nonzero element are above any rows of all zero.
2) The first nonzero element from the left (the pivot element) of a nonzero row is always strictly to the right of the leading coefficient of the row above it.
3) The leading coefficient of each nonzero row is 1.

The above examples, with minor rearrangement in the first three cases, are row echelon matrices. The minor rearrangement involve insuring 3) is obeyed by simply normalizing the row by division. It should be evident that the results are

1) Example 1.3.2

$$\left[\begin{array}{ccc|c} 1 & 3 & 1 & 1 \\ 0 & 1 & \frac{1}{5} & -\frac{3}{5} \\ 0 & 0 & 1 & 2 \end{array}\right] \tag{1.3.57}$$

2) Example 1.3.4

$$\left[\begin{array}{ccc|c} 1 & \frac{3}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & -1 & -5 \\ 0 & 0 & 0 & 0 \end{array}\right] \tag{1.3.58}$$

3) Example 1.3.5

$$\left[\begin{array}{ccc|c} 1 & \frac{3}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & -1 & 163 \\ 0 & 0 & 0 & 1 \end{array}\right] \tag{1.3.59}$$

4) Example 1.3.6

$$\left[\begin{array}{ccccc|c} 1 & 2 & -4 & 3 & 9 & 1 \\ 0 & 1 & -2 & 2 & 6 & 0 \\ 0 & 0 & 0 & 1 & 3 & 0 \end{array}\right] \tag{1.3.60}$$

The row echelon form is one step away from another upper triangular matrix we shall identify in later sections called a *reduced row echelon form*. These concepts, which are important, will be discussed in the following sections of this chapter.

Exercises

1.3.1 Complete the solution of (1.3.32). The answer is

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\dfrac{499}{23} \\ \dfrac{1}{46} \\ \dfrac{61}{23} \end{bmatrix} = \begin{bmatrix} -5.4239 \\ 0.0217 \\ 2.6522 \end{bmatrix} \tag{1.3.61}$$

1.3.2 Solve the system

$$\begin{aligned} x_2 + x_3 + x_4 &= 0 \\ 3x_1 + 3x_3 - 4x_4 &= 7 \\ x_1 + x_2 + x_3 + 2x_4 &= 6 \\ 2x_1 + 3x_2 + x_3 + 3x_4 &= 6 \end{aligned} \tag{1.3.62}$$

The answer is $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \\ 1 \\ 2 \end{bmatrix}$

1.3.3 Solve the system

$$\begin{aligned} -x_2 - x_3 + x_4 &= 0 \\ x_1 + x_2 + x_3 + x_4 &= 6 \\ 2x_1 + 4x_2 + x_3 - 2x_4 &= -1 \\ 3x_1 + x_2 - 2x_3 + 2x_4 &= 3 \end{aligned} \tag{1.3.63}$$

The answer is $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \\ 2 \end{bmatrix}$

1.3.4 Solve the system

$$\begin{aligned} x_1 + x_2 - x_3 + x_4 + x_5 &= 1 \\ -x_1 - x_2 + x_5 &= -1 \\ -2x_1 - 2x_2 + 3x_5 &= 1 \\ x_3 + x_4 + 3x_5 &= -1 \\ x_1 + x_2 + 2x_3 + 2x_4 + 4x_5 &= 1 \end{aligned} \tag{1.3.64}$$

The answer is that this system is inconsistent and, thus, has no solution.

Section 1.4. Elementary Row Operations, Elementary Matrices

The last section illustrated the Gauss method of elimination for finding the solution to systems of linear equations. The basic method that is implemented with the method is to perform row operations that are designed to build a row echelon matrix at the end of the process. If the system allows it, one builds an upper triangular matrix that allows the solution to be found by back substitution. As summarized at the start of Section 1.3, the row operations are simply creating equivalent systems of linear equations that, at the end of the process, are easier to solve than the original equations. The row operations utilized in the Gaussian Elimination method are

a) switching two rows,
b) multiplying one of the rows by a nonzero constant
c) multiply one row by a nonzero constant and adding it to another row, or
d) combinations of a),b) and c).

The first three of these row operations are call *elementary row operations*. They are the building blocks for the fourth operation. It is useful for theoretical and other purposes to implement the elementary row operations by a matrix multiplication operation utilizing so called *elementary matrices*.

Elementary matrices are square matrices. We shall introduce these matrices in the special case of a system of $M=3$ equations in $N=4$ unknowns. The generalization to different size systems should be evident. The augmented matrix for a 3×4 a system is

$$(A|\mathbf{b}) \equiv \left[\begin{array}{cccc|c} A_{11} & A_{12} & A_{13} & A_{14} & b_1 \\ A_{21} & A_{22} & A_{23} & A_{24} & b_2 \\ A_{31} & A_{32} & A_{33} & A_{34} & b_3 \end{array}\right] \tag{1.4.1}$$

If we wish to implement a row operation, for example, that switches the first and second row, we can form the product

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left[\begin{array}{cccc|c} A_{11} & A_{12} & A_{13} & A_{14} & b_1 \\ A_{21} & A_{22} & A_{23} & A_{24} & b_2 \\ A_{31} & A_{32} & A_{33} & A_{34} & b_3 \end{array}\right] \tag{1.4.2}$$

When the multiplication in (1.4.2) is performed, the result is

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left[\begin{array}{cccc|c} A_{11} & A_{12} & A_{13} & A_{14} & b_1 \\ A_{21} & A_{22} & A_{23} & A_{24} & b_2 \\ A_{31} & A_{32} & A_{33} & A_{34} & b_3 \end{array}\right] = \left[\begin{array}{cccc|c} A_{21} & A_{22} & A_{23} & A_{24} & b_2 \\ A_{11} & A_{12} & A_{13} & A_{14} & b_1 \\ A_{31} & A_{32} & A_{33} & A_{34} & b_3 \end{array}\right] \tag{1.4.3}$$

The result of the multiplication is the original augmented matrix except that its first two rows are switched. The matrix that achieved this row operation, $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, is an example of an elementary matrix. Note that the elementary matrix that switches rows is no more than the identity matrix with its two rows switched. This is a general property of elementary matrices. If we were wished to multiply the second row of $\left(A|\mathbf{b}\right)$ by a constant, say λ, we would multiply it by the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 1 \end{bmatrix}$. If we wish to define an elementary matrix that adds the second row to the and third row of a matrix we would multiply $\left(A|\mathbf{b}\right)$by the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$. The conclusion is that row operations can be implemented by multiplications by elementary matrices.

Example 1.4.1: In Example 1.3.3, we looked at the system (1.3.35), repeated,

$$\begin{aligned} 2x_1 + 3x_2 + x_3 &= 1 \\ x_1 + x_2 + x_3 &= 3 \\ 3x_1 + 4x_2 + 2x_3 &= 4 \end{aligned} \tag{1.4.4}$$

This example was worked with a set of three elementary row operations. These operations can be displayed in terms of elementary matrices by the formula

$$
\left[\begin{array}{ccc|c} 2 & 3 & 1 & 1 \\ 0 & -\frac{1}{2} & \frac{1}{2} & \frac{5}{2} \\ 0 & 0 & 0 & 0 \end{array}\right] = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}}_{\substack{\text{Subtracts row 2} \\ \text{from row 3}}} \left[\begin{array}{ccc|c} 2 & 3 & 1 & 1 \\ 0 & -\frac{1}{2} & \frac{1}{2} & \frac{5}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} & \frac{5}{2} \end{array}\right]
$$

$$
= \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}}_{\substack{\text{Subtracts row 2} \\ \text{from row 3}}} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{3}{2} & 0 & 1 \end{bmatrix}}_{\substack{\frac{3}{2}\times\text{row 1 subtracted} \\ \text{from row 3}}} \left[\begin{array}{ccc|c} 2 & 3 & 1 & 1 \\ 0 & -\frac{1}{2} & \frac{1}{2} & \frac{5}{2} \\ 3 & 4 & 2 & 4 \end{array}\right]
$$

$$
= \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}}_{\substack{\text{Subtracts row 2} \\ \text{from row 3}}} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{3}{2} & 0 & 1 \end{bmatrix}}_{\substack{\frac{3}{2}\times\text{row 1 subtracted} \\ \text{from row 3}}} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\substack{\frac{1}{2}\times\text{row 1} \\ \text{subtracted} \\ \text{from row 2}}} \left[\begin{array}{ccc|c} 2 & 3 & 1 & 1 \\ 1 & 1 & 1 & 3 \\ 3 & 4 & 2 & 4 \end{array}\right] \tag{1.4.5}
$$

The final step in creating the row echelon form from $\left[\begin{array}{ccc|c} 2 & 3 & 1 & 1 \\ 0 & -\frac{1}{2} & \frac{1}{2} & \frac{5}{2} \\ 0 & 0 & 0 & 0 \end{array}\right]$ is to normalize the first row by a division by 2 and the second row by a division by $-\frac{1}{2}$. The result is

$$\left[\begin{array}{ccc|c} 1 & \frac{3}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & -1 & -5 \\ 0 & 0 & 0 & 0 \end{array}\right] = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\substack{\text{Divide row 2} \\ \text{by } -\frac{1}{2}}} \underbrace{\begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\substack{\text{Divide row 1} \\ \text{by 2}}} \times$$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}}_{\substack{\text{Subtracts row 2} \\ \text{from row 3}}} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{3}{2} & 0 & 1 \end{bmatrix}}_{\substack{\frac{3}{2}\times\text{row 1 subtracted} \\ \text{from row 3}}} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\substack{\frac{1}{2}\times\text{row 1} \\ \text{subtracted} \\ \text{from row 2}}} \left[\begin{array}{ccc|c} 2 & 3 & 1 & 1 \\ 1 & 1 & 1 & 3 \\ 3 & 4 & 2 & 4 \end{array}\right] \tag{1.4.6}$$

From a computational standpoint, the method of solution utilized in Section 1.3 is preferred. It achieves the final row echelon form without the necessity of identifying the elementary matrices. However, as indicated above, it is a useful theoretical result that

> A $M \times N$ matrix A can be converted to a $M \times N$ matrix in *row echelon form* by multiplication of A by a finite number of $M \times M$ elementary matrices.

Equation (1.4.6) illustrates this assertion in the particular case where the matrix A is an augmented matrix associated with finding the solution of a system of $M \times N$ equations.

Example 1.4.2: In Example 1.3.6, we looked at the system (1.3.48), repeated,

$$\begin{aligned} x_1 + 2x_2 - 4x_3 + 3x_4 + 9x_5 &= 1 \\ 4x_1 + 5x_2 - 10x_3 + 6x_4 + 18x_5 &= 4 \\ 7x_1 + 8x_2 - 16x_3 &= 7 \end{aligned} \tag{1.4.7}$$

This example was worked with a set of five elementary row operations. These operations can be displayed in terms of elementary matrices by the formula

$$\underbrace{\left[\begin{array}{ccccc|c} 1 & 2 & -4 & 3 & 9 & 1 \\ 0 & 1 & -2 & 2 & 6 & 0 \\ 0 & 0 & 0 & 1 & 3 & 0 \end{array}\right]}_{\text{Row echelon form}} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{9} \end{bmatrix}}_{\text{Divide row 3 by -9}} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{Divide row 2 by -3}} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}}_{\substack{\text{Subtract 2×row2} \\ \text{from row3}}}$$
$$\times \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -7 & 0 & 1 \end{bmatrix}}_{\substack{\text{Subtract 7×row1} \\ \text{from row3}}} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\substack{\text{Subtract 4×row1} \\ \text{from row2}}} \underbrace{\left[\begin{array}{ccccc|c} 1 & 2 & -4 & 3 & 9 & 1 \\ 4 & 5 & -10 & 6 & 18 & 4 \\ 7 & 8 & -16 & 0 & 0 & 7 \end{array}\right]}_{\text{Given augmented matrix}} \tag{1.4.8}$$

Exercises:

1.4.1 Find the row echelon form of the matrix

$$\left(A|b\right) = \left[\begin{array}{cccc|c} 0 & 1 & 1 & 1 & 0 \\ 3 & 0 & 3 & -4 & 7 \\ 1 & 1 & 1 & 2 & 6 \\ 2 & 3 & 1 & 3 & 6 \end{array}\right] \tag{1.4.9}$$

Express the result in terms of elementary matrices.

1.4.2 Find the row echelon form of the matrix

$$\left(A|b\right) = \left[\begin{array}{cccc|c} 0 & -1 & -1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 6 \\ 2 & 4 & 1 & -2 & -1 \\ 3 & 1 & -2 & 2 & 3 \end{array}\right] \tag{1.4.10}$$

Express the result in terms of elementary matrices

1.4.3 Find the row echelon form of the matrix

$$\left(A|b\right) = \left[\begin{array}{ccccc|c} 1 & 1 & -1 & 1 & 1 & 1 \\ -1 & -1 & 0 & 0 & 1 & -1 \\ -2 & -2 & 0 & 0 & 3 & 1 \\ 0 & 0 & 1 & 1 & 3 & -1 \\ 1 & 1 & 2 & 2 & 4 & 1 \end{array}\right] \tag{1.4.11}$$

Express the result in terms of elementary matrices

Section 1.5. Gauss-Jordan Elimination, Reduced Row Echelon Form

In Section 1.3, we introduced the Gaussian Elimination method and identified the row echelon form as the final form of the elimination method that one reaches by the method. In this section, we shall extend the method by what is known as the Gauss-Jordan elimination method and identify the so called *reduced row echelon* form of the augmented matrix.

We shall continue to discuss the problem that led to (1.3.28), namely, a system of $M = N$ equations and N unknowns. Equation (1.3.28), repeated, is

$$\left[\begin{array}{ccccccc|c} A_{11} & A_{12} & A_{13} & \cdot & \cdot & \cdot & A_{1N} & b_1 \\ 0 & A_{22}^{(1)} & A_{23}^{(1)} & \cdot & \cdot & \cdot & A_{2N}^{(1)} & b_2^{(1)} \\ 0 & 0 & A_{33}^{(2)} & \cdot & \cdot & \cdot & A_{2N}^{(2)} & b_3^{(2)} \\ \cdot & & \cdot & \cdot & & & & \cdot \\ \cdot & & \cdot & & \cdot & & & \cdot \\ \cdot & & \cdot & & & A_{N-1,N-1}^{(N-2)} & A_{N-1,N}^{(N-2)} & b_{N-1}^{(N-2)} \\ 0 & 0 & 0 & 0 & \cdot & \cdot & A_{NN}^{(N-1)} & b_N^{(N-1)} \end{array}\right] \tag{1.5.1}$$

In order to reach this result, we have assumed that the pivot process did not yield zeros as the lead element in any row. If this had been the case, the elimination scheme would have not reached the triangular form shown in (1.5.1). The next step in the Gaussian Elimination method is to utilize back substitution to find the solution. The Gauss-Jordan elimination scheme is a refinement of the Gauss elimination method. It avoids back substitution by implementing additional row operations which zero the elements in the upper triangular part of the matrix. This scheme is best illustrated by an example.

Example 1.5.1: In Section 1.3, we worked Example 1.3.2. This example involved finding the solution of

$$\begin{aligned} x_1 + 3x_2 + x_3 &= 1 \\ 2x_1 + x_2 + x_3 &= 5 \\ -2x_1 + 2x_2 - x_3 &= -8 \end{aligned} \tag{1.5.2}$$

The augmented matrix is given by (1.3.14) and the row echelon form, which in this case, was an upper triangular matrix is given by (1.3.57). This row echelon form is

$$\left[\begin{array}{ccc|c} 1 & 3 & 1 & 1 \\ 2 & 1 & 1 & 5 \\ -2 & 2 & -1 & -8 \end{array}\right] \xrightarrow[\text{Row Operations}]{\text{Series of}} \left[\begin{array}{ccc|c} 1 & 3 & 1 & 1 \\ 0 & 1 & \frac{1}{5} & -\frac{3}{5} \\ 0 & 0 & 1 & 2 \end{array}\right] \tag{1.5.3}$$

The Gauss-Jordan process begins with (1.5.3) and proceeds as follows

$$\left[\begin{array}{ccc|c} 1 & 3 & 1 & 1 \\ 0 & 1 & \frac{1}{5} & -\frac{3}{5} \\ 0 & 0 & 1 & 2 \end{array}\right] \xrightarrow[\text{from row 2}]{\text{Subtract } \frac{1}{5}\times\text{row 3}} \left[\begin{array}{ccc|c} 1 & 3 & 1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right] \xrightarrow[\text{from row 1}]{\text{Subtract row 3}} \left[\begin{array}{ccc|c} 1 & 3 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$
$$\xrightarrow[\text{from row 1}]{\text{Subtract } 3\times\text{row 2}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right] \tag{1.5.4}$$

The final matrix in (1.5.4) shows that the solution is

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} \tag{1.5.5}$$

which is the result (1.3.12). The matrix $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right]$ in (1.5.4), as the end result of the Gauss-Jordan elimination process, is the reduced row echelon matrix in this example.

Example 1.5.2: In Section 1.3, we considered Example 1.3.3. The augmented matrix in this example is given by (1.3.34), repeated,

$$(A|\mathbf{b}) = \left[\begin{array}{ccc|c} 4 & 6 & 7 & -3 \\ 2 & -3 & 6 & 5 \\ 0 & 2 & 3 & 8 \end{array}\right] \tag{1.5.6}$$

The Gauss elimination portion of this solution is

$$\left[\begin{array}{ccc|c} 4 & 6 & 7 & -3 \\ 2 & -3 & 6 & 5 \\ 0 & 2 & 3 & 8 \end{array}\right] \xrightarrow[\text{from row 2}]{\text{Subtract } \frac{2}{4}\times\text{row 1}} \left[\begin{array}{ccc|c} 4 & 6 & 7 & -3 \\ 0 & -6 & \frac{5}{2} & \frac{13}{2} \\ 0 & 2 & 3 & 8 \end{array}\right] \xrightarrow[\text{to row 3}]{\text{Add } \frac{2}{6}\times\text{row 2}} \left[\begin{array}{ccc|c} 4 & 6 & 7 & -3 \\ 0 & -6 & \frac{5}{2} & \frac{13}{2} \\ 0 & 0 & \frac{23}{6} & \frac{61}{6} \end{array}\right]$$

$$\xrightarrow[\text{row 3 by } \frac{23}{6}]{\text{Divide row 2 by -6 and}} \left[\begin{array}{ccc|c} 4 & 6 & 7 & -3 \\ 0 & 1 & -\frac{5}{12} & -\frac{13}{12} \\ 0 & 0 & 1 & \frac{61}{23} \end{array}\right] \tag{1.5.7}$$

The Gauss-Jordan portion of the solution picks up from (1.5.7) with the steps

$$\left[\begin{array}{ccc|c} 4 & 6 & 7 & -3 \\ 0 & 1 & -\frac{5}{12} & -\frac{13}{12} \\ 0 & 0 & 1 & \frac{61}{23} \end{array}\right] \xrightarrow[\text{to row 2}]{\text{Add } \frac{5}{12}\times\text{row 3}} \left[\begin{array}{ccc|c} 4 & 6 & 7 & -3 \\ 0 & 1 & 0 & \frac{1}{46} \\ 0 & 0 & 1 & \frac{61}{23} \end{array}\right]$$

$$\xrightarrow[\text{from row 1}]{\text{Subtract } 7\times\text{row 3}} \left[\begin{array}{ccc|c} 4 & 6 & 0 & -\frac{496}{23} \\ 0 & 1 & 0 & \frac{1}{46} \\ 0 & 0 & 1 & \frac{61}{23} \end{array}\right] \xrightarrow[\text{from row 1}]{\text{Subtract } 6\times\text{row 2}} \left[\begin{array}{ccc|c} 4 & 0 & 0 & -\frac{499}{23} \\ 0 & 1 & 0 & \frac{1}{46} \\ 0 & 0 & 1 & \frac{61}{23} \end{array}\right]$$

$$\xrightarrow[\text{by 4}]{\text{Divide row 1}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -\frac{499}{92} \\ 0 & 1 & 0 & \frac{1}{46} \\ 0 & 0 & 1 & \frac{61}{23} \end{array}\right] \tag{1.5.8}$$

Therefore, the reduced row echelon form of the augmented matrix is the last matrix in (1.5.8) and the solution is

$$\mathbf{x}=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} -\dfrac{499}{92} \\ \dfrac{1}{46} \\ \dfrac{61}{23} \end{bmatrix} \tag{1.5.9}$$

The Gauss-Jordan elimination method illustrated above is easily applied to cases of underdetermined systems and overdetermined systems. The following illustrates the method for an underdetermined system.

Example 1.5.3: In Section 1.3, we considered Example 1.3.6. We looked at the same example in Section 1.4 when we worked Example 1.4.2. The end of the Gaussian Elimination process produced the augmented matrix(1.3.60), repeated,

$$\left[\begin{array}{ccccc|c} 1 & 2 & -4 & 3 & 9 & 1 \\ 0 & 1 & -2 & 2 & 6 & 0 \\ 0 & 0 & 0 & 1 & 3 & 0 \end{array}\right] \tag{1.5.10}$$

The special form of (1.5.10) makes the Gauss-Jordan part of the elimination simply from a numerical standpoint. Consider the following steps

$$\begin{gathered}\left[\begin{array}{ccccc|c} 1 & 2 & -4 & 3 & 9 & 1 \\ 0 & 1 & -2 & 2 & 6 & 0 \\ 0 & 0 & 0 & 1 & 3 & 0 \end{array}\right] \xrightarrow[\substack{2\times\text{row 3} \\ \text{subtracted from} \\ \text{row 2}}]{} \left[\begin{array}{ccccc|c} 1 & 2 & -4 & 3 & 9 & 1 \\ 0 & 1 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3 & 0 \end{array}\right] \\ \xrightarrow[\substack{3\times\text{row 3} \\ \text{subtracted from} \\ \text{row 1}}]{} \left[\begin{array}{ccccc|c} 1 & 2 & -4 & 0 & 0 & 1 \\ 0 & 1 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3 & 0 \end{array}\right] \xrightarrow[\substack{2\times\text{row 2} \\ \text{subtracted from} \\ \text{row 1}}]{} \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3 & 0 \end{array}\right]\end{gathered} \tag{1.5.11}$$

The final result, which is in reduced row echelon form, displays the solution (1.3.52), repeated,

$$\begin{aligned} x_1 &= 1 \\ x_2 - 2x_3 &= 0 \\ x_4 + 3x_5 &= 0 \end{aligned} \tag{1.5.12}$$

The *reduced row echelon matrix*, that is determined after completion of the Gauss-Jordan elimination method, is defined formally as follows:

Definition: A $M\times N$ matrix A is in is in *reduced row echelon form* if

1) Rows with at least one nonzero element are above any rows of all zero.

2) The first nonzero element from the left (the pivot element) of a nonzero row is always strictly to the right of the leading coefficient of the row above it.
3) The leading coefficient of each nonzero row is 1 and is the only nonzero entry in its column.

If this definition is compared to that of the row echelon form of a matrix given in Section 1.3, then a reduced row echelon form of a matrix is a row echelon form with the property that the entries above and below the leading coefficient are all zero. In Section 1.4 it was explained how the row echelon form of a matrix can be found by a series of multiplications by elementary matrices. If the additional row operations that implement the Gauss-Jordan elimination are represented by multiplications by row operations, we have the equivalent result

> A $M \times N$ matrix A can be converted to a $M \times N$ matrix in *reduced row echelon form* by multiplication of A by a finite number of $M \times M$ elementary matrices.

Exercises:

1.5.1 Use row operations to find the reduced row echelon form of the matrix

$$\left(A|b\right)=\left[\begin{array}{cccc|c} 0 & 1 & 1 & 1 & 0 \\ 3 & 0 & 3 & -4 & 7 \\ 1 & 1 & 1 & 2 & 6 \\ 2 & 3 & 1 & 3 & 6 \end{array}\right] \tag{1.5.13}$$

This augmented matrix arose earlier in Exercise 1.3.2. The answer is $\begin{bmatrix} 1 & 0 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$.

1.5.2 Use row operations to find the reduced row echelon form of the matrix

$$\left(A|\mathbf{b}\right)=\begin{bmatrix} 1 & 2 & -3 & -4 & 6 \\ 1 & 3 & 1 & -2 & 4 \\ 2 & 5 & -2 & -5 & 10 \end{bmatrix} \tag{1.5.14}$$

The answer is $\begin{bmatrix} 1 & 0 & -11 & 0 & 10 \\ 0 & 1 & 4 & 0 & -2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$.

1.5.3 Use row operations to find the reduced row echelon form of the matrix

$$\left(A|b\right)=\left[\begin{array}{cccc|c}0 & -1 & -1 & 1 & 0\\ 1 & 1 & 1 & 1 & 6\\ 2 & 4 & 1 & -2 & -1\\ 3 & 1 & -2 & 2 & 3\end{array}\right] \tag{1.5.15}$$

This augmented matrix arose in Exercise 1.3.3. The answer is $\begin{bmatrix}1 & 0 & 0 & 0 & 2\\ 0 & 1 & 0 & 0 & -1\\ 0 & 0 & 1 & 0 & 3\\ 0 & 0 & 0 & 1 & 2\end{bmatrix}$.

1.5.4 Use row operations to find the reduced row echelon form of the matrix

$$\left(A|b\right)=\left[\begin{array}{ccccc|c}1 & 1 & -1 & 1 & 1 & 1\\ -1 & -1 & 0 & 0 & 1 & -1\\ -2 & -2 & 0 & 0 & 3 & 1\\ 0 & 0 & 1 & 1 & 3 & -1\\ 1 & 1 & 2 & 2 & 4 & 1\end{array}\right] \tag{1.5.16}$$

This augmented matrix arose in Exercise 1.3.4. The answer is $\begin{bmatrix}1 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 1\end{bmatrix}$.

1.5.5 Find the solution or solutions of the following system of equations

$$\begin{aligned}&2x_1 - x_2 + x_3 = -4\\ &x_2 - 2x_3 = 2\\ &-2x_1 - 2x_2 + 5x_3 = -2\end{aligned} \tag{1.5.17}$$

1.5.6: Find the solution or solutions of the following system of equations

$$\begin{aligned}&x_1 + x_2 + x_3 = 6\\ &2x_1 + 5x_2 - 2x_3 = 6\\ &x_1 + 7x_2 - 7x_3 = -6\end{aligned} \tag{1.5.18}$$

1.5.7: Find the solution or solutions of the following system of equations

$$
\begin{aligned}
x_1 + 2x_2 - 3x_3 - 4x_4 &= 6 \\
x_1 + 3x_2 + x_3 - 2x_4 &= 4 \\
2x_1 + 5x_2 - 2x_3 - 5x_4 &= 10
\end{aligned}
\tag{1.5.19}
$$

Section 1.6. Elementary Matrices-More Properties

In this section we shall look deeper into the idea of an elementary matrix. This concept was introduced in Section 1.4. In that section, we explained that an elementary matrix is a $M \times M$ matrix that when it multiplies a $M \times N$ matrix A will achieve one of the following operations on A:

a) switch two rows,
b) multiply one of the rows by a nonzero constant,
c) multiply one row by a nonzero constant and add it to another row.

The objective of the elementary matrices is to cause row operations which can transform a matrix, first, into its row echelon form and, second, to its reduced row echelon form. In Sections 1.4 and Sections 1.5, this fact was summarized with the statement

> A $M \times N$ matrix A can be converted to a $M \times N$ in row echelon form and its reduced row echelon form by multiplication of A by a finite number of $M \times M$ elementary matrices.

The row echelon form as the result of multiplication by elementary matrices was illustrated with examples in Section 1.4. The additional multiplications by elementary matrices that convert a matrix from its row echelon form to its reduced row echelon form are illustrated by the following example.

Example 1.6.1: Example 1.4.2, which originated from a desire to solve the system of equations,

$$\begin{aligned} x_1 + 2x_2 - 4x_3 + 3x_4 + 9x_5 &= 1 \\ 4x_1 + 5x_2 - 10x_3 + 6x_4 + 18x_5 &= 4 \\ 7x_1 + 8x_2 - 16x_3 &= 7 \end{aligned} \tag{1.6.1}$$

The solution process produced (1.4.8), repeated,

$$\underbrace{\left[\begin{array}{ccccc|c} 1 & 2 & -4 & 3 & 9 & 1 \\ 0 & 1 & -2 & 2 & 6 & 0 \\ 0 & 0 & 0 & 1 & 3 & 0 \end{array}\right]}_{\text{Row echelon form}} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{9} \end{bmatrix}}_{\text{Divide row 3 by -9}} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{Divide row 2 by -3}} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}}_{\substack{\text{Subtract } 2\times\text{row2} \\ \text{from row3}}}$$
$$\times \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -7 & 0 & 1 \end{bmatrix}}_{\substack{\text{Subtract } 7\times\text{row1} \\ \text{from row3}}} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\substack{\text{Subtract } 4\times\text{row1} \\ \text{from row2}}} \underbrace{\left[\begin{array}{ccccc|c} 1 & 2 & -4 & 3 & 9 & 1 \\ 4 & 5 & -10 & 6 & 18 & 4 \\ 7 & 8 & -16 & 0 & 0 & 7 \end{array}\right]}_{\text{Given augmented matrix}} \tag{1.6.2}$$

The problem is to find the elementary matrices that will convert the matrix $\underbrace{\left[\begin{array}{ccccc|c} 1 & 2 & -4 & 3 & 9 & 1 \\ 0 & 1 & -2 & 2 & 6 & 0 \\ 0 & 0 & 0 & 1 & 3 & 0 \end{array}\right]}_{\text{Row echelon form}}$ into is reduced row echelon form. As explained in Section 1.4, the elementary matrices are derived from the identity matrix by applying the desired row operation to the identity matrix. The three row operations that achieve this step are shown in equation (1.5.11). Therefore the elementary matrices that achieve these steps are as follows:

1) $2\times$ row 3 subtracted from row 2

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[\text{from row 2}]{2\times\text{row 3 subtracted}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \tag{1.6.3}$$

2) $3\times$ row 3 subtracted from row 1

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[\text{from row 1}]{3\times\text{row 3 subtracted}} \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1.6.4}$$

3) $2\times$ row 2 subtracted from row 1

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[\text{from row 1}]{2\times\text{row 2 subtracted}} \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1.6.5}$$

Therefore, the reduced row echelon form is given by

$$
\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3 & 0 \end{array}\right] = \underbrace{\underbrace{\begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\substack{2\times\text{row 2 subtracted}\\ \text{from row 1}}} \underbrace{\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\substack{3\times\text{row 3 subtracted}\\ \text{from row 1}}} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 1 & 1 \end{bmatrix}}_{\substack{2\times\text{row 3 subtracted}\\ \text{from row 2}}}}_{\substack{\text{Calculates reduced row echelon form}\\ \text{from row echelon form}}} \times
$$

$$
\underbrace{\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{9} \end{bmatrix}}_{\text{Divide row 3 by -9}} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{Divide row 2 by -3}} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}}_{\substack{\text{Subtract } 2\times\text{row2}\\ \text{from row3}}} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -7 & 0 & 1 \end{bmatrix}}_{\substack{\text{Subtract } 7\times\text{row1}\\ \text{from row3}}} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\substack{\text{Subtract } 4\times\text{row1}\\ \text{from row2}}} \underbrace{\left[\begin{array}{ccccc|c} 1 & 2 & -4 & 3 & 9 & 1 \\ 4 & 5 & -10 & 6 & 18 & 4 \\ 7 & 8 & -16 & 0 & 0 & 7 \end{array}\right]}_{\text{Given augmented matrix}}}_{\text{Calculates row echelon form}} \tag{1.6.6}
$$

It is a property of elementary matrices that they are nonsingular. We shall illustrate this assertion by consideration of three examples.

Example 1.6.2: If you are given the elementary matrix that corresponds to switching the first and second row, i.e.,

$$
E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1.6.7}
$$

Then the inverse of E_1 must obey

$$
E_1^{-1} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1.6.8}
$$

The matrix

$$
E_1^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1.6.9}
$$

can be substituted into (1.6.8) and verify that it is the inverse. Just as (1.6.7) corresponds to switching the first and second row, its inverse, (1.6.8), corresponds to switching them again to return to the original matrix.

Example 1.6.3: If you are given the elementary matrix that corresponds to multiplying the second row by a nonzero constant, i.e.,

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1.6.10}$$

Then the inverse of E_2 must obey

$$E_2^{-1}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 1 \end{bmatrix} E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1.6.11}$$

The matrix

$$E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \dfrac{1}{\lambda} & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1.6.12}$$

can be substituted into (1.6.11) and verify that it is the inverse. Just as (1.6.10) corresponds to multiplying the second row by the nonzero constant λ, its inverse, (1.6.12), corresponds to dividing the second row by the nonzero constant λ.

Example 1.6.4: If you are given the elementary matrix that corresponds to multiplying the third row by a constant and adding the result to the first, i.e.,

$$E_3 = \begin{bmatrix} 1 & 0 & \lambda \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1.6.13}$$

Then the inverse of E_2 must obey

$$E_3^{-1}\begin{bmatrix} 1 & 0 & \lambda \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & \lambda \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1.6.14}$$

The matrix

$$E_3^{-1} = \begin{bmatrix} 1 & 0 & -\lambda \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1.6.15}$$

can be substituted into (1.6.14) and verify that it is the inverse. Just as (1.6.13) corresponds to multiplying the third row by a nonzero constant and adding the result to the first, (1.6.15) corresponds to multiplying the third row by a nonzero constant and subtracting the result from the first row.

In summary, we have the following *two facts* about elementary matrices:

1) An elementary matrix is a $M \times M$ matrix obtained from the $M \times M$ identity matrix by an elementary row operation.
2) If E is an elementary matrix, then E is nonsingular and, its inverse, E^{-1} is an elementary matrix of the same type.

Definition: A $M \times N$ matrix B is *row equivalent* to a $M \times N$ matrix A if there exist a finite number of elementary matrices $E_1, E_2, ..., E_k$ such that

$$B = E_k \cdots E_2 E_1 A \tag{1.6.16}$$

This definition is an adoption of the idea of equivalence to matrices which we introduced in Section 1.3 for systems of equations. It is possible to use the definition (1.6.16) to establish the following two important properties of row equivalence:

1) If B is equivalent to A, then A is equivalent to B.
2) If B is equivalent to A and A equivalent to C, then B is equivalent to C.

The proof of the first property follows directly from the definition (1.6.16). The details are as follows. Because each elementary matrix is nonsingular, we can repeatedly use the identity (1.1.36) and establish from (1.6.16) that

$$\begin{aligned} A &= \left(E_k \cdots E_2 E_1\right)^{-1} B \\ &= \left(E_{k-1} \cdots E_2 E_1\right)^{-1} E_k^{-1} B \\ &= \left(E_{k-2} \cdots E_2 E_1\right)^{-1} E_{k-1}^{-1} E_k^{-1} B \\ &\cdot \\ &\cdot \\ &\cdot \\ &= E_1^{-1} E_2^{-1} \cdots E_k^{-1} B \end{aligned} \tag{1.6.17}$$

Since the inverse elementary matrices are themselves elementary matrices, this result establishes property 1). The proof of the second property follows from a similar line of reasoning.

If we are given that A is a square matrix, we can identify three properties of A that relate to whether or not it is nonsingular.

Theorem 1.6.1: The following three conditions are equivalent for a square matrix A.

a) A is nonsingular.
b) The equation $A\mathbf{x}=\mathbf{0}$ only has the solution $\mathbf{x}=\mathbf{0}$.
c) A is row equivalent to the identity matrix I.

Proof: The proof requires that we accept any one of the propositions as true and, from that proposition, prove that the other two are also true. We begin by accepting a) and showing that a) implies b).

a)$\Rightarrow$b): If A is nonsingular, the equation $A\mathbf{x}=\mathbf{0}$ can be multiplied by its inverse to obtain $A^{-1}A\mathbf{x}=I\mathbf{x}=\mathbf{x}=\mathbf{0}$. Thus b) is established.

Given b), we shall next show that it implies c).

b)$\Rightarrow$c): We are given that the system $A\mathbf{x}=\mathbf{0}$ only has the solution $\mathbf{x}=\mathbf{0}$. Let $E_1,E_2,...,E_k$ be elementary matrices selected such that

$$U=E_k\cdots E_2E_1A \tag{1.6.18}$$

is in reduced row echelon form. It follows from $A\mathbf{x}=\mathbf{0}$ and (1.6.18) that

$$U\mathbf{x}=E_k\cdots E_2E_1A\mathbf{x}=\mathbf{0}$$

Because $\mathbf{x}=\mathbf{0}$ is the only solution allowed by the equation $U\mathbf{x}=\mathbf{0}$, the matrix U cannot have a zero on its diagonal. If it did, for example, in the NN position, this would allow $x_N\neq 0$ which would violate the condition that $\mathbf{x}=\mathbf{0}$. Because the reduced row echelon form of the matrix which has nonzero diagonal elements has the identity I for its reduced row echelon form, the result is established.

Given c), we shall next show that it implies a).

c) $\Rightarrow$a): We are given that A is row equivalent to the identity matrix I. Therefore, there exists elementary matrices $E_1,E_2,...,E_k$ such that

$$I=E_k\cdots E_2E_1A \tag{1.6.19}$$

Because each elementary matrix is nonsingular, this last result implies that the matrix A is given by

$$A = \left(E_k \cdots E_2 E_1\right)^{-1} = E_1^{-1} E_2^{-1} \cdots E_k^{-1} \tag{1.6.20}$$

where the identity (1.1.36) has been used. Given (1.6.20), it follows that A^{-1} is given by

$$\begin{aligned} A^{-1} &= \left(E_1^{-1} E_2^{-1} \cdots E_k^{-1}\right)^{-1} \\ &= \left(E_k^{-1}\right)^{-1} \cdots \left(E_2^{-1}\right)^{-1} \left(E_1^{-1}\right)^{-1} \\ &= E_k \cdots E_2 E_1 \end{aligned} \tag{1.6.21}$$

where we have again used the identity (1.1.36). Equation (1.6.21) establishes that A is nonsingular.

It is a corollary to the last result that *the system of N equations with N unknowns $A\mathbf{x} = \mathbf{b}$ has a unique solution if and only if A* is *nonsingular*. The argument to prove this corollary is as follows: First, assume the square matrix A is nonsingular, it then follows from $A\mathbf{x} = \mathbf{b}$ that the solution exists and is uniquely given by $\mathbf{x} = A^{-1}\mathbf{b}$. Conversely, we need to prove that if only unique solutions exist, the square matrix A must be nonsingular. The proof of this part of the corollary, like virtually all uniqueness proofs, begins with the assumption that the solution is not unique and then establishes condition that will force uniqueness. We begin with the assumption that there exist two solutions, $\mathbf{x}_1$ and $\mathbf{x}_2$, that obey

$$A\mathbf{x}_1 = \mathbf{b} \tag{1.6.22}$$

and

$$A\mathbf{x}_2 = \mathbf{b} \tag{1.6.23}$$

Given (1.6.22) and (1.6.23), it is true that

$$A\left(\mathbf{x}_1 - \mathbf{x}_2\right) = A\mathbf{x}_1 - A\mathbf{x}_2 = \mathbf{b} - \mathbf{b} = \mathbf{0} \tag{1.6.24}$$

Equation (1.6.24) and part b) of Theorem 1.6.1 tell us that $\mathbf{x}_1 - \mathbf{x}_2 = \mathbf{0}$ if and only if A is nonsingular,

Theorem 1.6.1 tells us that in those cases where A is nonsingular we can construct the inverse A^{-1} by finding the elementary matrices $E_1, E_2, ..., E_k$ which satisfy (1.6.19). When these elementary matrices are known, we can calculate A^{-1} from (1.6.21).

Example 1.6.5: As an example, consider the matrix

$$A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 1 & 1 \\ -2 & 2 & -1 \end{bmatrix} \tag{1.6.25}$$

which is the coefficient matrix in Example 1.3.2. This is also the matrix utilized in Example 1.5.1. Consider the following sequence of elementary matrices:

Step 1:

$$E_1 A = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\substack{2\times\text{row 1} \\ \text{subtracted} \\ \text{from row 2}}} \begin{bmatrix} 1 & 3 & 1 \\ 2 & 1 & 1 \\ -2 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 1 \\ 0 & -5 & -1 \\ -2 & 2 & -1 \end{bmatrix} \tag{1.6.26}$$

Step 2:

$$E_2 (E_1 A) = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}}_{\substack{2\times\text{row 1 added} \\ \text{to row 3}}} \begin{bmatrix} 1 & 3 & 1 \\ 0 & -5 & -1 \\ -2 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 1 \\ 0 & -5 & -1 \\ 0 & 8 & 1 \end{bmatrix} \tag{1.6.27}$$

Step 3:

$$E_3 (E_2 E_1 A) = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{8}{5} & 1 \end{bmatrix}}_{\substack{\frac{8}{5}\times\text{row 2} \\ \text{added} \\ \text{to row 3}}} \begin{bmatrix} 1 & 3 & 1 \\ 0 & -5 & -1 \\ 0 & 8 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 1 \\ 0 & -5 & -1 \\ 0 & 0 & -\frac{3}{5} \end{bmatrix} \tag{1.6.28}$$

Step 4

$$E_4\left(E_3E_2E_1A\right)=\underbrace{\begin{bmatrix}1&0&0\\0&1&0\\0&0&-\frac{5}{3}\end{bmatrix}}_{\substack{\text{multiple row}\\ 3\text{ by }-\frac{5}{3}}}\begin{bmatrix}1&3&1\\0&-5&-1\\0&0&-\frac{3}{5}\end{bmatrix}=\begin{bmatrix}1&3&1\\0&-5&-1\\0&0&1\end{bmatrix} \tag{1.6.29}$$

Step 5

$$E_5\left(E_4E_3E_2E_1A\right)=\underbrace{\begin{bmatrix}1&0&0\\0&1&1\\0&0&1\end{bmatrix}}_{\substack{\text{Add row 3}\\ \text{to row 2}}}\begin{bmatrix}1&3&1\\0&-5&-1\\0&0&1\end{bmatrix}=\begin{bmatrix}1&3&1\\0&-5&0\\0&0&1\end{bmatrix} \tag{1.6.30}$$

Step 6

$$E_6\left(E_5E_4E_3E_2E_1A\right)=\underbrace{\begin{bmatrix}1&0&-1\\0&1&0\\0&0&1\end{bmatrix}}_{\substack{\text{Subtract row 3}\\ \text{from row 1}}}\begin{bmatrix}1&3&1\\0&-5&0\\0&0&1\end{bmatrix}=\begin{bmatrix}1&3&0\\0&-5&0\\0&0&1\end{bmatrix} \tag{1.6.31}$$

Step 7

$$E_7\left(E_6E_5E_4E_3E_2E_1A\right)=\underbrace{\begin{bmatrix}1&0&0\\0&-\frac{1}{5}&0\\0&0&1\end{bmatrix}}_{\substack{\text{Divide row 5}\\ \text{by -5}}}\begin{bmatrix}1&3&0\\0&-5&0\\0&0&1\end{bmatrix}=\begin{bmatrix}1&3&0\\0&1&0\\0&0&1\end{bmatrix} \tag{1.6.32}$$

Step 8

$$E_8\left(E_7E_6E_5E_4E_3E_2E_1A\right)=\underbrace{\begin{bmatrix}1&-3&0\\0&1&0\\0&0&1\end{bmatrix}}_{\substack{\text{Subtract }3\times\text{row 2}\\ \text{from row 1}}}\begin{bmatrix}1&3&0\\0&1&0\\0&0&1\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=I \tag{1.6.33}$$

Given equation (1.6.33), it follows from (1.6.21) that

$$
\begin{aligned}
A^{-1} &= E_8E_7E_6E_5E_4E_3E_2E_1 \\
&= \underbrace{\begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E_8}
\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E_7}
\underbrace{\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E_6}
\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}}_{E_5} \times \\
&\quad \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{5}{3} \end{bmatrix}}_{E_4}
\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{8}{5} & 1 \end{bmatrix}}_{E_3}
\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}}_{E_2}
\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E_1}
\end{aligned}
\tag{1.6.34}
$$

If the multiplications shown in (1.6.34) are carried out, the result is

$$
A^{-1} = \begin{bmatrix} -1 & \frac{5}{3} & \frac{2}{3} \\ 0 & \frac{1}{3} & \frac{1}{3} \\ 2 & -\frac{8}{3} & -\frac{5}{3} \end{bmatrix}
\tag{1.6.35}
$$

Example 1.6.6: As an additional example, consider the following matrix A and the following sequence of elementary matrices:

$$
A = \begin{bmatrix} 1 & 4 & 3 \\ -1 & -2 & 0 \\ 2 & 2 & 3 \end{bmatrix}
\tag{1.6.36}
$$

Step 1:

$$
E_1A = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\substack{\text{Add row 1} \\ \text{to row two}}}
\begin{bmatrix} 1 & 4 & 3 \\ -1 & -2 & 0 \\ 2 & 2 & 3 \end{bmatrix}
= \begin{bmatrix} 1 & 4 & 3 \\ 0 & 2 & 3 \\ 2 & 2 & 3 \end{bmatrix}
\tag{1.6.37}
$$

Step 2:

$$E_2\left(E_1A\right)=\underbrace{\begin{bmatrix}1&0&0\\0&1&0\\-2&0&1\end{bmatrix}}_{\substack{\text{Multiply row 1}\\\text{by -2 and add}\\\text{to row 3}}}\begin{bmatrix}1&4&3\\0&2&3\\2&2&3\end{bmatrix}=\begin{bmatrix}1&4&3\\0&2&3\\0&-6&-3\end{bmatrix}\tag{1.6.38}$$

Step 3

$$E_3\left(E_2E_1A\right)=\underbrace{\begin{bmatrix}1&0&0\\0&\frac{1}{2}&0\\0&0&1\end{bmatrix}}_{\text{Multiply row 2 by }1/2}\begin{bmatrix}1&4&3\\0&2&3\\0&-6&-3\end{bmatrix}=\begin{bmatrix}1&4&3\\0&1&\frac{3}{2}\\0&-6&-3\end{bmatrix}\tag{1.6.39}$$

Step 4

$$E_4\left(E_3E_2E_1A\right)=\underbrace{\begin{bmatrix}1&0&0\\0&1&0\\0&0&-\frac{1}{6}\end{bmatrix}}_{\substack{\text{Multiply row 3}\\\text{by -}\frac{1}{6}}}\begin{bmatrix}1&4&3\\0&1&\frac{3}{2}\\0&-6&-3\end{bmatrix}=\begin{bmatrix}1&4&3\\0&1&\frac{3}{2}\\0&1&\frac{1}{2}\end{bmatrix}\tag{1.6.40}$$

Step 5:

$$E_5\left(E_4E_3E_2E_1A\right)=\underbrace{\begin{bmatrix}1&0&0\\0&1&0\\0&-1&1\end{bmatrix}}_{\substack{\text{Subtract row 2}\\\text{from row 3}}}\begin{bmatrix}1&4&3\\0&1&\frac{3}{2}\\0&1&\frac{1}{2}\end{bmatrix}=\begin{bmatrix}1&4&3\\0&1&\frac{3}{2}\\0&0&-1\end{bmatrix}\tag{1.6.41}$$

Step 6:

$$E_6\left(E_5E_4E_3E_2E_1A\right)=\underbrace{\begin{bmatrix}1&0&0\\0&1&0\\0&0&-1\end{bmatrix}}_{\text{Multiply row 3 by -1}}\begin{bmatrix}1&4&3\\0&1&\frac{3}{2}\\0&0&-1\end{bmatrix}=\begin{bmatrix}1&4&3\\0&1&\frac{3}{2}\\0&0&1\end{bmatrix}\tag{1.6.42}$$

Step 7

$$E_7\left(E_6E_5E_4E_3E_2E_1A\right)=\underbrace{\begin{bmatrix}1&-4&0\\0&1&0\\0&0&1\end{bmatrix}}_{\substack{\text{Multiply row 2 by 4}\\\text{and subtract from}\\\text{row 1}}}\begin{bmatrix}1&4&3\\0&1&\frac{3}{2}\\0&0&1\end{bmatrix}=\begin{bmatrix}1&0&-3\\0&1&\frac{3}{2}\\0&0&1\end{bmatrix} \tag{1.6.43}$$

Step 8:

$$E_8\left(E_7E_6E_5E_4E_3E_2E_1A\right)=\underbrace{\begin{bmatrix}1&0&0\\0&1&-\frac{3}{2}\\0&0&1\end{bmatrix}}_{\substack{\text{Multiply row 3}\\\text{by }-3/2\text{ and}\\\text{add to row 2}}}\begin{bmatrix}1&0&-3\\0&1&\frac{3}{2}\\0&0&1\end{bmatrix}=\begin{bmatrix}1&0&-3\\0&1&0\\0&0&1\end{bmatrix} \tag{1.6.44}$$

Step 9:

$$E_9\left(E_8E_7E_6E_5E_4E_3E_2E_1A\right)=\underbrace{\begin{bmatrix}1&0&3\\0&1&0\\0&0&1\end{bmatrix}}_{\substack{\text{Multiply row 3}\\\text{by 3 and}\\\text{add to row 1}}}\begin{bmatrix}1&0&-3\\0&1&0\\0&0&1\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=I \tag{1.6.45}$$

Therefore,

$$\begin{aligned}A^{-1}&=E_9E_8E_7E_6E_5E_4E_3E_2E_1\\&=\underbrace{\begin{bmatrix}1&0&3\\0&1&0\\0&0&1\end{bmatrix}}_{E_9}\underbrace{\begin{bmatrix}1&0&0\\0&1&-\frac{3}{2}\\0&0&1\end{bmatrix}}_{E_8}\underbrace{\begin{bmatrix}1&-4&0\\0&1&0\\0&0&1\end{bmatrix}}_{E_7}\underbrace{\begin{bmatrix}1&0&0\\0&1&0\\0&0&-1\end{bmatrix}}_{E_6}\times\\&\quad\underbrace{\begin{bmatrix}1&0&0\\0&1&0\\0&-1&0\end{bmatrix}}_{E_5}\underbrace{\begin{bmatrix}1&0&0\\0&1&0\\0&0&-\frac{1}{6}\end{bmatrix}}_{E_4}\underbrace{\begin{bmatrix}1&0&0\\0&\frac{1}{2}&0\\0&0&1\end{bmatrix}}_{E_3}\underbrace{\begin{bmatrix}1&0&0\\0&1&0\\-2&0&1\end{bmatrix}}_{E_2}\underbrace{\begin{bmatrix}1&0&0\\1&1&0\\0&0&1\end{bmatrix}}_{E_1}\end{aligned} \tag{1.6.46}$$

If this multiplication is performed, the following result is obtained for A^{-1}

$$A^{-1}=\begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{6} \end{bmatrix} \tag{1.6.47}$$

The above two examples illustrates how the elementary matrices generate the inverse for a nonsingular matrix. These examples are illustrations of the theoretical formula (1.6.21). As a practical matter, we are typically only interested in the inverse and not the recording of the individual elementary matrices. A computational algorithm based upon the augmented matrix approach gives the answer more directly. We shall illustrate this algorithm for the matrix in the second example above, i.e. the matrix defined by (1.6.36). The procedure is as follows:

First, form the augmented matrix $\left(A|I\right)$:

$$\left(A|I\right)=\left[\underbrace{\begin{matrix} 1 & 4 & 3 \\ -1 & -2 & 0 \\ 2 & 2 & 3 \end{matrix}}_{A}\left|\underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{I}\right.\right] \tag{1.6.48}$$

We next perform row operations on this auxiliary matrix until an identity appears in the left slot. The step by step process is the following:

Step 1(Add row 1 to row 2.)

$$\left[\underbrace{\begin{matrix} 1 & 4 & 3 \\ 0 & 2 & 3 \\ 2 & 2 & 3 \end{matrix}}_{E_1A}\left|\underbrace{\begin{matrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{E_1}\right.\right] \tag{1.6.49}$$

Step 2:(Multiply row 1 by -2 and add to row 3.)

$$\left[\underbrace{\begin{matrix} 1 & 4 & 3 \\ 0 & 2 & 3 \\ 0 & -6 & -3 \end{matrix}}_{E_2E_1A}\left|\underbrace{\begin{matrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -2 & 0 & 1 \end{matrix}}_{E_2E_1}\right.\right] \tag{1.6.50}$$

Step 3(Multiply row 2 by 1/2.)

$$\left[\begin{array}{c|c} \underbrace{\begin{matrix} 1 & 4 & 3 \\ 0 & 1 & \frac{3}{2} \\ 0 & -6 & -3 \end{matrix}}_{E_3E_2E_1A} & \underbrace{\begin{matrix} 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ -2 & 0 & 1 \end{matrix}}_{E_3E_2E_1} \end{array}\right] \tag{1.6.51}$$

Step 4(Multiply row 3 by – 1/6.)

$$\left[\begin{array}{c|c} \underbrace{\begin{matrix} 1 & 4 & 3 \\ 0 & 1 & \frac{3}{2} \\ 0 & 1 & \frac{1}{2} \end{matrix}}_{E_4E_3E_2E_1A} & \underbrace{\begin{matrix} 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{3} & 0 & -\frac{1}{6} \end{matrix}}_{E_4E_3E_2E_1} \end{array}\right] \tag{1.6.52}$$

Step 5:(Subtract row 2 from row 3)

$$\left[\begin{array}{c|c} \underbrace{\begin{matrix} 1 & 0 & -3 \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & -1 \end{matrix}}_{E_5E_4E_3E_2E_1A} & \underbrace{\begin{matrix} 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{6} & -\frac{1}{2} & -\frac{1}{6} \end{matrix}}_{E_5E_4E_3E_2E_1} \end{array}\right] \tag{1.6.53}$$

Step 6:(Multiply row 3 by -1.)

$$\left[\begin{array}{c|c} \underbrace{\begin{matrix} 1 & 4 & 3 \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & 1 \end{matrix}}_{E_6E_5E_4E_3E_2E_1A} & \underbrace{\begin{matrix} 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{6} \end{matrix}}_{E_6E_5E_4E_3E_2E_1} \end{array}\right] \tag{1.6.54}$$

Step 7:(Multiply row 2 by 4 and subtract from row 1.)

$$\left[\underbrace{\begin{matrix}1 & 0 & -3\\ 0 & 1 & \frac{3}{2}\\ 0 & 0 & 1\end{matrix}}_{E_7E_6E_5E_4E_3E_2E_1A}\left|\underbrace{\begin{matrix}-1 & 0 & 0\\ \frac{1}{2} & \frac{1}{2} & 0\\ \frac{1}{6} & \frac{1}{2} & \frac{1}{6}\end{matrix}}_{E_7E_6E_5E_4E_3E_2E_1}\right.\right] \tag{1.6.55}$$

Step 8:(Multiply row 3 by – 3/2 and add to row 2.)

$$\left[\underbrace{\begin{matrix}1 & 0 & -3\\ 0 & 1 & 0\\ 0 & 0 & 1\end{matrix}}_{E_8E_7E_6E_5E_4E_3E_2E_1A}\left|\underbrace{\begin{matrix}-1 & -2 & 0\\ \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4}\\ \frac{1}{6} & \frac{1}{2} & \frac{1}{6}\end{matrix}}_{E_8E_7E_6E_5E_4E_3E_2E_1}\right.\right] \tag{1.6.56}$$

Step 9:(Multiply row 3 by 3 and add to row 1.)

$$\left[\underbrace{\begin{matrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{matrix}}_{I=E_9E_8E_7E_6E_5E_4E_3E_2E_1A}\left|\underbrace{\begin{matrix}-\frac{1}{2} & -\frac{1}{2} & \frac{1}{2}\\ \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4}\\ \frac{1}{6} & \frac{1}{2} & \frac{1}{6}\end{matrix}}_{A^{-1}=E_9E_7E_6E_5E_4E_3E_2E_1}\right.\right] \tag{1.6.57}$$

If the matrix A is singular, the above calculation process will not reduce to the identity matrix in the left slot. In the singular case, of course, the formula (1.6.21), i.e., $A^{-1} = E_k \cdots E_2E_1$, is not valid.

Exercises:

1.6.1 You are given the matrix

$$A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix} \tag{1.6.58}$$

and its inverse $A^{-1} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$. Express A^{-1} as the product of a finite number of elementary matrices.

1.6.2 Use the computational algorithm illustrated above to determine the inverse of

$$A = \begin{bmatrix} 2 & 1 & 0 \\ -2 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix} \tag{1.6.59}$$

1.6.3 Use the computational algorithm illustrated above to determine the inverse of

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 3 & 0 & 3 & -4 \\ 1 & 1 & 1 & 2 \\ 2 & 3 & 1 & 3 \end{bmatrix} \tag{1.6.60}$$

The answer is

$$A^{-1} = -\frac{1}{20}\begin{bmatrix} 17 & -2 & -8 & -3 \\ -10 & 0 & 20 & -10 \\ -13 & -2 & -8 & 7 \\ 3 & 2 & -12 & 3 \end{bmatrix} \tag{1.6.61}$$

Section 1.7. LU Decomposition

At this point in our understanding of Matrix Algebra, we have two closely related approaches for solving systems of linear algebraic equations. One is based upon *Gaussian Elimination*, and the other method is based upon *Gauss-Jordan Elimination*. There is another class of solution methods based upon *decompositions* of the matrix A. Decomposition is the ability to start with an $M \times N$ matrix A, and derive from A two or more matrices which allow A to be decomposed into the product

$$A = A_1 A_2 \cdots A_k \tag{1.7.1}$$

Decomposition methods are useful because the factors in the decomposition have properties which make the subsequent solution of

$$A\mathbf{x} = \mathbf{b} \tag{1.7.2}$$

easier. One such method is called the *LU Decomposition*. For our purposes, there are two kinds of LU decompositions. For our purposes here, we shall call the first kind the *elementary LU decomposition*. After we discuss this calculation, we shall discuss the second kind, which we shall call the generalized *LU decomposition.*

The Elementary LU Decomposition

The question is under what circumstance can an $M \times N$ matrix A, can be decomposed in the form

$$A = LU \tag{1.7.3}$$

where

$U =$ An upper triangular $M \times N$ matrix

and

$L =$ A lower triangular nonsingular $M \times M$ matrix with 1s down the diagonal.[3]

While not sufficient, it is certainly going to be necessary for the number of unknown elements of L and U to be equal to the given number of elements of A, namely MN. It is helpful to do a little counting of these various elements in order to characterize the cases where this necessary condition is obeyed. For the square $M \times M$ matrix L:

---

[3] In some discussions of the LU decomposition, the diagonal elements of L are not required to be unity. It can be shown that this requirement insures that when the decomposition exists, it is unique.

$$\text{Number of Unknown Elements of } L = \frac{1}{2}M\left(M-1\right)$$

For the $M \times N$ matrix U, there are two cases

$$\text{Number of Unknown Elements of } U = \begin{cases} \frac{1}{2}M\left(M+1\right)+\left(N-M\right)M & \text{for} \quad N \geq M \\ \frac{1}{2}N\left(N+1\right) & \text{for} \quad N < M \end{cases}$$

The total number of unknown elements in L and U is, therefore,

$$\text{Total Number of Unknown Elements}=\begin{cases} \frac{1}{2}M\left(M-1\right)+\frac{1}{2}M\left(M+1\right)+\left(N-M\right)M & \text{for} \quad N \geq M \\ \frac{1}{2}M\left(M-1\right)+\frac{1}{2}N\left(N+1\right) & \text{for} \quad N < M \end{cases}$$

These totals can be rearranged to yield

$$\text{Total Number of Unknown Elements}=\begin{cases} MN & \text{for} \quad N \geq M \\ MN+\frac{1}{2}\left(M-\left(N+1\right)\right)\left(M-N\right) & \text{for} \quad N < M \end{cases}$$

In the case $N \geq M$, the necessary condition is clearly satisfied. In the case $N < M$ it is generally not satisfied *unless* $M = N+1$. These two pieces of information mean that we shall base our following discussion on the cases where N, the number of columns of A, obeys $N \geq M-1$. In other words, the number of columns can be one less than the number of rows or it can be equal to or greater than the number of rows. Most, but not all, of our examples will be for *square matrices*. We shall briefly discuss later in this section the issues that arise when $N < M-1$.

Example 1.7.1: If A is the matrix we have used before

$$A = \begin{bmatrix} 2 & 4 & 2 \\ 1 & 5 & 2 \\ 4 & -1 & 9 \end{bmatrix} \tag{1.7.4}$$

then the following is an elementary LU Decomposition

$$\underbrace{\begin{bmatrix} 2 & 4 & 2 \\ 1 & 5 & 2 \\ 4 & -1 & 9 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 2 & -3 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 2 & 4 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & 8 \end{bmatrix}}_{U} \tag{1.7.5}$$

Later in this section, we shall develop the computational procedure that constructs the decomposition (1.7.3).

The example (1.7.5) shows that elementary LU Decompositions do exist. However, one of the many questions about LU decompositions is whether or not they always exist. If it does not always exist for every $M \times N$ matrix with $N \geq M-1$, then can we characterize those situations when it does? The answer is that it does not always exist. It is this fact that will cause us to look at the generalized LU decomposition mentioned earlier. Returning to the elementary LU decomposition, it turns out that the factorization (1.7.3) always exists if the following is true:

> If the $M \times N$ matrix A, with $N \geq M-1$, can be reduced to upper triangular form *without* using partial pivoting (i.e. row switching) then A has an elementary LU decomposition.

We shall give an example below where the elementary LU decomposition does not exist. For the moment, we shall proceed and see what problems arises as we attempt the construction.

It is useful to note at this point one of the reasons the LU decomposition is useful. If we have the decomposition, (1.7.3), then the system of linear equations $A\mathbf{x} = \mathbf{b}$ can be written

$$LU\mathbf{x} = \mathbf{b} \tag{1.7.6}$$

Because L, is nonsingular, we can multiply on the left by L^{-1} and obtain

$$L^{-1}LU\mathbf{x} = IU\mathbf{x} = U\mathbf{x} = L^{-1}\mathbf{b} \tag{1.7.7}$$

Thus, our problem is reduced to solving

$$U\mathbf{x} = L^{-1}\mathbf{b} \tag{1.7.8}$$

Because U is an upper triangular matrix, (1.7.8) can be solved, for example, by back substitution or Gauss-Jordan elimination. Another benefit of the LU Decomposition is that it depends only on the properties of A. In other words, the decomposition does not depend upon $\mathbf{b}$. This means that we can perform the decomposition and then solve for $\mathbf{x}$ for a variety of choices of $\mathbf{b}$. The methods we have used to date, Gaussian Elimination and Gauss-Jordan Elimination, involved manipulations of the augmented matrix, $\left(A|\mathbf{b}\right)$, and, as a consequence, the intermediate calculations depended upon the specific $\mathbf{b}$.

The source of the method to create the elementary LU Decomposition is actually Gaussian Elimination. When the decomposition can be achieved, Gaussian Elimination will give us the matrix U. Our challenge is to discover how to find L such that $A = LU$. The construction of U begins with certain row operations on A. As these operations are conducted, either with elementary matrices or by row operations, we shall see that we build the matrix L.

As a motivation of how Gaussian Elimination plays a role in the derivation of $A = LU$, it is instructive to try what is essentially a brute force method. We shall briefly illustrate this method in the case where A is a 4×3 matrix. In this case, the equation we hope to derive, namely (1.7.3), can be written in components as

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \\ A_{41} & A_{42} & A_{43} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ L_{21} & 1 & 0 & 0 \\ L_{31} & L_{32} & 1 & 0 \\ L_{41} & L_{42} & L_{43} & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \\ 0 & 0 & 0 \end{bmatrix}}_{U} \tag{1.7.9}$$

Equation (1.7.9) connects the given twelve components of A on the left side to the unknown six components of L and the six components of U on the right side. If we expand the product on the right hand side, the result is

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \\ A_{41} & A_{42} & A_{43} \end{bmatrix} = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ L_{21}U_{11} & L_{21}U_{12} + U_{22} & L_{21}U_{13} + U_{23} \\ L_{31}U_{11} & L_{31}U_{12} + L_{32}U_{22} & L_{31}U_{13} + L_{32}U_{23} + U_{33} \\ L_{41}U_{11} & L_{41}U_{12} + L_{42}U_{22} & L_{41}U_{13} + L_{42}U_{23} + L_{43}U_{33} \end{bmatrix} \tag{1.7.10}$$

Our goal is to calculate the twelve unknown quantities that appear on the right side of (1.7.10). As we proceed with this calculation, we shall see explicitly how the calculation might fail and give some motivation for the generalized LU decomposition.

Our first step in the determination of L and U from (1.7.10) is to equate like elements on both sides of (1.7.10), the first row yields

$$\begin{aligned} U_{11} &= A_{11} \\ U_{12} &= A_{12} \\ U_{13} &= A_{13} \end{aligned} \tag{1.7.11}$$

Given $(1.7.11)_1$, we can equate the remaining elements in the first column of the two matrices and conclude

$$\begin{aligned} A_{21} &= L_{21}U_{11} \\ A_{31} &= L_{31}U_{11} \\ A_{41} &= L_{41}U_{11} \end{aligned} \tag{1.7.12}$$

If we assume A_{11}, which by $(1.7.11)_1$ equals U_{11}, is nonzero, it follows from (1.7.12) that

$$\begin{aligned} L_{21} &= \frac{A_{21}}{U_{11}} = \frac{A_{21}}{A_{11}} \\ L_{31} &= \frac{A_{31}}{U_{11}} = \frac{A_{31}}{A_{11}} \\ L_{41} &= \frac{A_{41}}{U_{11}} = \frac{A_{41}}{A_{11}} \end{aligned} \tag{1.7.13}$$

The fact that we have committed to the special case $U_{11} = A_{11} \neq 0$ represents one way that our calculation is special. Next, we equate the remaining unknown elements in the second row of the two matrices and obtain

$$\left.\begin{aligned} A_{22} &= L_{21}U_{12} + U_{22} = L_{21}A_{12} + U_{22} \\ A_{23} &= L_{21}U_{13} + U_{23} = L_{21}A_{13} + U_{23} \end{aligned}\right\} \left\{\begin{aligned} &\text{2 Eqs and 2 Unknowns} \\ &(U_{22} \text{ and } U_{23}) \end{aligned}\right. \tag{1.7.14}$$

Because we have assumed $A_{11} \neq 0$, the two unknowns in (1.7.14) are given by

$$\begin{aligned} U_{22} &= A_{22} - L_{21}A_{12} = A_{22} - \frac{A_{21}}{A_{11}}A_{12} \\ U_{23} &= A_{23} - L_{21}A_{13} = A_{23} - \frac{A_{21}}{A_{11}}A_{13} \end{aligned} \tag{1.7.15}$$

Equating the remaining unknown elements in the second column yields

$$\begin{aligned} A_{32} &= L_{31}U_{12} + L_{32}U_{22} \\ A_{42} &= L_{41}U_{12} + L_{42}U_{22} \end{aligned} \tag{1.7.16}$$

If we assume $U_{22} \neq 0$, which is a further specialization of this calculation, equations (1.7.16) yield

$$
\begin{aligned}
L_{32} &= \frac{A_{32} - L_{31}U_{12}}{U_{22}} = \frac{A_{32} - \left(\dfrac{A_{31}}{A_{11}}\right)A_{12}}{A_{22} - \dfrac{A_{21}}{A_{11}}A_{12}} \\
L_{42} &= \frac{A_{42} - L_{41}U_{12}}{U_{22}} = \frac{A_{42} - \left(\dfrac{A_{41}}{A_{11}}\right)A_{12}}{A_{22} - \dfrac{A_{21}}{A_{11}}A_{12}}
\end{aligned}
\tag{1.7.17}
$$

Equating the remaining unknown element in the third row yields

$$A_{33} = L_{31}U_{13} + L_{32}U_{23} + U_{33} \tag{1.7.18}$$

$$U_{33} = A_{33} - L_{31}A_{13} - L_{32}U_{23} = A_{33} - \left(\frac{A_{31}}{A_{11}}\right)A_{13} - \frac{A_{32} - \left(\dfrac{A_{31}}{A_{11}}\right)A_{12}}{A_{22} - \dfrac{A_{21}}{A_{11}}A_{12}}\left(A_{23} - \frac{A_{21}}{A_{11}}A_{13}\right) \tag{1.7.19}$$

Next, we assume $U_{33} \neq 0$ in this special case and equate the 43 elements of (1.7.10) to obtain

$$L_{43} = \frac{A_{43} - L_{41}U_{13} - L_{42}U_{23}}{U_{33}} = \frac{A_{43} - \dfrac{A_{41}}{A_{11}}A_{12} - \left(\dfrac{A_{42} - \left(\dfrac{A_{41}}{A_{11}}\right)A_{12}}{A_{22} - \dfrac{A_{21}}{A_{11}}A_{12}}\right)\left(A_{23} - \dfrac{A_{21}}{A_{11}}A_{13}\right)}{A_{33} - \left(\dfrac{A_{31}}{A_{11}}\right)A_{13} - \dfrac{A_{32} - \left(\dfrac{A_{31}}{A_{11}}\right)A_{12}}{A_{22} - \dfrac{A_{21}}{A_{11}}A_{12}}\left(A_{23} - \dfrac{A_{21}}{A_{11}}A_{13}\right)} \tag{1.7.20}$$

In summary, if we assume $A_{11} \neq 0$, $U_{22} = A_{22} - L_{21}A_{12} = A_{22} - \dfrac{A_{21}}{A_{11}}A_{12} \neq 0$ and

$U_{33} = A_{33} - L_{31}A_{13} - L_{32}U_{23} = A_{33} - \left(\dfrac{A_{31}}{A_{11}}\right)A_{13} - \dfrac{A_{32} - \left(\dfrac{A_{31}}{A_{11}}\right)A_{12}}{A_{22} - \dfrac{A_{21}}{A_{11}}A_{12}}\left(A_{23} - \dfrac{A_{21}}{A_{11}}A_{13}\right) \neq 0$, then the

elementary LU decomposition (1.7.9) is given by[4]

[4] In the case of an arbitrary $M \times N$ matrix with $M \geq N$, the above results generalize to

$$
A=\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \\ A_{41} & A_{42} & A_{43} \end{bmatrix}
=\underbrace{\begin{bmatrix}
1 & 0 & 0 & 0 \\
\dfrac{A_{21}}{A_{11}} & 1 & 0 & 0 \\
\dfrac{A_{31}}{A_{11}} & \dfrac{A_{32}-\dfrac{A_{31}}{A_{11}}A_{12}}{A_{22}-\dfrac{A_{21}}{A_{11}}A_{12}} & 1 & 0 \\
\dfrac{A_{41}}{A_{11}} & \dfrac{A_{42}-\left(\dfrac{A_{41}}{A_{11}}\right)A_{12}}{A_{22}-\dfrac{A_{21}}{A_{11}}A_{12}} & \dfrac{A_{43}-\dfrac{A_{41}}{A_{11}}A_{12}-\left(\dfrac{A_{42}-\left(\dfrac{A_{41}}{A_{11}}\right)A_{12}}{A_{22}-\dfrac{A_{21}}{A_{11}}A_{12}}\right)\left(A_{23}-\dfrac{A_{21}}{A_{11}}A_{13}\right)}{A_{33}-\left(\dfrac{A_{31}}{A_{11}}\right)A_{13}-\dfrac{A_{32}-\left(\dfrac{A_{31}}{A_{11}}\right)A_{12}}{A_{22}-\dfrac{A_{21}}{A_{11}}A_{12}}\left(A_{23}-\dfrac{A_{21}}{A_{11}}A_{13}\right)} & 1
\end{bmatrix}}_{L}\times
$$

$$
\underbrace{\begin{bmatrix}
A_{11} & A_{12} & A_{13} \\
0 & A_{22}-\dfrac{A_{21}}{A_{11}}A_{12} & A_{23}-\dfrac{A_{21}}{A_{11}}A_{13} \\
0 & 0 & A_{33}-\left(\dfrac{A_{31}}{A_{11}}\right)A_{13}-\dfrac{A_{32}-\left(\dfrac{A_{31}}{A_{11}}\right)A_{12}}{A_{22}-\dfrac{A_{21}}{A_{11}}A_{12}}\left(A_{23}-\dfrac{A_{21}}{A_{11}}A_{13}\right) \\
0 & 0 & 0
\end{bmatrix}}_{U}
\tag{1.7.21}
$$

$$U_{jk}=A_{jk}-\sum_{q=1}^{j-1}L_{jq}U_{qk} \qquad \text{for} \quad k=1,2,\ldots,N \quad \text{and} \quad j\le k$$

for the unknown components of U, and

$$L_{jk}=\frac{A_{jk}-\sum_{q=1}^{k-1}L_{jq}U_{qk}}{U_{kk}} \qquad \text{for} \quad k=1,2,\ldots,N \ \text{ and } \ k<j\le M$$

for the unknown components of L

It is a simple calculation to specialize the above to the case of a 3×3 matrix and use the resulting formulas to produce the result (1.7.5) shown in Example 1.7.1. This derivation also shows that not all matrices have an elementary LU decomposition like (1.7.3). An example where the above formulas cannot be used is the following:

Example 1.7.2:

$$A=\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 2\\ 0 & 1 & -1\end{bmatrix} \tag{1.7.22}$$

Because of the two zeros in the second column of the matrix (1.7.22), $(1.7.15)_3$ shows that $U_{22}=0$ which invalidates the derivation of (1.7.17). Thus, the elementary LU decomposition as defined by (1.7.3) does not exist for (1.7.22). When we discuss the generalized LU decomposition, we shall see how it avoids the problems that cause this example to fail.

In those cases where $A_{11}\neq 0$, $U_{22}\neq 0$ and $U_{33}\neq 0$ are valid, it should be noted that the steps dictated by the above formulas implement the following rearrangements to

$$A=\begin{bmatrix}A_{11} & A_{12} & A_{13}\\ A_{21} & A_{22} & A_{23}\\ A_{31} & A_{32} & A_{33}\\ A_{41} & A_{42} & A_{43}\end{bmatrix} \text{ to reach } \begin{bmatrix}U_{11} & U_{12} & U_{13}\\ 0 & U_{22} & U_{23}\\ 0 & 0 & U_{33}\\ 0 & 0 & 0\end{bmatrix}:$$

$$\begin{bmatrix}A_{11} & A_{12} & A_{13}\\ A_{21} & A_{22} & A_{23}\\ A_{31} & A_{32} & A_{33}\\ A_{41} & A_{42} & A_{43}\end{bmatrix}\longrightarrow$$

$$\begin{bmatrix}U_{11} & U_{12} & U_{13}\\ 0 & U_{22} & U_{23}\\ 0 & 0 & U_{33}\\ 0 & 0 & 0\end{bmatrix}=\underbrace{\begin{bmatrix}A_{11} & A_{12} & A_{13}\\ 0 & A_{22}-\dfrac{A_{21}}{A_{11}}A_{12} & A_{23}-\dfrac{A_{21}}{A_{11}}A_{13}\\ 0 & 0 & A_{33}-\left(\dfrac{A_{31}}{A_{11}}\right)A_{13}-\dfrac{A_{32}-\left(\dfrac{A_{31}}{A_{11}}\right)A_{12}}{A_{22}-\dfrac{A_{21}}{A_{11}}A_{12}}\left(A_{23}-\dfrac{A_{21}}{A_{11}}A_{13}\right)\\ 0 & 0 & 0\end{bmatrix}}_{U} \tag{1.7.23}$$

This result shows that the second row of U is simply the second row of the matrix equivalent to A obtained by Gauss elimination. If one studies the above formulas one can also conclude that the third row is also what one would obtain by Gaussian Elimination. These facts reveal an alternate way of generating the LU Decomposition.

Recall that we implemented Gaussian Elimination by performing row operations.

a) switching two rows,
b) multiplying one of the rows by a nonzero constant
c) multiply one row by a nonzero constant and adding it to another row.

We mainly used c). The operation b) was used when we chose to normalize a row such as occurs when finding the row echelon form of a matrix. The operation a) was used when the occurrence of a zero made it necessary to change the pivot row. An important fact is that the method of finding building the LU Decomposition will *only* make use of c). It will not proceed with row operations that produce the row echelon form or the reduced row echelon. It will proceed to the point where the coefficient matrix is an upper triangular form as with the Gaussian Elimination examples discussed in Section 1.3. Of course, we can view these row operations in the equivalent way as elementary matrix operations.

The key to finding the matrix L with the specified properties is to simply set up a tracking system as the matrix U is derived. The key to the tracking system is equation (1.7.8). It is a feature of the calculation that we shall actually construct L^{-1}. After it is calculated, it can be inverted to yield the L in the decomposition $A = LU$.

We begin the calculation with the equation $A\mathbf{x} = \mathbf{b}$. We shall perform row operations of the type c) above (or elementary matrix multiplications). These row operations, providing we do not confront a division by zero, will transform the matrix A into an upper triangular matrix we shall call U. It turns out that when we introduced the augmented matrix $\left(A|\mathbf{b}\right)$ and did row operations, we were, implicitly building the matrix L^{-1}. The construction was masked because we, in effect, multiplied each step by $\mathbf{b}$. To make this step explicit, we shall first write the equation $A\mathbf{x} = \mathbf{b}$ as

$$A\mathbf{x} = I\mathbf{b} \tag{1.7.24}$$

where I is the $M \times M$ identity matrix.

As we do row operations on A, we shall perform the exact same row operation on I. At the end of the set of operations which convert A to U, I will be converted to L^{-1}. Just as we facilitated the calculation of U, by use of the augmented matrix $\left(A|\mathbf{b}\right)$, we shall facilitate the calculation of U *and* L by starting the calculation with the $M \times \left(N + M\right)$ matrix $\left(A|I\right)$ defined by

$$\left(A|I\right)=\left[\begin{array}{cccccc|cccccc} A_{11} & A_{12} & \cdot & \cdot & \cdot & A_{1N} & 1 & 0 & \cdot & \cdot & \cdot & 0 \\ A_{21} & A_{22} & \cdot & \cdot & \cdot & A_{2N} & 0 & 1 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & & & \cdot & \cdot & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & \cdot & & \cdot & \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & & \cdot & \cdot & \cdot & \cdot & & & \cdot & \cdot \\ \multicolumn{6}{c|}{\underbrace{A_{M1} \quad A_{M2} \quad \cdot \quad \cdot \quad \cdot \quad A_{MN}}_{A \text{ an } M\times N \text{ matrix}}} & \multicolumn{6}{c}{\underbrace{0 \quad 0 \quad \cdot \quad \cdot \quad \cdot \quad 1}_{I \text{ an } M\times M \text{ matrix}}} \end{array}\right] \tag{1.7.25}$$

We shall perform row operations of the type c) on $\left(A|I\right)$ until we achieve the conversion

$$\left(A|I\right)\xrightarrow[\text{Operations}]{\text{Row}}\left(U|L^{-1}\right) \tag{1.7.26}$$

The following example will help illustrate the process just described.

Example 1.7.3: An example of the LU Decomposition for a matrix where it does exist is the following. We are given the matrix

$$A=\begin{bmatrix} 2 & 1 & 3 \\ 6 & 4 & 10 \\ 4 & 1 & 5 \end{bmatrix} \tag{1.7.27}$$

The first step is to form the augmented matrix $\left(A|I\right)$. Therefore,

$$\left(A|I\right)=\left[\begin{array}{ccc|ccc} 2 & 1 & 3 & 1 & 0 & 0 \\ 6 & 4 & 10 & 0 & 1 & 0 \\ \multicolumn{3}{c|}{\underbrace{4 \quad 1 \quad 5}_{A}} & \multicolumn{3}{c}{\underbrace{0 \quad 0 \quad 1}_{I}} \end{array}\right] \tag{1.7.28}$$

Next, we wish to perform row operations of the type c) that convert A to the upper triangular matrix that we shall call U. It is important to stress again that we shall *only* use row operations of the type c). Therefore, we will not reach the row echelon form of A. Because, we are interested in the elementary matrices that are equivalent to these row operations, they will be tracked during the step by step operations on (1.7.28).

Step 1:

$$\left[\underbrace{\begin{matrix}2&1&3\\6&4&10\\4&1&5\end{matrix}}_{A}\middle|\underbrace{\begin{matrix}1&0&0\\0&1&0\\0&0&1\end{matrix}}_{I}\right]\xrightarrow[\substack{\text{3×row 1}\\\text{subtracted}\\\text{from row 2}}]{}\left[\underbrace{\begin{matrix}2&1&3\\0&1&1\\4&1&5\end{matrix}}_{E_1A}\middle|\underbrace{\begin{matrix}1&0&0\\-3&1&0\\0&0&1\end{matrix}}_{E_1I=E_1}\right] \tag{1.7.29}$$

where

$$E_1=\begin{bmatrix}1&0&0\\-3&1&0\\0&0&1\end{bmatrix} \tag{1.7.30}$$

Step 2:

$$\left[\underbrace{\begin{matrix}2&1&3\\0&1&1\\4&1&5\end{matrix}}_{E_1A}\middle|\underbrace{\begin{matrix}1&0&0\\-3&1&0\\0&0&1\end{matrix}}_{E_1I}\right]\xrightarrow[\substack{\text{2×row 1}\\\text{subtracted}\\\text{from row 3}}]{}\left[\underbrace{\begin{matrix}2&1&3\\0&1&1\\0&-1&-1\end{matrix}}_{E_2E_1A}\middle|\underbrace{\begin{matrix}1&0&0\\-3&1&0\\-2&0&1\end{matrix}}_{E_2E_1}\right] \tag{1.7.31}$$

where

$$E_2=\begin{bmatrix}1&0&0\\0&1&0\\-2&0&1\end{bmatrix} \tag{1.7.32}$$

Step 3:

$$\left[\underbrace{\begin{matrix}2&1&3\\0&1&1\\0&-1&-1\end{matrix}}_{E_2E_1A}\middle|\underbrace{\begin{matrix}1&0&0\\-3&1&0\\-2&0&1\end{matrix}}_{E_2E_1I}\right]\xrightarrow[\substack{\text{Add row 2}\\\text{to row 3}}]{}\left[\underbrace{\begin{matrix}2&1&3\\0&1&1\\0&0&0\end{matrix}}_{E_3E_2E_1A}\middle|\underbrace{\begin{matrix}1&0&0\\-3&1&0\\-5&1&1\end{matrix}}_{E_3E_2E_1}\right] \tag{1.7.33}$$

where

$$E_3=\begin{bmatrix}1&0&0\\0&1&0\\0&1&1\end{bmatrix} \tag{1.7.34}$$

The result of the last row operation is the upper triangular matrix

$$U = E_3E_2E_1A = \begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \tag{1.7.35}$$

and the matrix we shall denote by L^{-1} given by

$$L^{-1} = E_3E_2E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 1 \\ -5 & 1 & 1 \end{bmatrix} \tag{1.7.36}$$

Because the matrix we have denoted by L^{-1} is equal to the product of three elementary matrices, it is nonsingular as the notation suggests. We can use $(1.7.36)_1$ in $(1.7.35)_1$ and obtain

$$U = L^{-1}A \tag{1.7.37}$$

or, equivalently, the decomposition

$$A = LU \tag{1.7.38}$$

We can obtain the explicit formula for L in this example by inverting the matrix (1.7.36). With the tools we have developed thus far, the easiest way to find L is to use $(1.7.36)_1$ to write

$$\begin{aligned} L = E_1^{-1}E_2^{-1}E_3^{-1} &= \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix} \end{aligned} \tag{1.7.39}$$

where the formulas (1.7.30), (1.7.32) and (1.7.34) have been used. We have also used equations like (1.6.15) to construct the inverses in $(1.7.39)_2$. Therefore, for this example, the LU Decomposition of (1.7.27) is

$$\underbrace{\begin{bmatrix} 2 & 1 & 3 \\ 6 & 4 & 10 \\ 4 & 1 & 5 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}}_{U} \tag{1.7.40}$$

The general conclusion from the explanation above and the example just completed is that, providing we do not encounter the necessity to divide by zero, we can find a finite number of row operations of the type c) as represented by elementary matrices $E_1, E_2, ..., E_k$ such that a $M \times N$ matrix A has the decomposition

$$A = LU \tag{1.7.41}$$

where

$$U = E_k \cdots E_2 E_1 A \tag{1.7.42}$$

and

$$L = \left(E_k \cdots E_2 E_1\right)^{-1} = E_1^{-1} E_2^{-1} \cdots E_k^{-1} \tag{1.7.43}$$

As the example indicates, the row operations continue until U is in upper triangular form. The resulting L, as calculated from (1.7.43), is lower triangular with 1's for diagonal elements because of the special forms of the elementary matrices $E_1, E_2, ..., E_k$.

Our numerical examples thus far have been for the case where A is square. We have asserted that the calculation scheme works when $N \geq M-1$. A case where $N > M$ is the following example.

Example 1.7.4: In Example 1.3.6, we studied the solution of the system (1.3.48), repeated,

$$\begin{aligned} x_1 + 2x_2 - 4x_3 + 3x_4 + 9x_5 &= 1 \\ 4x_1 + 5x_2 - 10x_3 + 6x_4 + 18x_5 &= 4 \\ 7x_1 + 8x_2 - 16x_3 &= 7 \end{aligned} \tag{1.7.44}$$

The matrix of coefficients in this case is

$$A = \begin{bmatrix} 1 & 2 & -4 & 3 & 9 \\ 4 & 5 & -10 & 6 & 18 \\ 7 & 8 & -16 & 0 & 0 \end{bmatrix} \tag{1.7.45}$$

The matrix $\left(A|I\right)$ is

$$\left(A|I\right) = \left[\begin{array}{ccccc|ccc} 1 & 2 & -4 & 3 & 9 & 1 & 0 & 0 \\ 4 & 5 & -10 & 6 & 18 & 0 & 1 & 0 \\ 7 & 8 & -16 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \tag{1.7.46}$$

The row operations sufficient to reduce A to upper triangular form can be read off from equation (1.3.50). Utilizing this previous calculation, it follows from (1.7.46) that

$$\left[\begin{array}{ccccc|ccc} 1 & 2 & -4 & 3 & 9 & 1 & 0 & 0 \\ 4 & 5 & -10 & 6 & 18 & 0 & 1 & 0 \\ \multicolumn{5}{c|}{\underbrace{7 \quad 8 \quad -16 \quad 0 \quad 0}_{A}} & \multicolumn{3}{c}{\underbrace{0 \quad 0 \quad 1}_{I}} \end{array}\right] \xrightarrow[\text{from row2}]{\text{Subtract } 4\times\text{row1}} \left[\begin{array}{ccccc|ccc} 1 & 2 & -4 & 3 & 9 & 1 & 0 & 0 \\ 0 & -3 & 6 & -6 & -18 & -4 & 1 & 0 \\ \multicolumn{5}{c|}{\underbrace{7 \quad 8 \quad -16 \quad 0 \quad 0}_{E_1A}} & \multicolumn{3}{c}{\underbrace{0 \quad 0 \quad 1}_{E_1}} \end{array}\right]$$

$$\xrightarrow[\text{from row3}]{\text{Subtract } 7\times\text{row1}} \left[\begin{array}{ccccc|ccc} 1 & 2 & -4 & 3 & 9 & 1 & 0 & 0 \\ 0 & -3 & 6 & -6 & -18 & -4 & 1 & 0 \\ \multicolumn{5}{c|}{\underbrace{0 \quad -6 \quad 12 \quad -21 \quad -63}_{E_2E_1A}} & \multicolumn{3}{c}{\underbrace{-7 \quad 0 \quad 1}_{E_2E_1}} \end{array}\right] \tag{1.7.47}$$

$$\xrightarrow[\text{from row3}]{\text{Subtract } 2\times\text{row2}} \left[\begin{array}{ccccc|ccc} 1 & 2 & -4 & 3 & 9 & 1 & 0 & 0 \\ 0 & -3 & 6 & -6 & -18 & -4 & 1 & 0 \\ \multicolumn{5}{c|}{\underbrace{0 \quad 0 \quad 0 \quad -9 \quad -27}_{U=E_3E_2E_1A}} & \multicolumn{3}{c}{\underbrace{1 \quad -2 \quad 1}_{L^{-1}=E_3E_2E_1}} \end{array}\right]$$

where

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1.7.48}$$

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -7 & 0 & 1 \end{bmatrix} \tag{1.7.49}$$

and

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \tag{1.7.50}$$

Therefore, from (1.7.42)

$$U = E_3E_2E_1A = \begin{bmatrix} 1 & 2 & -4 & 3 & 9 \\ 0 & -3 & 6 & -6 & -18 \\ 0 & 0 & 0 & -9 & -27 \end{bmatrix} \tag{1.7.51}$$

and from (1.7.43)

$$L = \left(E_3 E_2 E_1\right)^{-1} = E_1^{-1} E_2^{-1} E_3^{-1} = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -7 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right)^{-1}$$
$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 7 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 7 & 2 & 1 \end{bmatrix} \tag{1.7.52}$$

Therefore, the LU Decomposition for this example is

$$\underbrace{\begin{bmatrix} 1 & 2 & -4 & 3 & 9 \\ 4 & 5 & -10 & 6 & 18 \\ 7 & 8 & -16 & 0 & 0 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 7 & 2 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & 2 & -4 & 3 & 9 \\ 0 & -3 & 6 & -6 & -18 \\ 0 & 0 & 0 & -9 & -27 \end{bmatrix}}_{U} \tag{1.7.53}$$

It is instructive to briefly discuss the case where $N < M - 1$. An example is the 4×2 matrix

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \\ A_{31} & A_{32} \\ A_{41} & A_{42} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ L_{21} & 1 & 0 & 0 \\ L_{31} & L_{32} & 1 & 0 \\ L_{41} & L_{42} & L_{43} & 1 \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} \\ 0 & U_{22} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} U_{11} & U_{12} \\ L_{21}U_{11} & L_{21}U_{12} + U_{22} \\ L_{31}U_{11} & L_{31}U_{12} + L_{32}U_{22} \\ L_{41}U_{11} & L_{41}U_{12} + L_{42}U_{22} \end{bmatrix} \tag{1.7.54}$$

There are eight given elements of the matrix A. Unfortunately, there are nine unknown elements in the two matrices L and U. The element L_{43} is simply not determined by A. In other words, it does not appear in the product LU. In this case, the convention seems to be to take L_{43} to be zero, and the resulting decomposition is

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \\ A_{31} & A_{32} \\ A_{41} & A_{42} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ L_{21} & 1 & 0 & 0 \\ L_{31} & L_{32} & 1 & 0 \\ L_{41} & L_{42} & 0 & 1 \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} \\ 0 & U_{22} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} U_{11} & U_{12} \\ L_{21}U_{11} & L_{21}U_{12} + U_{22} \\ L_{31}U_{11} & L_{31}U_{12} + L_{32}U_{22} \\ L_{41}U_{11} & L_{41}U_{12} + L_{42}U_{22} \end{bmatrix} \tag{1.7.55}$$

Because

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ L_{21} & 1 & 0 & 0 \\ L_{31} & L_{32} & 1 & 0 \\ L_{41} & L_{42} & 0 & 1 \end{bmatrix}\begin{bmatrix} U_{11} & U_{12} \\ 0 & U_{22} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ L_{21} & 1 \\ L_{31} & L_{32} \\ L_{41} & L_{42} \end{bmatrix}\begin{bmatrix} U_{11} & U_{12} \\ 0 & U_{22} \end{bmatrix} = \begin{bmatrix} U_{11} & U_{12} \\ L_{21}U_{11} & L_{21}U_{12} + U_{22} \\ L_{31}U_{11} & L_{31}U_{12} + L_{32}U_{22} \\ L_{41}U_{11} & L_{41}U_{12} + L_{42}U_{22} \end{bmatrix} \quad (1.7.56)$$

for the kind of case being discussed, we could relax the requirement that L be a square $M \times M$ and that U be $M \times M$ and generate a different kind of decomposition.

The Generalized LU Decomposition

The discussion of the elementary LU revealed a couple of restrictions. First, in order to satisfy the necessary condition that the number of unknown elements in L and U equal the number of elements of A, we restricted our discussion to cases where $N \geq M - 1$. We shall see that in the generalized case, this restriction is unchanged. We also, in the elementary case, had to limit our discussion to cases where the Gaussian Elimination process would proceed to an upper triangular form for U without row switching. This was achieved by assuming certain coefficients that arise in the elimination process were not zero. It is this restriction that we shall relax for the generalized case. We shall begin with (1.7.25) and proceed with row operations of both types a) and c) until we reach an upper triangular $M \times N$ matrix U in the first slot. In other words, we proceed with row operations of types a) and c), repeated,

$$(A|I) \xrightarrow[\text{Operations}]{\text{Row}} (U|D) \quad (1.7.57)$$

At this point the $M \times M$ matrix D is something the process will determine. How it relates to a lower triangular nonsingular matrix L is something we shall also determine.

We shall associate with each row operation of the type c) an elementary matrix of the type introduced in Example 1.6.4. Likewise, the row operation of the type a), the row switching, can be associated with an elementary matrix of type introduced in Example 1.6.2. In this discussion, we shall refer to row switching elementary matrices as *permutation matrices*. Permutation matrices shall be given the symbol P. In order to distinguish row operations of the type a) from those of type c), we shall denote the latter by the symbol E. As we learned in Section 1.6, elementary matrices are nonsingular. The inverse of a permutation matrix P is the transpose P^T.[5] Given these preliminaries, then the upper triangular $M \times N$ matrix U can always be written

$$U = E_{M-1}P_{M-1} \cdots E_2P_2E_1P_1A \quad (1.7.58)$$

where our convention for the elementary matrices is as follows

$P_j =$ Row switch that brings a non zero element into the jj position, for $j = 1, 2, ..., M - 1$

[5] The transpose of a matrix is defined in Section 1.9.

$E_j =$ Elementary matrix resulting from type c) row operation to elements below jj element for $j = 1, 2, ..., M-1$.

If a zero is not encountered which forces row switching, the associated permutation is the trivial one represented by the $M \times M$ identity matrix. Equation (1.7.35) is an illustration of (1.7.58) in the case where every permutation matrix is the identity matrix.

In the elementary LU decomposition case, we obtained the $M \times M$ matrix L from equations like (1.7.35). In the generalized case, we first define the overall permutation matrix P by

$$P = P_{M-1} \cdots P_2 P_1 \tag{1.7.59}$$

The elementary matrices $E_1, E_2 ..., E_{M-1}$ and $P_1, P_2, ..., P_{M-1}$ are known quantities. We *define* a $M \times M$ nonsingular matrix L by

$$L^{-1} = \left(E_{M-1}P_{M-1} \cdots E_1 P_1\right) P^T = \left(E_{M-1}P_{M-1} \cdots E_1 P_1\right)\left(P_1^T P_2^T \cdots P_{M-1}^T\right) \tag{1.7.60}$$

Given the definitions (1.7.60) and (1.7.59), it follows from (1.7.58) that

$$PA = LU \tag{1.7.61}$$

Equation (1.7.61) is the generalized LU decomposition. We know that P is an $M \times M$ permutation matrix, and we know that U is an upper triangular $M \times N$ matrix. We also know that the $M \times M$ matrix L, as defined by the inverse of (1.7.60) is nonsingular. What we do not know as yet, but what is true, is that L is a lower triangular and it has 1s down its diagonal. These properties follow from the specialized properties of the elementary matrices that make up the definition (1.7.60).

An elementary example of the property just described can be illustrated at this point by applying the above argument to the matrix that we considered in Example 1.7.2.

Example 1.7.5: Equation (1.7.22), repeated, is the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix} \tag{1.7.62}$$

In Example 1.7.2, we explained why this A does not have an elementary LU decomposition. Our first step in illustrating that it does have a generalized LU decomposition is to construct (1.7.58) for this case. First, there is not a need for a row switching in order to make A_{11} nonzero. Also, we already have zeros below A_{11}. As a result,

$$P_1 = E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1.7.63}$$

We do need a row switch

$$P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \tag{1.7.64}$$

in order to place a nonzero element in the 22 position. The result is

$$P_2 A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix} \tag{1.7.65}$$

which is in the desired upper triangular form. As a result,

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1.7.66}$$

These results and the definition (1.7.60) yield

$$L^{-1} = \left(E_2 P_2 E_1 P_1\right)\left(P_1^T P_2^T\right) = P_2 P_2^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1.7.67}$$

Also, the permutation matrix P is given by (1.7.59) which yields

$$P = P_2 P_1 = P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \tag{1.7.68}$$

Thus, the generalized LU decomposition in this case is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix} \tag{1.7.69}$$

An example that is more substantial is the following:

Example 1.7.6: We are given the matrix

$$A = \begin{bmatrix} 0 & 5 & 5 \\ 2 & 9 & 0 \\ 6 & 8 & 8 \end{bmatrix} \tag{1.7.70}$$

Because $A_{11} = 0$, this matrix will not have an elementary LU decomposition. We can built the matrix U by identifying the row operations that give us (1.7.58). We can switch the first and third rows with the permutation P_1 defined by

$$P_1 A = \underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}}_{P_1} \begin{bmatrix} 0 & 5 & 5 \\ 2 & 9 & 0 \\ 6 & 8 & 8 \end{bmatrix} = \begin{bmatrix} 6 & 8 & 8 \\ 2 & 9 & 0 \\ 0 & 5 & 5 \end{bmatrix} \tag{1.7.71}$$

The row operation that creates a zero in the 22 position is E_1 defined by

$$E_1 P_1 A = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{3} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E_1} \begin{bmatrix} 6 & 8 & 8 \\ 2 & 9 & 0 \\ 0 & 5 & 5 \end{bmatrix} = \begin{bmatrix} 6 & 8 & 8 \\ 0 & \frac{19}{3} & -\frac{8}{3} \\ 0 & 5 & 5 \end{bmatrix} \tag{1.7.72}$$

Because the 22 element of (1.7.72) is not zero, P_2 is the identify matrix. The row operation that creates a zero in the 32 position is E_2 defined by

$$E_2 P_2 E_1 P_1 A = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{15}{19} & 1 \end{bmatrix}}_{E_2} \begin{bmatrix} 6 & 8 & 8 \\ 0 & \frac{19}{3} & -\frac{8}{3} \\ 0 & 5 & 5 \end{bmatrix} = \begin{bmatrix} 6 & 8 & 8 \\ 0 & \frac{19}{3} & -\frac{8}{3} \\ 0 & 0 & \frac{135}{19} \end{bmatrix} \tag{1.7.73}$$

Therefore, from (1.7.58) and (1.7.73)

$$U = E_2P_2E_1P_1A = \begin{bmatrix} 6 & 8 & 8 \\ 0 & \frac{19}{3} & -\frac{8}{3} \\ 0 & 0 & \frac{135}{19} \end{bmatrix} \tag{1.7.74}$$

It follows from (1.7.60) that

$$\begin{aligned} L^{-1} &= \left(E_2P_2E_1P_1\right)\left(P_1^T P_2^T\right) \\ &= \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{15}{19} & 1 \end{bmatrix}}_{E_2} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{P_2} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{3} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E_1} \underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}}_{P_1} \underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}}_{P_1^T} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{P_2^T} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{3} & 1 & 0 \\ -\frac{5}{19} & \frac{15}{19} & 1 \end{bmatrix} \end{aligned} \tag{1.7.75}$$

The inverse of the matrix (1.7.75) yields

$$L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{3} & 1 & 0 \\ 0 & \frac{15}{19} & 1 \end{bmatrix} \tag{1.7.76}$$

and the generalized LU decomposition becomes

$$\underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} 0 & 5 & 5 \\ 2 & 9 & 0 \\ 6 & 8 & 8 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{3} & 1 & 0 \\ 0 & \frac{15}{19} & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 6 & 8 & 8 \\ 0 & \frac{19}{3} & -\frac{8}{3} \\ 0 & 0 & \frac{135}{19} \end{bmatrix}}_{U} \tag{1.7.77}$$

In addition to illustrating that L is a lower triangular matrix with $1s$ down its diagonal, Example 1.7.6 also illustrates that the generalized LU decomposition is not unique. A different permutation could have been used for P_1 in order to address the zero in the 11 position of the matrix (1.7.70). This would have started the calculation with

$$P_1A = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{P_1} \begin{bmatrix} 0 & 5 & 5 \\ 2 & 9 & 0 \\ 6 & 8 & 8 \end{bmatrix} = \begin{bmatrix} 2 & 9 & 0 \\ 0 & 5 & 5 \\ 6 & 8 & 8 \end{bmatrix} \tag{1.7.78}$$

which would have created a different set of row operations and a different U and L.

Exercises

1.7.1 Show that the elementary LU Decomposition for the matrix

$$A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 1 & 1 \\ -2 & 2 & -1 \end{bmatrix} \tag{1.7.79}$$

is

$$\underbrace{\begin{bmatrix} 1 & 3 & 1 \\ 2 & 1 & 1 \\ -2 & 2 & -1 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -2 & -\frac{8}{5} & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & 3 & 1 \\ 0 & -5 & -1 \\ 0 & 0 & -\frac{3}{5} \end{bmatrix}}_{U} \tag{1.7.80}$$

1.7.2 Show that the elementary LU Decomposition for the matrix

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 1 & 1 \\ 3 & 4 & 2 \end{bmatrix} \tag{1.7.81}$$

is

$$\underbrace{\begin{bmatrix} 2 & 3 & 1 \\ 1 & 1 & 1 \\ 3 & 4 & 2 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{3}{2} & 1 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 2 & 3 & 1 \\ 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}}_{U} \tag{1.7.82}$$

1.7.3 Show that the elementary LU Decomposition for the matrix

$$A = \begin{bmatrix} 2 & 1 & 2 & 1 \\ 3 & 0 & 1 & 1 \\ -1 & 2 & -2 & 1 \\ -3 & 2 & 3 & 1 \end{bmatrix} \tag{1.7.83}$$

is

$$\underbrace{\begin{bmatrix} 2 & 1 & 2 & 1 \\ 3 & 0 & 1 & 1 \\ -1 & 2 & -2 & 1 \\ -3 & 2 & 3 & 1 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{3}{2} & 1 & 0 & 0 \\ -\frac{1}{2} & -\frac{5}{3} & 1 & 0 \\ -\frac{3}{2} & -\frac{7}{3} & -\frac{4}{13} & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 2 & 1 & 2 & 1 \\ 0 & -\frac{3}{2} & -2 & -\frac{1}{2} \\ 0 & 0 & -\frac{13}{3} & \frac{2}{3} \\ 0 & 0 & 0 & \frac{20}{13} \end{bmatrix}}_{U} \tag{1.7.84}$$

1.7.4 Start with the permutation shown in (1.7.78) for the matrix

$$A = \begin{bmatrix} 0 & 5 & 5 \\ 2 & 9 & 0 \\ 6 & 8 & 8 \end{bmatrix} \tag{1.7.85}$$

and illustrate that the generalized LU decomposition that results is

$$\underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} 0 & 5 & 5 \\ 2 & 9 & 0 \\ 6 & 8 & 8 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & -\frac{19}{5} & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 2 & 9 & 0 \\ 0 & 5 & 5 \\ 0 & 0 & 27 \end{bmatrix}}_{U} \tag{1.7.86}$$

rather than the result derived in Example 1.7.6.

Section 1.8. Consistency Theorem for Linear Systems

The various examples discussed in Sections 1.2, 1.3 and 1.5 reveal some of the possibilities when one attempts to construction to systems of linear equations of the form (1.2.1), repeated,

$$\begin{aligned} &A_{11}x_1 + A_{12}x_2 + A_{13}x_3 + \cdots + A_{1N}x_N = b_1 \\ &A_{21}x_1 + A_{22}x_2 + A_{23}x_3 + \cdots + A_{2N}x_N = b_2 \\ &. \\ &. \\ &. \\ &A_{M1}x_1 + A_{M2}x_2 + A_{M3}x_3 + \cdots + A_{MN}x_N = b_M \end{aligned} \tag{1.8.1}$$

In the case where $M = N$, we encountered examples where the system had a unique solution, examples where the system had a solution that was not unique and a system where there was no solution. The corollary to Theorem 1.6.1 told us that the system $A\mathbf{x} = \mathbf{b}$ has only unique solutions if and only if A is nonsingular. For the case of an over determined system, i.e., when $M > N$, we gave an example where the system did not have a solution. Finally, for the case of an undetermined system, we gave an example where the system had a solution, but it was not unique.

One of our objectives in the study of linear systems is to find general theorems that will allow one to have information about the solution before a solution is attempted. In this short section, we shall look at one such theorem. It is called, the Consistency Theorem.

As we have done in several sections, we shall often write (1.8.1) in the matrix form

$$A\mathbf{x} = \mathbf{b} \tag{1.8.2}$$

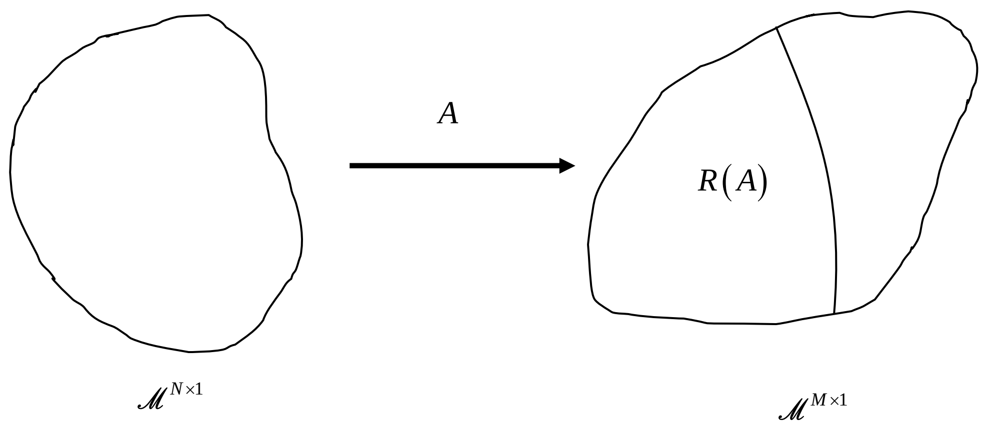
In Section 1.1, we introduced the notation $\mathscr{M}^{M\times N}$ for the set of $M \times N$ matrices. The matrix A is an element of the set $\mathscr{M}^{M\times N}$. This fact is expressed symbolically by writing $A \in \mathscr{M}^{M\times N}$. Viewed as a function, the matrix A is a function that maps column matrices in $\mathscr{M}^{N\times 1}$ into column matrices in $\mathscr{M}^{M\times 1}$. It is customary to express this functional relationship by writing

$$A : \mathscr{M}^{N\times 1} \to \mathscr{M}^{M\times 1} \tag{1.8.3}$$

The set $\mathscr{M}^{N\times 1}$ is the *domain* of the function A. The *range* of the function A is the set of all values of the function. In other words, the range is the set of possible values of $A\mathbf{x}$ generated for all possible values of $\mathbf{x}$ in $\mathscr{M}^{N\times 1}$. It is customary to use the symbol $R(A)$ for the range. A more formal way to introduce this notation is to write

$$R(A) = \left\{ A\mathbf{x} \,\middle|\, \mathbf{x} \in \mathscr{M}^{N\times 1} \right\} \tag{1.8.4}$$

Note that $R(A)$ is a subset of $\mathscr{M}^{M\times 1}$. As such, it is not necessarily true that all column matrices in $\mathscr{M}^{M\times 1}$ are also in the range. The following figure should be helpful.



The above definitions allow the statement of the following theorem:

Theorem 1.8.1: Given a matrix $A\in\mathscr{M}^{M\times N}$ and a vector $\mathbf{b}\in\mathscr{M}^{M\times 1}$, the system $A\mathbf{x}=\mathbf{b}$ has a solution if and only if $\mathbf{b}\in R(A)$.

Theorem 1.8.1 is known as the *consistency theorem* for linear systems. A less formal way of stating the same result involves writing the product $A\mathbf{x}$ as

$$
A\mathbf{x} = \begin{bmatrix} A_{11} & A_{12} & A_{13} & \cdot & \cdot & \cdot & A_{1N} \\ A_{21} & A_{22} & A_{23} & \cdot & \cdot & \cdot & A_{2N} \\ A_{31} & A_{32} & A_{33} & \cdot & \cdot & \cdot & \\ \cdot & & & & & & \\ \cdot & & & & & & \\ \cdot & & & & & & \\ A_{M1} & A_{M2} & A_{M3} & \cdot & \cdot & \cdot & A_{MN} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \cdot \\ \cdot \\ \cdot \\ x_N \end{bmatrix}
$$

$$
= \begin{bmatrix} A_{11}x_1 + A_{12}x_2 + \cdots + A_{1N}x_N \\ A_{21}x_1 + A_{22}x_2 + \cdots + A_{2N}x_N \\ A_{31}x_1 + A_{32}x_2 + \cdots + A_{3N}x_N \\ \cdot \\ \cdot \\ \cdot \\ A_{M1}x_1 + A_{M2}x_2 + \cdots + A_{MN}x_N \end{bmatrix}
$$

$$
= \underbrace{\begin{bmatrix} A_{11} \\ A_{21} \\ A_{31} \\ \cdot \\ \cdot \\ \cdot \\ A_{M1} \end{bmatrix} x_1 + \begin{bmatrix} A_{12} \\ A_{22} \\ A_{32} \\ \cdot \\ \cdot \\ \cdot \\ A_{M2} \end{bmatrix} x_2 + \begin{bmatrix} A_{13} \\ A_{23} \\ A_{33} \\ \cdot \\ \cdot \\ \cdot \\ A_{M3} \end{bmatrix} x_3 + \cdots + \begin{bmatrix} A_{1N} \\ A_{2N} \\ A_{3N} \\ \cdot \\ \cdot \\ \cdot \\ A_{MN} \end{bmatrix} x_N}_{\text{Linear Combination of Column Vectors}} \tag{1.8.5}
$$

As a further rearrangement, we shall use the symbol $\mathbf{a}_j$ for the j^{th} column of A. Therefore, we are writing

$$
\mathbf{a}_j = \begin{bmatrix} A_{1j} \\ A_{2j} \\ A_{3j} \\ \cdot \\ \cdot \\ \cdot \\ A_{Mj} \end{bmatrix} \qquad \text{for} \quad j = 1, 2, ..., N \tag{1.8.6}
$$

This definition allows (1.8.5) to be written

$$
A\mathbf{x} = \mathbf{a}_1 x_1 + \mathbf{a}_2 x_2 + \mathbf{a}_3 x_3 + \cdots + \mathbf{a}_N x_N \tag{1.8.7}
$$

Given (1.8.7) the linear system (1.8.2) is the equation

$$\mathbf{a}_1 x_1 + \mathbf{a}_2 x_2 + \mathbf{a}_3 x_3 + \cdots + \mathbf{a}_N x_N = \mathbf{b} \tag{1.8.8}$$

Given (1.8.8), we can restate Theorem 1.8.1 as

Theorem 1.8.1a: The linear system $A\mathbf{x} = \mathbf{b}$ is consistent (has a solution) if and only if $\mathbf{b}$ can be written as a linear combination of the column vectors of A.

Example 1.8.1: Example 1.2.4 is concerned with finding the solution of a system of linear equations given by (1.2.15), repeated,

$$\begin{aligned} 2x_1 + 3x_2 + x_3 &= 1 \\ 3x_1 + 4x_2 + 2x_3 &= -80 \\ x_1 + x_2 + x_3 &= 10 \end{aligned} \tag{1.8.9}$$

We discovered with this example that it did not have a solution. The above theorem indicates why it does not. It is not possible to find an x_1, x_2 and x_3 for which, forming (1.8.5), $\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} x_1 + \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} x_2 + \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} x_2$ will add to obtain $\begin{bmatrix} 1 \\ -80 \\ 10 \end{bmatrix}$. In the words of Theorem 1.8.1, the vector $\begin{bmatrix} 1 \\ -80 \\ 10 \end{bmatrix}$ is not in the range of $A = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 4 & 2 \\ 1 & 1 & 1 \end{bmatrix}$.

Section 1.9. The Transpose of a Matrix

In this section, we shall return to a discussion of matrix algebra. In particular, we shall briefly discuss the important operation of taking the transpose of a matrix. This operation begins with a matrix $A \in \mathscr{M}^{M\times N}$ given by

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} & \cdot & \cdot & \cdot & A_{1N} \\ A_{21} & A_{22} & A_{23} & \cdot & \cdot & \cdot & A_{2N} \\ A_{31} & A_{32} & A_{33} & \cdot & \cdot & \cdot & \\ \cdot & & & & & & \\ \cdot & & & & & & \\ \cdot & & & & & & \\ A_{M1} & A_{M3} & A_{M3} & \cdot & \cdot & \cdot & A_{MN} \end{bmatrix} \tag{1.9.1}$$

and constructs a matrix, called the *transpose* of A, in $\mathscr{M}^{N\times M}$ obtained by interchanging the rows and columns of A. The transpose of A is denoted by A^T. It is easily shown that

$$(A^T)^T = A \tag{1.9.2}$$

$$(\lambda A)^T = \lambda A^T \tag{1.9.3}$$

$$(A+B)^T = A^T + B^T \tag{1.9.4}$$

These properties are easily established from the definition of the transpose. Another important property of the transpose is

$$\left(AB\right)^T = B^T A^T \tag{1.9.5}$$

for matrices $A \in \mathscr{M}^{M\times N}$ and $B \in \mathscr{M}^{N\times L}$. The proof of (1.9.5) is a little tedious but straight forward. It involves no more than constructing the left side and the right side of (1.9.5) followed by a comparison of the results. Let

$$A = \left[A_{ij}\right] \qquad \text{and} \qquad B = \left[B_{ij}\right] \tag{1.9.6}$$

where $A \in \mathscr{M}^{M\times N}$ and $B \in \mathscr{M}^{N\times L}$. Then the definition of matrix multiplication introduced in Section 1.1 tells us that $C = AB$, a member of $\mathscr{M}^{M\times L}$, has components

$$C_{ij} = \sum_{k=1}^{N} A_{ik} B_{kj} \qquad i = 1, 2, \ldots, M \quad \text{and} \quad j = 1, 2, \ldots, L \tag{1.9.7}$$

Therefore,

$$C^T = (AB)^T = \begin{bmatrix} C_{11} & C_{21} & \cdot & \cdot & \cdot & C_{M1} \\ C_{12} & C_{22} & \cdot & \cdot & \cdot & C_{M2} \\ \cdot \\ \cdot \\ \cdot \\ C_{1L} & C_{2L} & & & & C_{ML} \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{k=1}^{N} A_{1k} B_{k1} & \sum_{k=1}^{N} A_{2k} B_{k1} & \cdot & \cdot & \cdot & \sum_{k=1}^{N} A_{Mk} B_{k1} \\ \sum_{k=1}^{N} A_{2k} B_{k1} & \sum_{k=1}^{N} A_{2k} B_{k2} & \cdot & \cdot & \cdot & \sum_{k=1}^{N} A_{Mk} B_{k2} \\ \cdot \\ \cdot \\ \cdot \\ \sum_{k=1}^{N} A_{1k} B_{kL} & \sum_{k=1}^{N} A_{2k} B_{kL} & & & & \sum_{k=1}^{N} A_{Mk} B_{kL} \end{bmatrix} \tag{1.9.8}$$

Next, we shall form $B^T A^T$.

$$B^T A^T = \begin{bmatrix} B_{11} & B_{21} & \cdot & \cdot & \cdot & B_{N1} \\ B_{12} & B_{22} & \cdot & \cdot & \cdot & B_{N2} \\ \cdot \\ \cdot \\ \cdot \\ B_{1L} & B_{2L} & & & & B_{NL} \end{bmatrix} \begin{bmatrix} A_{11} & A_{21} & \cdot & \cdot & \cdot & A_{M1} \\ A_{12} & A_{22} & \cdot & \cdot & \cdot & A_{M2} \\ \cdot \\ \cdot \\ \cdot \\ A_{1N} & A_{2N} & & & & A_{MN} \end{bmatrix} \tag{1.9.9}$$

If this product is multiplied by use of the rule for multiplying matrices, we find

$$B^T A^T = \begin{bmatrix} \sum_{k=1}^{N} A_{1k}B_{k1} & \sum_{k=1}^{N} A_{2k}B_{k1} & \cdot & \cdot & \cdot & \sum_{k=1}^{N} A_{Mk}B_{k1} \\ \sum_{k=1}^{N} A_{2k}B_{k1} & \sum_{k=1}^{N} A_{2k}B_{k2} & \cdot & \cdot & \cdot & \sum_{k=1}^{N} A_{Mk}B_{k2} \\ \cdot & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ \sum_{k=1}^{N} A_{1k}B_{kL} & \sum_{k=1}^{N} A_{2k}B_{kL} & & & & \sum_{k=1}^{N} A_{Mk}B_{kL} \end{bmatrix} \tag{1.9.10}$$

Equations (1.9.9) and (1.9.10) establish the result (1.9.5).

Example 1.9.1: It is perhaps useful to verify (1.9.5) with a specific example. You are given

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 3 & 5 \\ 2 & 4 & 1 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 1 \\ 5 & 4 & 1 \end{bmatrix} \tag{1.9.11}$$

Therefore

$$A^T = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 3 & 4 \\ 1 & 5 & 1 \end{bmatrix} \qquad \text{and} \qquad B^T = \begin{bmatrix} 1 & 2 & 5 \\ 0 & 1 & 4 \\ 3 & 1 & 1 \end{bmatrix} \tag{1.9.12}$$

and, as a result,

$$AB = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 3 & 5 \\ 2 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 1 \\ 5 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 10 & 6 & 6 \\ 34 & 23 & 17 \\ 15 & 8 & 11 \end{bmatrix} \tag{1.9.13}$$

and

$$(AB)^T = \begin{bmatrix} 10 & 6 & 6 \\ 34 & 23 & 17 \\ 15 & 8 & 11 \end{bmatrix}^T = \begin{bmatrix} 10 & 34 & 15 \\ 6 & 23 & 8 \\ 6 & 17 & 11 \end{bmatrix} \tag{1.9.14}$$

Next, evaluate the product

$$B^T A^T = \begin{bmatrix} 1 & 2 & 5 \\ 0 & 1 & 4 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 2 \\ 2 & 3 & 4 \\ 1 & 5 & 1 \end{bmatrix} = \begin{bmatrix} 10 & 34 & 15 \\ 6 & 23 & 8 \\ 6 & 17 & 11 \end{bmatrix} \tag{1.9.15}$$

which confirms (1.9.5) in this example.

Another important property of the transpose arises when $A \in \mathscr{M}^{N\times N}$ is nonsingular. In this special case, it is true that A^T is also nonsingular and its inverse is given by

$$\left(A^T\right)^{-1} = \left(A^{-1}\right)^T \tag{1.9.16}$$

The proof of (1.9.16) requires an application of the definition of inverse given in Section 1.1. In order to establish (1.9.16), we need to establish that

$$\left(A^{-1}\right)^T A^T = A^T \left(A^{-1}\right)^T = I \tag{1.9.17}$$

We start the proof of (1.9.17) with the definition of the inverse of A which is

$$AA^{-1} = A^{-1}A = I \tag{1.9.18}$$

The transpose of (1.9.18), after (1.9.5) is used, is

$$\left(A^{-1}\right)^T A^T = A^T \left(A^{-1}\right)^T = I \tag{1.9.19}$$

which confirms the result (1.9.17).

There are other definitions involving the transpose of square matrices that are useful to introduce at this point. A square matrix $A \in \mathscr{M}^{N\times N}$ is *symmetric* if $A = A^T$ and *skew symmetric* if $A = -A^T$. Therefore, for a symmetric matrix

$$A_{ij} = A_{ji} \tag{1.9.20}$$

and for a skew symmetric matrix

$$A_{ij} = -A_{ji} \tag{1.9.21}$$

Equation (1.9.21) implies that diagonal elements of a skew symmetric matrix are all zero.

It is an interesting fact that every square matrix A can be written uniquely as the sum of a symmetric matrix and a skew symmetric matrix. The proof of this assertion arises simply by writing the identity

$$A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T) \tag{1.9.22}$$

The first term is symmetric because

$$\left(\frac{1}{2}(A + A^T)\right)^T = \frac{1}{2}(A + A^T) \tag{1.9.23}$$

Likewise, the second term is skew symmetric because

$$\left(\frac{1}{2}(A - A^T)\right)^T = \frac{1}{2}(A^T - A) = -\left(\frac{1}{2}(A - A^T)\right) \tag{1.9.24}$$

There is another way to build a symmetric matrix. In this case, the matrix A need not be square. In particular, given a matrix $A \in \mathscr{M}^{M\times N}$, it is straight forward to use the definition of symmetry to show that AA^T is a symmetric matrix in $\mathscr{M}^{M\times M}$ and $A^T A$ is a symmetric matrix in $\mathscr{M}^{N\times N}$.

Section 1.10. The Determinant of a Square Matrix

The determinant of a matrix is a property of a *square matrix*. The determinant of a square matrix A shall be written $\det A$. On occasion, it will be convenient to use the alternate notation $|A|$. The determinant has many uses in matrix algebra. For our immediate purposes, the most important property of determinants is the fact that a square matrix A is nonsingular if and only if it has a nonzero determinant. We need to define a determinant of a square matrix and then establish this result.

There are many equivalent definitions of a determinant of a square matrix. Some depend upon more abstract ideas than we shall use in this introduction to matrix algebra. For our first discussion, we shall simply use a direct method of stating the formula for various cases. One of our goals later in this section is to provide a definition based upon more general linear algebraic concepts. We shall begin the discussion of a determinant as a series of definitions.

Definition: If A is a 1×1 matrix, i.e. a number, its determinant, written $\det A$, is defined by

$$\det A = A \tag{1.10.1}$$

Definition: If A is the 2×2 matrix

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \tag{1.10.2}$$

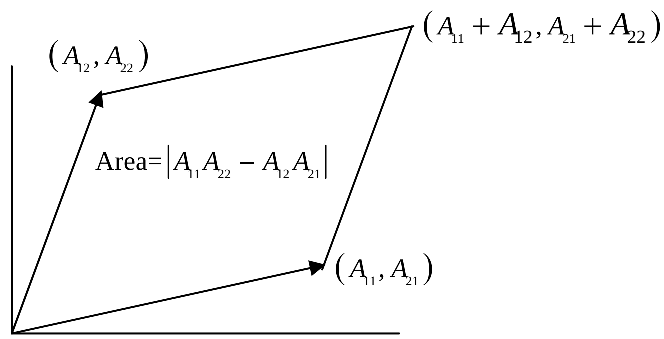
its determinant, written $\det A$, is defined by

$$\det A = A_{11}A_{22} - A_{12}A_{21} \tag{1.10.3}$$

It is customary to distinguish the matrix A which we write as in (1.10.2) from its determinant by writing

$$\det A = \begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} = A_{11}A_{22} - A_{12}A_{21} \tag{1.10.4}$$

It is possible to show that the absolute value of the determinant (1.10.4) is the area of the parallelogram



A couple of features of the definition (1.10.4), that are general properties of determinants of all sizes, are

a) The determinant is a linear function of each column.
b) The determinant is skew symmetric in its columns. Thus, if you switch the two columns you change the sign of the determinant.

For a 3×3 matrix, the definition of the determinant is given by

Definition: If A is the 3×3 matrix

$$A=\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \tag{1.10.5}$$

its determinant is defined by

$$\begin{aligned}
\det A = \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix} &= A_{11}\begin{vmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{vmatrix} - A_{21}\begin{vmatrix} A_{12} & A_{13} \\ A_{32} & A_{33} \end{vmatrix} + A_{31}\begin{vmatrix} A_{12} & A_{13} \\ A_{22} & A_{23} \end{vmatrix} \\
&= A_{11}\left(A_{22}A_{33}-A_{32}A_{23}\right) - A_{21}\left(A_{12}A_{33}-A_{32}A_{13}\right) + A_{31}\left(A_{12}A_{23}-A_{22}A_{13}\right) \qquad (1.10.6) \\
&= A_{11}A_{22}A_{33} - A_{11}A_{23}A_{32} - A_{12}A_{21}A_{33} + A_{13}A_{21}A_{32} + A_{12}A_{23}A_{31} - A_{13}A_{22}A_{31}
\end{aligned}$$

The absolute value of the determinant (1.10.6) is the volume of the parallelepiped formed from the three column vectors in (1.10.5).

Example 1.10.1: The determinant of the matrix

$$A = \begin{bmatrix} 2 & 1 & 2 \\ 3 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} \tag{1.10.7}$$

is, from the definition (1.10.6),

$$\begin{aligned} \det A = \begin{vmatrix} 2 & 1 & 2 \\ 3 & 2 & 2 \\ 1 & 2 & 3 \end{vmatrix} &= 2\begin{vmatrix} 2 & 2 \\ 2 & 3 \end{vmatrix} - 3\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} + 1\begin{vmatrix} 1 & 2 \\ 2 & 2 \end{vmatrix} \\ &= 2(2\times 3 - 2\times 2) - 3(1\times 3 - 2\times 2) + 1(1\times 2 - 2\times 2) \\ &= 2\times 2 - 3\times(-1) + 1\times(-2) = 4 + 3 - 2 = 5 \end{aligned} \tag{1.10.8}$$

Example 1.10.2: The determinants of the elementary matrices E_1, E_2 and E_3, given in equations (1.6.7), (1.6.10) and (1.6.13), can be showed from (1.10.6) to be

$$\det E_1 = \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -1 \tag{1.10.9}$$

$$\det E_2 = \begin{vmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 1 \end{vmatrix} = \lambda \tag{1.10.10}$$

and

$$\det E_3 = \begin{vmatrix} 1 & 0 & -\lambda \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \tag{1.10.11}$$

As with 2×2 matrices, the definition (1.10.6) reveals a couple of general properties of determinants. They are as follows:

a) The determinant is a linear function of each column.
b) The determinant is skew symmetric in its columns. Thus, if you switch the two columns you change the sign of the determinant.

A more formal display of the property a), utilizing the first column, is the formula

$$\begin{aligned}
\begin{vmatrix} A_{11}+\lambda B_{11} & A_{12} & A_{13} \\ A_{21}+\lambda B_{21} & A_{22} & A_{23} \\ A_{31}+\lambda B_{31} & A_{32} & A_{33} \end{vmatrix} &= \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix} + \begin{vmatrix} \lambda B_{11} & A_{12} & A_{13} \\ \lambda B_{21} & A_{22} & A_{23} \\ \lambda B_{31} & A_{32} & A_{33} \end{vmatrix} \\
&= \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix} + \lambda \begin{vmatrix} B_{11} & A_{12} & A_{13} \\ B_{21} & A_{22} & A_{23} \\ B_{31} & A_{32} & A_{33} \end{vmatrix}
\end{aligned} \tag{1.10.12}$$

for all real numbers $\lambda \in \mathscr{R}$. Likewise, b) is summarized by the formulas

$$\begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix} = -\begin{vmatrix} A_{12} & A_{11} & A_{13} \\ A_{22} & A_{21} & A_{23} \\ A_{32} & A_{31} & A_{33} \end{vmatrix} = \begin{vmatrix} A_{12} & A_{13} & A_{11} \\ A_{22} & A_{23} & A_{21} \\ A_{32} & A_{33} & A_{31} \end{vmatrix}, \text{ etc} \tag{1.10.13}$$

Relationships like (1.10.13) are often used to explain that a determinant is a *completely skew symmetric* linear function of the columns of A. It is a consequence of this skew symmetry that if two columns of a matrix are identical the determinant is zero. This simple fact follows if we form a matrix with two identical rows, switch the rows and obtain a determinant equal to minus itself. The only conclusion is that the determinant is zero.

It is useful to look at $(1.10.6)_4$ as a source of a greater insight into a more general definition of the determinant. The essential idea that one needs in order to interpret $(1.10.6)_4$ is the idea of a *permutation*. The idea of a permutation of the ordered set of the first three positive integers, $(1,2,3)$, is as follows: A permutation is a rearrangement of the three integers $(1,2,3)$ into a reordered set of the same three integers. Viewed as a function, which we shall denote by σ, the following are a list of possible permutations of $(1,2,3)$:

$$\begin{aligned}
(1,2,3) &\xrightarrow[\sigma]{} (1,2,3) \\
(1,2,3) &\xrightarrow[\sigma]{} (1,3,2) \\
(1,2,3) &\xrightarrow[\sigma]{} (2,1,3) \\
(1,2,3) &\xrightarrow[\sigma]{} (3,1,2) \\
(1,2,3) &\xrightarrow[\sigma]{} (2,3,1) \\
(1,2,3) &\xrightarrow[\sigma]{} (3,2,1)
\end{aligned} \tag{1.10.14}$$

As this example reflects, there are $3! = 6$ possible permutations of the ordered set $(1,2,3)$. The list (1.10.14) includes the identity permutation $(1.10.14)_1$. A permutation is *even* if it takes an even number of transpositions of the values in the above list to return the starting ordered set $(1,2,3)$. A permutation is *odd* if it takes an odd number of transpositions of the values in the above list to return the starting ordered set $(1,2,3)$. For example, the even and odd permutations in the list (1.10.14) are[5]

$$
\begin{aligned}
(1,2,3) &\xrightarrow[\sigma]{} \underbrace{(1,2,3)}_{\text{Even}} \\
(1,2,3) &\xrightarrow[\sigma]{} \underbrace{(1,3,2)}_{\text{Odd}} \\
(1,2,3) &\xrightarrow[\sigma]{} \underbrace{(2,1,3)}_{\text{Odd}} \\
(1,2,3) &\xrightarrow[\sigma]{} \underbrace{(3,1,2)}_{\text{Even}} \\
(1,2,3) &\xrightarrow[\sigma]{} \underbrace{(2,3,1)}_{\text{Even}} \\
(1,2,3) &\xrightarrow[\sigma]{} \underbrace{(3,2,1)}_{\text{Odd}}
\end{aligned}
\tag{1.10.15}
$$

The *parity* of a permutation is the even or odd designation of that permutation. Given the idea of parity, it is customary to utilize the symbol ε_σ defined as follows:

$$
\varepsilon_\sigma = \begin{cases} +1 & \text{if } \sigma \text{ is an even permutation of } (1,2,3) \\ -1 & \text{if } \sigma \text{ is an odd permutation of } (1,2,3) \end{cases}
\tag{1.10.16}
$$

This definition tells us that for the permutations in (1.10.15)

[5] Sometimes the permutation σ is displayed symbolically by the notation

$$
\sigma = \begin{pmatrix} 1 & 2 & 3 \\ \sigma(1) & \sigma(2) & \sigma(3) \end{pmatrix}
$$

where $\sigma(1), \sigma(2)$ and $\sigma(3)$ are distinct elements from the set $\{1,2,3\}$.

$$
\begin{aligned}
&(1,2,3) \xrightarrow[\sigma]{} \underbrace{(1,2,3)}_{\text{Even}} \Rightarrow \varepsilon_\sigma = 1 \\
&(1,2,3) \xrightarrow[\sigma]{} \underbrace{(1,3,2)}_{\text{Odd}} \Rightarrow \varepsilon_\sigma = -1 \\
&(1,2,3) \xrightarrow[\sigma]{} \underbrace{(2,1,3)}_{\text{Odd}} \Rightarrow \varepsilon_\sigma = -1 \\
&(1,2,3) \xrightarrow[\sigma]{} \underbrace{(3,1,2)}_{\text{Even}} \Rightarrow \varepsilon_\sigma = 1 \\
&(1,2,3) \xrightarrow[\sigma]{} \underbrace{(2,3,1)}_{\text{Even}} \Rightarrow \varepsilon_\sigma = 1 \\
&(1,2,3) \xrightarrow[\sigma]{} \underbrace{(3,2,1)}_{\text{Odd}} \Rightarrow \varepsilon_\sigma = -1
\end{aligned}
\tag{1.10.17}
$$

Given the idea of a permutation and the associated idea of its parity, the definition (1.10.16) allows us to write the definition of the determinant of a 3×3 matrix, equation $(1.10.6)_4$, as

$$
\det A = \sum_\sigma \varepsilon_\sigma A_{\sigma(1)1} A_{\sigma(2)2} A_{\sigma(3)3} \tag{1.10.18}
$$

where the sum is understood to be over *all* permutations of the ordered set $(1,2,3)$. The formula (1.10.18) is often the starting place for the definition of the determinant of a 3×3 matrix. It should be evident that the idea of permutation generalizes to permutations of ordered sets of the type $(1,2,3,...,N)$ and, for example, the definition (1.10.3) can be expressed in a fashion entirely similar to (1.10.18).

Returning to the result (1.10.6) or the more formal expression (1.10.18), there are important conclusions that can be reached about the determinant of a 3×3 matrix. The expression (1.10.6) for $\det A$ can be rearranged to yield an expansion in terms of the *first row* rather than the *first column*. The rearrangement takes the following form:

$$
\begin{aligned}
\det A &= \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix} \\
&= A_{11}A_{22}A_{33} - A_{11}A_{23}A_{32} - A_{12}A_{21}A_{33} + A_{13}A_{21}A_{32} + A_{12}A_{23}A_{31} - A_{13}A_{22}A_{31} \\
&= A_{11}\left(A_{22}A_{33} - A_{23}A_{32}\right) - A_{12}\left(A_{21}A_{33} - A_{23}A_{31}\right) + A_{13}\left(A_{21}A_{32} - A_{22}A_{31}\right) \\
&= A_{11}\begin{vmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{vmatrix} - A_{12}\begin{vmatrix} A_{21} & A_{23} \\ A_{31} & A_{33} \end{vmatrix} + A_{13}\begin{vmatrix} A_{21} & A_{22} \\ A_{31} & A_{32} \end{vmatrix}
\end{aligned}
\tag{1.10.19}
$$

In a notation like that used in (1.10.18), (1.10.19) can be written

$$\det A = \sum_{\sigma} \varepsilon_{\sigma} A_{1\sigma(1)} A_{2\sigma(2)} A_{3\sigma(3)} \tag{1.10.20}$$

The results (1.10.6) and (1.10.19) give us the interesting result that the determinant of a matrix and the determinant of its transpose are the same, i.e.,

$$\det A = \det A^T \tag{1.10.21}$$

The form of (1.10.18) and (1.10.20) reveal an interesting characteristic of determinants that one can expand about any row or column to obtain the answer. One simply must assign the proper sign to the column or row utilized. As an example of an expansion of a 3×3 expanded about its second column consider the following

$$\begin{aligned}
&A_{12}\begin{vmatrix} A_{21} & A_{23} \\ A_{31} & A_{33} \end{vmatrix} - A_{22}\begin{vmatrix} A_{11} & A_{13} \\ A_{31} & A_{33} \end{vmatrix} + A_{32}\begin{vmatrix} A_{11} & A_{13} \\ A_{21} & A_{23} \end{vmatrix} \\
&= A_{13}\left(A_{21}A_{33} - A_{23}A_{31}\right) - A_{22}\left(A_{11}A_{33} - A_{13}A_{31}\right) + A_{32}\left(A_{11}A_{22} - A_{13}A_{21}\right) \\
&= A_{13}A_{21}A_{33} - A_{13}A_{23}A_{31} - A_{22}A_{11}A_{33} + A_{22}A_{13}A_{31} + A_{32}A_{11}A_{22} - A_{32}A_{13}A_{21} \\
&= -A_{11}\left(A_{22}A_{33} - A_{32}A_{23}\right) + A_{21}\left(A_{12}A_{33} - A_{32}A_{13}\right) - A_{31}(A_{12}A_{23} - A_{22}A_{13}) \\
&= -A_{11}\begin{vmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{vmatrix} + A_{21}\begin{vmatrix} A_{12} & A_{13} \\ A_{32} & A_{33} \end{vmatrix} - A_{31}\begin{vmatrix} A_{12} & A_{13} \\ A_{22} & A_{23} \end{vmatrix} \\
&= -\begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix} = -\det A
\end{aligned} \tag{1.10.22}$$

This result simply reflects the fact that the determinant is also a linear function of each row and, as a function of its rows, it is completely skew symmetric.

The definition (1.10.18) generalizes to higher order square matrices. For example, for a 4×4 matrix, the definition is as follows:

Definition: If A is the 4×4 matrix

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{bmatrix} \tag{1.10.23}$$

its determinant is defined by

$$\det A = \sum_{\sigma} \varepsilon_{\sigma} A_{\sigma(1)1} A_{\sigma(2)2} A_{\sigma(3)3} A_{\sigma(4)4} \tag{1.10.24}$$

or, in expanded form

$$\det A = \begin{vmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{vmatrix} = A_{11}\begin{vmatrix} A_{22} & A_{23} & A_{24} \\ A_{32} & A_{33} & A_{34} \\ A_{42} & A_{43} & A_{44} \end{vmatrix} - A_{21}\begin{vmatrix} A_{12} & A_{13} & A_{14} \\ A_{32} & A_{33} & A_{34} \\ A_{42} & A_{43} & A_{44} \end{vmatrix}$$
$$+A_{31}\begin{vmatrix} A_{12} & A_{13} & A_{14} \\ A_{22} & A_{23} & A_{24} \\ A_{42} & A_{43} & A_{44} \end{vmatrix} - A_{41}\begin{vmatrix} A_{12} & A_{13} & A_{14} \\ A_{22} & A_{23} & A_{24} \\ A_{32} & A_{33} & A_{34} \end{vmatrix} \tag{1.10.25}$$

where the 3×3 determinates are evaluated by (1.10.6). Both of the formulas (1.10.24) and (1.10.25) reveal the feature identified above for 3×3 matrices that the determinant is linear in each column and completely skew symmetric in its columns.

The expansion in our definition (1.10.25) can be rearranged into an expansion by the first row of A just as was done for the 3×3 matrix in (1.10.19) above. The formula in this case looks like

$$\det A = \begin{vmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{vmatrix} = A_{11}\begin{vmatrix} A_{22} & A_{23} & A_{24} \\ A_{32} & A_{33} & A_{34} \\ A_{42} & A_{43} & A_{44} \end{vmatrix} - A_{12}\begin{vmatrix} A_{21} & A_{23} & A_{24} \\ A_{31} & A_{33} & A_{34} \\ A_{41} & A_{43} & A_{44} \end{vmatrix}$$
$$+A_{13}\begin{vmatrix} A_{21} & A_{22} & A_{24} \\ A_{31} & A_{32} & A_{34} \\ A_{41} & A_{42} & A_{44} \end{vmatrix} - A_{14}\begin{vmatrix} A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \\ A_{41} & A_{42} & A_{43} \end{vmatrix} \tag{1.10.26}$$

As an alternate to (1.10.26) as an expansion by rows formula, one can establish the generalization of (1.10.20), namely,

$$\det A = \sum_{\sigma} \varepsilon_{\sigma} A_{1\sigma(1)} A_{2\sigma(2)} A_{3\sigma(3)} A_{4\sigma(4)} \tag{1.10.27}$$

Just as (1.10.21) holds for 3×3, the last result shows that it also holds for 4×4, indeed, for $N\times N$ matrices.

The pattern illustrated by (1.10.4) for 2×2 matrices, by (1.10.6) for 3×3 matrices and by (1.10.26) for 4×4 matrices can be generalized to square matrices of arbitrary order. More formally, the expressions (1.10.18) and (1.10.24) generalize to

$$\det A = \sum_{\sigma} \varepsilon_{\sigma} A_{\sigma(1)1} A_{\sigma(2)2} A_{\sigma(3)3} \cdots A_{\sigma(N)N} \tag{1.10.28}$$

for the determinant of a $N\times N$ matrix. Again, we see that the determinant is linear in each column and it is completely skew symmetric in each column. It is also true that an expansion by rows will yield the same result, i.e.,

$$\det A=\sum_{\sigma}\varepsilon_{\sigma}A_{1\sigma(1)}A_{2\sigma(2)}A_{3\sigma(3)}\cdots A_{N\sigma(N)} \tag{1.10.29}$$

and, thus, again it is displayed that

$$\det A=\det A^{T} \tag{1.10.30}$$

Example 1.10.3: The identity matrix in $\mathscr{M}^{N\times N}$ is given by (1.1.27), repeated,

$$I=\begin{bmatrix} 1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 1 & \cdot & \cdot & \cdot & 0 \\ \cdot & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ 0 & 0 & \cdot & \cdot & \cdot & 1 \end{bmatrix} \tag{1.10.31}$$

It follows from (1.10.28) or (1.10.29) that

$$\det I=1 \tag{1.10.32}$$

Example 1.10.4: A matrix that arises in multiple applications is the Vandermonde matrix.[6] It is a square matrix that takes the form

$$V=\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & \cdot & \cdot & x_N \\ x_1^2 & x_2^2 & x_3^2 & \cdot & \cdot & x_N^2 \\ \cdot & & & \cdot & & \cdot \\ \cdot & & & & \cdot & \cdot \\ x_1^{N-1} & x_2^{N-1} & x_3^{N-1} & \cdot & \cdot & x_N^{N-1} \end{bmatrix} \tag{1.10.33}$$

It turns out that the determinant of this matrix is

$$\det V=\prod_{\substack{i,j=1\\ i>j}}^{N}\left(x_i-x_j\right) \tag{1.10.34}$$

[6] The Vondermonde matrix is named after Alexandre-Theophile Vandermonde, a French musician and chemist. Information about Vandermonde can be found, for example, at http://en.wikipedia.org/wiki/Alexandre-Th%C3%A9ophile_Vandermonde.

Given (1.10.34) it is evident that the determinant of V is nonzero if the numbers $x_1, x_2, ..., x_N$ are distinct. This fact is evident from (1.10.33) because if two of the numbers were the same, the determinant would have two equal columns and, as a result, have a zero determinant. We shall first illustrate (1.10.34) in the case where $N = 3$ and use the expansion (1.10.6). The result is

$$\begin{aligned}
\det V = \begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ x_1^2 & x_2^2 & x_3^2 \end{vmatrix} &= 1\begin{vmatrix} x_2 & x_3 \\ x_2^2 & x_3^2 \end{vmatrix} - x_1 \begin{vmatrix} 1 & 1 \\ x_2^2 & x_3^2 \end{vmatrix} + x_1^2 \begin{vmatrix} 1 & 1 \\ x_2 & x_3 \end{vmatrix} \\
&= x_2 x_3 \begin{vmatrix} 1 & 1 \\ x_2 & x_3 \end{vmatrix} - x_1 \begin{vmatrix} 1 & 1 \\ x_2^2 & x_3^2 \end{vmatrix} + x_1^2 \begin{vmatrix} 1 & 1 \\ x_2 & x_3 \end{vmatrix} \\
&= x_2 x_3 (x_3 - x_2) - x_1 (x_3^2 - x_2^2) + x_1^2 (x_3 - x_2) \\
&= (x_3 - x_2)\left(x_2 x_3 - x_1 (x_3 + x_2) + x_1^2\right) \\
&= (x_3 - x_2)(x_3 - x_1)(x_2 - x_1)
\end{aligned} \tag{1.10.35}$$

The same kind of straight forward expansion based upon (1.10.26) yields

$$\det V = \begin{vmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ x_1^2 & x_2^2 & x_3^2 & x_4^2 \\ x_1^3 & x_2^3 & x_3^3 & x_4^3 \end{vmatrix} = (x_4 - x_3)(x_4 - x_2)(x_4 - x_1)(x_3 - x_2)(x_3 - x_1)(x_2 - x_1) \tag{1.10.36}$$

Example 1.10.5: One can continue the individual expansions, as with the derivations of (1.10.35) and (1.10.36), and infer the general result (1.10.34). It is instructive to reach (1.10.34) by utilization of the property that a determinant is linear in each of its columns and derive a formula that works directly with the determinant

$$\det V = \begin{vmatrix} 1 & 1 & 1 & \cdot & \cdot & 1 \\ x_1 & x_2 & x_3 & \cdot & \cdot & x_N \\ x_1^2 & x_2^2 & x_3^2 & \cdot & \cdot & x_N^2 \\ \cdot & & & \cdot & & \cdot \\ \cdot & & & & \cdot & \cdot \\ x_1^{N-1} & x_2^{N-1} & x_3^{N-1} & \cdot & \cdot & x_N^{N-1} \end{vmatrix} \tag{1.10.37}$$

The first step is to subtract from (1.10.37) a series of determinants, each having values of zero, as follows:

$$\det V = \begin{vmatrix} 1 & 1 & 1 & \cdot & \cdot & 1 \\ x_1 & x_2 & x_3 & \cdot & \cdot & x_N \\ x_1^2 & x_2^2 & x_3^2 & \cdot & \cdot & x_N^2 \\ \cdot & & & \cdot & & \cdot \\ \cdot & & & & \cdot & \cdot \\ x_1^{N-1} & x_2^{N-1} & x_3^{N-1} & \cdot & \cdot & x_N^{N-1} \end{vmatrix} - \underbrace{\begin{vmatrix} 1 & 1 & 1 & \cdot & \cdot & 1 \\ x_1 & x_1 & x_1 & \cdot & \cdot & x_1 \\ x_1^2 & x_2^2 & x_3^2 & \cdot & \cdot & x_N^2 \\ \cdot & & & \cdot & & \cdot \\ \cdot & & & & \cdot & \cdot \\ x_1^{N-1} & x_2^{N-1} & x_3^{N-1} & \cdot & \cdot & x_N^{N-1} \end{vmatrix}}_{=0}$$

$$- \underbrace{\begin{vmatrix} 1 & 1 & 1 & \cdot & \cdot & 1 \\ x_1 & x_2 & x_3 & \cdot & \cdot & x_N \\ x_1^2 & x_1x_2 & x_1x_3 & \cdot & \cdot & x_1x_N \\ \cdot & & & \cdot & & \cdot \\ \cdot & & & & \cdot & \cdot \\ x_1^{N-1} & x_2^{N-1} & x_3^{N-1} & \cdot & \cdot & x_N^{N-1} \end{vmatrix}}_{=0} - \cdots - \underbrace{\begin{vmatrix} 1 & 1 & 1 & \cdot & \cdot & 1 \\ x_1 & x_2 & x_3 & \cdot & \cdot & x_N \\ x_1^2 & x_2^2 & x_3^2 & \cdot & \cdot & x_N^2 \\ \cdot & & & \cdot & & \cdot \\ \cdot & & & & \cdot & \cdot \\ x_1^{N-1} & x_1x_2^{N-2} & x_1x_3^{N-2} & \cdot & \cdot & x_1x_N^{N-2} \end{vmatrix}}_{=0} \tag{1.10.38}$$

The linearity property illustrated with equation (1.10.12) allows (1.10.38) to be written

$$\det V = \begin{vmatrix} 1 & 1 & 1 & \cdot & \cdot & 1 \\ 0 & x_2 - x_1 & x_3 - x_1 & \cdot & \cdot & x_N - x_1 \\ 0 & x_2^2 - x_1x_2 & x_3^2 - x_1x_3 & \cdot & \cdot & x_N^2 - x_1x_N \\ \cdot & & & \cdot & & \cdot \\ \cdot & & & & \cdot & \cdot \\ 0 & x_2^{N-1} - x_1x_2^{N-2} & x_3^{N-1} - x_1x_3^{N-2} & \cdot & \cdot & x_N^{N-1} - x_1x_N^{N-2} \end{vmatrix}$$

$$= (x_2 - x_1)(x_3 - x_1)\cdots(x_N - x_1)\begin{vmatrix} 1 & 1 & 1 & \cdot & \cdot & 1 \\ 0 & 1 & 1 & \cdot & \cdot & 1 \\ 0 & x_2 & x_3 & \cdot & \cdot & x_N \\ \cdot & & & \cdot & & \cdot \\ \cdot & & & & \cdot & \cdot \\ 0 & x_2^{N-2} & x_3^{N-2} & \cdot & \cdot & x_N^{N-2} \end{vmatrix}$$

$$= (x_2 - x_1)(x_3 - x_1)\cdots(x_N - x_1)\underbrace{\begin{vmatrix} 1 & 1 & \cdot & \cdot & 1 \\ x_2 & x_3 & & & x_N \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ x_2^{N-2} & x_3^{N-2} & \cdot & \cdot & x_N^{N-2} \end{vmatrix}}_{N\text{-1 dimensional Vandermonde Determinant}} \tag{1.10.39}$$

If the above process is repeated on the $N-1$ dimensional Vandermonde determinant, we obtain a product involving the $N-2$ dimensional Vandermonde determinant and so forth. This iterative scheme produces the result (1.10.34).

Another, equivalent, way of defining the determinant of a $N \times N$ matrix is to introduce the *minor* and the *cofactor* of a matrix.

Definition: Given a $N \times N$ matrix A, where $N > 1$, the determinant of the $(N-1)\times(N-1)$ matrix obtained from A by omitting the i^{th} row and the j^{th} column is the *minor* of the ij element of A.

It is customary to denote the minor of the ij element by M_{ij}.

Example 1.10.6: For a 3×3 matrix

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \tag{1.10.40}$$

it follows from the definition that

$$\begin{aligned} M_{11} &= \begin{vmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{vmatrix} & M_{12} &= \begin{vmatrix} A_{21} & A_{23} \\ A_{31} & A_{33} \end{vmatrix} & M_{13} &= \begin{vmatrix} A_{21} & A_{22} \\ A_{31} & A_{32} \end{vmatrix} \\ M_{21} &= \begin{vmatrix} A_{12} & A_{13} \\ A_{32} & A_{33} \end{vmatrix} & M_{22} &= \begin{vmatrix} A_{11} & A_{13} \\ A_{31} & A_{33} \end{vmatrix} & M_{23} &= \begin{vmatrix} A_{11} & A_{12} \\ A_{31} & A_{32} \end{vmatrix} \\ M_{31} &= \begin{vmatrix} A_{12} & A_{13} \\ A_{22} & A_{23} \end{vmatrix} & M_{32} &= \begin{vmatrix} A_{11} & A_{13} \\ A_{21} & A_{23} \end{vmatrix} & M_{33} &= \begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} \end{aligned} \tag{1.10.41}$$

Definition: The ij *cofactor* of the $n \times n$ matrix A is

$$\operatorname{cof} A_{ij} = (-1)^{i+j} M_{ij} \tag{1.10.42}$$

For the 3×3 example above the cofactors are

$$
\begin{aligned}
&\operatorname{cof} A_{11} = M_{11} = \begin{vmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{vmatrix}, \qquad \operatorname{cof} A_{12} = -M_{12} = -\begin{vmatrix} A_{21} & A_{23} \\ A_{31} & A_{33} \end{vmatrix}, \\
&\operatorname{cof} A_{13} = M_{13} = \begin{vmatrix} A_{21} & A_{22} \\ A_{31} & A_{32} \end{vmatrix}, \qquad \operatorname{cof} A_{21} = -M_{21} = -\begin{vmatrix} A_{12} & A_{13} \\ A_{32} & A_{33} \end{vmatrix}, \\
&\operatorname{cof} A_{22} = M_{22} = \begin{vmatrix} A_{11} & A_{13} \\ A_{31} & A_{33} \end{vmatrix}, \qquad \operatorname{cof} A_{23} = -M_{23} = -\begin{vmatrix} A_{11} & A_{12} \\ A_{31} & A_{32} \end{vmatrix}, \\
&\operatorname{cof} A_{31} = M_{31} = \begin{vmatrix} A_{12} & A_{13} \\ A_{22} & A_{23} \end{vmatrix}, \qquad \operatorname{cof} A_{32} = -M_{32} = -\begin{vmatrix} A_{11} & A_{13} \\ A_{21} & A_{23} \end{vmatrix}, \\
&\operatorname{cof} A_{33} = M_{33} = \begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix}
\end{aligned}
\tag{1.10.43}
$$

The results (1.10.43) allow (1.10.6), the formula for the determinant of a 3×3 matrix, to be rewritten

$$
\begin{aligned}
\det A &= A_{11}\begin{vmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{vmatrix} - A_{21}\begin{vmatrix} A_{12} & A_{13} \\ A_{32} & A_{33} \end{vmatrix} + A_{31}\begin{vmatrix} A_{12} & A_{13} \\ A_{22} & A_{23} \end{vmatrix} \\
&= A_{11}\operatorname{cof} A_{11} + A_{21}\operatorname{cof} A_{21} + A_{31}\operatorname{cof} A_{31}
\end{aligned}
\tag{1.10.44}
$$

Likewise, the formula (1.10.19) yields

$$
\begin{aligned}
\det A &= A_{11}\begin{vmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{vmatrix} - A_{12}\begin{vmatrix} A_{21} & A_{23} \\ A_{31} & A_{33} \end{vmatrix} + A_{13}\begin{vmatrix} A_{21} & A_{22} \\ A_{31} & A_{32} \end{vmatrix} \\
&= A_{11}\operatorname{cof} A_{11} + A_{12}\operatorname{cof} A_{12} + A_{13}\operatorname{cof} A_{13}
\end{aligned}
\tag{1.10.45}
$$

We can generalize (1.10.44) and (1.10.45) for $N\times N$ matrix A by the formulas

$$
\det A = \sum_{k=1}^{N} A_{k1}\operatorname{cof} A_{k1}
\tag{1.10.46}
$$

and

$$
\det A = \sum_{k=1}^{N} A_{1k}\operatorname{cof} A_{1k}
\tag{1.10.47}
$$

As with the 3×3 matrix discussed above, it is possible to write the determinant of a $N\times N$ matrix A as expansions about any column or any row. As a practical matter, one chooses the row or column so as to capitalize on as many zeros as possible. In any case, the formula which reflects this last assertion is

$$\det A = \sum_{k=1}^{N} A_{ki} \operatorname{cof} A_{ki} \qquad i = 1,\dots,N \tag{1.10.48}$$

for an expansion about the i^{th} column and

$$\det A = \sum_{k=1}^{N} A_{jk} \operatorname{cof} A_{jk} \qquad j = 1,\dots,N \tag{1.10.49}$$

for an expansion about the j^{th} row. The fact that a determinant vanishes when two of the columns or rows are the same tells us that

$$0 = \sum_{k=1}^{N} A_{ki} \operatorname{cof} A_{kj} \qquad i \neq j \tag{1.10.50}$$

for an expansion about the i^{th} column and

$$0 = \sum_{k=1}^{N} A_{jk} \operatorname{cof} A_{ik} \qquad j \neq i \tag{1.10.51}$$

for expansion about the j^{th} row. Fortunately, the four formulas (1.10.48) through (1.10.51) can be written as two formulas if we utilize the Kronecker delta defined by (1.1.29), repeated,

$$\delta_{ij} = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases} \tag{1.10.52}$$

With this definition, equations (1.10.48) and (1.10.50) can be written

$$\delta_{ij} \det A = \sum_{k=1}^{N} A_{ki} \operatorname{cof} A_{kj} \qquad i, j = 1,\dots,N \tag{1.10.53}$$

and equations (1.10.49) and (1.10.51) can be written

$$\delta_{ij} \det A = \sum_{k=1}^{N} A_{ik} \operatorname{cof} A_{jk} \qquad i, j = 1,\dots,N \tag{1.10.54}$$

The right sides of equations (1.10.53) and (1.10.54) are of the same general form as the formula, $\sum_{j=1}^{N} A_{ij} B_{js}$, $i = 1,\dots,M$, $s = 1,\dots,K$, introduced in Section 1.1 for the product of two matrices A and B. We shall next exploit this fact and rewrite (1.10.53) and (1.10.54) as the product of two matrices. We have already introduced the symbols $A = \left[A_{ij}\right]$ and $I = \left[\delta_{ij}\right]$ for the matrix whose components are A_{ij} and δ_{ij}, respectively. The matrix whose components are the cofactors, $\operatorname{cof} A_{jk}$, is an $N \times N$ matrix called the *adjugate* matrix of A.[7] It is given the symbol $\operatorname{adj} A$. In order to accommodate the different order of the indices in (1.10.53) and (1.10.54), over that used in the definition of matrix multiplication, the definition of the adjugate involves the transpose of the matrix of cofactors is follows

$$\operatorname{adj} A = \underbrace{\begin{bmatrix} \operatorname{cof} A_{11} & \operatorname{cof} A_{21} & \operatorname{cof} A_{31} & \cdot & \cdot & \operatorname{cof} A_{N1} \\ \operatorname{cof} A_{12} & \operatorname{cof} A_{22} & \operatorname{cof} A_{32} & \cdot & \cdot & \operatorname{cof} A_{N2} \\ \operatorname{cof} A_{13} & \operatorname{cof} A_{23} & \operatorname{cof} A_{33} & & & \cdot \\ \cdot & & & \cdot & & \cdot \\ \cdot & & & & \cdot & \cdot \\ \operatorname{cof} A_{1N} & \cdot & \cdot & \cdot & \cdot & \operatorname{cof} A_{NN} \end{bmatrix}}_{\text{Transposed Matrix of cofactors of } A} \tag{1.10.55}$$

Thus, the adjugate matrix is simply the *transposed matrix of cofactors*.

Example 1.10.7: If A is the matrix (1.10.7), then

$$\begin{aligned} \operatorname{adj} A &= \begin{bmatrix} \begin{vmatrix} 2 & 2 \\ 2 & 3 \end{vmatrix} & -\begin{vmatrix} 3 & 2 \\ 1 & 3 \end{vmatrix} & \begin{vmatrix} 3 & 2 \\ 1 & 2 \end{vmatrix} \\ -\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} & \begin{vmatrix} 2 & 2 \\ 1 & 3 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} \\ \begin{vmatrix} 1 & 2 \\ 2 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & 2 \\ 3 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} \end{bmatrix}^T \\ &= \begin{bmatrix} \begin{vmatrix} 2 & 2 \\ 2 & 3 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 2 & 2 \end{vmatrix} \\ -\begin{vmatrix} 3 & 2 \\ 1 & 3 \end{vmatrix} & \begin{vmatrix} 2 & 2 \\ 1 & 3 \end{vmatrix} & -\begin{vmatrix} 2 & 2 \\ 3 & 2 \end{vmatrix} \\ \begin{vmatrix} 3 & 2 \\ 1 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} \end{bmatrix} \\ &= \begin{bmatrix} 2 & 1 & -2 \\ -7 & 4 & 2 \\ 4 & -3 & 1 \end{bmatrix} \end{aligned} \tag{1.10.56}$$

The definition (1.10.55) allows us to write (1.10.53) as the matrix equation

[7] Many textbooks use the word *adjoint* for what we have called *adjugate*. The adjoint name is growing less common because of another operation, which we shall see later, given the same name.

$$(\operatorname{adj} A)A = (\det A)I \tag{1.10.57}$$

Likewise, we can write (1.10.54) as

$$A(\operatorname{adj} A) = (\det A)I \tag{1.10.58}$$

Thus, the adjugate matrix, $\operatorname{adj} A$, has the interesting property that when it multiplies A on the left or the right, one gets the diagonal matrix $(\det A)I$. These last two results lead us to an important result in matrix algebra. Namely, *when the determinant of a matrix A is nonzero, it is nonsingular*. This assertion follows by recalling from Section 1.1 that a matrix A is nonsingular if there exists a matrix, which we wrote as A^{-1}, that obeys

$$AA^{-1} = A^{-1}A = I \tag{1.10.59}$$

If $\det A \neq 0$, the two equations (1.10.57) and (1.10.58) can be written

$$\left(\frac{\operatorname{adj} A}{\det A}\right)A = A\left(\frac{\operatorname{adj} A}{\det A}\right) = I \tag{1.10.60}$$

Thus, A^{-1} not only exists when $\det A \neq 0$, it is given explicitly by

$$A^{-1} = \frac{\operatorname{adj} A}{\det A} \tag{1.10.61}$$

Example 1.10.8: For the matrix (1.10.7) whose determinant is given by (1.10.8) and whose adjugate matrix is given by (1.10.56), its inverse is given by

$$A^{-1} = \frac{\operatorname{adj} A}{\det A} = \begin{bmatrix} \frac{2}{5} & \frac{1}{5} & -\frac{2}{5} \\ -\frac{7}{5} & \frac{4}{5} & \frac{2}{5} \\ \frac{4}{5} & -\frac{3}{5} & \frac{1}{5} \end{bmatrix} \tag{1.10.62}$$

Example 1.10.9: You are given the 2×2 matrix

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \tag{1.10.63}$$

The determinant of this matrix is given by (1.10.4). When we assume that $\det A$ is not zero, we are assuming that

$$A_{11}A_{22} - A_{12}A_{21} \neq 0 \tag{1.10.64}$$

From the definition of cofactor and the definition of the matrix $\operatorname{adj} A$, it is given by

$$\operatorname{adj} A = \begin{bmatrix} \operatorname{cof} A_{11} & \operatorname{cof} A_{21} \\ \operatorname{cof} A_{12} & \operatorname{cof} A_{22} \end{bmatrix} = \begin{bmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{bmatrix} \tag{1.10.65}$$

and, from (1.10.61) the inverse of A is given by

$$A^{-1} = \underbrace{\frac{1}{A_{11}A_{22} - A_{12}A_{21}}}_{\det A} \underbrace{\begin{bmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{bmatrix}}_{\substack{\text{Swap position of diagonals} \\ \text{Switch sign of off diagonals}}} \tag{1.10.66}$$

The theoretical formula (1.10.66) was used in Example 1.1.2 to calculate the inverse given in equation (1.1.32). This formula was given without proof in Exercise 1.1.7.

The equation $A^{-1} = \dfrac{\operatorname{adj} A}{\det A}$ is a valuable theoretical result which allows one to determine when a matrix is nonsingular and to actually calculate the inverse. The converse result, namely, that if A is nonsingular, then its determinant is nonzero is also true. This is a result we shall develop as soon as we derive one more theoretical result.

The theoretical result we need concerns the determinant of the product of two matrices. If A and B are in $\mathscr{M}^{N\times N}$ matrices then the product AB is also in $\mathscr{M}^{N\times N}$. The theoretical result we need is

$$\det AB = \det A \det B \tag{1.10.67}$$

This interesting and simple result is either easy or hard to prove depending upon how one introduces the idea of a determinant. The essential features of the general proof are well illustrated in the special case where $N = 2$. For this case, the product AB is

$$AB = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{bmatrix} \tag{1.10.68}$$

The determinant of this product requires that we expand

$$\det AB = \begin{vmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{vmatrix} \tag{1.10.69}$$

The fact that the determinant of a matrix is linear in the column or row vectors allows us to write (1.10.69) as

$$
\begin{aligned}
\det AB &= \underbrace{\begin{vmatrix} A_{11}B_{11}+A_{12}B_{21} & A_{11}B_{12}+A_{12}B_{22} \\ A_{21}B_{11}+A_{22}B_{21} & A_{21}B_{12}+A_{22}B_{22} \end{vmatrix}}_{\text{Use linearity in first column}} \\
&= \underbrace{\begin{vmatrix} A_{11}B_{11} & A_{11}B_{12}+A_{12}B_{22} \\ A_{21}B_{11} & A_{21}B_{12}+A_{22}B_{22} \end{vmatrix} + \begin{vmatrix} A_{12}B_{21} & A_{11}B_{12}+A_{12}B_{22} \\ A_{22}B_{21} & A_{21}B_{12}+A_{22}B_{22} \end{vmatrix}}_{\text{Use linearity in second column}} \\
&= \underbrace{\begin{vmatrix} A_{11}B_{11} & A_{11}B_{12} \\ A_{21}B_{11} & A_{21}B_{12} \end{vmatrix}}_{\substack{\text{Factor constants}\\ \text{from columns}}} + \underbrace{\begin{vmatrix} A_{11}B_{11} & A_{12}B_{22} \\ A_{21}B_{11} & A_{22}B_{22} \end{vmatrix}}_{\substack{\text{Factor constants}\\ \text{from columns}}} \\
&\quad + \underbrace{\begin{vmatrix} A_{12}B_{21} & A_{11}B_{12} \\ A_{22}B_{21} & A_{21}B_{12} \end{vmatrix}}_{\substack{\text{Factor constants}\\ \text{from columns}}} + \underbrace{\begin{vmatrix} A_{12}B_{21} & A_{12}B_{22} \\ A_{22}B_{21} & A_{22}B_{22} \end{vmatrix}}_{\substack{\text{Factor constants}\\ \text{from columns}}} \\
&= B_{11}B_{12}\underbrace{\begin{vmatrix} A_{11} & A_{11} \\ A_{21} & A_{21} \end{vmatrix}}_{\substack{\text{Identical}\\ \text{columns}\\ \Rightarrow\det\ \text{zero}}} + B_{11}B_{22}\begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} \\
&\quad + B_{21}B_{12}\begin{vmatrix} A_{12} & A_{11} \\ A_{22} & A_{21} \end{vmatrix} + B_{21}B_{22}\underbrace{\begin{vmatrix} A_{12} & A_{12} \\ A_{22} & A_{22} \end{vmatrix}}_{\substack{\text{Identical}\\ \text{columns}\\ \Rightarrow\det\ \text{zero}}} \\
&= 0 + B_{11}B_{22}\begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} + B_{21}B_{12}\underbrace{\begin{vmatrix} A_{12} & A_{11} \\ A_{22} & A_{21} \end{vmatrix}}_{\substack{\text{Switch colunns}\\ \text{change sign}}} + 0 \\
&= \left(B_{11}B_{22} - B_{21}B_{12}\right)\begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} = \det B \det A
\end{aligned}
\tag{1.10.70}
$$

The generalization of the above argument to $N \times N$ matrices requires a little care, but follows the same type of argument just utilized for 2×2 matrices.

Next, we shall utilize the formula (1.10.67) to prove that *if a matrix is nonsingular, its determinant is nonzero*. We begin by assuming we are given a matrix A in $\mathscr{M}^{N\times N}$. Because, by assumption A is nonsingular, there exists a matrix A^{-1} in $\mathscr{M}^{N\times N}$ such that

$$AA^{-1} = A^{-1}A = I \tag{1.10.71}$$

If we calculate the determinant of this equation and use (1.10.67) and (1.10.32), we obtain

$$\det AA^{-1} = \det A^{-1} \det A = \det I = 1 \tag{1.10.72}$$

The equation $\det A^{-1} \det A = 1$ rules out the possibility that the determinant of a nonsingular matrix can be zero.

In summary, we have learned that a *nonsingular matrix* A *in* $\mathscr{M}^{N\times N}$ *is nonsingular if and only if its determinant is nonzero.*

Exercises

1.10.1: Show that the determinant of the matrix

$$A = \begin{bmatrix} 4 & 3 & 0 \\ 3 & 1 & 2 \\ 5 & -1 & -4 \end{bmatrix} \tag{1.10.73}$$

is $\det A = 58$.

1.10.2: Show that the inverse of the matrix (1.10.73) is

$$A^{-1} = \frac{1}{58}\begin{bmatrix} -2 & 12 & 6 \\ 22 & -16 & -8 \\ -8 & 19 & -5 \end{bmatrix} \tag{1.10.74}$$

1.10.3: Show that the determinant of the matrix

$$A = \begin{bmatrix} 2 & -1 & 2 \\ 1 & 3 & 2 \\ 5 & 1 & 6 \end{bmatrix} \tag{1.10.75}$$

is $\det A = 0$.

1.10.4: Show that the determinant of the matrix

$$A = \begin{bmatrix} 3 & 3 & 1 \\ 0 & 1 & 2 \\ 0 & 2 & 3 \end{bmatrix} \tag{1.10.76}$$

is $\det A = -3$.

1.10.5: Show that the inverse of the matrix (1.10.76) is

$$A^{-1} = -\frac{1}{3}\begin{bmatrix} -1 & -7 & 5 \\ 0 & 9 & -6 \\ 0 & -6 & 3 \end{bmatrix} \tag{1.10.77}$$

1.10.6: Show that the determinant of the matrix

$$A = \begin{bmatrix} 2 & -1 & 3 \\ -1 & 2 & -2 \\ 1 & 4 & 0 \end{bmatrix} \tag{1.10.78}$$

is $\det A = 0$.

1.10.7: Show that the determinant of the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & -1 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 7 & 3 \end{bmatrix} \tag{1.10.79}$$

is $\det A = 0$.

1.10.8: Show that the determinant of the matrix

$$A = \begin{bmatrix} 2 & 1 & 2 & 1 \\ 3 & 0 & 1 & 1 \\ -1 & 2 & -2 & 1 \\ -3 & 2 & 3 & 1 \end{bmatrix} \tag{1.10.80}$$

is $\det A = 20$.

1.10.9: Show that the inverse of the matrix (1.10.80) is

$$A^{-1} = \frac{1}{20}\begin{bmatrix} 10 & -5 & 0 & -5 \\ 26 & -23 & 4 & -7 \\ 4 & -2 & -4 & 2 \\ -34 & 37 & 4 & 13 \end{bmatrix} \tag{1.10.81}$$

1.10.10: If A is in $\mathscr{M}^{N\times N}$ and is an upper triangle matrix

$$A=\begin{bmatrix} A_{11} & A_{12} & A_{13} & \cdot & \cdot & A_{1N} \\ 0 & A_{22} & A_{23} & \cdot & \cdot & A_{2N} \\ \cdot & 0 & A_{33} & & & \cdot \\ \cdot & & & \cdot & & \cdot \\ 0 & & & & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & 0 & A_{NN} \end{bmatrix} \tag{1.10.82}$$

show that

$$\det A = A_{11}A_{22}A_{33}\cdots A_{NN} \tag{1.10.83}$$

Thus, the determinant of an upper triangular square matrix is simply the product of the diagonal elements. A similar result holds for lower triangular matrices and, trivially, for diagonal matrices.

1.10.11 If A is in $\mathscr{M}^{N\times N}$ and has the LU decomposition (1.7.3), show that

$$\det A = \det U \tag{1.10.84}$$

In certain numerical applications, it is numerically better to evaluate the determinant with (1.10.84) after a LU Decomposition.

1.10.12 If A and B are in $\mathscr{M}^{N\times N}$ show that

$$\operatorname{adj}(AB) = \operatorname{adj} B \operatorname{adj} A \tag{1.10.85}$$

1.10.13 If A is a nonsingular matrix, establish the following properties of the adjugate matrix:

$$\begin{aligned} \det(\operatorname{adj} A) &= (\det A)^{N-1} \\ \operatorname{adj}(\operatorname{adj} A) &= (\det A)^{N-2} A \\ \det(\operatorname{adj}(\operatorname{adj} A)) &= (\det A)^{(N-2)^2} \end{aligned} \tag{1.10.86}$$

1.10.14 If A is nonsingular, show that $\operatorname{adj} A$ is nonsingular and given by

$$(\operatorname{adj} A)^{-1} = \operatorname{adj} A^{-1} \tag{1.10.87}$$

1.10.15 Show that $\operatorname{adj} A^T$ is given by

$$\operatorname{adj} A^T = \left(\operatorname{adj} A\right)^T \tag{1.10.88}$$

1.10.16 Show that the adjugate and the inverse of the Vandermonde matrix

$$V = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ x_1^2 & x_2^2 & x_3^2 \end{bmatrix} \tag{1.10.89}$$

are given by

$$\operatorname{adj} V = \begin{bmatrix} x_2 x_3 \left(x_3 - x_2\right) & x_2^2 - x_3^2 & x_3 - x_2 \\ x_1 x_3 \left(x_1 - x_3\right) & x_3^2 - x_1^2 & x_1 - x_3 \\ x_1 x_2 \left(x_2 - x_1\right) & x_1^2 - x_2^2 & x_2 - x_1 \end{bmatrix} \tag{1.10.90}$$

and

$$V^{-1} = \begin{bmatrix} \dfrac{x_2 x_3}{\left(x_1 - x_2\right)\left(x_1 - x_3\right)} & -\dfrac{x_2 + x_3}{\left(x_1 - x_2\right)\left(x_1 - x_3\right)} & \dfrac{1}{\left(x_1 - x_2\right)\left(x_1 - x_3\right)} \\ -\dfrac{x_1 x_3}{\left(x_1 - x_2\right)\left(x_2 - x_3\right)} & \dfrac{x_1 + x_3}{\left(x_1 - x_2\right)\left(x_2 - x_3\right)} & -\dfrac{1}{\left(x_1 - x_2\right)\left(x_2 - x_3\right)} \\ \dfrac{x_1 x_2}{\left(x_1 - x_3\right)\left(x_2 - x_3\right)} & -\dfrac{x_1 + x_2}{\left(x_1 - x_3\right)\left(x_2 - x_3\right)} & \dfrac{1}{\left(x_1 - x_3\right)\left(x_2 - x_3\right)} \end{bmatrix} \tag{1.10.91}$$

1.10.17 There are applications where the elements of a matrix depend upon a parameter and it is necessary to differentiate the determinant with respect to this parameter. If, for example, $N = 3$ and the matrix is written

$$\det A(t) = \begin{vmatrix} A_{11}(t) & A_{12}(t) & A_{13}(t) \\ A_{21}(t) & A_{22}(t) & A_{23}(t) \\ A_{31}(t) & A_{32}(t) & A_{33}(t) \end{vmatrix} \tag{1.10.92}$$

show that

$$\begin{aligned}\frac{d\left(\det A(t)\right)}{dt} &= \begin{vmatrix} \dfrac{dA_{11}(t)}{dt} & A_{12}(t) & A_{13}(t) \\ \dfrac{dA_{21}(t)}{dt} & A_{22}(t) & A_{23}(t) \\ \dfrac{dA_{31}(t)}{dt} & A_{32}(t) & A_{33}(t) \end{vmatrix} + \begin{vmatrix} A_{11}(t) & \dfrac{dA_{12}(t)}{dt} & A_{13}(t) \\ A_{21}(t) & \dfrac{dA_{22}(t)}{dt} & A_{23}(t) \\ A_{31}(t) & \dfrac{dA_{32}(t)}{dt} & A_{33}(t) \end{vmatrix} + \begin{vmatrix} A_{11}(t) & A_{12}(t) & \dfrac{dA_{13}(t)}{dt} \\ A_{21}(t) & A_{22}(t) & \dfrac{dA_{23}(t)}{dt} \\ A_{31}(t) & A_{32}(t) & \dfrac{dA_{33}(t)}{dt} \end{vmatrix} \\ &= \operatorname{tr}\left(\left(\operatorname{adj} A(t)\right)\frac{dA(t)}{dt}\right)\end{aligned} \tag{1.10.93}$$

Section 1.11. Systems of Linear Equations: Cramer's Rule

The formula (1.10.61), repeated,

$$A^{-1} = \frac{\operatorname{adj} A}{\det A} \tag{1.11.1}$$

is useful when one wants to calculate the inverse. While they are numerically more useful algorithms for large systems, (1.11.1) is a formula with a lot of applications. One of these arises when one knows that a matrix $A \in \mathscr{M}^{N\times N}$ is nonsingular and the goal is to find the solution to the linear system

$$A\mathbf{x} = \mathbf{b} \tag{1.11.2}$$

We know from Section 1.6 that this system has a unique solution when the matrix A is nonsingular. Equation (1.11.1) allows us to write that solution in the explicit form

$$\mathbf{x} = A^{-1}\mathbf{b} = \frac{1}{\det A}\left(\operatorname{adj} A\right)\mathbf{b} \tag{1.11.3}$$

Example 1.11.1: In the special case $N = 2$, we can use the formula (1.10.66) to express the solution (1.11.3) as

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{A_{11}A_{22} - A_{12}A_{21}} \begin{bmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} \dfrac{b_1A_{22} - b_2A_{12}}{A_{11}A_{22} - A_{12}A_{21}} \\ \dfrac{-b_1A_{21} + b_2A_{11}}{A_{11}A_{22} - A_{12}A_{21}} \end{bmatrix} \tag{1.11.4}$$

One way the solution (1.11.4) is sometimes written is

$$x_1 = \frac{\begin{vmatrix} b_1 & A_{12} \\ b_2 & A_{22} \end{vmatrix}}{\begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix}} \qquad \text{and} \qquad x_2 = \frac{\begin{vmatrix} A_{11} & b_1 \\ A_{12} & b_2 \end{vmatrix}}{\begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix}} \tag{1.11.5}$$

Equation (1.11.5) is an example of Cramer's Rule[8]. One simply places in the numerator of each formula the determinant formed from the determinant of the matrix of coefficients except that the first column is replaced by the components of $\mathbf{b}$ in the formula for x_1 and the second column is replaced by the components of $\mathbf{b}$ in the formula for x_2.

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[8] This rule is named after the Swiss mathematician Gabriel Cramer. Information about Gabriel Cramer can be found, for example, at http://en.wikipedia.org/wiki/Gabriel_Cramer.

Just as (1.11.5) follows from (1.11.3), Cramer's rule for systems of arbitrary order also follow from (1.11.3). The first step is to express (1.11.3) in the component form

$$x_j = \frac{1}{\det A}\sum_{i=1}^{N}\left(\operatorname{cof} A_{ij}\right)b_i \tag{1.11.6}$$

Just as (1.10.53) expresses the determinant as a cofactor expansion, one can recognize the numerator of (1.11.6) as a determinant. Except that it is a determinant with the j^{th} column of A replaced by the components of $\mathbf{b}$. For example, in the case $N = 4$, the four solutions are given by

$$x_1 = \frac{\begin{vmatrix} b_1 & A_{12} & A_{13} & A_{14} \\ b_2 & A_{22} & A_{23} & A_{24} \\ b_3 & A_{32} & A_{33} & A_{34} \\ b_4 & A_{42} & A_{43} & A_{44} \end{vmatrix}}{\begin{vmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{vmatrix}} \qquad x_2 = \frac{\begin{vmatrix} A_{11} & b_1 & A_{13} & A_{14} \\ A_{21} & b_2 & A_{23} & A_{24} \\ A_{31} & b_3 & A_{33} & A_{34} \\ A_{41} & b_4 & A_{43} & A_{44} \end{vmatrix}}{\begin{vmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{vmatrix}}$$

$$x_3 = \frac{\begin{vmatrix} A_{11} & A_{12} & b_1 & A_{14} \\ A_{21} & A_{22} & b_2 & A_{24} \\ A_{31} & A_{32} & b_3 & A_{34} \\ A_{41} & A_{42} & b_4 & A_{44} \end{vmatrix}}{\begin{vmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{vmatrix}} \qquad x_4 = \frac{\begin{vmatrix} A_{11} & A_{12} & A_{13} & b_1 \\ A_{21} & A_{22} & A_{23} & b_2 \\ A_{31} & A_{32} & A_{33} & b_3 \\ A_{41} & A_{42} & A_{43} & b_4 \end{vmatrix}}{\begin{vmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{vmatrix}} \tag{1.11.7}$$

Example 1.11.2: In Example 1.3.1, we solved the system (1.3.2), repeated,

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 1 \\ 2x_1 - x_2 + x_3 &= 3 \\ -x_1 + 2x_2 + 3x_3 &= 7 \end{aligned} \tag{1.11.8}$$

Cremer's rule tells us that if $\det A \neq 0$ the solution is

$$x_1 = \frac{\begin{vmatrix} b_1 & A_{12} & A_{13} \\ b_2 & A_{22} & A_{23} \\ b_3 & A_{32} & A_{33} \end{vmatrix}}{\begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix}} = \frac{\begin{vmatrix} 1 & 2 & -1 \\ 3 & -1 & 1 \\ 7 & 2 & 3 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & -1 \\ 2 & -1 & 1 \\ -1 & 2 & 3 \end{vmatrix}} = \frac{1(-5)-3(8)+7(1)}{1(-5)-2(8)-1(1)} = \frac{-22}{-22} = 1$$

$$x_2 = \frac{\begin{vmatrix} A_{11} & b_1 & A_{13} \\ A_{21} & b_2 & A_{23} \\ A_{31} & b_3 & A_{33} \end{vmatrix}}{\begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix}} = \frac{\begin{vmatrix} 1 & 1 & -1 \\ 2 & 3 & 1 \\ -1 & 7 & 3 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & -1 \\ 2 & -1 & 1 \\ -1 & 2 & 3 \end{vmatrix}} = \frac{1(2)-2(10)-1(4)}{1(-5)-2(8)-1(1)} = \frac{-22}{-22} = 1$$

$$x_3 = \frac{\begin{vmatrix} A_{11} & A_{12} & b_1 \\ A_{21} & A_{22} & b_2 \\ A_{31} & A_{32} & b_3 \end{vmatrix}}{\begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix}} = \frac{\begin{vmatrix} 1 & 2 & 1 \\ 2 & -1 & 3 \\ -1 & 2 & 7 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & -1 \\ 2 & -1 & 1 \\ -1 & 2 & 3 \end{vmatrix}} = \frac{1(-13)-2(12)-1(7)}{1(-5)-2(8)-1(1)} = \frac{-44}{-22} = 2 \tag{1.11.9}$$

which, of course, is the earlier result (1.3.10).

Exercises

1.11.1: Utilize Cremer's rule to find the solution of the system

$$\begin{aligned} 2x_1 + 4x_2 &= 9 \\ -2x_1 + x_2 &= 6 \end{aligned} \tag{1.11.10}$$

1.11.2: Utilize Cremer's rule to find the solution of the system

$$\begin{aligned} 2x_1 + 3x_2 + x_3 &= 9 \\ x_1 + 2x_2 + 3x_3 &= 6 \\ 3x_1 + x_2 + 2x_3 &= 8 \end{aligned} \tag{1.11.11}$$

1.11.3: Utilize Cremer's rule to find the solution of the system

$$\begin{aligned} 3x_1 - x_2 + 2x_3 &= 6 \\ 2x_1 + x_2 + x_3 &= 6 \\ x_1 - 3x_2 &= 6 \end{aligned} \tag{1.11.12}$$

1.11.4: Utilize Cremer's rule to find the solution of the system introduced in Exercise 1.3.2, i.e., the system

$$\begin{aligned} x_1 + 3x_2 + x_3 &= 1 \\ 2x_1 + x_2 + x_3 &= 5 \\ -2x_1 + 2x_2 - x_3 &= -8 \end{aligned} \tag{1.11.13}$$

1.11.5: Utilize Cremer's rule to find the solution of the system introduced in Exercise 1.3.3, i.e., the system

$$\begin{aligned} 2x_2 + 3x_3 &= 8 \\ 4x_1 + 6x_2 + 7x_3 &= -3 \\ 2x_1 - 3x_2 + 6x_3 &= 5 \end{aligned} \tag{1.11.14}$$

1.11.6: Utilize Cremer's rule to show that the solution of the system

$$\begin{aligned} x_1 + 2x_2 + 1x_3 &= 5 \\ 2x_1 + 2x_2 + 1x_3 &= 6 \\ x_1 + 2x_2 + 3x_3 &= 9 \end{aligned} \tag{1.11.15}$$

is $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$.

1.11.7: Utilize Cremer's rule to show that the solution of the system

$$\begin{aligned} x_2 + x_3 + x_4 &= 0 \\ 3x_1 + 3x_3 - 4x_4 &= 7 \\ x_1 + x_2 + x_3 + 2x_4 &= 6 \\ 2x_1 + 3x_2 + x_3 + 3x_4 &= 6 \end{aligned} \tag{1.11.16}$$

is $\mathbf{x} = \begin{bmatrix} 4 \\ -3 \\ 1 \\ 2 \end{bmatrix}$

Chapter 2

VECTOR SPACES

Chapter 1 consisted of a quick summary of a lot of topics in matrix algebra. Matrices represent an excellent introduction and example of a concept known as a *vector space*. Another example is the geometric one in the form of a directed line segment. This one is usually represented graphically as a straight line with an arrowhead. The topics of this chapter involve abstracting and generalizing the matrix algebra concepts from Chapter 1 and the geometric concepts of a directed line segment. The generalization leads to the study of *Vector Spaces* or, equivalently, *Linear Spaces*. The concept of a vector space put forward in this chapter is purely algebraic. It is a set with a prescribed list of properties. After the abstract idea of a vector space is introduced, it will be illustrated with examples that should provide a good connection with the discussions in Chapter 1 plus a connection with that of a vector as a directed line segment. Other connections will be made with examples of vector spaces that arise in other subjects.

Section 2.1. The Axioms for a Vector Space

Before, we list the formal axioms that define a vector space, it is useful to introduce a few fundamental concepts. One of the basic building blocks to assigning structure to a set is a function called a *binary relation*. Prior to giving this definition, we need to be sure we understand the idea of the *Cartesian product* of two sets.

Definition: If $\mathscr{A}$ and $\mathscr{B}$ are two sets, their *Cartesian product* is a set denoted by $\mathscr{A}\times\mathscr{B}$ defined by

$$\mathscr{A}\times\mathscr{B}=\left\{(a,b)\middle| a\in\mathscr{A} \text{ and } b\in\mathscr{B}\right\} \tag{2.1.1}$$

In words, the Cartesian product is simply a set of *ordered pairs*, the first element from $\mathscr{A}$ and the next element from $\mathscr{B}$.

It should be evident how to generalize the above definition to the Cartesian product of an arbitrary number of sets. We have already seen this definition, for example, when we wrote $\mathscr{R}^N$ in equation (1.1.1) as short hand for what is really $\underbrace{\mathscr{R}\times\mathscr{R}\times\cdots\times\mathscr{R}}_{N \text{ times}}$.

Definition: A *binary relation* on a set $\mathscr{V}$ is a function $\mathscr{V}\times\mathscr{V}\to\mathscr{V}$.

In words, a binary relation on a set $\mathscr{V}$ is simply a function that takes a *pair* of elements of $\mathscr{V}$ and produces an element of $\mathscr{V}$.

A *vector space* is a triple, i.e. a list of three quantities, written $(\mathscr{V},\mathscr{F},f)$. The first quantity, denoted by $\mathscr{V}$, is a set known as an additive *Abelian* or *commutative* group. This is sort of a mouthful. More importantly, it is a set with a property call "addition" which we shall note by $+$. This property, or function, is a rule which takes two members of $\mathscr{V}$ and produces another member of $\mathscr{V}$ according to a certain set of rules. The rule is an example of the *binary relation* or *binary operation* just introduced. In the notation above, the $+$ function is the rule

$$+:\mathscr{V}\times\mathscr{V}\to\mathscr{V} \tag{2.1.2}$$

If $\mathbf{u},\mathbf{v}\in\mathscr{V}$, we write the value of the $+$ function as $\mathbf{u}+\mathbf{v}$.

This function has properties defined by the following four rules:

(a) There exists a binary operation in $\mathscr{V}$ called *addition* and denoted by $+$ such that

(1) $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$ for all $\mathbf{u},\mathbf{v},\mathbf{w}\in\mathscr{V}$.
(2) $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$ for all $\mathbf{u},\mathbf{v}\in\mathscr{V}$.
(3) There exists an element $\mathbf{0}\in\mathscr{V}$ such that $\mathbf{u}+\mathbf{0}=\mathbf{u}$ for all $\mathbf{u}\in\mathscr{V}$.
(4) For every $\mathbf{u}\in\mathscr{V}$ there exists an element $-\mathbf{u}\in\mathscr{V}$. such that $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$.

Rule (a1) is an *associative* rule, (a2) is a *commutative* rule, (a3) specifies the existence of an additive *identity* and (a4) specifies the existence of an additive *inverse*. . In this definition the vector $\mathbf{u}+\mathbf{v}$ in $\mathscr{V}$ is called the *sum* of $\mathbf{u}$ and $\mathbf{v}$ and, when needed, the *difference* of $\mathbf{u}$ and $\mathbf{v}$ is written $\mathbf{u}-\mathbf{v}$ and is defined by

$$\mathbf{u}-\mathbf{v}=\mathbf{u}+(-\mathbf{v}) \tag{2.1.3}$$

The second quantity, denoted by $\mathscr{F}$ in the definition of a vector space, is an algebraic structure known as a *field*. This structure was briefly introduced in Section 1.1. For the sake of completeness, we shall repeat the definitions here.

A field is a set of quantities $\mathscr{F}$ which is equipped with *two* binary operations $+$ and $\cdot$ such that

(b1) The binary operation + is a commutative group whose binary operation is called *addition* and whose members, therefore, obey

(1)	$\lambda+\beta=\beta+\lambda$	(commutative)
(2)	$(\lambda+\beta)+\gamma=\lambda+(\beta+\gamma)$	(associative)
(3)	$0+\alpha=\alpha$	(identity)
(4)	For every $\alpha\in\mathscr{F}$, there exists an element, written $-\alpha$, such that $\alpha+(-\alpha)=0$.	(inverse)

(b2) The binary operation $\cdot$ is a binary operation on $\mathscr{F}$, called *multiplication*, whose members obey

(1) $\lambda \cdot \beta = \beta \cdot \lambda$ (commutative)
(2) $(\lambda \cdot \beta) \cdot \gamma = \lambda \cdot (\beta \cdot \gamma)$ (associative)
(3) $1 \cdot \lambda = \lambda$ (identity)
(4) For every element $\alpha \in \mathscr{F}$, there exists an element, written α^{-1}, such that $\alpha \cdot (\alpha^{-1}) = (\alpha^{-1}) \cdot \alpha = 1$ (inverse)

(b3) The multiplication operation distributes over addition such that

(1) $\lambda \cdot (\beta + \gamma) = \lambda \cdot \beta + \lambda \cdot \gamma$
(2) $(\lambda + \beta) \cdot \gamma = \lambda \cdot \gamma + \beta \cdot \gamma$

It is conventional in the following not to indicate multiplication between elements of fields by a special symbol such as $\cdot$. In the following, our convention will be to write

$$\lambda\beta \equiv \lambda \cdot \beta \tag{2.1.4}$$

and the $\cdot$ will be understood.

Example 2.1.1: Examples of Fields

a) The two most important examples of fields we shall confront in this course are the two introduced in Chapter 1. Namely, set of real numbers $\mathscr{R}$ and the set of complex numbers $\mathscr{C}$.
b) The set of rational numbers can be shown to form a field. Recall that rational numbers are real numbers of the form α/β where α and β are integers ($\beta \neq 0$).

Example 2.1.2: An example of a subset of $\mathscr{R}$ that is *not* a field is the set of integers. They are not a field when you use the usual definitions of addition and multiplication. The problem is the inverse under multiplication. The integers do have inverses that are real numbers but they are not integers.

The binary operations we have introduced thus far are examples of *internal* binary operations. Addition in $\mathscr{V}$ is an operation on elements of $\mathscr{V}$. Likewise, the two operations we have defined on $\mathscr{F}$ are operations on elements of $\mathscr{F}$. The third element of the triple $(\mathscr{V}, \mathscr{F}, f)$ is the function f which takes an element of $\mathscr{F}$ and an element of $\mathscr{V}$ and, through a specific set of rules, produces an element of $\mathscr{V}$. More formally, f is a function $f: \mathscr{F} \times \mathscr{V} \to \mathscr{V}$, called *scalar multiplication*, such that

(c)

(1) $f\left(\lambda, f\left(\mu,\mathbf{v}\right)\right)= f\left(\lambda\mu,\mathbf{v}\right)$

(2) $f\left(\lambda+\mu,\mathbf{u}\right)= f\left(\lambda,\mathbf{u}\right)+ f\left(\mu,\mathbf{u}\right)$

(3) $f\left(\lambda,\mathbf{u}+\mathbf{v}\right)= f\left(\lambda,\mathbf{u}\right)+ f\left(\lambda,\mathbf{v}\right)$

(4) $f\left(1,\mathbf{v}\right)=\mathbf{v}$

for all $\lambda,\mu\in\mathscr{F}$ and all $\mathbf{u},\mathbf{v}\in\mathscr{V}$.

The operation $f:\mathscr{F}\times\mathscr{V}\rightarrow\mathscr{V}$ is an example of an *external* operation. It takes two elements from different sets in order to produce the result.

The elements of $\mathscr{V}$ are called *vectors*. The elements of the field $\mathscr{F}$ are called *scalars*. The notation $\left(\mathscr{V},\mathscr{F},f\right)$ for a vector space will be shortened to simply $\mathscr{V}$. It is also customary to use a simplified notation for the scalar multiplication function f. It is customary to write

$$f\left(\lambda,\mathbf{v}\right)=\lambda\mathbf{v} \tag{2.1.5}$$

and *also* regard $\lambda\mathbf{v}$ and $\mathbf{v}\lambda$ to be identical. In this simplified notation we shall now list in detail the axioms of a vector space.

Definition of Vector Space Restated:

Definition. Let $\mathscr{V}$ be a set and $\mathscr{F}$ a field. $\mathscr{V}$ is a *vector space* if it satisfies the following rules:

(a) There exists a binary operation in $\mathscr{V}$ called *addition* and denoted by $+$ such that

(1) $\left(\mathbf{u}+\mathbf{v}\right)+\mathbf{w}=\mathbf{u}+\left(\mathbf{v}+\mathbf{w}\right)$ for all $\mathbf{u},\mathbf{v},\mathbf{w}\in\mathscr{V}$.

(2) $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$ for all $\mathbf{u},\mathbf{v}\in\mathscr{V}$.

(3) There exists an element $\mathbf{0}\in\mathscr{V}$ such that $\mathbf{u}+\mathbf{0}=\mathbf{u}$ for all $\mathbf{u}\in\mathscr{V}$.

(4) For every $\mathbf{u}\in\mathscr{V}$ there exists an element $-\mathbf{u}\in\mathscr{V}$. such that $\mathbf{u}+\left(-\mathbf{u}\right)=\mathbf{0}$.

(b) There exists an operation called *scalar multiplication* in which every scalar $\lambda\in\mathscr{F}$ can be combined with every element $\mathbf{u}\in\mathscr{V}$ to give an element $\lambda\mathbf{u}\in\mathscr{V}$ such that

(1) $\lambda\left(\mu\mathbf{u}\right)=\left(\lambda\mu\right)\mathbf{u}$

(2) $\left(\lambda+\mu\right)\mathbf{u}=\lambda\mathbf{u}+\mu\mathbf{u}$

(3) $\lambda\left(\mathbf{u}+\mathbf{v}\right)=\lambda\mathbf{u}+\lambda\mathbf{v}$

(4) $1\mathbf{u}=\mathbf{u}$ for all $\lambda,\mu\in\mathscr{F}$ and all $\mathbf{u},\mathbf{v}\in\mathscr{V}$.

If the field $\mathscr{F}$ employed in a vector space is actually the field of real numbers $\mathscr{R}$, the space is called a *real vector space*. A *complex vector space* is similarly defined. For the most part, it is useful to think of the vector spaces being discussed here as complex. Many of our examples will be for real vectors spaces as a special case.

There are many and varied sets of objects that qualify as vector spaces. The following is a partial list:

a) A trivial example is the set consisting of the zero element $\mathbf{0}$ of any vector space. This single element is a vector space.

b) The set of complex numbers $\mathscr{C}$, with the usual definitions of addition and multiplication by an element of $\mathscr{R}$ forms a real vector space.

c) The vector space $\mathscr{C}^N$ is the set of all N-tuples of the form $\mathbf{u}=\left(\lambda_1,\lambda_2,...,\lambda_N\right)$, where N is a positive integer and $\lambda_1,\lambda_2,...,\lambda_N\in\mathscr{C}$. Since an N-tuple is an ordered set, if $\mathbf{v}=\left(\mu_1,\mu_2,...,\mu_N\right)$ is a second N-tuple, then $\mathbf{u}$ and $\mathbf{v}$ are equal if and only if

$$\mu_k=\lambda_k \quad \text{for all } k=1,2,...,N \tag{2.1.6}$$

The zero N-tuple is $\mathbf{0}=\left(0,0,...,0\right)$ and the negative of the N-tuple $\mathbf{u}$ is $-\mathbf{u}=\left(-\lambda_1,-\lambda_2,...,-\lambda_N\right)$. Addition and scalar multiplication of N-tuples are defined by the formulas

$$\mathbf{u}+\mathbf{v}=\left(\mu_1+\lambda_1,\mu_2+\lambda_2,...,\mu_N+\lambda_N\right) \tag{2.1.7}$$

and

$$\mu\mathbf{u}=\left(\mu\lambda_1,\mu\lambda_2,....,\mu\lambda_N\right) \tag{2.1.8}$$

respectively. The notation $\mathscr{C}^N$ is used for this vector space because it can be considered to be an N^{th} Cartesian product of $\mathscr{C}$.

d) The set $\mathscr{V}=\mathscr{M}^{M\times N}$ of all $N\times M$ complex matrices is a vector space with respect to the usual operation of matrix addition and multiplication by a complex number. Of course, the zero matrix and the negative matrix (additive inverse) are defined as in Chapter 1.

e) Let $\mathscr{H}$ be a vector space whose vectors are actually functions defined on a set $\mathscr{A}$ with values in $\mathscr{C}$. Thus, if $\mathbf{h}\in\mathscr{H}$, $x\in\mathscr{A}$ then $\mathbf{h}\left(x\right)\in\mathscr{C}$ and $\mathbf{h}:\mathscr{A}\rightarrow\mathscr{C}$. If $\mathbf{k}$ is another vector of $\mathscr{H}$ then equality of vectors (functions) is defined by

$$\mathbf{h} = \mathbf{k} \text{ if and only if } \mathbf{h}(x) = \mathbf{k}(x) \qquad \text{for all } x \in \mathscr{A} \tag{2.1.9}$$

The zero vector (function) is given the symbol $\mathbf{0}$ and is defined as the function on $\mathscr{A}$ whose value is zero for all x. Addition and scalar multiplication are defined by

$$(\mathbf{h}+\mathbf{k})(x) = \mathbf{h}(x) + \mathbf{k}(x) \qquad \text{for all } x \in \mathscr{A} \tag{2.1.10}$$

and

$$(\lambda\mathbf{h})(x) = \lambda(\mathbf{h}(x)) \qquad \text{for all } x \in \mathscr{A} \tag{2.1.11}$$

respectively.

f) Example 3 is frequently stated in a more elementary context. Recall that $[a,b] = \{x \mid a \le x \le b\}$ is a closed interval of the set of real numbers and $(a,b) = \{x \mid a < x < b\}$ is an open interval of the set of real numbers. Let $C[a,b]$ be the set of real valued functions defined by

$$C[a,b] = \{f \mid f \text{ continuous on every open subinterval of } [a,b]\} \tag{2.1.12}$$

i.e., the set of all real valued functions defined on every closed interval $[a,b]$ and continuous on (a,b). The set $C[a,b]$ is a vector space providing one defines addition and scalar multiplication as above with (2.1.10) and (2.1.11). In particular addition is defined by

$$(f+g)(x) = f(x) + g(x) \qquad \text{for all } x \in [a,b] \tag{2.1.13}$$

and scalar multiplication by

$$(\lambda f)(x) = \lambda(f(x)) \qquad \text{for all } x \in [a,b] \tag{2.1.14}$$

g) Let $C^2[a,b]$ be the set of real valued functions defined by

$$C^2[a,b] = \{f \mid f \text{ twice differentiable on every open subinterval of } [a,b]\} \tag{2.1.15}$$

If the above definitions (2.1.13) and (2.1.14) of addition and scalar multiplication are adopted, it is easily shown that $C^2[a,b]$ is a vector space.

h) Let $\mathscr{P}_N$ denote the set of all polynomials p of degree equal to or less than N defined, for all $x \in \mathscr{C}$, by

$$p(x) = \lambda_0 + \lambda_1 x + \lambda_2 x^2 + \cdots + \lambda_N x^N \tag{2.1.16}$$

where $\lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_N \in \mathscr{R}$. The zero polynomial in $\mathscr{P}_N$ is given the symbol 0 and, formally, is defined by

$$0(x) = 0 + 0x + 0x^2 + \cdots + 0x^N \tag{2.1.17}$$

The set $\mathscr{P}_N$ forms a vector space over the complex numbers $\mathscr{C}$ if addition and scalar multiplication of polynomials are defined in the usual way, i.e., by

$$(p_1 + p_2)(x) = p_1(x) + p_2(x) \text{ for all } x \in \mathscr{R} \tag{2.1.18}$$

and

$$(\lambda p)(x) = \lambda p(x) \qquad \text{for all } x \in \mathscr{R} \tag{2.1.19}$$

Exercises

2.1.1 Let $\mathscr{V}$ and $\mathscr{U}$ be vector spaces. Show that the set $\mathscr{V} \times \mathscr{U}$ is a vector space with the definitions

$$(\mathbf{u}, \mathbf{x}) + (\mathbf{v}, \mathbf{y}) = (\mathbf{u} + \mathbf{v}, \mathbf{x} + \mathbf{y})$$

and

$$\lambda(\mathbf{u}, \mathbf{x}) = (\lambda\mathbf{u}, \lambda\mathbf{x})$$

where $\mathbf{u}, \mathbf{v} \in \mathscr{V}$; $\mathbf{x}, \mathbf{y} \in \mathscr{U}$; and $\lambda \in \mathscr{F}$

2.1.2 Let $\mathscr{V}$ be a vector space and consider the set $\mathscr{V} \times \mathscr{V}$. Define addition in $\mathscr{V} \times \mathscr{V}$ by

$$(\mathbf{u}, \mathbf{v}) + (\mathbf{x}, \mathbf{y}) = (\mathbf{u} + \mathbf{x}, \mathbf{v} + \mathbf{y})$$

and multiplication by complex numbers by

$$(\lambda + i\mu)(\mathbf{u}, \mathbf{v}) = (\lambda\mathbf{u} - \mu\mathbf{v}, \mu\mathbf{u} + \lambda\mathbf{v})$$

where $\lambda, \mu \in \mathscr{R}$. Show that $\mathscr{V} \times \mathscr{V}$ is a vector space over the field of complex numbers.

Section 2.2. Some Properties of a Vector Space

Next, we shall prove certain results which follow by a systematic application of the above properties of a vector space. Aside from yielding important information, the proofs are excellent illustrations of how the axioms of a vector space are used to deduce additional conclusions. First, we shall prove that the zero vector, $\mathbf{0}$, and the additive inverse of an arbitrary vector $\mathbf{u}$, which we call $-\mathbf{u}$, are unique.

Theorem 2.2.1: The zero vector, $\mathbf{0}$, is unique.

Proof: As mentioned in Section 1.6, uniqueness theorems usually begin by the assumption of a lack of uniqueness. One then attempts to establish a contradiction. By Axiom (a3), if the zero is not unique, we must have, for all $\mathbf{u} \in \mathscr{V}$,

$$\underbrace{\mathbf{u}+\mathbf{0}=\mathbf{u}}_{\text{(a3)}} \qquad \text{and} \qquad \underbrace{\mathbf{u}+\mathbf{0}'=\mathbf{u}}_{\text{(a3)}} \tag{2.2.1}$$

where $\mathbf{0}$ and $\mathbf{0}'$ are the two zeros. Because $\mathbf{u}$ is arbitrary, in the first equation take $\mathbf{u}=\mathbf{0}'$ and in the second equation take $\mathbf{u}=\mathbf{0}$. The results are

$$\mathbf{0}'+\mathbf{0}=\mathbf{0}' \quad \text{and} \quad \mathbf{0}+\mathbf{0}'=\mathbf{0} \tag{2.2.2}$$

Because of (a2), $\mathbf{0}'+\mathbf{0}=\mathbf{0}+\mathbf{0}'$, and, as a result the only conclusion from (2.2.2) is

$$\mathbf{0}=\mathbf{0}' \tag{2.2.3}$$

Next, we want to establish that the additive inverse is unique. We shall approach this result by first establishing a more general result.

Theorem 2.2.2: For every pair of vectors $\mathbf{v},\mathbf{w} \in \mathscr{V}$, there exists a unique $\mathbf{u}$ such that $\mathbf{u}+\mathbf{v}=\mathbf{w}$.

Proof: As usual, assume a lack of uniqueness. As a result, for given $\mathbf{v},\mathbf{w} \in \mathscr{V}$,

$$\mathbf{u}+\mathbf{v}=\mathbf{w} \quad \text{and} \quad \mathbf{u}'+\mathbf{v}=\mathbf{w} \tag{2.2.4}$$

Our goal is to prove that $\mathbf{u}'=\mathbf{u}$. It follows from (2.2.4) that

$$\mathbf{u}+\mathbf{v}=\mathbf{u}'+\mathbf{v} \tag{2.2.5}$$

Next, let $-\mathbf{v}$ be *an* additive inverse of the element $\mathbf{v}$ and, in effect, add it to (2.2.5) to obtain

$$\mathbf{u}+\mathbf{v}+(-\mathbf{v})=\mathbf{u}'+\mathbf{v}+(-\mathbf{v}) \tag{2.2.6}$$

Properties (a4) and (a3), reduce the last equation to

$$\mathbf{u} = \mathbf{u}' \tag{2.2.7}$$

which establishes the uniqueness of the solution to $\mathbf{u} + \mathbf{v} = \mathbf{w}$.

Corollary: The element (the additive inverse), for every $\mathbf{v} \in \mathscr{V}$ is unique.

Proof: Make the choice $\mathbf{w} = \mathbf{0}$ in equation (2.2.4) of the last theorem. As a result, $\mathbf{u} + \mathbf{v} = \mathbf{0}$. The uniqueness of $\mathbf{u}$ establishes the result.

Theorem 2.2.2 established the uniqueness of solutions, $\mathbf{u}$, to $\mathbf{u} + \mathbf{v} = \mathbf{w}$. A related question relates to whether or a solution actually exists. The axioms of a Vector Space give us this existence. The proof is simple, one just postulates a solution and shows that it satisfies the given equation, $\mathbf{u} + \mathbf{v} = \mathbf{w}$. The postulated solution is

$$\mathbf{u} = \mathbf{w} + (-\mathbf{v}) \tag{2.2.8}$$

and it is elementary to substitute this assumed solution into $\mathbf{u} + \mathbf{v} = \mathbf{w}$ and use appropriate axioms to establish the validity of the solution. Recall that earlier we defined the difference in two vectors $\mathbf{w}$ and $\mathbf{v}$ by the formula $\mathbf{w} - \mathbf{v} = \mathbf{w} + (-\mathbf{v})$

The next theorems relate to relationships between the zero scalar and the zero vector.

Theorem 2.2.3:

$$\lambda\mathbf{u} = \mathbf{0} \text{ if and only if } \lambda = 0 \quad \text{or} \quad \mathbf{u} = 0 \tag{2.2.9}$$

Proof: The proof of this theorem requires the proof of the following three assertions: a) $0\mathbf{u} = \mathbf{0}$, b) $\lambda\mathbf{0} = \mathbf{0}$ and c) $\lambda\mathbf{u} = \mathbf{0} \Rightarrow \lambda = 0$ or $\mathbf{u} = \mathbf{0}$. We shall establish these results in the order listed.

Part (a): We shall prove that $0\mathbf{u} = \mathbf{0}$.

Proof: Take $\mu = 0$ and $\lambda = 1$ in Axiom (b2) for a vector space; then

$$\underbrace{\mathbf{u} = \mathbf{u} + 0\mathbf{u}}_{\text{(b2) and (b4)}} \tag{2.2.10}$$

Therefore, after utilizing $\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v}))$

$$\mathbf{u}-\mathbf{u}=(\mathbf{u}+0\mathbf{u})-\mathbf{u}=\underbrace{(\mathbf{u}-\mathbf{u})+0\mathbf{u}}_{\text{(a1) and (a2)}} \tag{2.2.11}$$

where Axioms (a1) and (a2) have been used. Next, by Axiom (a4) the last result is

$$\mathbf{0}=\mathbf{0}+0\mathbf{u} \tag{2.2.12}$$

Next, we can use (a3) and conclude the result $\mathbf{0}=0\mathbf{u}$.

Part (b): We shall prove that $\lambda\mathbf{0}=\mathbf{0}$.

Proof: Set $\mathbf{v}=\mathbf{0}$ in Axiom (b3), then

$$\underbrace{\lambda\mathbf{u}=\lambda\mathbf{u}+\lambda\mathbf{0}}_{\mathbf{v}=\mathbf{0}\text{ in (b3)}} \tag{2.2.13}$$

Therefore

$$\lambda\mathbf{u}-\lambda\mathbf{u}=\lambda\mathbf{u}+\lambda\mathbf{0}-\lambda\mathbf{u}=\underbrace{\lambda\mathbf{u}-\lambda\mathbf{u}}_{\text{(a1) and (a2)}}+\lambda\mathbf{0} \tag{2.2.14}$$

and by Axiom (a4)

$$\underbrace{\mathbf{0}}_{\text{(a4)}}=\underbrace{\mathbf{0}}_{\text{(a4)}}+\lambda\mathbf{0}=\underbrace{\lambda\mathbf{0}}_{\text{(a3)}} \tag{2.2.15}$$

Part (c): We shall prove that $\lambda\mathbf{u}=\mathbf{0}\Rightarrow\lambda=0$ or $\mathbf{u}=\mathbf{0}$**.**

Proof: We begin with the assumption that $\lambda\mathbf{u}=\mathbf{0}$. If $\lambda=0$, we know from (a) that the equation $\lambda\mathbf{u}=\mathbf{0}$ is satisfied. If $\lambda\neq 0$, then we show that $\mathbf{u}$ must be zero as follows:

$$\underbrace{\mathbf{u}=1\mathbf{u}}_{\text{(b4)}}=\underbrace{\lambda\left(\frac{1}{\lambda}\right)}_{\substack{\text{Mult inverse}\\ \text{property in}\\ \text{definition of Field}}}\mathbf{u}=\underbrace{\frac{1}{\lambda}(\lambda\mathbf{u})}_{\substack{\text{Commutative}\\ \text{property in}\\ \text{definition of}\\ \text{Field, and (b1)}}}=\frac{1}{\lambda}(\mathbf{0})=\mathbf{0} \tag{2.2.16}$$

Theorem 2.2.4:

$$(-1)\mathbf{u}=-\mathbf{u} \tag{2.2.17}$$

Proof:

$$\underbrace{\mathbf{u}+(-1)\mathbf{u}=(1-1)\mathbf{u}}_{\text{(b2) with }\lambda=1\text{ and }\mu=-1\text{ and (b4)}}=\underbrace{0\mathbf{u}=\mathbf{0}}_{\text{Part (a)}} \tag{2.2.18}$$

Because the negative vector is unique, equation (2.2.18) and (a4) yield the result (2.2.17).

Two other theorems of interest are the following:

Theorem 2.2.5:

$$\underbrace{(-\lambda)\mathbf{u}}_{\substack{\text{Scalar multiplication}\\ \text{of }(-\lambda)\text{ times }\mathbf{u}}}=\underbrace{-\lambda\mathbf{u}}_{\substack{\text{Negative of}\\ \text{vector }\lambda\mathbf{u}}} \tag{2.2.19}$$

Proof. Let $\mu=0$ and replace λ by $-\lambda$ in Axiom (b2) for a vector space and this result follows directly.

Theorem 2.2.6:

$$\underbrace{-\lambda\mathbf{u}}_{\substack{\text{Negative of}\\ \text{vector }\lambda\mathbf{u}}}=\lambda\underbrace{(-\mathbf{u})}_{\substack{\text{Negative of}\\ \text{vector }\mathbf{u}}} \tag{2.2.20}$$

Proof: Take $\mu=-1$ in (b1) to obtain

$$\lambda(-\mathbf{u})=(-\lambda)\mathbf{u} \tag{2.2.21}$$

and use (2.2.20).

Finally, we note that the concepts of *length* and *angle* have not been introduced. They are *not* part of the definition of a vector space. However, they can be introduced as additional algebraic structure. These ideas will be introduced later in Chapter 4.

Section 2.3. Subspace of a Vector Space

In this section, we shall introduce the idea of a subspace of vector space. The idea is elementary. A subspace is simply a subset of a vector space that is itself a vector space. A more precise definition is as follows:

Definition: A non empty subset $\mathscr{U}$ of a vector space $\mathscr{V}$ is a *subspace* if:

(a) $\mathbf{u},\mathbf{w}\in\mathscr{U}$ implies $\mathbf{u}+\mathbf{w}\in\mathscr{U}$ for all $\mathbf{u},\mathbf{w}\in\mathscr{U}$.
(b) $\mathbf{u}\in\mathscr{U}$ implies $\lambda\mathbf{u}\in\mathscr{U}$ for all $\lambda\in\mathscr{C}$

Conditions (a) and (b) in this definition can be replaced by the equivalent condition:

(a') $\mathbf{u},\mathbf{w}\in\mathscr{U}$ implies $\lambda\mathbf{u}+\mu\mathbf{w}\in\mathscr{U}$ for all $\lambda\in\mathscr{C}$.

A few examples of subspaces of vector spaces are as follows:

Example 2.3.1: Trivially, any vector space $\mathscr{V}$ is a subspace of itself.

Example 2.3.2: Another trivial example is the set consisting of the zero vector $\{\mathbf{0}\}$ is a subspace of $\mathscr{V}$.

As stated in these two examples, the vector spaces $\{\mathbf{0}\}$ and $\mathscr{V}$ itself are considered to be *trivial* subspaces of the vector space $\mathscr{V}$. If $\mathscr{U}$ is not a trivial subspace, it is said to be a *proper subspace* of $\mathscr{V}$.

Example 2.3.3: The subset of the vector space $\mathscr{C}^N$ of all *N*-tuples of the form $(0,\lambda_2,\lambda_3,...,\lambda_N)$ is a subspace of $\mathscr{C}^N$.

Example 2.3.4: The set of real numbers $\mathscr{R}$ can be viewed as a subspace of the vector space of complex numbers $\mathscr{C}$.

Example 2.3.5: Consider the vector space $\mathscr{M}^{N\times N}$ of square matrices with real elements. Let $\mathscr{S}$ be the *subset* of $\mathscr{M}^{N\times N}$ defined by

$$\mathscr{S}=\left\{A\middle|A\in\mathscr{M}^{N\times N},A=A^T\right\} \tag{2.3.1}$$

The question is whether or not $\mathscr{S}$, the subset of symmetric matrices, is a *subspace* of $\mathscr{M}^{N\times N}$. Because

$$(\lambda A+\mu B)^T=\lambda A+\mu B \tag{2.3.2}$$

for all $\lambda,\mu\in\mathscr{R}$ and all $A,B\in\mathscr{S}$, then from the above definition $\mathscr{S}$ is a subspace.

An elementary property of subspaces is given in the following theorem:

Theorem 2.3.1: If $\mathscr{U}$ is a subspace of $\mathscr{V}$, then $\mathbf{0}\in\mathscr{U}$.

Proof. The proof of this theorem follows easily from (b) in the definition of a subspace above by placing $\lambda=0$.

If one is given a set of N vectors $\{\mathbf{u}_1,\mathbf{u}_2,...,\mathbf{u}_N\}\subset\mathscr{V}$, then one can construct a subspace as by essentially constructing all possible linear combinations of the elements of the set. More formally, this construction is as follows:

Definition: Let $\{\mathbf{u}_1,\mathbf{u}_2,...,\mathbf{u}_N\}$ be a set of vectors in a vector space $\mathscr{V}$. A sum of the form $\alpha_1\mathbf{u}_1+\alpha_2\mathbf{u}_2+\cdots+\alpha_N\mathbf{u}_N$ is a *linear combination* of $\mathbf{u}_1,\mathbf{u}_2,...,\mathbf{u}_N$

Definition: The set of *all* linear combinations of $\mathbf{u}_1,\mathbf{u}_2,...,\mathbf{u}_N$ is called the *span* of $\mathbf{u}_1,\mathbf{u}_2,...,\mathbf{u}_N$. It is denoted by $\operatorname{Span}(\mathbf{u}_1,\mathbf{u}_2,...,\mathbf{u}_N)$

Theorem 2.3.2: The $\operatorname{Span}(\mathbf{u}_1,\mathbf{u}_2,...,\mathbf{u}_N)$ is a subspace of $\mathscr{V}$.

The proof follows directly from the definition of subspace.

Definition: The subset $\{\mathbf{u}_1,\mathbf{u}_2,...,\mathbf{u}_N\}$ is a *spanning set* for $\mathscr{V}$ if every vector in $\mathscr{V}$ can be written as a linear combination of $\mathbf{u}_1,\mathbf{u}_2,...,\mathbf{u}_N$.

Example 2.3.6: Consider the vector space of 3-tuples, $\mathscr{R}^3$. A typical element of $\mathscr{R}^3$ can be written

$$\mathbf{u}=(u_1,u_2,u_3) \tag{2.3.3}$$

Consider the set of two vectors $\{\mathbf{i}_1,\mathbf{i}_2\}$ in $\mathscr{R}^3$ defined by

$$\mathbf{i}_1=(1,0,0) \tag{2.3.4}$$

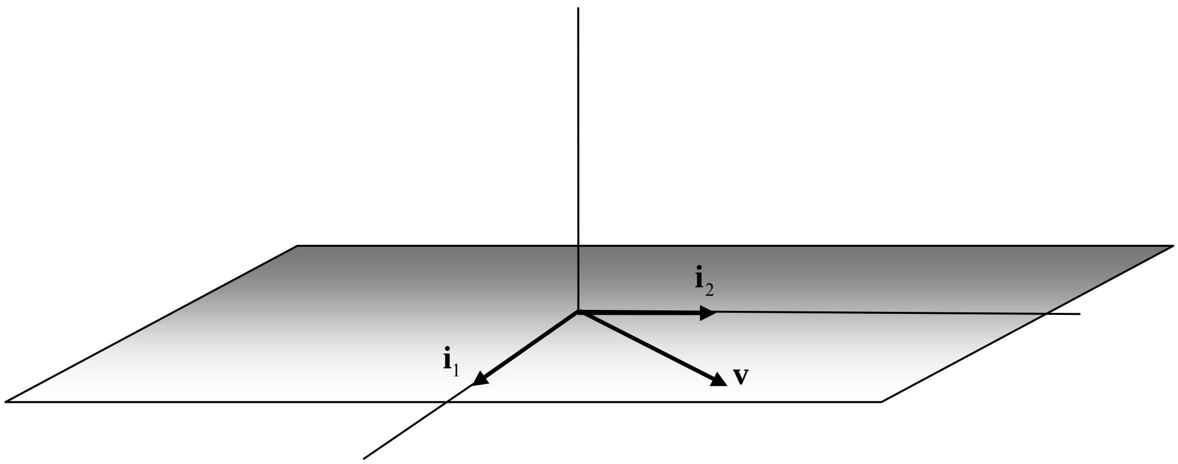
and

$$\mathbf{i}_2=(0,1,0) \tag{2.3.5}$$

The span of $\{\mathbf{i}_1, \mathbf{i}_2\}$, written $\text{Span}(\mathbf{i}_1, \mathbf{i}_2)$, consists of all possible linear combinations of the form $\alpha_1\mathbf{i}_1 + \alpha_2\mathbf{i}_2$. Given (2.3.4) and (2.3.5), a vector $\mathbf{v} \in \text{Span}(\mathbf{i}_1, \mathbf{i}_2)$ will necessarily be of the form

$$\mathbf{v} = \upsilon_1\mathbf{i}_1 + \upsilon_2\mathbf{i}_2 = \upsilon_1(1,0,0) + \upsilon_2(0,1,0) = (\upsilon_1, \upsilon_2, 0) \tag{2.3.6}$$

Therefore, the subspace $\text{Span}(\mathbf{i}_1, \mathbf{i}_2)$ consists of all of those vectors in $\mathscr{R}^3$ that have zero in their third position. This simple example of a subspace in $\mathscr{R}^3$ is usually illustrated by the simple figure



Thus, $\text{Span}(\mathbf{i}_1, \mathbf{i}_2)$ is the horizontal plane through the point $(0,0,0)$.

Example 2.3.7: Consider the vector space $\mathscr{P}_2$ of polynomials of degree less than or equal to 2. A typical element of $\mathscr{P}_2$ can be written

$$p(x) = \lambda_0 + \lambda_1 x + \lambda_2 x^2 \tag{2.3.7}$$

Next, consider the subset of $\mathscr{P}_2$ defined by $\{1 - x^2, x + 2, x^2\}$. The span of $\{1 - x^2, x + 2, x^2\}$, $\text{Span}(1 - x^2, x + 2, x^2)$, is the set of polynomials of the form

$$\upsilon(x) = \upsilon_1(1 - x^2) + \upsilon_2(x + 2) + \upsilon_3 x^2 = (\upsilon_1 + 2\upsilon_2) + \upsilon_2 x + (\upsilon_3 - \upsilon_1)x^2 \tag{2.3.8}$$

We wish to determine the is the relationship between the subspace, $\text{Span}(1 - x^2, x + 2, x^2)$, and the vector space $\mathscr{P}_2$. They are the same because if we force

$$\begin{aligned}\lambda_0 &= \upsilon_1 + 2\upsilon_2 \\ \lambda_1 &= \upsilon_2 \\ \lambda_2 &= \upsilon_3 - \upsilon_1\end{aligned} \tag{2.3.9}$$

Then one can solve (2.3.9) and obtain

$$\begin{aligned}\upsilon_1 &= \lambda_0 - 2\lambda_1 \\ \upsilon_2 &= \lambda_1 \\ \upsilon_3 &= \lambda_2 + \lambda_0 - 2\lambda_1\end{aligned} \tag{2.3.10}$$

Therefore $\text{Span}\left(1-x^2, x+2, x^2\right) = \mathscr{P}_2$.

Example 2.3.8: Consider the vector space $C^2\left[a,b\right]$ of twice differentiable functions on every open subset of $\left[a,b\right]$. This vector space was defined by equation (2.1.15). The subset of functions in $C^2\left[a,b\right]$ that obey the differential equation

$$\frac{d^2 f\left(x\right)}{dx^2} + f\left(x\right) = 0 \tag{2.3.11}$$

can be shown to be a subspace of $C^2\left[a,b\right]$

Example 2.3.9: Consider the set of functions $C^\infty\left[a,b\right]$. These functions are real valued functions defined on $\left[a,b\right]$ that have derivatives of arbitrary order on every open subset of $\left[a,b\right]$. This set is a vector space when one defines addition and scalar multiplication by (2.1.13) and (2.1.14). The vector space $\mathscr{P}_N$ was introduced in Section 2.1. Each element is also an element of $C^\infty\left[a,b\right]$. It is elementary to show that $\mathscr{P}_N$ is a subspace of $C^\infty\left[a,b\right]$.

Section 2.4. Linear Independence

This section is concerned with the idea of a *linearly independent* set of vectors. It arises from the desire to identify those vector spaces which have a minimal spanning set, i.e. a spanning set with no unnecessary elements. If such a set can be found, then from the definition of spanning set, every vector in the vector space would have the representation as a linear combination of members of this minimum spanning set.

It is helpful to consider the following example as motivation of the idea of linear independence and the related idea of linear dependence.

Example 2.4.1: Let $\mathscr{V} = \mathscr{M}^{3\times 1}$, the vector space of column vectors of order 3. You are given the following three vectors in $\mathscr{V}$:

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \qquad \mathbf{u}_2 = \begin{bmatrix} -4 \\ 7 \\ 2 \end{bmatrix}, \qquad \mathbf{u}_3 = \begin{bmatrix} -1 \\ 4 \\ 8 \end{bmatrix} \tag{2.4.1}$$

Let $\mathscr{S} = \mathrm{Span}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$ be the subspace of $\mathscr{V} = \mathscr{M}^{3\times 1}$ spanned by $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$. The question is whether or not $\mathscr{S}$ can be generated by less than the three vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$. The answer is yes because

$$\mathbf{u}_3 = \begin{bmatrix} -1 \\ 4 \\ 8 \end{bmatrix} = \begin{bmatrix} 3-4 \\ -3+7 \\ 6+2 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ 6 \end{bmatrix} + \begin{bmatrix} -4 \\ 7 \\ 2 \end{bmatrix} = 3\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + \begin{bmatrix} -4 \\ 7 \\ 2 \end{bmatrix} = 3\mathbf{u}_1 + \mathbf{u}_2 \tag{2.4.2}$$

This equation is an example of what is known as *linear dependence*, i.e. one of the vectors can be written as a linear combination of the others. On the basis of this dependence, any linear combination of the three vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ can be expressed, by (2.4.2), as a linear combination of $\mathbf{u}_1, \mathbf{u}_2$. The conclusion is that

$$\mathscr{S} = \mathrm{Span}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3) = \mathrm{Span}(\mathbf{u}_1, \mathbf{u}_2) \tag{2.4.3}$$

The choice of the two vectors $\mathbf{u}_1, \mathbf{u}_2$ rather than, say, $\mathbf{u}_1, \mathbf{u}_3$ is arbitrary. One can solve the equation $\mathbf{u}_3 = 3\mathbf{u}_1 + \mathbf{u}_2$ for $\mathbf{u}_2 = \mathbf{u}_3 - 3\mathbf{u}_1$ and formally eliminate $\mathbf{u}_2$. The bottom line is that

$$\mathscr{S} = \mathrm{Span}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3) = \mathrm{Span}(\mathbf{u}_1, \mathbf{u}_2) = \mathrm{Span}(\mathbf{u}_1, \mathbf{u}_3) = \mathrm{Span}(\mathbf{u}_2, \mathbf{u}_3) \tag{2.4.4}$$

The question naturally arises whether or not, for example, one can express $\mathbf{u}_2$ in terms of $\mathbf{u}_1$ and further reduce the spanning set. If one forces a relationship

$$\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 = \mathbf{0} \tag{2.4.5}$$

Then (2.4.1) and (2.4.4) would require

$$\alpha_1\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + \alpha_2\begin{bmatrix} -4 \\ 7 \\ 2 \end{bmatrix} = \mathbf{0} \tag{2.4.6}$$

As a matrix equation, (2.4.6) requires

$$\begin{aligned} \alpha_1 - 4\alpha_2 &= 0 \\ -\alpha_1 + 7\alpha_2 &= 0 \\ 2\alpha_1 + 2\alpha_2 &= 0 \end{aligned} \tag{2.4.7}$$

which only has the solution $\alpha_1 = \alpha_2 = 0$ which renders the relationship (2.4.5) trivial. The conclusion is that one cannot express $\mathbf{u}_2$ in terms of $\mathbf{u}_1$ and, as a consequence, $\{\mathbf{u}_1, \mathbf{u}_2\}$ is the minimum spanning set for $\mathscr{S}$.

The concept of *linear independence* is introduced by first defining what is meant by *linear dependence* in a set of vectors, and then defining a set of vectors that is not linearly dependent to be linearly independent. The general definition of linear dependence of a set of *N* vectors is an algebraic generalization and abstraction of the concepts of co-linearity from elementary geometry.

Definition. A finite set of $N\left(N \geq 1\right)$ vectors $\left\{\mathbf{v}_{1,}\mathbf{v}_{2,}...,\mathbf{v}_N\right\}$ in a vector space $\mathscr{V}$ is said to be *linearly dependent* if there exists a set of scalars $\{\alpha_1, \alpha_2, ..., \alpha_N\}$, not all zero, such that

$$\sum_{j=1}^{N} \alpha_j \mathbf{v}_j = \mathbf{0} \tag{2.4.8}$$

It is a trivial consequence of this definition that every set of vectors which contain the zero vector is linearly dependent.

The essential content of the definition of linear dependence is that at least one of the vectors in the set $\left\{\mathbf{v}_{1,}\mathbf{v}_{2,}...,\mathbf{v}_N\right\}$ can be expressed as a linear combination of the other vectors.

Example 2.4.2: Consider the set of four vectors in $\mathscr{M}^{3\times 1}$ $\{\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3, \mathbf{v}\}$ where

$$\mathbf{i}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{i}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{i}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \tag{2.4.9}$$

The question is whether or not we can find four scalars $\alpha_1, \alpha_2, \alpha_3$ and α_4 , not all zero, such that

$$\alpha_1 \mathbf{i}_1 + \alpha_2 \mathbf{i}_2 + \alpha_3 \mathbf{i}_3 + \alpha_4 \mathbf{v} = \mathbf{0} \tag{2.4.10}$$

If we use the explicit forms of the four vectors given in (2.4.9), then we need to find $\alpha_1, \alpha_2, \alpha_3$ and α_4 such that

$$\alpha_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \alpha_4 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{2.4.11}$$

These equations require that the $\alpha_1, \alpha_2, \alpha_3$ and α_4 obey

$$\begin{aligned} \alpha_1 + \alpha_4 &= 0 \\ \alpha_2 + 2\alpha_4 &= 0 \\ \alpha_3 + 3\alpha_4 &= 0 \end{aligned} \tag{2.4.12}$$

Therefore,

$$\begin{aligned} \alpha_1 &= -\alpha_4 \\ \alpha_2 &= -2\alpha_4 \\ \alpha_3 &= -3\alpha_4 \end{aligned} \tag{2.4.13}$$

If these equations are substituted back into (2.4.10), the result is

$$-\alpha_4 \mathbf{i}_1 - 2\alpha_4 \mathbf{i}_2 - 3\alpha_4 \mathbf{i}_3 + \alpha_4 \mathbf{v} = \mathbf{0} \tag{2.4.14}$$

or, since α_4 is nonzero,

$$\mathbf{v} = \mathbf{i}_1 + 2\mathbf{i}_2 + 3\mathbf{i}_3 \tag{2.4.15}$$

and the conclusion that the given set is linearly dependent.

A more or less obvious consequence of the definition of linear dependence is the following theorem:

Theorem 2.4.1: If the set of vectors $\{\mathbf{v}_{1,}\mathbf{v}_{2,}\cdots,\mathbf{v}_N\}$ is linearly dependent, then every other finite set of vectors containing $\{\mathbf{v}_{1,}\mathbf{v}_{2,}\cdots,\mathbf{v}_N\}$ is linearly dependent.

Definition: A finite set of $N\left(N\geq 1\right)$ vectors $\{\mathbf{v}_{1,}\mathbf{v}_{2,}...,\mathbf{v}_N\}$ in a vector space $\mathscr{V}$ is said to be *linearly independent* if they are not linearly dependent.

Equivalent Definition: A finite set of $N\left(N\geq 1\right)$ vectors $\{\mathbf{v}_{1,}\mathbf{v}_{2,}...,\mathbf{v}_N\}$ in a vector space $\mathscr{V}$ is said to be *linearly independent* if the only scalars $\{\alpha_1,\alpha_2,...,\alpha_N\}$ which obey

$$\sum_{j=1}^{N}\alpha_j\mathbf{v}_j=\mathbf{0} \tag{2.4.16}$$

are $\alpha_1=\alpha_2=\cdots=\alpha_N=0$

An evident consequence of this definition is the following theorem.

Theorem 2.4.2: Every non empty *subset* of a linearly independent set is a linearly independent set.

There is a relationship between linearly independent column vectors and matrices that is useful to establish at this point. Let $\{\mathbf{u}_1,\mathbf{u}_2,...,\mathbf{u}_N\}\subset\mathscr{M}^{M\times 1}$ be a set of N column vectors in $\mathscr{M}^{M\times 1}$. The test for linear independence or dependence requires determining whether or not the equation

$$\alpha_1\mathbf{u}_1+\alpha_2\mathbf{u}_2+\cdots+...+\alpha_N\mathbf{u}_N=\mathbf{0} \tag{2.4.17}$$

has nonzero solutions for some of the coefficients $\alpha_1,\alpha_2,...,\alpha_N$. We can write (2.4.17) as a matrix equation, as follows:

$$\alpha_1\underbrace{\begin{bmatrix}u_{11}\\u_{21}\\\cdot\\\cdot\\\cdot\\u_{M1}\end{bmatrix}}_{\mathbf{u}_1}+\alpha_2\underbrace{\begin{bmatrix}u_{12}\\u_{22}\\\cdot\\\cdot\\\cdot\\u_{M2}\end{bmatrix}}_{\mathbf{u}_2}+\cdots+\alpha_N\underbrace{\begin{bmatrix}u_{1N}\\u_{2N}\\\cdot\\\cdot\\\cdot\\u_{MN}\end{bmatrix}}_{\mathbf{u}_N}=\mathbf{0} \tag{2.4.18}$$

or,

$$\underbrace{\begin{bmatrix} u_{11} & u_{12} & \cdot & \cdot & \cdot & u_{1N} \\ u_{21} & u_{22} & \cdot & \cdot & \cdot & u_{2N} \\ \cdot & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & & \cdot & \cdot \\ u_{M1} & u_{M2} & \cdot & \cdot & \cdot & u_{MN} \end{bmatrix}}_{M\times N} \underbrace{\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \cdot \\ \cdot \\ \cdot \\ \alpha_N \end{bmatrix}}_{N\times 1} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}}_{M\times 1} \tag{2.4.19}$$

We know from the matrix algebra discussion in Chapter 1, that whether or not (2.4.19) has a nonzero solution depends upon the properties of the matrix of coefficients. The next formal step in the calculation would be to reduce the augmented matrix to reduced row echelon form and see whether or not (2.4.19) allows a nonzero solution for some of the unknowns $\alpha_1, \alpha_2, ..., \alpha_N$.

If the number of vectors and the size of the column vectors agree, i.e., $M = N$, then the matrix of coefficients in (2.4.19) is square. In this case, the system has nonzero solutions if and only if the matrix of coefficients is singular. Thus, the set $\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_N\}$ is linearly dependent if and only if the $N \times N$ matrix of coefficients is singular.

Example 2.4.3: Consider the following three column matrices in $\mathscr{M}^{3\times 1}$

$$\mathbf{u}_1 = \begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 2 \\ -5 \\ 3 \end{bmatrix} \tag{2.4.20}$$

The next step is to form the sum (2.4.17)

$$\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \alpha_3 \mathbf{u}_3 = \mathbf{0} \tag{2.4.21}$$

Given the definitions (2.4.20), the matrix form of (2.4.21) is

$$\begin{bmatrix} 4 & 2 & 2 \\ 2 & 3 & -5 \\ 3 & 1 & 3 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = 0 \tag{2.4.22}$$

As explained above, the linear independence of the three column matrices (2.4.20) depends upon whether or not the coefficients $\alpha_1, \alpha_2, \alpha_3$ are all zero. We know from the results of Theorem 1.6.1 that (2.4.22) has the zero solution if and only if the matrix of coefficients is nonsingular. We know from the results of Section 1.10 that the matrix is nonsingular if and only if its determinant is nonzero. The determinant of the matrix of coefficients in (2.4.22) is

$$\begin{vmatrix} 4 & 2 & 2 \\ 2 & 3 & -5 \\ 3 & 1 & 3 \end{vmatrix} = 4(14) - 2(4) + 3(-16) = 0 \tag{2.4.23}$$

Therefore, the matrix is singular and the system (2.4.22) has a nonzero solution. As a result, the three column matrices (2.4.20) form a linear dependent set. More detailed information about this example is obtained if (2.4.22) is converted to reduced row echelon form. The result turns out to be

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = 0 \tag{2.4.24}$$

Therefore,

$$\alpha_1 = -2\alpha_3 \tag{2.4.25}$$

and

$$\alpha_2 = 3\alpha_3 \tag{2.4.26}$$

As a result of (2.4.25) and (2.4.26), the three vectors in (2.4.21) are related by

$$-2\mathbf{u}_1 + 3\mathbf{u}_2 + \mathbf{u}_3 = \mathbf{0} \tag{2.4.27}$$

It is helpful to observe in passing that equation (2.4.27) not only defines the relationship between the three columns of the matrix of coefficients in (2.4.22). It also defines the relationship between the three columns of the reduced row echelon form that appears in (2.4.24). This is a general feature we shall see in our other examples. Our theoretical results later will show the origin or this result.

In Section 2.1 we introduced the vector space $\mathscr{P}_N$ consisting of the set of polynomials of degree less than or equal to N. Given this vector space, it is useful to establish a procedure for whether or not a subset of polynomials in $\mathscr{P}_N$ is linear independent. If we are given a subset $\{p_1,..., p_K\}$ of $\mathscr{P}_N$. The check for linear independence, according to the above definition is whether or not the equation

$$\alpha_1 p_1 + \alpha_2 p_2 + \cdots + \alpha_K p_K = 0 \tag{2.4.28}$$

implies that the scalars $\alpha_1, \alpha_2, ..., \alpha_K$ are necessarily zero. Each polynomial in (2.4.28) has the representation

$$p_j(x) = \lambda_{0j} + \lambda_{1j}x + \cdots + \lambda_{Nj}x^N \qquad \text{for} \qquad j = 1, 2, ..., K \tag{2.4.29}$$

Therefore, (2.4.28) can be written

$$\begin{aligned} &\alpha_1 p_1 + \alpha_2 p_2 + \cdots + \alpha_K p_K \\ &\quad = \alpha_1\left(\lambda_{01} + \lambda_{11}x + \cdots + \lambda_{N1}x^N\right) \\ &\qquad + \alpha_2\left(\lambda_{02} + \lambda_{12}x + \cdots + \lambda_{N2}x^N\right) \\ &\qquad\quad + \cdots + \alpha_K\left(\lambda_{0K} + \lambda_{1K}x + \cdots + \lambda_{NK}x^N\right) = 0 \end{aligned} \tag{2.4.30}$$

The terms in (2.4.30) can be grouped in like powers of x to yield

$$\begin{aligned} &\left(\alpha_1\lambda_{01} + \alpha_2\lambda_{02} + \cdots + \alpha_K\lambda_{0K}\right) \\ &\quad + \left(\alpha_1\lambda_{11} + \alpha_2\lambda_{12} + \cdots + \alpha_K\lambda_{1K}\right)x \\ &\qquad + \left(\alpha_1\lambda_{21} + \alpha_2\lambda_{22} + \cdots + \alpha_K\lambda_{2K}\right)x^2 \\ &\qquad\quad + \cdots + \left(\alpha_1\lambda_{N1} + \alpha_2\lambda_{N2} + \cdots + \alpha_K\lambda_{NK}\right)x^N = 0 \end{aligned} \tag{2.4.31}$$

Because (2.4.31) must hold for all x, it implies the following $N+1$ equations, written in matrix form, which the K coefficients, $\alpha_1, ..., \alpha_K$, must obey

$$\underbrace{\begin{bmatrix} \lambda_{01} & \lambda_{02} & \cdot & \cdot & \cdot & \lambda_{0K} \\ \lambda_{11} & \lambda_{12} & \cdot & \cdot & \cdot & \lambda_{1K} \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ \lambda_{N1} & & & & & \lambda_{NK} \end{bmatrix}}_{(N+1)\times K} \underbrace{\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \cdot \\ \cdot \\ \cdot \\ \alpha_K \end{bmatrix}}_{K\times 1} = 0 \tag{2.4.32}$$

If this system only has the zero solution, then the set of polynomials are linearly independent. Otherwise, they are dependent.

Example 2.4.4: Consider the vector space $\mathscr{P}_3$ and the subset of $\mathscr{P}_3$ consisting of the following three polynomials:

$$
\begin{aligned}
p_1(x) &= 3 - 2x + x^2 \\
p_2(x) &= 8 + x + 2x^2 \\
p_3(x) &= 7 + 8x + x^2
\end{aligned}
\tag{2.4.33}
$$

As with the discussion just completed, the test for linear dependence or independence of this subset requires that we look for implications of the following linear combination:

$$
\begin{aligned}
&\alpha_1 p_1(x) + \alpha_2 p_2(x) + \alpha_3 p_3(x) \\
&\quad = \alpha_1\left(3 - 2x + x^2\right) + \alpha_2\left(8 + x + 2x^2\right) + \alpha_3\left(7 + 8x + x^2\right) = 0
\end{aligned}
\tag{2.4.34}
$$

In order for (2.4.34) to hold for all x, like powers of x must be placed to zero. The results are

$$
\begin{aligned}
3\alpha_1 + 8\alpha_2 + 7\alpha_3 &= 0 \qquad \text{for } x^0 \\
-2\alpha_1 + \alpha_2 + 8\alpha_3 &= 0 \qquad \text{for } x^1 \\
\alpha_1 + 2\alpha_2 + \alpha_3 &= 0 \qquad \text{for } x^2
\end{aligned}
\tag{2.4.35}
$$

As a matrix equation, these three equations can be written

$$
\begin{bmatrix} 3 & 8 & 7 \\ -2 & 1 & 8 \\ 1 & 2 & 1 \end{bmatrix}
\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = 0
\tag{2.4.36}
$$

If you calculate the determinant of the matrix of coefficient, you find it is zero. The conclusion is that the matrix of coefficients is singular and, as a consequence, the coefficients $\alpha_1, \alpha_2, \alpha_3$ are not all necessarily zero. As such, the set of three polynomials defined above are linearly dependent. More detailed information about this example is obtained if (2.4.36) is converted to reduced row echelon form. The result turns out to be

$$
\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}
\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = 0
\tag{2.4.37}
$$

Therefore, $\alpha_1 = 3\alpha_3, \alpha_2 = -2\alpha_3$. These results when substituted into $(2.4.34)_1$ yield the following relationship between the elements of the set $\{p_1, p_2, p_3\}$

$$
3p_1 - 2p_2 + p_3 = 0
\tag{2.4.38}
$$

Example 2.4.5: Consider again the vector space $\mathscr{P}_3$. In this case, we are given the subset of $\mathscr{P}_3$ consisting of the following four polynomials:

$$\begin{aligned} p_1(x) &= 1 \\ p_2(x) &= x \\ p_3(x) &= x^2 \\ p_4(x) &= x^3 \end{aligned} \tag{2.4.39}$$

The test for linear dependence or independence of this subset requires that we look for implications of the following linear combination:

$$\begin{aligned} &\alpha_1 p_1(x) + \alpha_2 p_2(x) + \alpha_3 p_3(x) + \alpha_4 p_4(x) \\ &= \alpha_1 + \alpha_2 x + \alpha_3 x^2 + \alpha_4 x^3 = 0 \end{aligned} \tag{2.4.40}$$

In order for (2.4.34) to hold for all x, we quickly conclude that $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$ and the set $\{p_1, p_2, p_3, p_4\}$ is linearly independent.

The set $\mathscr{P}_N$ is an example of a set of functions. In Section 2.1 this set was shown to be a vector space. Also, in Section 2.1, we introduced other sets of real valued functions by symbols such as:

$C[a,b] = \{ f \mid f$ continuous on all open subsets of [a,b]$\}$

$C^2[a,b] = \{ f \mid f$ continuous and with continuous second derivatives on all open subsets of [a,b]$\}$

We explained that, with the usual definitions of addition of functions and multiplication by scalars, these sets are examples of vector spaces.

In the next discussion, we are interested in finding a condition that characterizes when real valued functions in a vector space $C^{N-1}[a,b]$, i.e. the set of real valued functions that are continuous and have continuous $N-1$ derivatives on every open subset of $[a,b]$, are linearly independent. This question is one that arises, for example, in the study of the solutions of ordinary differential equations. Let $\{f_1, ..., f_N\} \subset C^{N-1}[a,b]$. The question about linear independence or dependence, as with any vector space, requires that we determine the implication of a relationship of the form

$$\alpha_1 f_1(x) + \alpha_2 f_2(x) + \cdots + \alpha_N f_N(x) = 0 \quad \text{for all } x \in [a,b] \tag{2.4.41}$$

on the coefficients $\alpha_1, \alpha_2, ..., \alpha_N$. Given that these functions have $N-1$ continuous derivatives, we can differentiate the above equation $N-1$ times to obtain the following $N-1$ equations:

$$\begin{aligned} \alpha_1 f_1'(x) + \alpha_2 f_2'(x) + \cdots + \alpha_N f_N'(x) &= 0 \\ \alpha_1 f_1''(x) + \alpha_2 f_2''(x) + \cdots + \alpha_N f_N''(x) &= 0 \\ &\;\cdot \\ &\;\cdot \\ &\;\cdot \\ \alpha_1 f_1^{N-1}(x) + \alpha_2 f_2^{N-1}(x) + \cdots + \alpha_N f_N^{N-1}(x) &= 0 \end{aligned} \tag{2.4.42}$$

Equations (2.4.41) and (2.4.42), like (2.4.31), must hold for all $x \in [a,b]$. Expressed as a matrix equation, (2.4.41) and (2.4.42) can be written

$$\begin{bmatrix} f_1(x) & f_2(x) & \cdot & \cdot & \cdot & f_N(x) \\ f_1'(x) & f_2'(x) & & & & f_N'(x) \\ f_1''(x) & f_2''(x) & & & & f_N''(x) \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ f_1^{N-1}(x) & f_2^{N-1}(x) & \cdot & \cdot & \cdot & f_N^{N-1}(x) \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \cdot \\ \cdot \\ \cdot \\ \alpha_N \end{bmatrix} = 0 \tag{2.4.43}$$

This matrix equation must hold for all $x \in [a,b]$. Given what we know about the solution of systems of $N \times N$ equations, we can conclude that if there exists an x, say x_0 in $[a,b]$, where

$$\det \begin{bmatrix} f_1(x_0) & f_2(x_0) & \cdot & \cdot & \cdot & f_N(x_0) \\ f_1'(x_0) & f_2'(x_0) & & & & f_N'(x_0) \\ f_1''(x_0) & f_2''(x_0) & & & & f_N''(x_0) \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ f_1^{N-1}(x_0) & f_2^{N-1}(x_0) & \cdot & \cdot & \cdot & f_N^{N-1}(x_0) \end{bmatrix} \neq 0 \tag{2.4.44}$$

then $\alpha_1 = \alpha_2 = \cdots = \alpha_N = 0$ and the given set of functions is linearly independent. If no such x_0 exists, the set of functions is linearly dependent. The determinant above is called the *Wronskian* and is of fundamental importance when one tries to establish the linear independence of solutions to ordinary differential equations.[1] It is usually given the symbol W and written

[1] The Wronskian is named after the Polish mathematician Józef Hoëne-Wroński. Information about Jozef Hoene-Wronski can be found, for example, at http://en.wikipedia.org/wiki/J%C3%B3zef_Maria_Hoene-Wro%C5%84ski.

$$W[f_1, f_2, \ldots, f_N](x) = \begin{vmatrix} f_1(x) & f_2(x) & \cdot & \cdot & \cdot & f_N(x) \\ f_1'(x) & f_2'(x) & & & & f_N'(x) \\ f_1''(x) & f_2''(x) & & & & f_N''(x) \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ f_1^{N-1}(x) & f_2^{N-1}(x) & \cdot & \cdot & \cdot & f_N^{N-1}(x) \end{vmatrix} \tag{2.4.45}$$

Example 2.4.6: Consider three of the simplest kinds of functions on the interval $[-\pi,\pi]$, namely

$$f_1(x) = 1 \tag{2.4.46}$$

$$f_2(x) = \cos x \tag{2.4.47}$$

and

$$f_3(x) = \sin x \tag{2.4.48}$$

Because these functions have derivatives of *all order*, they are elements of the set $C^\infty[-\pi,\pi]$. The question is whether or not set of functions $\{1,\cos(x),\sin(x)\}$ is linearly independent. In this case, the Wronskian (2.4.45) is

$$\begin{aligned} W[f_1, f_2, f_3](x) &= \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ f_1'(x) & f_2'(x) & f_3'(x) \\ f_1''(x) & f_2''(x) & f_3''(x) \end{vmatrix} = \begin{vmatrix} 1 & \cos x & \sin x \\ 0 & -\sin x & \cos x \\ 0 & -\cos x & -\sin x \end{vmatrix} \\ &= \sin^2 x + \cos^2 x = 1 \end{aligned} \tag{2.4.49}$$

Because the Wronskian is nonzero, the set of functions $\{1,\cos x,\sin x\}$ is a linearly independent set.

Example 2.4.7: Consider the set $\{1,\cos x,\sin x,\cos 2x,\sin 2x\}$ in $C^\infty[-\pi,\pi]$. The question of linear dependence or independence of $\{1,\cos x,\sin x,\cos 2x,\sin 2x\}$ depends upon whether or not the Wronskian

$$W[f_1,f_2,f_3,f_4,f_5](x)=\begin{vmatrix} 1 & \cos x & \sin x & \cos 2x & \sin 2x \\ 0 & -\sin x & \cos x & -2\sin 2x & 2\cos 2x \\ 0 & -\cos x & -\sin x & -4\cos 2x & -4\sin 2x \\ 0 & \sin x & -\cos x & 8\sin 2x & -8\cos 2x \\ 0 & \cos x & \sin x & 16\cos 2x & 16\sin 2x \end{vmatrix} = \begin{vmatrix} -\sin x & \cos x & -2\sin 2x & 2\cos 2x \\ -\cos x & -\sin x & -4\cos 2x & -4\sin 2x \\ \sin x & -\cos x & 8\sin 2x & -8\cos 2x \\ \cos x & \sin x & 16\cos 2x & 16\sin 2x \end{vmatrix} \tag{2.4.50}$$

is zero or not for some $x_0 \in [-\pi,\pi]$. Rather than try to expand this 4×4 determinant, we can evaluate it at $x=0$ to obtain

$$W[f_1,f_2,f_3,f_4,f_5](0)=\begin{vmatrix} 0 & 1 & 0 & 2 \\ -1 & 0 & -4 & 0 \\ 0 & -1 & 0 & -8 \\ 1 & 0 & 16 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 2 \\ -1 & 0 & -8 \\ 0 & 16 & 0 \end{vmatrix} - \begin{vmatrix} 1 & 0 & 2 \\ 0 & -4 & 0 \\ -1 & 0 & -8 \end{vmatrix}$$

$$= -16\begin{vmatrix} 1 & 2 \\ -1 & -8 \end{vmatrix} + 4\begin{vmatrix} 1 & 2 \\ -1 & -8 \end{vmatrix} = 16(6)-4(6)=72$$

The fact that the Wronskian is nonzero at $x=0$ tells us that the functions in the set $\{1,\cos x,\sin x,\cos 2x,\sin 2x\}$ are linearly independent.

Many more examples of the type just discussed can be generated by simply looking at sets $\mathscr{S}$ in $C^\infty[-\pi,\pi]$ of the type

$$\mathscr{S}=\{1,\cos x,\sin x,\cos 2x,\sin 2x,...,\cos mx,\sin mx\} \tag{2.4.51}$$

where m is an *arbitrary* positive integer. The bottom line is that that in $C^\infty[-\pi,\pi]$ there are subsets of linearly independent vectors (functions) with an *arbitrary* number of elements. By choice of m, one can constructs sets of linearly independent vectors *of any size desired*. We shall have more to say about this example later. However, if you happen to be familiar with the theory of what is called a Fourier Series you know that certain classes of functions can be represented as infinite series of the form

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx \tag{2.4.52}$$

where

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos x dx \qquad n = 0,1,2,3,... \tag{2.4.53}$$

and

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin x dx \qquad n = 1,2,3,... \tag{2.4.54}$$

In some sense, the infinite set of vectors (functions)

$$\mathscr{S} = \{1, \cos x, \sin x, \cos 2x, \sin 2x, \cos 3x, \sin 3x, ...\} \tag{2.4.55}$$

span $C^\infty[-\pi,\pi]$.

Next, we need to record a theorem which will prove useful later.

Theorem 2.4.3: Let $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_N\}$ be a set of vectors in a vector space $\mathscr{V}$. Also, let $\text{Span}(\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_N)$ be the span of the set $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_N\}$. Then a vector $\mathbf{v} \in \text{Span}(\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_N)$ has a *unique* representation as a linear combination of $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_N\}$ if and only if the set $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_N\}$ is linearly independent.

Proof: First, assume $\mathbf{v} \in \text{Span}(\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_N)$ is not unique, i.e., it has two different representations

$$\mathbf{v} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_N\mathbf{v}_N \tag{2.4.56}$$

and

$$\mathbf{v} = \beta_1\mathbf{v}_1 + \beta_2\mathbf{v}_2 + \cdots + \beta_N\mathbf{v}_N \tag{2.4.57}$$

The difference of these two equations yields

$$(\alpha_1 - \beta_1)\mathbf{v}_1 + (\alpha_2 - \beta_2)\mathbf{v}_2 + \cdots + (\alpha_N - \beta_N)\mathbf{v}_N = \mathbf{0} \tag{2.4.58}$$

If we add the given condition that the set $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_N\}$ is linearly independent, the definition of linear independence implies that (2.4.58) can only be satisfied if

$$\begin{aligned} \alpha_1 &= \beta_1 \\ \alpha_2 &= \beta_2 \\ &\cdot \\ &\cdot \\ &\cdot \\ \alpha_N &= \beta_N \end{aligned} \tag{2.4.59}$$

Equation (2.4.59) establishes the uniqueness of the representation for $\mathbf{v} \in \operatorname{Span}(\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_N)$. If the set $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_N\}$ is linearly dependent, the definition of linear dependence asserts that not all of the coefficients of in the equation

$$(\alpha_1 - \beta_1)\mathbf{v}_1 + (\alpha_2 - \beta_2)\mathbf{v}_2 + \cdots + (\alpha_N - \beta_N)\mathbf{v}_N = \mathbf{0} \tag{2.4.60}$$

can be zero. Thus, the representation for $\mathbf{v} \in \operatorname{Span}(\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_N)$ is not unique.

Exercises:

2.4.1 Given the vector space $\mathscr{P}_4$ and the subset $\mathscr{S} = \{p_1, p_2, p_3, p_4, p_5\}$ of $\mathscr{P}_4$ defined by

$$\begin{aligned} p_1(x) &= 1 \\ p_2(x) &= x - a \\ p_3(x) &= (x-a)(x-b) \\ p_4(x) &= (x-a)(x-b)(x-c) \\ p_5(x) &= (x-a)(x-b)(x-c)(x-d) \end{aligned} \tag{2.4.61}$$

where a, b, c, d are real numbers. Show that the set $\mathscr{S}$ is linearly independent. Polynomials of the form (2.4.61) are the building blocks of a form of interpolation known as Newton Interpolation.

2.4.2 Given the vector space $\mathscr{P}_4$ and the subset $\mathscr{S} = \{p_1, p_2, p_3, p_4, p_5\}$ of $\mathscr{P}_4$ defined by

$$
\begin{aligned}
p_1(x) &= \frac{(x-b)(x-c)(x-d)(x-e)}{(a-b)(a-c)(a-d)(a-e)} \\
p_2(x) &= \frac{(x-a)(x-c)(x-d)(x-e)}{(b-a)(b-c)(b-d)(b-e)} \\
p_3(x) &= \frac{(x-a)(x-b)(x-d)(x-e)}{(c-a)(c-b)(c-d)(c-e)} \\
p_4(x) &= \frac{(x-a)(x-b)(x-c)(x-e)}{(d-a)(d-b)(d-c)(d-e)} \\
p_5(x) &= \frac{(x-a)(x-b)(x-c)(x-d)}{(e-a)(e-b)(e-c)(e-d)}
\end{aligned}
\tag{2.4.62}
$$

where a,b,c,d,e are distinct real numbers. Show that the set $\mathscr{S}$ is linearly independent. Polynomials of the form (2.4.62) are the building blocks of a form of interpolation known as Lagrange Interpolation.

2.4.3 Determine whether or not the vectors $e^x, \sin x$ and e^{-x} are linearly independent in $C^\infty[-\pi,\pi]$.

2.4.4 Use the more or less obvious generalization to 4×4 determinants the formula for the derivative of a determinant given in Exercise 1.10.16 and show that the derivative of the Wronskian in equation $(2.4.50)_2$ is zero. The conclusion is that the determinant, in this case, does not actually depend upon x.

Section 2.5. Basis and Dimension

In this section, we introduce two important properties of vector spaces. These properties are basis and dimension. In rough terms, a basis is a set of vectors that form building elements for the other vectors in the vector space. Also in rough terms, the dimension of a vector space is the number of vectors that form this basis. A more formal approach to these concepts begins with the following definition:

Definition: A *minimal spanning set* is the set containing the smallest number of vectors of $\mathscr{V}$ whose span is $\mathscr{V}$.

A minimal spanning set has an important characteristic that might not be evident from this definition. Elements of a minimal spanning set are necessarily *linearly independent*. If they were not, some could be eliminated in favor of others in the set and, such elimination, would violate the idea of the spanning set being minimal.

Definition: A *basis* for a vector space $\mathscr{V}$ is a minimal spanning set. In other words, a basis for a vector space $\mathscr{V}$ is a set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_N\}$ such that

a) The set $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_N\}$ is linearly independent.

b) $\operatorname{Span}(\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_N) = \mathscr{V}$

Example 2.5.1: Consider the vector space $\mathscr{V} = \mathscr{M}^{4\times 1}$ consisting of the set of 4×1 column vectors. An arbitrary vector $\mathbf{v} \in \mathscr{V}$ has the representation

$$\mathbf{v} = \begin{bmatrix} \upsilon_1 \\ \upsilon_2 \\ \upsilon_3 \\ \upsilon_4 \end{bmatrix} \tag{2.5.1}$$

By rearranging (2.5.1), we can write

$$\mathbf{v} = \begin{bmatrix} \upsilon_1 \\ \upsilon_2 \\ \upsilon_3 \\ \upsilon_4 \end{bmatrix} = \upsilon_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \upsilon_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \upsilon_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \upsilon_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{2.5.2}$$

This equation shows that every vector $\mathbf{v} \in \mathscr{V}$ is in the span of

$$\mathbf{i}_1=\begin{bmatrix}1\\0\\0\\0\end{bmatrix},\mathbf{i}_2=\begin{bmatrix}0\\1\\0\\0\end{bmatrix},\mathbf{i}_3=\begin{bmatrix}0\\0\\1\\0\end{bmatrix}\text{ and }\mathbf{i}_4=\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\tag{2.5.3}$$

Because these four column vectors (2.5.3) are linearly independent, the set $\{\mathbf{i}_1,\mathbf{i}_2,\mathbf{i}_3,\mathbf{i}_4\}$ is a basis for $\mathscr{V}=\mathscr{M}^{4\times1}$. This particular basis is sometimes called the *standard basis* of $\mathscr{V}=\mathscr{M}^{4\times1}$.

Example 2.5.2: Consider the vector space $\mathscr{V}=\mathscr{M}^{2\times2}$. Since $A\in\mathscr{V}$, it has the representation

$$A=\begin{bmatrix}A_{11}&A_{12}\\A_{21}&A_{22}\end{bmatrix}=A_{11}\begin{bmatrix}1&0\\0&0\end{bmatrix}+A_{12}\begin{bmatrix}0&1\\0&0\end{bmatrix}+A_{21}\begin{bmatrix}0&0\\1&0\end{bmatrix}+A_{22}\begin{bmatrix}0&0\\0&1\end{bmatrix}\tag{2.5.4}$$

the 2×2 matrices

$$\mathbf{i}_{11}=\begin{bmatrix}1&0\\0&0\end{bmatrix},\mathbf{i}_{12}=\begin{bmatrix}0&1\\0&0\end{bmatrix},\mathbf{i}_{21}=\begin{bmatrix}0&0\\1&0\end{bmatrix}\text{ and }\mathbf{i}_{22}=\begin{bmatrix}0&0\\0&1\end{bmatrix}\tag{2.5.5}$$

form a basis for $\mathscr{V}=\mathscr{M}^{2\times2}$. Like the first example above, this basis is sometimes called the *standard basis* of $\mathscr{V}=\mathscr{M}^{2\times2}$.

Theorem 2.5.1: If N is a positive integer and $\{\mathbf{v}_1,\mathbf{v}_2,...,\mathbf{v}_N\}$ is a basis for a vector space $\mathscr{V}$, then any set of vectors $\{\mathbf{u}_1,\mathbf{u}_2,...,\mathbf{u}_M\}$, where M is a positive (finite) integer greater than N, is linearly dependent.

Proof: Since $\{\mathbf{v}_1,\mathbf{v}_2,...,\mathbf{v}_N\}$ is a basis of $\mathscr{V}$, every vector in the set $\{\mathbf{u}_1,\mathbf{u}_2,...,\mathbf{u}_M\}$ can be written

$$\mathbf{u}_i=A_{i1}\mathbf{v}_1+A_{i2}\mathbf{v}_2+\cdots+A_{iN}\mathbf{v}_N=\sum_{j=1}^{N}A_{ij}\mathbf{v}_j\qquad\text{for } i=1,2,...,M\tag{2.5.6}$$

The theorem asserts the set $\{\mathbf{u}_1,\mathbf{u}_2,...,\mathbf{u}_M\}$ is linear dependent when $M>N$. The test for linear independence or dependence requires that we examine a linear relationship of the form

$$\alpha_1\mathbf{u}_1+\alpha_2\mathbf{u}_2+\cdots+\alpha_M\mathbf{u}_M=\sum_{i=1}^{M}\alpha_i\mathbf{u}_i=\mathbf{0}\tag{2.5.7}$$

If the vectors $\{\mathbf{u}_1,\mathbf{u}_2,...,\mathbf{u}_M\}$ are linear dependent when $M>N$, as the theorem asserts, some of the coefficients $\alpha_1,\alpha_2,...,\alpha_M$ must be nonzero. Equation (2.5.7) can be expressed in terms of the basis $\{\mathbf{v}_1,\mathbf{v}_2,...,\mathbf{v}_N\}$ by use (2.5.6). The result of utilizing (2.5.6) in (2.5.7) is

$$\sum_{i=1}^{M} \alpha_i \mathbf{u}_i = \sum_{i=1}^{M} \alpha_i \sum_{j=1}^{N} A_{ij} \mathbf{v}_j = \sum_{j=1}^{N} \left(\sum_{i=1}^{M} \alpha_i A_{ij} \right) \mathbf{v}_j = \mathbf{0} \tag{2.5.8}$$

We are given that the set $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_N\}$ is linearly independent. As a result, the coefficients in (2.5.8) must be zero. The resulting N equations for the $M > N$ coefficients $\alpha_1, \alpha_2, ..., \alpha_M$ are

$$\sum_{i=1}^{M} \alpha_i A_{ij} = 0 \qquad \text{for} \qquad j = 1, 2, ..., N \tag{2.5.9}$$

The proof of the theorem comes down to asking whether or not (2.5.9) has nonzero solutions for some of the coefficients $\alpha_1, \alpha_2, ..., \alpha_M$. As an undetermined system, i.e., a system with more unknowns than equations, it is the case that the $\alpha_1, \alpha_2, ..., \alpha_M$ are not all necessary zero and, as a result, the set $\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_M\}$ is linearly dependent.

Corollary: If $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_N\}$ and $\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_M\}$ are both bases of a vector space $\mathscr{V}$, then $M = N$.

Proof: Theorem 1.5.1 says that if $M > N$, the set $\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_M\}$ must be linearly dependent. Since we have postulated that $\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_M\}$ is linearly *independent*, we must conclude that $M \le N$. If we now reverse the roles of $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_N\}$ and $\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_N\}$, we are led to conclude that $N \le M$. These two conditions force the conclusion that $M = N$.

We have defined a basis as a minimal spanning set. We have just established with Theorem 2.5.1 that when a basis has a finite number of elements it is also a maximal set in the sense that it not a proper subset of any other linearly independent set. The above corollary allows us to conclude, for the case where the basis has a finite number of elements, that number is a *property* of the vector space. It is given a name.

Definition: Let $\mathscr{V}$ be a vector space. If $\mathscr{V}$ has a basis of N vectors, where N is a positive integer, then $\mathscr{V}$ is said to have *dimension* $N = \dim \mathscr{V}$.

Definition: The subspace $\{\mathbf{0}\}$ of $\mathscr{V}$ has, by definition, the dimension 0.

Definition: The vector space $\mathscr{V}$ is *finite dimensional* if the basis has a finite number of members. If this is not finite dimensional, it is *infinite dimensional*.

Exclusive of a few examples, the vector spaces we shall study will be finite dimensional. The following two theorems are most useful in the applications that will follow later in this textbook;

Theorem 2.5.2: If $\mathscr{U}$ is a subspace of a vector space $\mathscr{V}$, then

$$\dim \mathscr{U} \le \dim \mathscr{V} \tag{2.5.10}$$

Proof: This result is more or less obvious. If $\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_M\}$ is a basis for $\mathscr{U}$ and $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_N\}$ is a basis for $\mathscr{V}$, then we need to rule out the case $M > N$. If $M > N$, the basis for $\mathscr{V}$, $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_N\}$, would not be a minimum spanning set.

Theorem 2.5.3: If $\mathscr{U}$ is a subspace of a vector space $\mathscr{V}$, then $\dim \mathscr{U} = \dim \mathscr{V}$ if and only if $\mathscr{U} = \mathscr{V}$.

Proof: If $\mathscr{U} = \mathscr{V}$, then $\dim \mathscr{U} = \dim \mathscr{V}$. Conversely, if $\dim \mathscr{U} = \dim \mathscr{V}$, a basis for the subspace $\mathscr{U}$ is a basis for $\mathscr{V}$ which implies $\mathscr{U} = \mathscr{V}$

Example 2.5.3: Examples 2.5.1 and 2.5.2 tell us that $\dim \mathscr{M}^{3\times 1} = 3$ and $\dim \mathscr{M}^{2\times 2} = 4$. It is a fact that

$$\dim \mathscr{M}^{M\times N} = MN \tag{2.5.11}$$

Example 2.5.4: Every element of the set $\mathscr{P}_N$ is a polynomial of the form (2.1.16), repeated,

$$p(x) = \lambda_0 + \lambda_1 x + \lambda_2 x^2 + \cdots + \lambda_N x^N \tag{2.5.12}$$

An elementary generalization of Example 2.4.5 shows that the set $\{1, x, x^2, ..., x^N\}$ is linearly independent. It is perhaps evident that if this set is augmented by any other polynomial of order less than or equal to N the resulting set will be linearly dependent. The conclusion from this observation is that

$$\dim \mathscr{P}_N = N + 1 \tag{2.5.13}$$

Example 2.5.5: This example is a vector space that is not finite dimensional. Let $\mathscr{P}_\infty$ be the vector space of *all* polynomials. We assume it is finite dimensional and look for confirmation or a contradiction. If it is finite dimensional, say of dimension N, then we know from Theorem 2.5.1 that a set of $N+1$ polynomials in $\mathscr{P}_\infty$ would be linearly dependent. Consider the subset of $\mathscr{P}_\infty$ defined by

$$\mathscr{S} = \{1, x, x^2, ..., x^N\} \tag{2.5.14}$$

The subset $\mathscr{S}$ of $\mathscr{P}_\infty$ contains $N+1$ polynomials. Theorem 2.5.1 asserts that if $\mathscr{P}_\infty$ has dimension N, then $\mathscr{S}$ is a linearly dependent set. A convenient test of this linear dependence is given by whether or not the Wronskian, equation (2.4.45), is zero for some $x \in \mathscr{R}$. It follows from (2.4.45) that the Wronskian of this set of functions in (2.5.14) is

$$W[1,x,x^2,...,x^N](x)=\begin{vmatrix} 1 & x & x^2 & \cdot & \cdot & x^N \\ 0 & 1 & 2x & & & Nx^{N-1} \\ 0 & 0 & 2 & & & N(N-1)x^{N-2} \\ \cdot & \cdot & 0 & & & \cdot \\ \cdot & \cdot & \cdot & & & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & N! \end{vmatrix}=(2!)(3!)\cdots(N!)>0 \qquad (2.5.15)$$

Thus, the set $\mathscr{S}$ must also be *linearly independent*. This conclusion contradicts the assertion that the dimension of $\mathscr{P}_\infty$ is N. Because N is arbitrary, we can only conclude that $\mathscr{P}_\infty$ is not finite dimensional, i.e. it is infinite dimensional.

Example 2.5.6: The set $C^\infty[-\pi,\pi]$ is also infinite dimensional. The proof is like the last one except that one starts with the set

$$\mathscr{S}=\{1,\cos x,\sin x,\cos 2x,\sin 2x,...,\cos Nx,\sin Nx\} \qquad (2.5.16)$$

for some prescribed N. On the presumption that the dimension of $C^\infty[-\pi,\pi]$ is $2N+1$ (the number of members of $\mathscr{S}$), the next step is to consider the set

$$\mathscr{S}_1=\{1,\cos x,\sin x,\cos 2x,\sin 2x,...,\cos Nx,\sin Nx,\cos(N+1)x,\sin(N+1)x\} \qquad (2.5.17)$$

and examine whether or not this set is linearly independent or independent. It turns out that it is linearly independent for all N. The fact that N is arbitrary leads to the conclusion that $C^\infty[-\pi,\pi]$ is infinite dimensional.

Example 2.5.6 explains that $C^\infty[-\pi,\pi]$ is an infinite dimensional subspace. For given N, the set $\mathscr{S}$ is linearly independent because its Wronskian is non zero at, for example, $x=0$. The span of $\mathscr{S}$, Span$\mathscr{S}$, is a vector space of dimension N. It is an example of a finite dimensional subspace of an infinite dimensional vector space. The theory of Fourier Series is, in effect, an approximation of an arbitrary vector in $C^\infty[-\pi,\pi]$ by a finite dimensional vector in $\mathscr{S}$. Our interest is primarily in the study of finite dimensional vector spaces, thus we will not pursue the mathematical ideas behind Fourier Series in this chapter.

Exercises

2.5.1 What is the dimension of the subspace spanned by the set $\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$, where

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ -3 \\ 4 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 3 \\ -1 \\ 2 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ -5 \\ 8 \\ -7 \end{bmatrix}$$

Section 2.6. Change of Basis

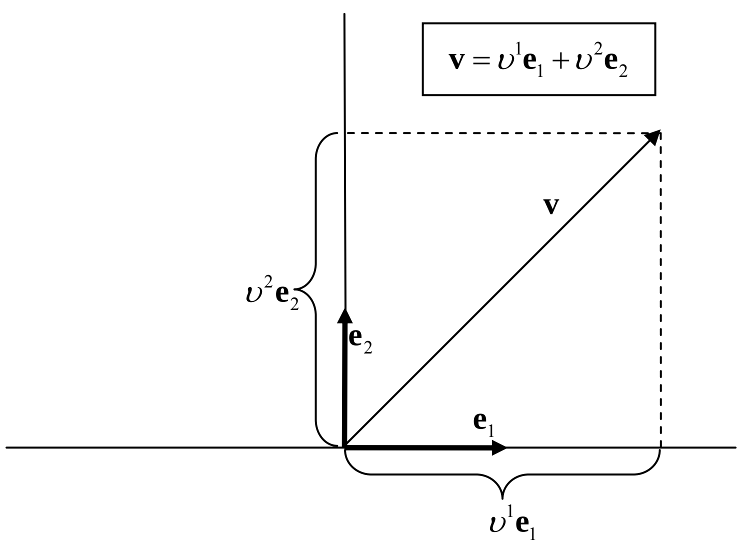
In Theorem 2.5.1 we established that if $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_N\}$ is a basis for a vector space $\mathscr{V}$, then any set of vectors $\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_M\}$ in $\mathscr{V}$, $M > N$, is linearly dependent. We also established the Corollary to Theorem 2.5.1 that said that if two different sets of vectors were a basis for $\mathscr{V}$, then each set had to have the same number of elements. This number is the dimension of $\mathscr{V}$ which we wrote $\dim \mathscr{V}$.

Given Theorem 2.5.1, and a basis $\{\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_N\}$ for a finite dimensional vector space $\mathscr{V}$, we can conclude that an arbitrary vector $\mathbf{v} \in \mathscr{V}$ has the representation[2]

$$\mathbf{v} = \upsilon^1 \mathbf{e}_1 + \upsilon^2 \mathbf{e}_2 + \cdots + \upsilon^N \mathbf{e}_N = \sum_{j=1}^{N} \upsilon^j \mathbf{e}_j \tag{2.6.1}$$

The scalars $\upsilon^1, \upsilon^2, ..., \upsilon^N$ are called the *components of* $\mathbf{v}$ *with respect to the basis* $\{\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_N\}$. If we apply the results of Theorem 2.4.3, we can conclude that the components of $\mathbf{v}$ with respect to the basis $\{\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_N\}$ are unique.

While (2.6.1) holds in great generality for every vector space of finite dimension, it is important to connect it with geometric ideas that are a part of our experience with elementary mathematics. For example, consider the two dimensional vector space of geometric vectors in the plane. We can visualize vectors in the plane with the following figure:

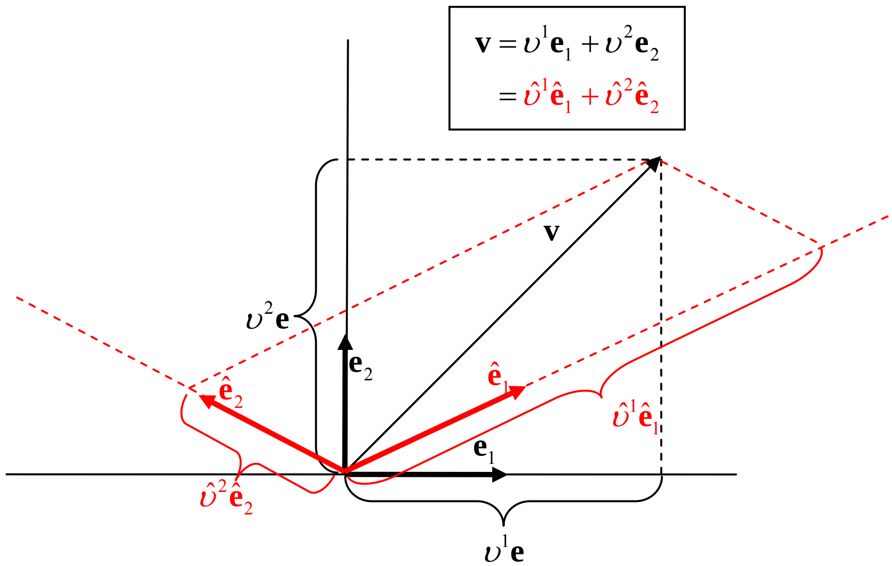


[2] The scalars are indexed with superscripts in order to follow a notation conventional in Linear Algebra. For much of what we do, there is no loss of generality if one wanted to stick strictly with subscripts.

If we choose a different basis $\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2\}$, we can also expand the vector $\mathbf{v}$ in the form

$$\mathbf{v} = \hat{\upsilon}^1 \hat{\mathbf{e}}_1 + \hat{\upsilon}^2 \hat{\mathbf{e}}_2 \tag{2.6.2}$$

The geometric representation of this equation can be displayed by superimposing it on the above figure. The result is



This figure is a geometric representation of a *change of basis*. The fact that a finite dimensional vector space has multiple ways to represent the same vector gives rise to the idea of a *change of basis* in a vector space. In the study of geometric vectors, this change is usually characterized as a change of coordinates. In any case, we shall now formalize the *transformation rules* of vectors under a change of basis.

Let $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_N\}$ and $\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \ldots, \hat{\mathbf{e}}_N\}$ be two bases for an N dimensional vector space $\mathscr{V}$. Because $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_N\}$ is a basis, we can express any vector in $\mathscr{V}$, including those in $\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \ldots, \hat{\mathbf{e}}_N\}$, as an expansion in the vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_N$. Therefore, for the j^{th} vector of $\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \ldots, \hat{\mathbf{e}}_N\}$,

$$\hat{\mathbf{e}}_j = \sum_{k=1}^{N} T_j^k \mathbf{e}_k \quad \text{for } j = 1, 2, \ldots, N \tag{2.6.3}$$

One way to look at (2.6.3) is that the coefficient T_j^k is the k^{th} component of $\hat{\mathbf{e}}_j$ with respect to the basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_N\}$.

If we reverse the roles of $\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, ..., \hat{\mathbf{e}}_N\}$ and $\{\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_N\}$, then we can also write

$$\mathbf{e}_k = \sum_{s=1}^{N} \hat{T}_k^s \hat{\mathbf{e}}_s \qquad \text{for } k = 1,2,...,N \tag{2.6.4}$$

As one would suspect, the coefficients $\hat{T}_k^s$, $s,k = 1,...,N$, are related to the coefficients T_j^k, $k, j = 1,...,N$. To see this relationship, we can substitute (2.6.4) into (2.6.3) and obtain

$$\hat{\mathbf{e}}_j = \sum_{k=1}^{N} T_j^k \mathbf{e}_k = \sum_{k=1}^{N} T_j^k \sum_{s=1}^{N} \hat{T}_k^s \hat{\mathbf{e}}_s = \sum_{s=1}^{N} \left(\sum_{k=1}^{N} \hat{T}_k^s T_j^k \right) \hat{\mathbf{e}}_s \tag{2.6.5}$$

Likewise, we can substitute (2.6.3) into (2.6.4) and obtain

$$\mathbf{e}_k = \sum_{s=1}^{N} \hat{T}_k^s \hat{\mathbf{e}}_s = \sum_{s=1}^{N} \hat{T}_k^s \sum_{q=1}^{N} T_s^q \mathbf{e}_q = \sum_{q=1}^{N} \left(\sum_{s=1}^{N} T_s^q \hat{T}_k^s \right) \mathbf{e}_q \tag{2.6.6}$$

These equations force the following relationships on the coefficients $\hat{T}_k^j$ and T_j^k, $k, j = 1,...,N$

$$\sum_{k=1}^{N} \hat{T}_k^s T_j^k = \delta_j^s \quad \text{and} \quad \sum_{s=1}^{N} T_s^q \hat{T}_k^s = \delta_k^q \tag{2.6.7}$$

Where the Kronecker delta, defined by equation (1.1.28), has been redefined with the notation

$$\delta_j^k = \begin{cases} 1 & \text{for } k = j \\ 0 & \text{for } k \neq j \end{cases} \tag{2.6.8}$$

It is helpful to write (2.6.7) in matrix notation by introduction of the notation[3]

$$T = \left[T_j^k \right] \qquad \text{and} \qquad \hat{T} = \left[\hat{T}_k^j \right] \tag{2.6.9}$$

These definitions allow the two equations (2.6.7) to be written as matrix equations

$$\hat{T}T = I \qquad \text{and} \qquad T\hat{T} = I \tag{2.6.10}$$

These two equations tell us that the matrix T is nonsingular and that $\hat{T} = T^{-1}$. The matrix T is called the *transition matrix*.

---

[3] When the superscript-subscript notation is used as with T_j^k the convention is that the superscript denotes the row of the matrix and the subscript the column.

Example 2.6.1: Let $\mathscr{V}$ be a vector space of dimension 3 with basis $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$. You are given a second basis $\{\hat{\mathbf{e}}_1,\hat{\mathbf{e}}_2,\hat{\mathbf{e}}_3\}$ defined by the formulas

$$\begin{aligned}\hat{\mathbf{e}}_1 &= 2\mathbf{e}_1-\mathbf{e}_2-\mathbf{e}_3\\ \hat{\mathbf{e}}_2 &= -\mathbf{e}_1+\mathbf{e}_3\\ \hat{\mathbf{e}}_3 &= 4\mathbf{e}_1-\mathbf{e}_2+6\mathbf{e}_3\end{aligned} \tag{2.6.11}$$

If (2.6.11) and (2.6.3) are compared, we see that the transition matrix is

$$T=\left[T_k^j\right]=\begin{bmatrix} T_1^1 & T_2^1 & T_3^1\\ T_1^2 & T_2^2 & T_3^2\\ T_1^3 & T_2^3 & T_3^3\end{bmatrix}=\begin{bmatrix} 2 & -1 & 4\\ -1 & 0 & -1\\ -1 & 1 & 6\end{bmatrix} \tag{2.6.12}$$

The inverse of the transition matrix turns out to be

$$\hat{T}=\left[\hat{T}_k^j\right]=\begin{bmatrix} \hat{T}_1^1 & \hat{T}_2^1 & \hat{T}_3^1\\ \hat{T}_1^2 & \hat{T}_2^2 & \hat{T}_3^2\\ \hat{T}_1^3 & \hat{T}_2^3 & \hat{T}_3^3\end{bmatrix}=\begin{bmatrix} 2 & -1 & 4\\ -1 & 0 & -1\\ -1 & 1 & 6\end{bmatrix}^{-1}=\begin{bmatrix} -\frac{1}{9} & -\frac{10}{9} & -\frac{1}{9}\\ -\frac{7}{9} & -\frac{16}{9} & \frac{2}{9}\\ \frac{1}{9} & \frac{1}{9} & \frac{1}{9}\end{bmatrix} \tag{2.6.13}$$

Therefore, (2.6.4) takes the form

$$\begin{aligned}\mathbf{e}_1 &= \sum_{j=1}^{N}\hat{T}_1^j\hat{\mathbf{e}}_j=\hat{T}_1^1\hat{\mathbf{e}}_1+\hat{T}_1^2\hat{\mathbf{e}}_2+\hat{T}_1^3\hat{\mathbf{e}}_3=-\frac{1}{9}\hat{\mathbf{e}}_1-\frac{7}{9}\hat{\mathbf{e}}_2+\frac{1}{9}\hat{\mathbf{e}}_2\\ \mathbf{e}_2 &= \sum_{j=1}^{N}\hat{T}_2^j\hat{\mathbf{e}}_j=\hat{T}_2^1\hat{\mathbf{e}}_1+\hat{T}_2^2\hat{\mathbf{e}}_2+\hat{T}_2^3\hat{\mathbf{e}}_3=-\frac{10}{9}\hat{\mathbf{e}}_1-\frac{16}{9}\hat{\mathbf{e}}_2+\frac{1}{9}\hat{\mathbf{e}}_2\\ \mathbf{e}_3 &= \sum_{j=1}^{N}\hat{T}_3^j\hat{\mathbf{e}}_j=\hat{T}_3^1\hat{\mathbf{e}}_1+\hat{T}_3^2\hat{\mathbf{e}}_2+\hat{T}_3^3\hat{\mathbf{e}}_3=-\frac{1}{9}\hat{\mathbf{e}}_1+\frac{2}{9}\hat{\mathbf{e}}_2+\frac{1}{9}\hat{\mathbf{e}}\end{aligned} \tag{2.6.14}$$

Example 2.6.2: Consider the vector space of polynomials of degree less than or equal to 3, i.e., $\mathscr{P}_3$. Elements of $\mathscr{P}_3$ consist of polynomials of degree 3 or less and from (2.1.16) are defined by

$$p(x)=\lambda_0+\lambda_1 x+\lambda_2 x^2+\lambda_3 x^3 \tag{2.6.15}$$

Example 2.4.5 established that a basis for this four dimensional vector space are the polynomials (2.4.39), repeated,

$$\begin{aligned} p_1(x) &= 1 \\ p_2(x) &= x \\ p_3(x) &= x^2 \\ p_4(x) &= x^3 \end{aligned} \tag{2.6.16}$$

If a,b,c are given real numbers, one can establish that the four polynomials

$$\begin{aligned} \hat{p}_1(x) &= 1 \\ \hat{p}_2(x) &= x-a \\ \hat{p}_3(x) &= (x-a)(x-b) \\ \hat{p}_4(x) &= (x-a)(x-b)(x-c) \end{aligned} \tag{2.6.17}$$

are also linearly independent and thus a basis. Polynomials of the form (2.6.17) are the building blocks of a form of interpolation known as Newton Interpolation. The change of basis from $\{p_1, p_2, p_3, p_4\}$ to $\{\hat{p}_1, \hat{p}_2, \hat{p}_3, \hat{p}_4\}$ is defined by (2.6.3). In the notation being used for the basis elements, (2.6.3) is

$$\hat{p}_j = \sum_{k=1}^{4} T_j^k p_k \quad \text{for } j = 1,2,...,4 \tag{2.6.18}$$

At each $x \in \mathscr{R}$, (2.6.18) becomes

$$\hat{p}_j(x) = \sum_{k=1}^{4} T_j^k p_k(x) \quad \text{for } j = 1,2,...,4 \tag{2.6.19}$$

The components of the transition matrix, $T^k{}_j$, for $j,k = 1,2,...,4$ are obtained by substitution of (2.6.16) and (2.6.17) into (2.6.19). The resulting four equations are

$$\begin{aligned} 1 &= T_1^1 + T_1^2 x + T_1^3 x^2 + T_1^4 x^3 \\ x-a &= T_2^1 + T_2^2 x + T_2^3 x^2 + T_2^4 x^3 \\ (x-a)(x-b) &= T_3^1 + T_3^2 x + T_3^3 x^2 + T_3^4 x^3 \\ (x-a)(x-b)(x-c) &= T_4^1 + T_4^2 x + T_4^3 x^2 + T_4^4 x^3 \end{aligned} \tag{2.6.20}$$

If these four equations are forced to hold for all $x \in \mathscr{R}$, it readily follows that the transition matrix is given by

$$T = \left[T_k^j \right] = \begin{bmatrix} T_1^1 & T_2^1 & T_3^1 & T_4^1 \\ T_1^2 & T_2^2 & T_3^2 & T_4^2 \\ T_1^3 & T_2^3 & T_3^3 & T_4^3 \\ T_1^4 & T_2^4 & T_3^4 & T_4^4 \end{bmatrix} = \begin{bmatrix} 1 & -a & ab & -abc \\ 0 & 1 & -(a+b) & ab+ac+bc \\ 0 & 0 & 1 & -(a+b+c) \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.6.21}$$

The inverse transition matrix turns out to be

$$\hat{T} = \left[\hat{T}_k^j \right] = \begin{bmatrix} \hat{T}_1^1 & \hat{T}_2^1 & \hat{T}_3^1 & \hat{T}_4^1 \\ \hat{T}_1^2 & \hat{T}_2^2 & \hat{T}_3^2 & \hat{T}_4^2 \\ \hat{T}_1^3 & \hat{T}_2^3 & \hat{T}_3^3 & \hat{T}_{44}^3 \\ \hat{T}_1^4 & \hat{T}_2^4 & \hat{T}_3^4 & \hat{T}_{44}^4 \end{bmatrix} = \begin{bmatrix} 1 & -a & ab & -abc \\ 0 & 1 & -(a+b) & ab+ac+bc \\ 0 & 0 & 1 & -(a+b+c) \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & a & a^2 & a^3 \\ 0 & 1 & a+b & a^2+ab+b^2 \\ 0 & 0 & 1 & a+b+c \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.6.22}$$

Example 2.6.3: Consider again the vector space of polynomials of degree less than or equal to 3, i.e., $\mathscr{P}_3$. If a,b,c,d are *distinct* real numbers, one can establish that the four polynomials

$$\begin{aligned} \hat{p}_1(x) &= \frac{(x-b)(x-c)(x-d)}{(a-b)(a-c)(a-d)} \\ \hat{p}_2(x) &= \frac{(x-a)(x-c)(x-d)}{(b-a)(b-c)(b-d)} \\ \hat{p}_3(x) &= \frac{(x-a)(x-b)(x-d)}{(c-a)(c-b)(c-d)} \\ \hat{p}_4(x) &= \frac{(x-a)(x-b)(x-c)}{(d-a)(d-b)(d-c)} \end{aligned} \tag{2.6.23}$$

are also linearly independent and thus a basis. Polynomials of the form (2.6.23) are the building blocks of a form of interpolation known as Lagrange Interpolation. In any case, the change of basis from $\{p_1, p_2, p_3, p_4\}$ to $\{\hat{p}_1, \hat{p}_2, \hat{p}_3, \hat{p}_4\}$ is again defined by(2.6.18). The same kind of calculation used in Example 2.6.2 yields the transition matrix

$$
T = \begin{bmatrix}
-\frac{bcd}{(a-b)(a-c)(a-d)} & -\frac{acd}{(b-a)(b-c)(b-d)} & -\frac{abd}{(c-a)(c-b)(c-d)} & -\frac{abc}{(d-a)(d-b)(d-c)} \\
\frac{cd+bd+bc}{(a-b)(a-c)(a-d)} & \frac{ad+cd+ac}{(b-a)(b-c)(b-d)} & \frac{ab+ad+bd}{(c-a)(c-b)(c-d)} & \frac{ab+ac+bc}{(d-a)(d-b)(d-c)} \\
-\frac{b+c+d}{(a-b)(a-c)(a-d)} & -\frac{a+c+d}{(b-a)(b-c)(b-d)} & -\frac{a+b+d}{(c-a)(c-b)(c-d)} & -\frac{a+b+c}{(d-a)(d-b)(d-c)} \\
\frac{1}{(a-b)(a-c)(a-d)} & \frac{1}{(b-a)(b-c)(b-d)} & \frac{1}{(c-a)(c-b)(c-d)} & \frac{1}{(d-a)(d-b)(d-c)}
\end{bmatrix}
\tag{2.6.24}
$$

The inverse transition matrix turns out to be

$$
\hat{T} = \begin{bmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{bmatrix} \tag{2.6.25}
$$

If (2.6.25) is compared to the Vandermonde matrix given in (1.10.33), we see that it is a Vandermonde matrix transposed.

The change of basis from $\{\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_N\}$ to $\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, ..., \hat{\mathbf{e}}_N\}$ is characterized by the transition matrix through equations (2.6.3) and (2.6.4). If we are given a vector $\mathbf{v} \in \mathscr{V}$, it can be expanded in the basis $\{\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_N\}$ or, equivalently, in the basis $\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, ..., \hat{\mathbf{e}}_N\}$. The result is two *equivalent* representations of the same vector of the forms

$$
\mathbf{v} = \sum_{j=1}^{N} \upsilon^j \mathbf{e}_j \tag{2.6.26}
$$

and

$$
\mathbf{v} = \sum_{j=1}^{N} \hat{\upsilon}^j \hat{\mathbf{e}}_j \tag{2.6.27}
$$

As explained at the start of this Section, the set of scalars $\{\upsilon^1, \upsilon^2, ..., \upsilon^N\}$ are the components of $\mathbf{v}$ with respect to the basis $\{\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_N\}$. Likewise, the set of scalars $\{\hat{\upsilon}^1, \hat{\upsilon}^2, ..., \hat{\upsilon}^N\}$ are the components of $\mathbf{v}$ with respect to the basis $\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, ..., \hat{\mathbf{e}}_N\}$. The connection between the two bases as given by (2.6.3) and (2.6.4). We shall now use these formulas to derive a transformation rule for the components resulting from the basis change from $\{\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_N\}$ to $\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, ..., \hat{\mathbf{e}}_N\}$. The first step is to equate (2.6.26) and (2.6.27) and use (2.6.3) to eliminate the basis $\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, ..., \hat{\mathbf{e}}_N\}$. Therefore,

$$\mathbf{v} = \sum_{j=1}^{N} \upsilon^j \mathbf{e}_j = \sum_{j=1}^{N} \hat{\upsilon}^j \hat{\mathbf{e}}_j = \sum_{j=1}^{N} \hat{\upsilon}^j \sum_{k=1}^{N} T_j^k \mathbf{e}_k = \sum_{k=1}^{N} \sum_{j=1}^{N} T_j^k \hat{\upsilon}^j \mathbf{e}_k \tag{2.6.28}$$

In order to extract additional information from (2.6.28), it is rewritten

$$\sum_{j=1}^{N} \upsilon^j \mathbf{e}_j = \sum_{k=1}^{N} \sum_{j=1}^{N} T_j^k \hat{\upsilon}^j \mathbf{e}_k \tag{2.6.29}$$

Next, we simply change the name of the summation index on the left side from j to k and rewrite the result as

$$\sum_{k=1}^{N} \left(\upsilon^k - \sum_{j=1}^{N} T_j^k \hat{\upsilon}^j \right) \mathbf{e}_k = \mathbf{0} \tag{2.6.30}$$

Because the basis $\{\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_N\}$ is a linearly independent set, (2.6.30) yields the desired component transformation rule

$$\upsilon^k = \sum_{j=1}^{N} T_j^k \hat{\upsilon}^j \tag{2.6.31}$$

If we were to have eliminated $\{\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_N\}$ in favor of $\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, ..., \hat{\mathbf{e}}_N\}$, we would have obtained the transformation rule

$$\hat{\upsilon}^k = \sum_{j=1}^{N} \hat{T}_j^k \upsilon^j \tag{2.6.32}$$

which, of course, is just what one obtains when the system (2.6.31) is inverted. Equations (2.6.31) and (2.6.32) represent the *transformation rules* for the components of vectors $\mathbf{v} \in \mathscr{V}$.

Example 2.6.4: Given the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ for a vector space $\mathscr{V}$ and a vector $\mathbf{v} \in \mathscr{V}$ defined by

$$\mathbf{v} = \mathbf{e}_1 + 2\mathbf{e}_2 + 3\mathbf{e}_3 \tag{2.6.33}$$

The components with respect to $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ are

$$\begin{aligned} \upsilon^1 &= 1 \\ \upsilon^2 &= 2 \\ \upsilon^3 &= 3 \end{aligned} \tag{2.6.34}$$

You are given a second basis $\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3\}$ defined by (2.6.11) of Example 2.6.1. It follows from (2.6.32) and (2.6.13) that

$$\begin{bmatrix} \hat{\upsilon}^1 \\ \hat{\upsilon}^2 \\ \hat{\upsilon}^3 \end{bmatrix} = \begin{bmatrix} \hat{T}_1^1 & \hat{T}_2^1 & \hat{T}_3^1 \\ \hat{T}_1^2 & \hat{T}_2^2 & \hat{T}_3^2 \\ \hat{T}_1^3 & \hat{T}_2^3 & \hat{T}_3^3 \end{bmatrix} \begin{bmatrix} \upsilon^1 \\ \upsilon^2 \\ \upsilon^3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{9} & -\frac{10}{9} & -\frac{1}{9} \\ -\frac{7}{9} & -\frac{16}{9} & \frac{2}{9} \\ \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -\frac{8}{3} \\ -\frac{11}{3} \\ \frac{2}{3} \end{bmatrix} \tag{2.6.35}$$

Therefore, with respect to the basis $\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3\}$,

$$\mathbf{v} = -\frac{8}{3}\hat{\mathbf{e}}_1 - \frac{11}{3}\hat{\mathbf{e}}_2 + \frac{2}{3}\hat{\mathbf{e}}_3 \tag{2.6.36}$$

Example 2.6.5: Given the basis $\{p_1, p_2, p_3, p_4\}$ of the vector space $\mathscr{P}_3$ defined by (2.6.16) and a member of $\mathscr{P}_3$, i.e., a third order polynomial given by

$$p(x) = \upsilon^1 p_1(x) + \upsilon^2 p_2(x) + \upsilon^3 p_3(x) + \upsilon^4 p_4(x) = 4 - 3x - 2x^2 + x^3 \tag{2.6.37}$$

Therefore, the components with respect to $\{p_1, p_2, p_3, p_4\}$ are

$$\begin{aligned} \upsilon^1 &= 4 \\ \upsilon^2 &= -3 \\ \upsilon^3 &= -2 \\ \upsilon^4 &= 1 \end{aligned} \tag{2.6.38}$$

You are given a second basis $\{\hat{p}_1, \hat{p}_2, \hat{p}_3, \hat{p}_4\}$ defined by the following special case of (2.6.17)

$$\begin{aligned} \hat{p}_1(x) &= 1 \\ \hat{p}_2(x) &= x - 1 \\ \hat{p}_3(x) &= (x-1)(x-2) \\ \hat{p}_4(x) &= (x-1)(x-2)(x-3) \end{aligned} \tag{2.6.39}$$

It follows from (2.6.32), (2.6.38), the inverse transition matrix (2.6.22) and the choices $a=1, b=2, c=3$ that

$$\begin{aligned}\begin{bmatrix}\hat{\upsilon}^1\\ \hat{\upsilon}^2\\ \hat{\upsilon}^3\\ \hat{\upsilon}^4\end{bmatrix}&=\begin{bmatrix}1 & a & a^2 & a^3\\ 0 & 1 & a+b & a^2+ab+b^2\\ 0 & 0 & 1 & a+b+c\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\upsilon^1\\ \upsilon^2\\ \upsilon^3\\ \upsilon^4\end{bmatrix}\\ &=\begin{bmatrix}1 & 1 & 1 & 1\\ 0 & 1 & 3 & 7\\ 0 & 0 & 1 & 5\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}4\\ -3\\ -2\\ 1\end{bmatrix}=\begin{bmatrix}0\\ -2\\ 4\\ 1\end{bmatrix}\end{aligned} \tag{2.6.40}$$

Therefore, with respect to the basis $\{\hat{p}_1, \hat{p}_2, \hat{p}_3, \hat{p}_4\}$, the polynomial p is given by

$$\begin{aligned}p(x)&=\hat{\upsilon}^1\hat{p}_1(x)+\hat{\upsilon}^2\hat{p}_2(x)+\hat{\upsilon}^3\hat{p}_3(x)+\hat{\upsilon}^4\hat{p}_4(x)\\ &=-2(x-1)+4(x-1)(x-2)+(x-1)(x-2)(x-3)\end{aligned} \tag{2.6.41}$$

Exercises:

2.6.1 Given the following two bases of $\mathscr{M}^{2\times 1}$

$$\mathbf{e}_1=\begin{bmatrix}1\\ 0\end{bmatrix}, \mathbf{e}_2=\begin{bmatrix}0\\ 1\end{bmatrix} \tag{2.6.42}$$

and

$$\hat{\mathbf{e}}_1=\begin{bmatrix}5\\ 3\end{bmatrix}, \hat{\mathbf{e}}_2=\begin{bmatrix}3\\ 2\end{bmatrix} \tag{2.6.43}$$

Find the transition matrix.

2.6.2 If $\mathbf{v}$ is defined with respect to the basis (2.6.42)

$$\mathbf{v}=10\mathbf{e}_1+7\mathbf{e}_2 \tag{2.6.44}$$

determine the components of $\mathbf{v}$ with respect to the basis $\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2\}$ defined by (2.6.43).

2.6.3 The vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ are a basis for a three dimensional vector space $\mathscr{V}$. The set of vectors $\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3\}$, defined by,

$$\begin{aligned}
\hat{\mathbf{e}}_1 &= \mathbf{e}_1 + 2\mathbf{e}_2 + \mathbf{e}_3 \\
\hat{\mathbf{e}}_2 &= \mathbf{e}_1 - \mathbf{e}_2 + 2\mathbf{e}_3 \\
\hat{\mathbf{e}}_3 &= \mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3
\end{aligned} \tag{2.6.45}$$

represent a change of basis. Determine the transition matrix for the basis change $\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3\} \to \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. Also, if $\mathbf{v} = 3\mathbf{e}_1 + 2\mathbf{e}_3$, determine the components of $\mathbf{v}$ with respect to the basis $\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3\}$.

2.6.4 Let $\mathscr{V}$ be a vector space of dimension 3 with basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. You are given a second basis $\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3\}$ defined by the formulas

$$\begin{aligned}
\hat{\mathbf{e}}_1 &= -\frac{4}{3}\mathbf{e}_1 + \frac{4}{3}\mathbf{e}_2 \\
\hat{\mathbf{e}}_2 &= -\frac{10}{3}\mathbf{e}_1 + \frac{7}{3}\mathbf{e}_2 \\
\hat{\mathbf{e}}_3 &= \mathbf{e}_3
\end{aligned} \tag{2.6.46}$$

If a vector $\mathbf{v} \in \mathscr{V}$ is defined by

$$\mathbf{v} = \mathbf{e}_1 + 2\mathbf{e}_2 + 3\mathbf{e}_3 \tag{2.6.47}$$

determine the components of $\mathbf{v} \in \mathscr{V}$ with respect to the basis $\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3\}$.

2.6.5 Let $\mathscr{V}$ be a vector space of dimension 3 with basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. You are given a second basis $\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3\}$ defined by the formulas

$$\begin{aligned}
\hat{\mathbf{e}}_1 &= -\frac{4i}{5}\mathbf{e}_1 + \frac{3}{5}\mathbf{e}_2 \\
\hat{\mathbf{e}}_2 &= -\frac{7i}{3}\mathbf{e}_1 + \frac{7}{3}\mathbf{e}_2 \\
\hat{\mathbf{e}}_3 &= \mathbf{e}_3
\end{aligned} \tag{2.6.48}$$

If a vector $\mathbf{v} \in \mathscr{V}$ is defined by

$$\mathbf{v} = 2i\mathbf{e}_1 + 2\mathbf{e}_2 + 5i\mathbf{e}_3 \tag{2.6.49}$$

determine the components of $\mathbf{v} \in \mathscr{V}$ with respect to the basis $\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3\}$.

2.6.6 A four dimensional vector space $\mathscr{V}$ has a basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$. A vector $\mathbf{v} \in \mathscr{V}$ has the component representation

$$\mathbf{v} = 5\mathbf{e}_1 + 3\mathbf{e}_2 + 6\mathbf{e}_3 + 2\mathbf{e}_4 \tag{2.6.50}$$

You are also given a change of basis to a new basis for $\mathscr{V}$, defined by

$$\begin{aligned} \mathbf{e}_1 &= \hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3 + 3\hat{\mathbf{e}}_4 \\ \mathbf{e}_2 &= \hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3 + 2\hat{\mathbf{e}}_4 \\ \mathbf{e}_3 &= \hat{\mathbf{e}}_3 + \hat{\mathbf{e}}_4 \\ \mathbf{e}_4 &= -\hat{\mathbf{e}}_1 \end{aligned} \tag{2.6.51}$$

Determine the transition matrix associated with this basis change. Also, determine the components of the vector $\mathbf{v}$ with respect to the basis $\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3, \hat{\mathbf{e}}_4\}$ of $\mathscr{V}$.

Section 2.7. Image Space, Rank and Kernel of a Matrix

The ideas introduced in Sections 2.1 through 2.7 contain useful information relative to the problem of solving the matrix equation (1.2.1), repeated,

$$\begin{aligned}
&A_{11}x_1 + A_{12}x_2 + A_{13}x_3 + \cdots + A_{1N}x_N = b_1 \\
&A_{21}x_1 + A_{22}x_2 + A_{23}x_3 + \cdots + A_{2N}x_N = b_2 \\
&\cdot \\
&\cdot \\
&\cdot \\
&A_{M1}x_1 + A_{M2}x_2 + A_{M3}x_3 + \cdots + A_{MN}x_N = b_M
\end{aligned} \tag{2.7.1}$$

Equivalently, we can write (2.7.1) in its matrix form equation (1.2.2), repeated,

$$\begin{bmatrix} A_{11} & A_{12} & \cdot & \cdot & \cdot & A_{1N} \\ A_{21} & A_{22} & & & & A_{2N} \\ \cdot & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ A_{M1} & A_{M2} & \cdot & \cdot & \cdot & A_{MN} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_N \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ \cdot \\ b_M \end{bmatrix} \tag{2.7.2}$$

or, equivalently, as (1.2.3), repeated,

$$A\mathbf{x} = \mathbf{b} \tag{2.7.3}$$

where, as usual, A is the matrix

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdot & \cdot & \cdot & A_{1N} \\ A_{21} & A_{22} & & & & \\ \cdot & & \cdot & & & \\ \cdot & & & \cdot & & \\ \cdot & & & & \cdot & \\ A_{M1} & A_{M2} & & & & A_{MN} \end{bmatrix} \tag{2.7.4}$$

The matrix A is an element of the vector space $\mathscr{M}^{M\times N}$, the column matrix $\mathbf{x}$ is a member of the vector space $\mathscr{M}^{N\times 1}$ and $\mathbf{b}$ is a member of the vector space $\mathscr{M}^{M\times 1}$. Recall from Section 2.5 that $\dim\mathscr{M}^{M\times N} = MN$, $\dim\mathscr{M}^{N\times 1} = N$ and $\dim\mathscr{M}^{M\times 1} = M$. In equation (1.8.3), we stressed the view of $A\in\mathscr{M}^{M\times N}$ as a function

$$A:\mathscr{M}^{N\times 1}\rightarrow\mathscr{M}^{M\times 1} \tag{2.7.5}$$

As a function whose *domain* is the vector space $\mathscr{M}^{N\times 1}$ and whose values lie in the vector space $\mathscr{M}^{M\times 1}$, the usual matrix operations imply

$$A(\mathbf{v}_1+\mathbf{v}_2)=A\mathbf{v}_1+A\mathbf{v}_2 \tag{2.7.6}$$

for all vectors $\mathbf{v}_1,\mathbf{v}_2\in\mathscr{M}^{N\times 1}$ and
and

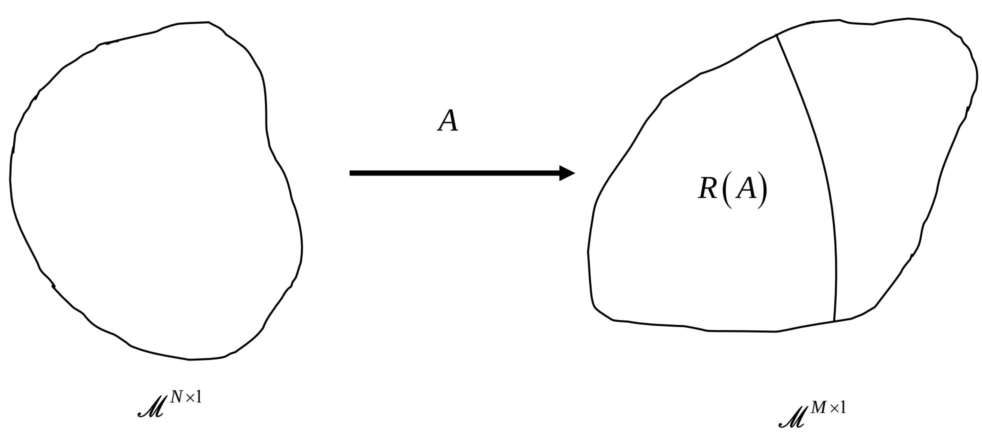
$$A(\lambda\mathbf{v})=\lambda A(\mathbf{v}) \tag{2.7.7}$$

for all vectors $\mathbf{v}\in\mathscr{M}^{N\times 1}$ and $\lambda\in\mathscr{C}$. Functions defined on vector spaces that obey rules like (2.7.6) and (2.7.7) are called *linear transformatio*ns. The matrix $A\in\mathscr{M}^{M\times N}$ is but one example of a linear transformation. These functions will be studies in greater generality in Chapter 3. In this section, we are interested in recording properties of this particular kind of linear transformation.

As explained in Section 1.8, the *range* of the function A is the set of all values of the function. In other words, the range is the set of possible values of $A\mathbf{x}$ generated for all possible values of $\mathbf{x}$ in $\mathscr{M}^{N\times 1}$. In Section 1.8, we gave this quantity the symbol $R(A)$. It was defined formally in equation (1.8.4), repeated,

$$R(A)=\left\{A\mathbf{x}\middle|\ \mathbf{x}\in\mathscr{M}^{N\times 1}\right\} \tag{2.7.8}$$

The following figure should be helpful.



It is a fact that the set $R(A)$ is a *subspace* of $\mathscr{M}^{M\times 1}$. The proof of this assertion, like all such assertions about subspaces, simply requires that the definition of subspace be satisfied. If $\mathbf{v}_1$ and $\mathbf{v}_2$ are two members of $\mathscr{M}^{N\times 1}$, then $A\mathbf{v}_1$ and $A\mathbf{v}_2$ are members of $R(A)$. We need to prove that their sum, $A\mathbf{v}_1 + A\mathbf{v}_2$, is also in $R(A)$. The proof follows from (2.7.6), repeated,

$$A\mathbf{v}_1 + A\mathbf{v}_2 = A(\mathbf{v}_1 + \mathbf{v}_2) \tag{2.7.9}$$

Since $\mathbf{v}_1 + \mathbf{v}_2 \in \mathscr{M}^{N\times 1}$, $A(\mathbf{v}_1 + \mathbf{v}_2) \in R(A)$ and, by (2.7.9), $A\mathbf{v}_1 + A\mathbf{v}_2 \in R(A)$. Thus, the first part of the definition of a subspace is established. An entirely similar manipulation establishes that $\lambda(A\mathbf{v}) \in R(A)$ for all λ, and, thus, $R(A)$ is a subspace. In order to stress the fact that the range is a subspace, we shall begin to refer to the range as the *image space*. There are two other important concepts involving the matrix $A:\mathscr{M}^{N\times 1} \to \mathscr{M}^{M\times 1}$ that we will now introduce.

Definition: A matrix $A:\mathscr{M}^{N\times 1} \to \mathscr{M}^{M\times 1}$ is said to be *onto* if it has the property that $R(A) = \mathscr{M}^{M\times 1}$.

Definition: A matrix $A:\mathscr{M}^{N\times 1} \to \mathscr{M}^{M\times 1}$ is *one to one* if

$$A\mathbf{v}_1 = A\mathbf{v}_2 \quad \text{implies} \quad \mathbf{v}_1 = \mathbf{v}_2 \tag{2.7.10}$$

Onto and one to one matrices have special properties which we will characterize later in this section.

In Section 1.8, we assigned the columns of the matrix (2.7.4) the symbols $\mathbf{a}_j$, $j = 1,2,...,N$, by the formulas

$$\mathbf{a}_j = \begin{bmatrix} A_{1j} \\ A_{2j} \\ \cdot \\ \cdot \\ \cdot \\ A_{Mj} \end{bmatrix} \qquad \text{for} \qquad j = 1,...,N \tag{2.7.11}$$

The set of column vectors $\{\mathbf{a}_1, \mathbf{a}_2, ..., \mathbf{a}_N\}$ consists of vectors in the set $\mathscr{M}^{M\times 1}$. As established in Section 1.8, for an arbitrary vector $\mathbf{x} \in \mathscr{M}^{N\times 1}$, equation (1.8.7), repeated, tells us that

$$A\mathbf{x} = \mathbf{a}_1 x_1 + \mathbf{a}_2 x_2 + \mathbf{a}_3 x_3 + \cdots + \mathbf{a}_N x_N \tag{2.7.12}$$

Equation (2.7.12) establishes that every vector in the image space $R(A)$ has the representation as a linear combination of the vectors in the set $\{\mathbf{a}_1,\mathbf{a}_2,...,\mathbf{a}_N\}$. This fact is summarized by the formula

$$R(A)=\operatorname{Span}(\mathbf{a}_1,\mathbf{a}_2,...,\mathbf{a}_N) \tag{2.7.13}$$

Because of the result (2.7.13), the image space $R(A)$ is also known as the *column space*. We next defined the *rank* of the matrix A.

Definition: The *rank* of $A\in\mathscr{M}^{M\times N}$ is $\dim R(A)$.

Given that the result (2.7.13), an *equivalent* definition of rank is as follows:

Definition: The *rank* of $A\in\mathscr{M}^{M\times N}$ is the number of linearly independent columns of A.[4]

An interesting and important result can be established about the rank of $A\in\mathscr{M}^{M\times N}$ and that of any matrix $B\in\mathscr{M}^{M\times N}$ that is row equivalent to $A\in\mathscr{M}^{M\times N}$. Recall from Section 1.6, a $M\times N$ matrix $B\in\mathscr{M}^{M\times N}$ is *row equivalent* to a $M\times N$ matrix $A\in\mathscr{M}^{M\times N}$ if there exist a finite number of elementary matrices $E_1,E_2,...,E_k\in\mathscr{M}^{M\times M}$ such that

$$B=E_k\cdots E_2E_1A \tag{2.7.14}$$

The image space of $A\in\mathscr{M}^{M\times N}$ is given by (2.7.13). Likewise, the image space of $B\in\mathscr{M}^{M\times N}$ is given by

$$R(B)=\operatorname{Span}(\mathbf{b}_1,\mathbf{b}_2,...,\mathbf{b}_N) \tag{2.7.15}$$

where $\mathbf{b}_1,\mathbf{b}_2,...,\mathbf{b}_N$ are the column vectors of the matrix B. It follows from (2.7.14) that the two sets of column vectors $\{\mathbf{a}_1,\mathbf{a}_2,...,\mathbf{a}_N\}$ and $\{\mathbf{b}_1,\mathbf{b}_2,...,\mathbf{b}_N\}$ are connected by the formulas

$$\mathbf{b}_j=E_k\cdots E_2E_1\mathbf{a}_j \qquad \text{for } j=1,2,...,N \tag{2.7.16}$$

Because the product of elementary matrices, $E_k\cdots E_2E_1$, is a nonsingular matrix, it follows that $A\in\mathscr{M}^{M\times N}$ and the matrix row equivalent to A, $B\in\mathscr{M}^{M\times N}$, have the same number of linearly independent columns. As a result, the matrix A and its row equivalent B have the same rank. The formal relationship that reflects this equality is

$$\dim R(A)=\dim R(B) \tag{2.7.17}$$

[4] The rank as defined by the number of linearly independent columns is sometimes called the *column rank*.

for row equivalent matrices A and B in $\mathscr{M}^{M\times N}$. It is important to observe that we have *not asserted* that $R(A)=R(B)$. This equality is simply not true as we shall show with an example below.

It is possible to give some general information about the rank of a matrix $A\in\mathscr{M}^{M\times N}$. If we apply the results of Theorem 2.5.2, the theorem that says the dimension of a subspace is less than or equal to the dimension of the containing vector space, then

$$\dim R(A)\leq \dim\mathscr{M}^{M\times 1}=M \tag{2.7.18}$$

It also follows from (2.7.13) that the dimension of $R(A)$ can never be larger than $N=\dim\mathscr{M}^{N\times 1}$, the number of vectors in the set $\{\mathbf{a}_1,\mathbf{a}_2,...,\mathbf{a}_N\}$. Therefore, the rank, $\dim R(A)$, is bounded by

$$\dim R(A)\leq \min\left(\dim\mathscr{M}^{M\times 1},\dim\mathscr{M}^{N\times 1}\right)=\min(M,N) \tag{2.7.19}$$

Therefore, *the rank is less than or equal to the smallest of N and M.*

Example 2.7.1: Determine the rank of the matrix of coefficients in Example 1.2.3 (also in Example 1.3.4). From equation (1.2.13), the matrix is

$$A=\begin{bmatrix}2 & 3 & 1\\ 1 & 1 & 1\\ 3 & 4 & 2\end{bmatrix} \tag{2.7.20}$$

We immediately know from (2.7.19) that the rank is less than or equal to three. Step one in finding the actual rank involves utilizing the definition to identify the three column vectors whose span generates the image space of A. As (2.7.20) shows, these column vectors are

$$\mathbf{a}_1=\begin{bmatrix}2\\1\\3\end{bmatrix},\mathbf{a}_2=\begin{bmatrix}3\\1\\4\end{bmatrix},\mathbf{a}_3=\begin{bmatrix}1\\1\\2\end{bmatrix} \tag{2.7.21}$$

From the definition, the rank of A is the dimension of the span of the three column vectors (2.7.21). This dimension is equal to the number of linearly independent vectors in the set $\{\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3\}$. In Section 2.4, we established a procedure for establishing whether or not a set of column vectors is linearly independent. If we utilize (2.7.21), the test for linear independence or dependence, i.e.,

$$\alpha_1\mathbf{a}_1+\alpha_2\mathbf{a}_2+\alpha_3\mathbf{a}_3=\mathbf{0} \tag{2.7.22}$$

takes the form of finding whether or not

$$\begin{bmatrix} 2 & 3 & 1 \\ 1 & 1 & 1 \\ 3 & 4 & 2 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = 0 \tag{2.7.23}$$

has at least one nonzero solution for the coefficients $\alpha_1, \alpha_2, \alpha_3$. We can quickly determine if (2.7.23) forces the coefficients $\alpha_1, \alpha_2, \alpha_3$ to be zero by calculating the determinant of the matrix of coefficients. It is an elementary calculation to show that

$$\begin{aligned} \det \begin{bmatrix} 2 & 3 & 1 \\ 1 & 1 & 1 \\ 3 & 4 & 2 \end{bmatrix} &= \begin{vmatrix} 2 & 3 & 1 \\ 1 & 1 & 1 \\ 3 & 4 & 2 \end{vmatrix} \\ &= 2\begin{vmatrix} 1 & 1 \\ 4 & 2 \end{vmatrix} - 1\begin{vmatrix} 3 & 1 \\ 4 & 2 \end{vmatrix} + 3\begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix} = -4 - 2 + 6 = 0 \end{aligned} \tag{2.7.24}$$

Thus, one cannot conclude that all of the coefficients $\alpha_1, \alpha_2, \alpha_3$ are zero. Therefore, the rank of (2.7.20) is less than 3. The actual numerical value of the rank is obtained by first reducing the matrix (2.7.20) to reduced row echelon form by a series of row operations. If one implements the row operations by the use of elementary matrices, the reduced row echelon form can be shown to be given by

$$\begin{aligned} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} &= \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \times \\ &\quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{3}{2} & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 1 \\ 1 & 1 & 1 \\ 3 & 4 & 2 \end{bmatrix} \end{aligned} \tag{2.7.25}$$

Therefore, (2.7.23) can be replaced by

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = 0 \tag{2.7.26}$$

Therefore, $\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \alpha_3 \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$. This result, combined with (2.7.22) and the fact that α_3 is arbitrary, yields,

$$-2\mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3 = \mathbf{0} \tag{2.7.27}$$

and we concluded that,

$$R(A) = \text{Span}(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3) = \text{Span}(\mathbf{a}_1, \mathbf{a}_2) \tag{2.7.28}$$

Therefore, the dimension of $R(A)$ i.e. the rank of A is 2.

As with Example 2.4.3, equation (2.7.27) defines the relationship between the three columns of the matrix of coefficients in (2.7.23). It again defines the relationship between the three columns of the reduced row echelon form that appears in (2.7.26). This feature is a theoretical consequence of (2.7.16). If there are linear relationships between the column vectors of a matrix A as, for example, with (2.7.27), it follows from (2.7.16) that the same relationships exist between the columns of matrices row equivalent to A. In addition, we can reach the number for the rank of (2.7.20) rather quickly if we simply use (2.7.17). For example, it follows from (2.7.26) that the reduced row echelon form of (2.7.20) is

$$U = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \tag{2.7.29}$$

Of course, the reduced row echelon form is row equivalent to the matrix A (2.7.20). The simplicity of the reduced row echelon form reveals the two linearly independent columns and, as a result, the rank of 2.

It mentioned above that we have *not* asserted that the image space $R(A)$ and the image space of a row equivalent matrix, such as $R(B)$ are the same. In fact, they are not. Example 2.7.1 and the later examples illustrate this observation. For example, if we look at the matrix studied in Example 2.7.1, namely, the matrix given in equation (2.7.20)

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 1 & 1 \\ 3 & 4 & 2 \end{bmatrix} \tag{2.7.30}$$

We showed that the image space of A is the two dimensional subspace spanned by the two vectors

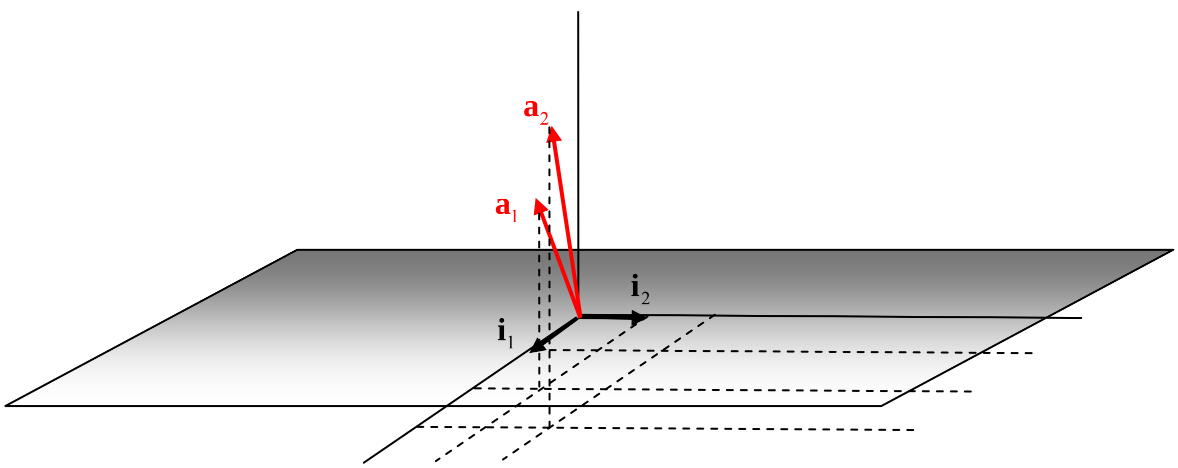
$$\mathbf{a}_1 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \quad \text{and} \quad \mathbf{a}_2 = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} \tag{2.7.31}$$

given in equations $(2.7.21)_1$ and $(2.7.21)_2$. The reduced row echelon form of A is given by (2.7.29)

It should be evident that the image space of the matrix (2.7.29) is the two dimensional subspace spanned by

$$\mathbf{i}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{i}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \tag{2.7.32}$$

Because a vector in $R(U)$ cannot have a nonzero element in its third position, a subspace spanned by (2.7.31) is necessarily *not* the same as one spanned by (2.7.32). The following figure shows the plane that represents the image space $R(U)$ and the vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ that span $R(A)$.



Example 2.7.2: Determine the rank of the matrix

$$A = \begin{bmatrix} 1 & 1 & -2 & 1 & 3 \\ 2 & -1 & 2 & 2 & 6 \\ 3 & 2 & -4 & -3 & -9 \end{bmatrix} \tag{2.7.33}$$

For this particular problem, equation (2.7.19) tells us that the rank of the matrix of coefficients obeys

$$\dim R(A) \le \min\left(\dim \mathscr{M}^{M\times 1}, \dim \mathscr{M}^{N\times 1}\right) = \min(3,5) = 3 \tag{2.7.34}$$

If we utilize (2.7.33), the test for linear independence or dependence takes the form of finding whether or not

$$\alpha_1\mathbf{a}_1 + \alpha_2\mathbf{a}_2 + \alpha_3\mathbf{a}_3 + \alpha_4\mathbf{a}_4 + \alpha_5\mathbf{a}_5 = 0 \tag{2.7.35}$$

where

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \mathbf{a}_2 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \mathbf{a}_3 = \begin{bmatrix} -2 \\ 2 \\ -4 \end{bmatrix}, \mathbf{a}_4 = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \mathbf{a}_5 = \begin{bmatrix} 3 \\ 6 \\ -9 \end{bmatrix} \tag{2.7.36}$$

has nonzero solutions for the coefficients $\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5$. As usual, (2.7.36) can be written as a matrix equation of the form

$$\begin{bmatrix} 1 & 1 & -2 & 1 & 3 \\ 2 & -1 & 2 & 2 & 6 \\ 3 & 2 & -4 & -3 & -9 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \end{bmatrix} = 0 \tag{2.7.37}$$

As with Example 2.7.1, we can reduce (2.7.37) by row operations to obtain the reduced row echelon form

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \end{bmatrix} = 0 \tag{2.7.38}$$

Thus, $\alpha_1 = 0, \alpha_2 = 2\alpha_3, \alpha_4 = -3\alpha_5$ and (2.7.35) reduces to

$$\alpha_3\left(2\mathbf{a}_2 + \mathbf{a}_3\right) - \alpha_5\left(3\mathbf{a}_4 - \mathbf{a}_5\right) = 0 \tag{2.7.39}$$

It follows from (2.7.39) and the fact that α_3 and α_4 are arbitrary, that

$$\mathbf{a}_3 = -2\mathbf{a}_2 \quad \text{and} \quad \mathbf{a}_5 = 3\mathbf{a}_4 \tag{2.7.40}$$

Equations (2.7.40), which are obvious from (2.7.36), show that there are three linearly independent vectors in the set $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5\}$ and, thus, the rank of (2.7.33) is 3. If we were to use the theoretical result (2.7.17), the rank of 3 is evident from the reduced row echelon form displayed in equation (2.7.38). As with our previous examples, the relationships between the column vectors of A, (2.7.40), are also obeyed for the columns of the reduced row echelon form of A shown in (2.7.38).

A related concept to rank is the concept of *row rank*. For our purposes, the most direct way to introduce this concept is to recall the definition of the transpose of a matrix discussed in Section 1.9 and define the row rank by the following definition:

Definition: The *row rank* of $A \in \mathscr{M}^{M\times N}$ is $\dim R\left(A^T\right)$.

In other words, the row rank of $A \in \mathscr{M}^{M\times N}$ is the dimension of the span of the column vectors of $A^T \in \mathscr{M}^{N\times M}$ Equivalently, the row rank is the dimension of the subspace of $\mathscr{M}^{1\times N}$ spanned by the row vectors of $A \in \mathscr{M}^{M\times N}$. The *row space* of $A \in \mathscr{M}^{M\times N}$ is that subspace. The row space can also be thought of as the image space of the transpose of $A \in \mathscr{M}^{M\times N}$, $R\left(A^T\right)$. Technically, $R\left(A^T\right)$ is a subspace of $\mathscr{M}^{N\times 1}$ and not $\mathscr{M}^{1\times N}$. However, these two different vector spaces are in one to one correspondence. In any case, it is a theorem, that we shall establish later, that the rank and the row rank are the same. The proof utilizes an interesting and useful result that connects the image space of the transpose, A^T, to the image space of B^T, where B is a matrix row equivalent to A. This result, which we shall now establish, is that

$$R\left(A^T\right) = R\left(B^T\right) \tag{2.7.41}$$

for any matrix $B \in \mathscr{M}^{M\times N}$ that is row equivalent to $A \in \mathscr{M}^{M\times N}$. Recall from Section 1.6 and as was mentioned above, a $M \times N$ matrix $B \in \mathscr{M}^{M\times N}$ is *row equivalent* to a $M \times N$ matrix $A \in \mathscr{M}^{M\times N}$ if there exist a finite number of elementary matrices $E_1, E_2, ..., E_k \in \mathscr{M}^{M\times M}$ such that (2.7.14), repeated,

$$B = E_k \cdots E_2 E_1 A \tag{2.7.42}$$

This definition and the rule for transposing matrix products, equation (1.9.5), yields

$$B^T = A^T E_1^T E_2^T \cdots E_k^T \tag{2.7.43}$$

The definition of image space, equation (2.7.8), applied to this case yields

$$\begin{aligned} R\left(B^T\right) &= \left\{B^T\mathbf{x} \,\middle|\, \mathbf{x}\in\mathscr{M}^{M\times 1}\right\} = \left\{A^T E_1^T E_2^T \cdots E_k^T\mathbf{x} \,\middle|\, \mathbf{x}\in\mathscr{M}^{M\times 1}\right\} \\ &= \left\{A^T\mathbf{x} \,\middle|\, \mathbf{x}\in\mathscr{M}^{M\times 1}\right\} = R\left(A^T\right) \end{aligned} \tag{2.7.44}$$

where the fact that the elementary matrices are nonsingular has been used. The result (2.7.41) is usually most useful when B is the reduced row echelon form of A. One can easily illustrate the validity of (2.7.41) utilizing the results in Examples 2.7.1 and 2.7.2.

In the case where we are given a matrix $A\in\mathscr{M}^{M\times N}$ with a reduced row echelon form U, it follows from (2.7.17) that

$$\dim R(A) = \dim R(U) \tag{2.7.45}$$

and, from (2.7.41),

$$\dim R\left(A^T\right) = \dim R\left(U^T\right) \tag{2.7.46}$$

Equations (2.7.45) says that the rank of A and that of U are the same. Likewise, (2.7.46) says that the row rank of A and that of U are the same. Next, we shall prove that the rank and the row rank are the same. This result follows from the form of the reduced row echelon form U. From its definition, as given in Section 1.5, the reduced row echelon form of a matrix $A\in\mathscr{M}^{M\times N}$ will be of the form

$$U = \begin{array}{c} \begin{array}{cc} \overbrace{\hphantom{1 \quad 0 \quad \cdots \quad 0}}^{R} & \overbrace{\hphantom{D_{1,R+1} \quad \cdots \quad D_{1N}}}^{N-R} \end{array} \\ \left[\begin{array}{cccccc} 1 & 0 & \cdots & 0 & D_{1,R+1} & \cdots & D_{1N} \\ 0 & 1 & \cdots & 0 & D_{2,R+1} & \cdots & D_{2N} \\ \vdots & & & \vdots & \vdots & & \vdots \\ 0 & \cdots & \cdots & 1 & D_{R,R+1} & \cdots & D_{RN} \\ \hdashline 0 & \cdots & \cdots & 0 & \cdots & \cdots & 0 \\ \vdots & & & \vdots & & & \vdots \\ 0 & \cdots & \cdots & 0 & \cdots & \cdots & 0 \end{array}\right] \end{array} \begin{array}{l} \left.\vphantom{\begin{array}{c}1\\0\\\vdots\\0\end{array}}\right\} R \\ \left.\vphantom{\begin{array}{c}0\\\vdots\\0\end{array}}\right\} M\text{-}R \end{array} \tag{2.7.47}$$

Equations that illustrate the generic row echelon form (2.7.47) are (2.4.24), (2.7.29) and (2.7.38). Several other examples can be found in Section 1.5 and in the Exercises at the end of Section 1.5. The row rank of (2.7.47) is the number of nonzero rows of U. We have given this number the symbol R in (2.7.47). The first R columns of U are clearly linearly independent. Thus, we have established that

$$\dim R(U) \le \dim R\left(U^T\right) \tag{2.7.48}$$

This result and (2.7.45) and (2.7.46) yield

$$\dim R(A) \leq \dim R(A^T) \tag{2.7.49}$$

If we repeat the above construction, but apply it to A^T, the result is

$$\dim R(A^T) \leq \dim R(A) \tag{2.7.50}$$

The two results (2.7.49) and (2.7.50) combine to yield the asserted result

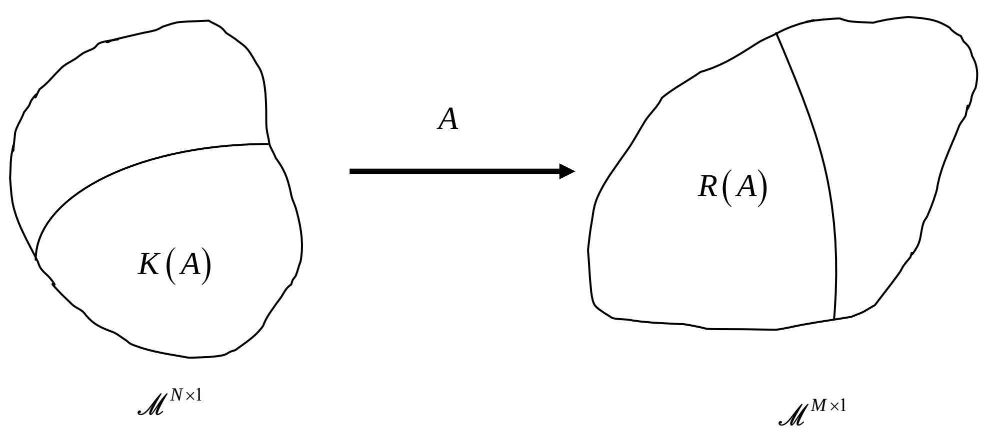
$$\dim R(A) = \dim R(A^T) \tag{2.7.51}$$

The next property of $A \in \mathscr{M}^{M\times N}$ we wish to introduce is the *kernel*.

Definition: The *kernel* of $A \in \mathscr{M}^{M\times N}$ is the subset of $\mathscr{M}^{N\times 1}$ defined by

$$K(A) = \{\mathbf{v} | A\mathbf{v} = \mathbf{0}\} \tag{2.7.52}$$

It is easy to establish that $K(A)$ is a *subspace* of $\mathscr{M}^{N\times 1}$. It consists of those vectors in $\mathscr{M}^{N\times 1}$ mapped to zero by the matrix $A \in \mathscr{M}^{M\times N}$. The figure below should be helpful.



Definition: The *nullity* of $A \in \mathscr{M}^{M\times N}$ is $\dim K(A)$.

Because $K(A)$, is a subspace

$$\dim K(A) \leq \dim \mathscr{M}^{N\times 1} = N \tag{2.7.53}$$

Example 2.7.3: The kernel of the matrix used above in Example 2.7.1 is, from (2.7.20), the set of vectors that obey

$$A\mathbf{v} = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 1 & 1 \\ 3 & 4 & 2 \end{bmatrix} \begin{bmatrix} \upsilon_1 \\ \upsilon_2 \\ \upsilon_3 \end{bmatrix} = 0 \tag{2.7.54}$$

Equation (2.7.54) is equation (2.7.23) that we solved in Example 2.7.1. The solution is again

$$\mathbf{v} = \begin{bmatrix} \upsilon_1 \\ \upsilon_2 \\ \upsilon_3 \end{bmatrix} = \upsilon_3 \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \tag{2.7.55}$$

Therefore, for this example, the kernel is the one dimensional subspace of $\mathscr{M}^{3\times 1}$ spanned by $\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$.

This example reveals an important relationship between the calculation that was used with Examples 2.7.1 and 2.7.2 to determine the image space and rank and the calculation in Example 2.7.3 to determine the kernel and nullity. These examples illustrate that the scalars $\alpha_1, \alpha_2, ..., \alpha_N$ that appear in the test for linear independence of the column vectors $\mathbf{a}_1, \mathbf{a}_2, ..., \mathbf{a}_N$, i.e., in the formula

$$\alpha_1 \mathbf{a}_1 + \alpha_2 \mathbf{a}_2 + \cdots + \alpha_N \mathbf{a}_N = \mathbf{0} \tag{2.7.56}$$

are the components of a column vector $\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \\ \\ \alpha_N \end{bmatrix}$ in $K(A)$. This relationship is also the origin or another feature illustrated by this example. In this example, the rank of the matrix A is two. Thus, the rank plus the nullity equals the dimension of the domain of A. This is a general result, a result we shall prove after the next example.

Example 2.7.4: The objective is to find the kernel of the matrix $A \in \mathscr{M}^{2\times 4}$ defined by

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix} \tag{2.7.57}$$

The kernel of (2.7.57) consists of those column vectors $\mathbf{v} \in \mathscr{M}^{4\times 2}$ that obey

$$A\mathbf{v} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \upsilon_1 \\ \upsilon_2 \\ \upsilon_3 \\ \upsilon_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{2.7.58}$$

The reduced row echelon form of (2.7.57) is obtained from

$$\begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix} \tag{2.7.59}$$

It therefore follows from (2.7.58) that

$$\begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} \upsilon_1 \\ \upsilon_2 \\ \upsilon_3 \\ \upsilon_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \upsilon_1 \\ \upsilon_2 \\ \upsilon_3 \\ \upsilon_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{2.7.60}$$

Therefore,

$$\mathbf{v} = \begin{bmatrix} \upsilon_1 \\ \upsilon_2 \\ \upsilon_3 \\ \upsilon_4 \end{bmatrix} = \begin{bmatrix} \upsilon_3 - \upsilon_4 \\ -2\upsilon_3 + \upsilon_4 \\ \upsilon_3 \\ \upsilon_4 \end{bmatrix} = \upsilon_3 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + \upsilon_4 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \tag{2.7.61}$$

The kernel of (2.7.57) is the subspace spanned by the two linearly independent vectors $\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$. Therefore, the nullity of (2.7.57) is 2. Also, one can see from the reduced row echelon form shown in (2.7.60) that the rank of (2.7.57) is 2, so again we see the result we wish to prove below that the rank plus nullity equals the dimension of the domain of A, which is 4 in this case.

The last example illustrates another general result that is important to mention. We started with equation (2.7.58), repeated,

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \upsilon_1 \\ \upsilon_2 \\ \upsilon_3 \\ \upsilon_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{2.7.62}$$

and replaced it with $(2.7.60)_1$, repeated,

$$\begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} \upsilon_1 \\ \upsilon_2 \\ \upsilon_3 \\ \upsilon_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{2.7.63}$$

which utilized the reduced row echelon form of the original matrix (2.7.57). Explicit in the relationships (2.7.62) and (2.7.63) is that *the kernel of a matrix and that of a matrix row equivalent to it are the same*.

The general result just stated is a consequence of the relationship between matrices and those it is row equivalent to, namely equation (1.6.16), and the definition of kernel. As explained in Section 1.6 and utilized twice in this section, a $M \times N$ matrix $B \in \mathscr{M}^{M\times N}$ is *row equivalent* to a $M \times N$ matrix $A \in \mathscr{M}^{M\times N}$ if there exist a finite number of elementary matrices $E_1, E_2, ..., E_k \in \mathscr{M}^{M\times M}$ such that (2.7.14) or, equivalently, (2.7.42) hold. The *kernel* of B equals the *kernel* of A because

$$K(B) = \{\mathbf{v} | B\mathbf{v} = \mathbf{0}\} = \{\mathbf{v} | E_k E_{k-1} \cdots E_1 A\mathbf{v} = \mathbf{0}\} = \{\mathbf{v} | A\mathbf{v} = \mathbf{0}\} = K(A) \tag{2.7.64}$$

where we have again made use of the fact that the elementary matrices are nonsingular.

In the examples above it was observed that the sum of rank plus nullity equaled the dimension of the domain. The formal theorem, which we shall now prove, is as follows:

Theorem 2.7.1:

$$\underbrace{\dim \mathscr{M}^{N\times 1}}_{N} = \underbrace{\dim R(A)}_{\text{Rank}=R} + \underbrace{\dim K(A)}_{\text{Nullity}=P} \tag{2.7.65}$$

This theorem is usually referred to as the rank-nullity theorem.

Proof: Because of (2.7.17), applied in the case where the row equivalent matrix B is the reduced row echelon form of A, it is true that

$$\dim R(A) = \dim R(U) \tag{2.7.66}$$

Likewise, because of (2.7.64), it is true that

$$\dim K(A) = \dim K(U) \tag{2.7.67}$$

Therefore, because of (2.7.66) and (2.7.67), we can establish the result (2.7.65) if we can prove that

$$\dim \mathscr{M}^{N\times 1} = \dim R(U) + \dim K(U) \tag{2.7.68}$$

The result (2.7.68) is a consequence of the form of the reduced row echelon form. From its definition, as given in Section 1.5, the reduced row echelon form of a matrix $A \in \mathscr{M}^{M\times N}$ will be of the form

$$U = \left[\begin{array}{cccccc} \multicolumn{4}{c}{\overbrace{\hphantom{1\ 0\ \cdots\ 0}}^{R}} & \multicolumn{2}{c}{\overbrace{\hphantom{D_{1,R+1}\ \cdots\ D_{1N}}}^{N-R}} \\ 1 & 0 & \cdots & 0 & D_{1,R+1} \cdots & D_{1N} \\ 0 & 1 & \cdots & 0 & D_{2,R+1} \cdots & D_{2N} \\ \vdots & & & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & 1 & D_{R,R+1} \cdots & D_{RN} \\ \hdashline 0 & \cdots & \cdots & 0 & \cdots \quad \cdots & 0 \\ \vdots & & & \vdots & & \vdots \\ 0 & \cdots & \cdots & 0 & \cdots \quad \cdots & 0 \end{array}\right] \begin{array}{l} \\ \left.\vphantom{\begin{array}{c}1\\0\\\vdots\\0\end{array}}\right\} R \\ \left.\vphantom{\begin{array}{c}0\\\vdots\\0\end{array}}\right\} M\text{-}R \end{array} \tag{2.7.69}$$

The rank of the reduced row echelon form (2.7.69) is the number of linearly independent columns and, as shown above, the number of linearly independent rows. As shown, the rank is $R = \dim R(U) = \dim R(U^T)$. The kernel of U, which by (2.7.64) is the kernel of A, consists of those column vectors $\mathbf{v} \in \mathscr{M}^{N\times 1}$ that obey

$$U\mathbf{v} = \mathbf{0} \tag{2.7.70}$$

The next step in the calculation is illustrated in Example 2.7.4. The same argument that produced (2.7.61) will yield from (2.7.70)

$$\mathbf{v}=\begin{bmatrix}\upsilon_1\\ \upsilon_2\\ \vdots\\ \upsilon_R\\ \upsilon_{R+1}\\ \vdots\\ \vdots\\ \upsilon_N\end{bmatrix}=\begin{bmatrix}-D_{1,R+1} & \cdots & -D_{1N}\\ -D_{2,R+1} & & -D_{2N}\\ \vdots & & \vdots\\ -D_{R,R+1} & \cdots & -D_{RN}\\ 1 & \cdots & 0\\ \vdots & & \vdots\\ \vdots & & \vdots\\ 0 & \cdots & 1\end{bmatrix}\begin{bmatrix}\upsilon_{R+1}\\ \vdots\\ \upsilon_N\end{bmatrix}=\upsilon_{R+1}\begin{bmatrix}-D_{1,R+1}\\ -D_{2,R+1}\\ \vdots\\ -D_{R,R+1}\\ 1\\ \vdots\\ \vdots\\ 0\end{bmatrix}+\upsilon_{R+2}\begin{bmatrix}-D_{1,R+2}\\ -D_{2,R+2}\\ \vdots\\ -D_{R,R+2}\\ 0\\ 1\\ \vdots\\ 0\end{bmatrix}+\cdots+\upsilon_N\begin{bmatrix}-D_{1N}\\ -D_{2N}\\ \vdots\\ -D_{RN}\\ 0\\ \vdots\\ \vdots\\ 1\end{bmatrix}\tag{2.7.71}$$

The $N-R$ column vectors on the right side of (2.7.71) are clearly linearly independent. As a result,

$$\dim K(U)=N-R=\dim\mathscr{M}^{N\times 1}-\dim R(U)\tag{2.7.72}$$

Equation (2.7.72) is the result (2.7.68) which, in turn, yields the asserted result (2.7.65)

Equation (2.7.65), written in the form

$$R=\dim R(A)=\dim\mathscr{M}^{N\times 1}-\dim K(A)\tag{2.7.73}$$

improves on the inequality (2.7.19), repeated,

$$R=\dim R(A)\le\min\left(\dim\mathscr{M}^{M\times 1},\dim\mathscr{M}^{N\times 1}\right)\tag{2.7.74}$$

Example 2.7.5: Given a matrix $A\in\mathscr{M}^{M\times N}$, we can associate with this matrix its reduced roe echelon form which we have denoted by U. It follows from (2.7.17) that

$$\dim R(A)=\dim R(U)\tag{2.7.75}$$

but, as observed earlier in this section,

$$R(A)\ne R(U)\tag{2.7.76}$$

In equation (2.7.41) we showed that

$$R\left(A^T\right)=R\left(U^T\right)\tag{2.7.77}$$

Finally, in equation (2.7.64) we showed that

$$K(A)=K(U)\tag{2.7.78}$$

These theoretical formulas are illustrated by the examples given in this Section. As an additional example, the matrix

$$A = \begin{bmatrix} 2 & 1 & -2 & 0 & 2 \\ -7 & 4 & -8 & 0 & -7 \\ 4 & -3 & 6 & 0 & 4 \end{bmatrix} \tag{2.7.79}$$

can be shown to have a reduced row echelon form of

$$U = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{2.7.80}$$

Therefore,

$$\dim R(A) = \dim R(U) = 2 \tag{2.7.81}$$

$$R(A) = \operatorname{span}\left(\begin{bmatrix} 2 \\ -7 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix} \right) \neq R(U) = \operatorname{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right) \tag{2.7.82}$$

$$R(A^T) = \operatorname{span}\left(\begin{bmatrix} 2 \\ 1 \\ -2 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -7 \\ 4 \\ -8 \\ 0 \\ -7 \end{bmatrix} \right) = R(U^T) = \operatorname{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \\ 0 \\ 0 \end{bmatrix} \right) \tag{2.7.83}$$

and

$$K(A) = K(U) = \operatorname{span}\left(\begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right) \tag{2.7.84}$$

This example illustrates a possible point of confusion. The matrix U^T is *not* the reduced row echelon form of A^T. This fact is evident from the formula

$$U^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \tag{2.7.85}$$

which is not even in reduced row echelon form. If one forms the matrix A^T and then reduces this matrix to its reduced row echelon form, the result turns out to be

$$V = \begin{bmatrix} 1 & 0 & -\frac{1}{3} \\ 0 & 1 & -\frac{2}{3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{2.7.86}$$

We now return to the problem of finding solutions to (2.7.3). In Section 1.8, we stated and proved the *consistency theorem for linear systems*. This theorem repeated is

Theorem 1.8.1 (Repeated): Given a matrix $A \in \mathscr{M}^{M\times N}$ and a vector $\mathbf{b} \in \mathscr{M}^{M\times 1}$, the system $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{b} \in R(A)$.

If $\mathbf{b} \in R(A)$, then, by definition, it is expressible as a linear combination of the column vectors of A. This fact, gives the following simple test to determine whether or not $\mathbf{b} \in R(A)$:

Theorem 2.7.2: Given a matrix $A \in \mathscr{M}^{M\times N}$, the vector $\mathbf{b} \in \mathscr{M}^{M\times 1}$ is in $R(A)$ if and only if the rank of A and the augmented matrix $(A|\mathbf{b})$ are the same.

Theorem 1.8.1 tells us that if $\mathbf{b} \in R(A)$, then the equation $A\mathbf{x} = \mathbf{b}$ has a solution. It does not tell you that the solution is unique. Next, we will look at conditions sufficient to insure that the solution is unique. The first result we need is the following theorem.

Theorem 2.7.3: A matrix $A : \mathscr{M}^{N\times 1} \to \mathscr{M}^{M\times 1}$ is one to one, if and only if $K(A) = \{\mathbf{0}\}$.

Proof: As The key to the proof is the relationship $A\mathbf{v}_1 = A\mathbf{v}_2$, which by linearity can be written $A(\mathbf{v}_1 - \mathbf{v}_2) = \mathbf{0}$. Thus, if $K(A) = \{\mathbf{0}\}$, then $A\mathbf{v}_1 = A\mathbf{v}_2$ implies $\mathbf{v}_1 = \mathbf{v}_2$. Conversely, assume A is one to one. Since $K(A)$ is a subspace it must contain the $\mathbf{0} \in \mathscr{M}^{N\times 1}$. Therefore, $A\mathbf{0} = \mathbf{0}$. If $K(A)$

contains any other element $\mathbf{v}$, as an element of the kernel it would be true that $A\mathbf{v}=\mathbf{0}$. This formula contradicts the fact that A is one to one.

Next we shall prove:

Theorem 2.7.4: If $\mathbf{b}\in R(A)$, then the solution to $A\mathbf{x}=\mathbf{b}$ is unique if and only if $K(A)=\{\mathbf{0}\}$.

Proof: To prove this theorem, we follow the usual procedure and assume a lack of uniqueness. Given $\mathbf{b}\in R(A)$, then let $\mathbf{v}_1$ and $\mathbf{v}_2$ be solutions, i.e.

$$A\mathbf{v}_1=\mathbf{b} \qquad \text{and} \qquad A\mathbf{v}_2=\mathbf{b} \tag{2.7.87}$$

Therefore,

$$A\mathbf{v}_1=A\mathbf{v}_2 \tag{2.7.88}$$

and, as a result, $\mathbf{v}_1-\mathbf{v}_2\in K(A)$. If we take $K(A)=\{\mathbf{0}\}$, it follows then that $\mathbf{v}_1=\mathbf{v}_2$ and thus the solution is unique. If we assume the solution is unique, we need to prove that the subspace $K(A)$ only contains the $\mathbf{0}$. Let $\mathbf{v}$ be the unique solution, i.e., $A\mathbf{v}=\mathbf{b}$ and assume $K(A)$ contains, in addition to $\mathbf{0}$, a vector $\mathbf{w}$. As an element of $K(A)$, it would have to be true that $A\mathbf{w}=\mathbf{0}$. This fact and $A\mathbf{v}=\mathbf{b}$ would yield $A(\mathbf{v}+\mathbf{w})=\mathbf{b}$ which would mean $\mathbf{v}+\mathbf{w}$ is also a solution in contradiction to the assumed uniqueness. Thus, $K(A)=\{\mathbf{0}\}$.

If $K(A)$ contains more vectors than $\mathbf{0}$, then the situation is more complicated. There is a representation which is useful:

Theorem 2.7.5: If $\mathbf{b}\in R(A)$, and if $\mathbf{x}_0$ is a *particular solution* of $A\mathbf{x}=\mathbf{b}$, i.e. a vector in $\mathscr{M}^{N\times 1}$ which obeys $A\mathbf{x}_0=\mathbf{b}$, then the solution to the matrix equation $A\mathbf{x}=\mathbf{b}$ has the representation

$$\mathbf{x}=\mathbf{x}_h+\mathbf{x}_0 \tag{2.7.89}$$

where $\mathbf{x}_h\in K(A)$, i.e., $A\mathbf{x}_h=\mathbf{0}$.

The proof of this theorem uses $A\mathbf{x}_0=\mathbf{b}$ to write the equation $A\mathbf{x}=\mathbf{b}$ as $A\mathbf{x}=A\mathbf{x}_0$, which, in turn, can be written

$$A(\mathbf{x}-\mathbf{x}_0)=\mathbf{0} \tag{2.7.90}$$

which implies $\mathbf{x}_h\equiv\mathbf{x}-\mathbf{x}_0\in K(A)$. Therefore, the representation (2.7.89) is valid.

The equation $A\mathbf{x}_h = \mathbf{0}$ is the *homogeneous equation* associated with the problem of solving $A\mathbf{x} = \mathbf{b}$.

Example 2.7.6: You are given the following system of three equations and five unknowns:

$$\begin{bmatrix} 1 & 1 & -2 & 1 & 3 \\ 2 & -1 & 2 & 2 & 6 \\ 3 & 2 & -4 & -3 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \tag{2.7.91}$$

From Example 2.7.2, we learned that the rank of the matrix of coefficients in (2.7.91) is 3. We can conclude in advance whether or not the system (2.7.91) has a solution by application of Theorem 2.7.2. In particular, we need to determine whether or not the augmented matrix

$$\left(A|\mathbf{b}\right) = \left[\begin{array}{ccccc|c} 1 & 1 & -2 & 1 & 3 & 1 \\ 2 & -1 & 2 & 2 & 6 & 2 \\ 3 & 2 & -4 & -3 & -9 & 3 \end{array}\right] \tag{2.7.92}$$

also has rank 3. By the usual method, the reduced row echelon form of (2.7.92) is

$$\left[\begin{array}{ccccc|c} 1 & 1 & -2 & 1 & 3 & 1 \\ 2 & -1 & 2 & 2 & 6 & 2 \\ 3 & 2 & -4 & -3 & -9 & 3 \end{array}\right] \longrightarrow \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3 & 0 \end{array}\right] \tag{2.7.93}$$

Equation $(2.7.93)_2$ shows that the augmented matrix has three linearly independent columns and, thus, a rank of 3. The equality of the rank of A and $\left(A|\mathbf{b}\right)$, from Theorem 2.7.2, tells us that $\mathbf{b} \in R\left(A\right)$. This result and Theorem 1.8.1 tell us that the system (2.7.91) has a solution.

Of course, we are also interested in constructing the actual solution. In particular, we are interested in showing that the solution has the representation prescribed by Theorem 2.7.4. The manipulations required to show that the solution exists are, for this example, almost bring us to the solution. If we use the reduced row echelon form of (2.7.91) given in $(2.7.93)_2$, the system (2.7.91) is replaced by

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{2.7.94}$$

Therefore, the solution is

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 2x_3 \\ x_3 \\ -3x_5 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 2x_3 \\ x_3 \\ -3x_5 \\ x_5 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \underbrace{x_3 \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ 0 \\ 0 \\ -3 \\ 1 \end{bmatrix}}_{\mathbf{x}_h \in K(A)} + \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{x}_p} \tag{2.7.95}$$

The solution (2.7.95) reflects the decomposition asserted in Theorem 2.7.4. The number of *free parameters* in the solution, x_3 and x_5, is the same as the dimension of the kernel of the matrix of coefficients.

The case where the matrix A is square is of special importance. When $A \in \mathscr{M}^{N\times N}$, we are talking about situations where the number of equations and the number of unknowns are the same. Equation (2.7.65), repeated, is

$$\underbrace{\dim \mathscr{M}^{N\times 1}}_{N} = \underbrace{\dim R(A)}_{\text{Rank}=R} + \underbrace{\dim K(A)}_{\text{Nullity}=P} \tag{2.7.96}$$

We also have (2.7.19), repeated,

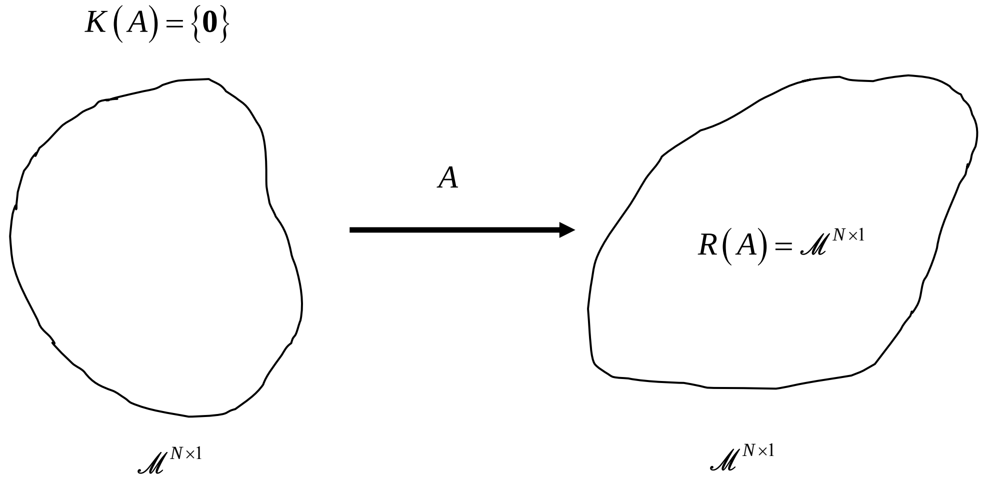
$$R = \dim R(A) \le \min\left(\dim \mathscr{M}^{M\times 1}, \dim \mathscr{M}^{N\times 1}\right) = \min(M, N) \tag{2.7.97}$$

Equations (2.7.96), (2.7.97) and the additional requirement $M = N$ allow us to reach the following conclusions:

When the rank, $R = \dim R(A)$, equals N,

a) the nullity, $P = \dim K(A)$, is zero which shows that $K(A) = \{\mathbf{0}\}$. As a result, the matrix A is *one to one*.
b) the image space $R(A)$ equals $\mathscr{M}^{N\times 1}$. As a result, the matrix A is *onto*.

The situation characterized by these two cases is illustrated in the following figure



Note that for a one to one matrix $A:\mathscr{M}^{N\times 1}\to\mathscr{M}^{M\times 1}$, Theorem 2.7.3 tells us that $K(A)=\{\mathbf{0}\}$. It follows, in this case, from (2.7.65), that $\dim\mathscr{M}^{N\times 1}=\dim R(A)$. If we require, in addition that $A:\mathscr{M}^{N\times 1}\to\mathscr{M}^{M\times 1}$ be onto, the image space $R(A)$ equals $\mathscr{M}^{M\times 1}$ and, of course, $\dim\mathscr{M}^{M\times 1}=\dim R(A)$. Therefore, when the matrix $A:\mathscr{M}^{N\times 1}\to\mathscr{M}^{M\times 1}$ is both one to one and onto, it is necessary that $\dim\mathscr{M}^{M\times 1}=\dim\mathscr{M}^{N\times 1}$. Therefore, one to one onto matrices necessarily are functions that map column vectors between spaces of column vectors of the same size. In more simple terms, it is necessary that $M=N$.

If we continue to focus on the case where $M=N$, when $A:\mathscr{M}^{N\times 1}\to\mathscr{M}^{N\times 1}$ is onto, every member of $\mathscr{M}^{N\times 1}$ is in $R(A)$. Theorem 1.8.1 tells us that the system $A\mathbf{x}=\mathbf{b}$ has a solution. When $A:\mathscr{M}^{N\times 1}\to\mathscr{M}^{N\times 1}$ is also one to one, Theorem 2.7.3 tells us the solution is unique. The one to one correspondence of a one to one onto function $A:\mathscr{M}^{N\times 1}\to\mathscr{M}^{N\times 1}$, was reached by the two assumptions $M=N$ and $\dim R(A)=N$. The one to one correspondence means that $A:\mathscr{M}^{N\times 1}\to\mathscr{M}^{N\times 1}$ has an inverse $A^{-1}:\mathscr{M}^{N\times 1}\to\mathscr{M}^{N\times 1}$. The summary conclusion is that when $M=N$ and $\dim R(A)=N$, then A is *nonsingular*.

Because an equivalent definition of the rank is the number of linearly independent columns of A, another way to state the last result is that when $M=N$ and the N columns of A are linearly independent, then A is nonsingular. Conversely, if $M=N$ and A is nonsingular, then the N columns of A are linearly independent.

Exercises

2.7.1 In Example 2.7.1, we started with the matrix

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 1 & 1 \\ 3 & 4 & 2 \end{bmatrix} \tag{2.7.98}$$

and showed that its reduced row echelon form is

$$U = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \tag{2.7.99}$$

Confirm the theoretical result (2.7.41). Namely, that

$$R\left(A^T\right) = R\left(U^T\right) \tag{2.7.100}$$

2.7.2 In Example 2.7.2, we started with the matrix

$$A = \begin{bmatrix} 1 & 1 & -2 & 1 & 3 \\ 2 & -1 & 2 & 2 & 6 \\ 3 & 2 & -4 & -3 & -9 \end{bmatrix} \tag{2.7.101}$$

and showed that its reduced row echelon form is

$$U = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix} \tag{2.7.102}$$

Confirm the theoretical result (2.7.41). Namely, that

$$R\left(A^T\right) = R\left(U^T\right) \tag{2.7.103}$$

2.7.3 For the matrix

$$A = \begin{bmatrix} 7 & 3 & 2 \\ 2 & 2 & 4 \\ 4 & 7 & 9 \end{bmatrix} \tag{2.7.104}$$

Find a basis for the column space, $R(A)$, the row space, $R\left(A^T\right)$ and the kernel, $K(A)$.

2.7.4 For the matrix

$$A = \begin{bmatrix} -3 & 2 & 3 & 1 \\ 1 & 4 & -1 & -3 \\ -3 & 7 & 4 & 2 \end{bmatrix} \tag{2.7.105}$$

Find a basis for the column space, $R(A)$, the row space, $R(A^T)$ and the kernel, $K(A)$.

2.7.5 For the matrix

$$A = \begin{bmatrix} 1 & 1 & 2 & 1 \\ 2 & 3 & -1 & -2 \\ 4 & 5 & 3 & 0 \end{bmatrix} \tag{2.7.106}$$

Find a basis for the column space, $R(A)$, the row space, $R(A^T)$ and the kernel, $K(A)$.

2.7.6 For the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & 0 & 2 & 1 \\ 2 & -5 & 1 & 2 \end{bmatrix} \tag{2.7.107}$$

Find a basis for the column space, $R(A)$, the row space, $R(A^T)$ and the kernel, $K(A)$.

2.7.7 For the matrix

$$A = \begin{bmatrix} 1 & 2 & -4 & 3 & 9 \\ 4 & 5 & -10 & 6 & 18 \\ 7 & 8 & -16 & 0 & 0 \end{bmatrix} \tag{2.7.108}$$

Find a basis for the column space, $R(A)$, the row space, $R(A^T)$ and the kernel, $K(A)$.

2.7.8 For the matrix

$$A = \begin{bmatrix} 1 & -1 & 5 \\ 2 & 4 & 4 \\ 3 & 2 & 10 \\ 4 & -5 & 21 \end{bmatrix} \tag{2.7.109}$$

Find a basis for the column space, $R(A)$, the row space, $R(A^T)$ and the kernel, $K(A)$.

2.7.9 You are given the system of equations

$$\begin{aligned} 36x_1 - 9x_2 + 18x_3 + 9x_4 &= 36 \\ 42x_1 - 7x_2 + 19x_3 + 8x_4 &= 53 \\ 48x_1 - 5x_2 + 20x_3 + 7x_4 &= 70 \\ 58x_1 - 25x_2 + 35x_3 + 22x_4 &= 25 \end{aligned} \tag{2.7.110}$$

a) Utilize Theorem 2.7.2 to determine whether or not the solution exists for the system (2.7.110).
b) If the solution to the system (2.7.110) exists, express the solution in the form predicted by Theorem 2.7.5.

2.7.10 You are given a matrix $A \in \mathscr{M}^{3\times5}$ defined by

$$A = \begin{bmatrix} 1 & 2 & -2 & 1 & 3 \\ 2 & 4 & -3 & 0 & 1 \\ 1 & 2 & 1 & 5 & 1 \end{bmatrix} \tag{2.7.111}$$

What is the rank of the matrix (2.7.111)? Also, determine a basis for the kernel of the matrix (2.7.111).

2.7.11 A certain matrix $A \in \mathscr{M}^{3\times5}$ has the reduced row echelon form

$$U = \begin{bmatrix} 1 & 2 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \tag{2.7.112}$$

Determine the rank of, the basis of the kernel and the basis of the row space of $A \in \mathscr{M}^{3\times5}$.

2.7.12 Let $A \in \mathscr{M}^{6\times5}$ be a matrix with three linearly independent column vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$. The two remaining column vectors obey

$$\begin{aligned} \mathbf{a}_4 &= \mathbf{a}_1 + 3\mathbf{a}_2 + \mathbf{a}_3 \\ \mathbf{a}_5 &= 2\mathbf{a}_1 - \mathbf{a}_3 \end{aligned} \tag{2.7.113}$$

Determine a basis for the kernel, $K(A)$. Also, determine the reduced row echelon form of . $A \in \mathscr{M}^{6\times5}$.

Chapter 3

LINEAR TRANSFORMATIONS

In Section 2.7, we looked at a matrix $A \in \mathscr{M}^{M \times N}$ as a function $A: \mathscr{M}^{N\times 1} \to \mathscr{M}^{M\times 1}$. Because of the usual rules for matrix addition and matrix multiplication, it was pointed out that

$$A(\mathbf{v}_1 + \mathbf{v}_2) = A\mathbf{v}_1 + A\mathbf{v}_2$$

for all vectors $\mathbf{v}_1, \mathbf{v}_2 \in \mathscr{M}^{N\times 1}$ and

$$A(\lambda \mathbf{v}) = \lambda A(\mathbf{v})$$

for all vectors $\mathbf{v} \in \mathscr{M}^{N\times 1}$ and $\lambda \in \mathscr{C}$. In this Chapter, we shall study *linear transformations*. As explained in Section 2.7, matrices such as $A \in \mathscr{M}^{M\times N}$ are examples of linear transformations. It is important to observe that linear transformations are more general mathematical objects than are matrices. While all matrices are linear transformations, not all linear transformations are matrices. It is this generality that we hope to capture in this chapter.

Section 3.1. Definition of a Linear Transformation

Linear transformations are functions defined on a vector space with values in a vector space. Let $\mathscr{V}$ and $\mathscr{U}$ be two vector spaces. Most of our examples will be for finite dimensional vector spaces. When we need to make explicit the dimensions, we shall use the convention that

$$N = \dim \mathscr{V} \qquad \text{and} \qquad M = \dim \mathscr{U} \tag{3.1.1}$$

We shall continue to denote the scalar field by $\mathscr{C}$, the set of complex numbers. When we intend the scalar field to be the set of real numbers, we shall continue to denote this set by $\mathscr{R}$.

The formal definition of a linear transformation is as follows:

Definition: If $\mathscr{V}$ and $\mathscr{U}$ are vector spaces, a *linear transformation* is a function $\mathbf{A}: \mathscr{V} \to \mathscr{U}$ such that

1) $$\mathbf{A}(\mathbf{u}+\mathbf{v}) = \mathbf{A}(\mathbf{u}) + \mathbf{A}(\mathbf{v}) \tag{3.1.2}$$

2) $$\mathbf{A}(\lambda \mathbf{u}) = \lambda \mathbf{A}(\mathbf{u}) \tag{3.1.3}$$

for all $\mathbf{u},\mathbf{v}\in\mathscr{V}$ and $\lambda\in\mathscr{C}$.

Observe that the $+$ symbol on the left side of (3.1.2) denotes addition in $\mathscr{V}$, while on the right side it denotes addition in $\mathscr{U}$. Likewise the scalar multiplication on the left side of (3.1.3) is in $\mathscr{V}$, while on the right side the scalar multiplication is in $\mathscr{U}$. It would be extremely cumbersome to adopt different symbols for these quantities. Further, it is customary to omit the parentheses and write simply $\mathbf{Au}$ for $\mathbf{A}(\mathbf{u})$ when $\mathbf{A}$ is a linear transformation.

The two parts of the definition of a linear transformation can be combined as follows:

Alternate Definition of a Linear Transformation: If $\mathscr{V}$ and $\mathscr{U}$ are vector spaces, a *linear transformation* is a function $\mathbf{A}:\mathscr{V}\rightarrow\mathscr{U}$ such that

$$\mathbf{A}(\lambda\mathbf{u}+\mu\mathbf{v})=\lambda\mathbf{Au}+\mu\mathbf{Av} \tag{3.1.4}$$

for all $\mathbf{u},\mathbf{v}\in\mathscr{V}$ and $\lambda,\mu\in\mathscr{C}$.

It is also possible to show that

$$\mathbf{A}(\lambda_1\mathbf{v}_1+\lambda_2\mathbf{v}_2+\cdots+\lambda_R\mathbf{v}_R)=\lambda_1\mathbf{Av}_1+\lambda_2\mathbf{Av}_2+\cdots+\lambda_R\mathbf{Av}_R \tag{3.1.5}$$

for all $\mathbf{v}_1,...,\mathbf{v}_R\in\mathscr{V}$ and $\lambda_1,...,\lambda_R\in\mathscr{C}$. From (3.1.3) of the definition, we can take $\lambda=0$ and see that

$$\mathbf{A0}=\mathbf{0} \tag{3.1.6}$$

Of course, we have used the same symbol for the zero in $\mathscr{V}$ as for the zero in $\mathscr{U}$. When we need to distinguish these zeros, we will write $\mathbf{0}_{\mathscr{V}}$ and $\mathbf{0}_{\mathscr{U}}$, respectively. If we make the choice $\lambda=-1$, (3.1.3) yields

$$\mathbf{A}(-\mathbf{v})=-\mathbf{Av} \tag{3.1.7}$$

Linear transformations arise in virtually all areas of mathematics, pure and applied. A few examples are as follows:

Example 3.1.1: Define a linear transformation $\mathbf{I}:\mathscr{V}\rightarrow\mathscr{V}$ by

$$\mathbf{Iv}=\mathbf{v}\quad\text{for all }\mathbf{v}\in\mathscr{V} \tag{3.1.8}$$

This linear transformation is the *identity* linear transformation.

Example 3.1.2: Define a linear transformation $\mathbf{A}:\mathscr{V}\rightarrow\mathscr{V}$ by

$$\mathbf{A}\mathbf{v}=\alpha\mathbf{v} \qquad \text{for all } \mathbf{v}\in\mathscr{V} \text{ and for a given } \alpha\in\mathscr{C} \tag{3.1.9}$$

This linear transformation simply takes a $\mathbf{v}\in\mathscr{V}$ and produces the same vector amplified by the scalar α.

Example 3.1.3: We select for $\mathscr{V}$ the vector space $C^{\infty}[0,\infty]$. Next, we define a function $\mathbf{A}:\mathscr{V}\rightarrow\mathscr{V}$ by the formal rule

$$\left(\mathbf{A}f\right)\left(s\right)=\int_{t=0}^{\infty}e^{-st}f\left(t\right)dt \tag{3.1.10}$$

The fact that the function defined is a linear transformation should be evident. Some of you will recognize this definition as that of the *Laplace Transform* of a function.

Example 3.1.4: We again select for $\mathscr{V}$ to be the vector space $C^{\infty}[0,\infty]$, and define a function $\mathbf{A}:\mathscr{V}\rightarrow\mathscr{V}$ by the formal rule

$$\left(\mathbf{A}f\right)\left(x\right)=\alpha\frac{d^{2}f\left(x\right)}{dx^{2}}+\beta\frac{df\left(x\right)}{dx}+\gamma f\left(x\right) \text{ for all } x\in\left(0,\infty\right) \tag{3.1.11}$$

where α,β and γ are scalars. The definition (3.1.11) shows that $\mathbf{A}$ is a linear transformation. It is the usual second order differential operator one encounters in elementary courses on ordinary differential equations.

The next few examples arise in the study of various topics in theoretical mechanics.

Example 3.1.5: When one studies heat and mass transfer in solids, one of the models is based upon the so called Fourier's Law.[1] Basically, it proposes that the flow of heat is caused by temperature gradients in the solid. The formal mathematic relationship for *Fourier's Law* is

$$\mathbf{q}=-\mathbf{K}\mathbf{g} \tag{3.1.12}$$

where $\mathbf{q}$ is the *heat flux vector*, $\mathbf{g}$ is the *temperature gradient* and $\mathbf{K}$ is a linear transformation known as the *conductivity*. The heat flux vector and the temperature gradient are vectors in a three dimensional vector space $\mathscr{V}$ and the conductivity is a linear transformation $\mathbf{K}:\mathscr{V}\rightarrow\mathscr{V}$. For the special kinds of heat conductors known as isentropic, the conductivity takes the special form $\mathbf{K}=k\mathbf{I}$ where k is a positive number.

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[1] Additional information about Jean Baptiste Joseph Fourier can be found at http://en.wikipedia.org/wiki/Joseph_Fourier.

Example 3.1.6: When one studies rigid body dynamics like arises in the study of gyroscopes and the study of space vehicles, one encounters the concept of *moment of inertia*. The moment of inertia is a linear transformation that connects the *angular velocity* to the *angular momentum* in certain special cases. The formal relationship is

$$\boldsymbol{\ell} = \mathbf{I}\boldsymbol{\omega} \tag{3.1.13}$$

where $\boldsymbol{\omega}$ is the *angular velocity*, $\boldsymbol{\ell}$ is the *angular momentum* and $\mathbf{I}$ is a linear transformation known as the *moment of inertia*. The angular velocity and the angular momentum are vectors in a three dimensional vector space $\mathscr{V}$ and the moment of inertia is a linear transformation $\mathbf{I}:\mathscr{V}\to\mathscr{V}$.

Example 3.1.7: In the study of the electrodynamics of magnets, electrical conductors and other electrical materials, one encounters a need to model a relationship between *current* and *voltage*. The relationship that is often used is known as *Ohm's law*.[2] It usually takes the form

$$\mathbf{j} = \mathbf{C}\mathbf{e} \tag{3.1.14}$$

where $\mathbf{e}$ is the *electrical field strength*, $\mathbf{j}$ is the *current density* and $\mathbf{C}$ is a linear transformation known as the *electrical conductivity*. The electrical field strength and the current density are vectors in a three dimensional vector space $\mathscr{V}$ and the electrical conductivity is a linear transformation $\mathbf{C}:\mathscr{V}\to\mathscr{V}$.

Example 3.1.7: In the study of continuum mechanics, one is concerned with the result of forces on deformable media. One category of force is the so called contact force. This force is characterized by a quantity known as a *stress vector* and it is calculated by a result known as Cauchy's Theorem.[3] The mathematical form of this theorem is

$$\mathbf{t} = \mathbf{T}\mathbf{n} \tag{3.1.15}$$

where $\mathbf{t}$ is the *stress vector* representing the force per unit area on a surface with *unit normal* $\mathbf{n}$. The linear transformation $\mathbf{T}$ is usually called the *stress tensor*. It represents a linear transformation that when multiplied by the unit normal yields the local force on the surface with that unit normal.

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[2] Additional information about Georg Simond Ohm can be found at http://en.wikipedia.org/wiki/Georg_Ohm.
[3] Additional information about Augustin-Louis Cauchy can be found at http://en.wikipedia.org/wiki/Augustin-Louis_Cauchy.

Section 3.2. Matrix Representation of a Linear Transformation

We first encountered the idea of a linear transformation in Section 2.7 when we were discussing matrices. Many of the examples of linear transformations given here involve matrices. In this section, we explore this relationship and try to establish to what extent a linear transformation is equivalent to a matrix. The short version of the answer is that when we restrict ourselves to finite dimensional vector spaces there is a *one to one correspondence between a linear transformation and a matrix*. The advantages of this fact and the disadvantages will be discussed.

The first idea we shall introduce is that of the *components of a linear transformation*. This idea arises only in the case where $\mathscr{V}$ and $\mathscr{U}$ are *finite dimensional*. Let $\{\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_N\}$ be a basis for $\mathscr{V}$ and $\{\mathbf{b}_1, \mathbf{b}_2, ..., \mathbf{b}_M\}$ be a basis for $\mathscr{U}$. Given a linear transformation, $\mathbf{A}: \mathscr{V} \to \mathscr{U}$, it is true that the set $\{\mathbf{Ae}_1, \mathbf{Ae}_2, ..., \mathbf{Ae}_N\}$ is a set of vectors in $\mathscr{U}$. As such, they can be *expanded* in the basis of $\mathscr{U}$, i.e. in the set $\{\mathbf{b}_1, \mathbf{b}_2, ..., \mathbf{b}_M\}$. Therefore, we can write

$$\mathbf{Ae}_k = A^1{}_k \mathbf{b}_1 + A^2{}_k \mathbf{b}_2 + \cdots + A^M{}_k \mathbf{b}_M \quad \text{for} \quad k = 1, 2, ..., N \tag{3.2.1}$$

In this equation, $A^j{}_k$ represents the j^{th} component of $\mathbf{Ae}_k$ with respect to the basis for $\mathscr{U}$. We shall typically write (3.2.1) in the more compact form

$$\mathbf{Ae}_k = \sum_{j=1}^{M} A^j{}_k \mathbf{b}_j \quad k = 1, 2, ..., N \tag{3.2.2}$$

Definition: The *components of a linear transformation* $\mathbf{A}: \mathscr{V} \to \mathscr{U}$ with respect to the bases $\{\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_N\}$ and $\{\mathbf{b}_1, \mathbf{b}_2, ..., \mathbf{b}_M\}$ are the MN scalars $A^j{}_k$, for $j = 1, 2, ..., M$ and $k = 1, 2, ..., N$.

Example 3.2.1: Let $\mathbf{A}: \mathscr{V} \to \mathscr{U}$, and $\{\mathbf{e}_1, \mathbf{e}_2\}$ be a basis for $\mathscr{V}$ and $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ a basis for $\mathscr{U}$. Define a linear transformation by the rule

$$\mathbf{Av} = \upsilon^1 \mathbf{b}_1 + \upsilon^2 \mathbf{b}_2 + \left(\upsilon^1 + \upsilon^2\right) \mathbf{b}_3 \tag{3.2.3}$$

for all $\mathbf{v} \in \mathscr{V}$. We wish to find the components of $\mathbf{A}$ with respect to the basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ and the basis $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$. We need to apply the definition (3.2.1), which in this case reduces to

$$\begin{aligned} \mathbf{Ae}_1 &= A^1{}_1 \mathbf{b}_1 + A^2{}_1 \mathbf{b}_2 + A^3{}_1 \mathbf{b}_3 \\ \mathbf{Ae}_2 &= A^1{}_2 \mathbf{b}_1 + A^2{}_2 \mathbf{b}_2 + A^3{}_2 \mathbf{b}_3 \end{aligned} \tag{3.2.4}$$

In order to find the coefficients $A^j{}_k$, $j = 1,2$ and $k = 1,2,3$, we need to use the defining equation (3.2.3). As a vector in $\mathscr{V}$, $\mathbf{v}$, it has the component representation

$$\mathbf{v} = \upsilon^1 \mathbf{e}_1 + \upsilon^2 \mathbf{e}_2 \tag{3.2.5}$$

Therefore, the choice $\mathbf{v} = \mathbf{e}_1$ implies $\upsilon^1 = 1$ and $\upsilon^2 = 0$. Therefore,

$$\mathbf{A}\mathbf{e}_1 = \mathbf{b}_1 + \mathbf{b}_3 \tag{3.2.6}$$

and the choice $\mathbf{v} = \mathbf{e}_2$ implies $\upsilon^1 = 0$ and $\upsilon^2 = 1$. Therefore,

$$\mathbf{A}\mathbf{e}_2 = \mathbf{b}_2 + \mathbf{b}_3 \tag{3.2.7}$$

If these equations are compared to (3.2.4), we see that

$$\begin{gathered} A^1{}_1 = 1, A^2{}_1 = 0, A^3{}_1 = 1 \\ \\ A^1{}_2 = 0, A^2{}_2 = 1, A^3{}_2 = 1 \end{gathered} \tag{3.2.8}$$

As one would anticipate, and as the notation suggests, it is convenient to arrange the components of a linear transformation in a matrix. This idea results in the following definition:

Definition: If $\mathbf{A}: \mathscr{V} \to \mathscr{U}$, is a linear transformation, and if $\mathscr{V}$ and $\mathscr{U}$ are finite dimensional, the *matrix of* $\mathbf{A}: \mathscr{V} \to \mathscr{U}$ *with respect to the basis* $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_N\}$ *of* $\mathscr{V}$ *and* $\{\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_M\}$ *of* $\mathscr{U}$ is the $M \times N$ matrix

$$M\left(\mathbf{A}, \mathbf{e}_k, \mathbf{b}_j\right) = \begin{bmatrix} A^1{}_1 & A^1{}_2 & \cdot & \cdot & \cdot & A^1{}_N \\ A^2{}_1 & A^2{}_2 & & & & A^2{}_N \\ A^3{}_1 & & A^3{}_3 & & & \cdot \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ A^M{}_1 & A^M{}_2 & \cdot & \cdot & \cdot & A^M{}_N \end{bmatrix} \tag{3.2.9}$$

It is customary to give the *matrix of a linear transformation* a different symbol than that of the linear transformation it represents. Equation (3.2.9) shows the notation $M\left(\mathbf{A}, \mathbf{e}_k, \mathbf{b}_j\right)$ for the matrix. This notation has the advantage of showing that the matrix depends upon the choice of the bases for the two vector spaces. Later, we shall show how the components of a linear transformation transform when the two bases are changed. In those cases where the choices of bases do not need to be stressed, it is sometimes conventional to write $[\mathbf{A}]$ for the matrix.

The above construction shows a correspondence, for finite dimensional vector spaces, between a linear transformation and the matrix of a linear transformation. The fact that one must select a basis for $\mathscr{V}$ and a basis for $\mathscr{U}$ in order to display the matrix means that the physical quantity represented by the linear transformation is now *dependent* upon the choice of basis. Physical quantities have intrinsic meaning and do not depend upon how one chooses to project them into a basis. Thus, it is advantageous to study linear transformations as mathematical objects and not complicate the picture by the arbitrary selection of bases. Of course, if we are not talking about infinite dimensional vector spaces. For infinite dimensional vector spaces, the idea of a matrix is meaningless.

Example 3.2.2: A linear transformation $\mathbf{A}:\mathscr{V}\to\mathscr{U}$, is defined, for all vectors $\mathbf{v}\in\mathscr{V}$, by

$$\begin{aligned}\mathbf{A}\mathbf{v} &= \left(\upsilon^2+2\upsilon^3+3\upsilon^4\right)\mathbf{b}_1+\left(-\upsilon^1+\upsilon^2+4\upsilon^3+\upsilon^4\right)\mathbf{b}_2\\ &\quad+\left(-\upsilon^1+\upsilon^2+\upsilon^3-2\upsilon^4\right)\mathbf{b}_3+\left(\upsilon^1+\upsilon^2-2\upsilon^3+2\upsilon^4\right)\mathbf{b}_4\\ &=\upsilon^1\left(-\mathbf{b}_2-\mathbf{b}_3+\mathbf{b}_4\right)+\upsilon^2\left(\mathbf{b}_1+\mathbf{b}_2+\mathbf{b}_3+\mathbf{b}_4\right)\\ &\quad+\upsilon^3\left(2\mathbf{b}_1+4\mathbf{b}_2+\mathbf{b}_3-2\mathbf{b}_4\right)+\upsilon^4\left(3\mathbf{b}_1+\mathbf{b}_2-2\mathbf{b}_3+2\mathbf{b}_4\right)\end{aligned} \tag{3.2.10}$$

where $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3,\mathbf{e}_4\}$ is a basis for $\mathscr{V}$ and $\{\mathbf{b}_1,\mathbf{b}_2,\mathbf{b}_3,\mathbf{b}_4\}$ is a basis for $\mathscr{U}$. It follows from this definition that

$$\begin{aligned}\mathbf{A}\mathbf{e}_1 &= -\mathbf{b}_2-\mathbf{b}_3+\mathbf{b}_4\\ \mathbf{A}\mathbf{e}_2 &= \mathbf{b}_1+\mathbf{b}_2+\mathbf{b}_3+\mathbf{b}_4\\ \mathbf{A}\mathbf{e}_3 &= 2\mathbf{b}_1+4\mathbf{b}_2+\mathbf{b}_3-2\mathbf{b}_4\\ \mathbf{A}\mathbf{e}_4 &= 3\mathbf{b}_1+\mathbf{b}_2-2\mathbf{b}_3+2\mathbf{b}_4\end{aligned} \tag{3.2.11}$$

Next, we need to utilize (3.2.2) which for this case reduces to

$$\begin{aligned}\mathbf{A}\mathbf{e}_1 &= A^1{}_1\mathbf{b}_1+A^2{}_1\mathbf{b}_2+A^3{}_1\mathbf{b}_3+A^4{}_1\mathbf{b}_4\\ \mathbf{A}\mathbf{e}_2 &= A^1{}_2\mathbf{b}_1+A^2{}_2\mathbf{b}_2+A^3{}_2\mathbf{b}_3+A^4{}_2\mathbf{b}_4\\ \mathbf{A}\mathbf{e}_3 &= A^1{}_3\mathbf{b}_1+A^2{}_3\mathbf{b}_2+A^3{}_3\mathbf{b}_3+A^4{}_3\mathbf{b}_4\\ \mathbf{A}\mathbf{e}_4 &= A^1{}_4\mathbf{b}_1+A^2{}_4\mathbf{b}_2+A^3{}_4\mathbf{b}_3+A^4{}_4\mathbf{b}_4\end{aligned} \tag{3.2.12}$$

If (3.2.12) is compared to (3.2.11), we see that

$$M\left(\mathbf{A},\mathbf{e}_k,\mathbf{b}_j\right)=\begin{bmatrix} A^1{}_1 & A^1{}_2 & A^1{}_3 & A^1{}_4 \\ A^2{}_1 & A^2{}_2 & A^2{}_3 & A^2{}_4 \\ A^3{}_1 & A^3{}_2 & A^3{}_3 & A^3{}_4 \\ A^4{}_1 & A^4{}_2 & A^4{}_3 & A^4{}_4 \end{bmatrix}=\begin{bmatrix} 0 & 1 & 2 & 3 \\ -1 & 1 & 4 & 1 \\ -1 & 1 & 1 & -2 \\ 1 & 1 & -2 & 2 \end{bmatrix} \tag{3.2.13}$$

Exercises

3.2.1 You are given a linear transformation $\mathbf{A}:\mathscr{V}\to\mathscr{U}$ defined by

$$\begin{aligned}\mathbf{A}\mathbf{v}&=\upsilon^1\left(\mathbf{b}_1+2\mathbf{b}_2+\mathbf{b}_3\right)+\upsilon^2\left(4\mathbf{b}_1+\mathbf{b}_2-2\mathbf{b}_3\right)\\&+\upsilon^3\left(\mathbf{b}_1+\mathbf{b}_2+\mathbf{b}_3\right)+\upsilon^4\left(\mathbf{b}_2+2\mathbf{b}_3\right)\end{aligned} \tag{3.2.14}$$

for all $\mathbf{v}=\upsilon^1\mathbf{e}_1+\upsilon^2\mathbf{e}_2+\upsilon^3\mathbf{e}_3+\upsilon^4\mathbf{e}_4\in\mathscr{V}$. Determine the matrix of $\mathbf{A}:\mathscr{V}\to\mathscr{U}$ with respect to the bases $\left\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3,\mathbf{e}_4\right\}$ and $\left\{\mathbf{b}_1,\mathbf{b}_2,\mathbf{b}_3\right\}$.

3.2.2 You are given a linear transformation $\mathbf{B}:\mathscr{U}\to\mathscr{W}$ defined by

$$\mathbf{B}\mathbf{u}=u^1\left(\mathbf{d}_1+5\mathbf{d}_2+\mathbf{d}_3\right)+u^2\left(2\mathbf{d}_1-\mathbf{d}_3\right)+u^3\left(-3\mathbf{d}_1+2\mathbf{d}_2+\mathbf{d}_3\right) \tag{3.2.15}$$

for all $\mathbf{u}=u^1\mathbf{b}_1+u^2\mathbf{b}_2+u^3\mathbf{b}_3\in\mathscr{U}$. Determine the matrix of $\mathbf{B}:\mathscr{U}\to\mathscr{W}$ with respect to the bases $\left\{\mathbf{b}_1,\mathbf{b}_2,\mathbf{b}_3\right\}$ and $\left\{\mathbf{d}_1,\mathbf{d}_2,\mathbf{d}_3\right\}$.

3.3.3 A linear transformation $\mathbf{A}:\mathscr{V}\to\mathscr{U}$ is defined, for all vectors $\mathbf{v}\in\mathscr{V}$, by

$$\begin{aligned}\mathbf{A}\mathbf{v}&=\left(\upsilon^1-i\upsilon^2+7\upsilon^3\right)\mathbf{b}_1+\left(-8i\upsilon^1+9\upsilon^2+8i\upsilon^3\right)\mathbf{b}_2\\&+\left(4\upsilon^1-9\upsilon^2+15i\upsilon^3\right)\mathbf{b}_3+\left(7i\upsilon^1+4i\upsilon^2\right)\mathbf{b}_4\end{aligned} \tag{3.2.16}$$

for all vectors $\mathbf{v}=\upsilon^1\mathbf{e}_1+\upsilon^2\mathbf{e}_2+\upsilon^3\mathbf{e}_3\in\mathscr{V}$, where $\left\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\right\}$ is a basis for $\mathscr{V}$ and $\left\{\mathbf{b}_1,\mathbf{b}_2,\mathbf{b}_3,\mathbf{b}_4\right\}$ is a basis for $\mathscr{U}$. Determine the matrix of $\mathbf{A}:\mathscr{V}\to\mathscr{U}$ with respect to the bases $\left\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\right\}$ and $\left\{\mathbf{b}_1,\mathbf{b}_2,\mathbf{b}_3,\mathbf{b}_4\right\}$.

3.3.4 A linear transformation $\mathbf{A}:\mathscr{V}\to\mathscr{R}_3$ is defined, for all vectors $\mathbf{v}\in\mathscr{V}$, by

$$\begin{aligned}\mathbf{Av} &= \left(3\upsilon^1 - 2\upsilon^2 - 2\upsilon^4\right) p_1 + \left(-\upsilon^1 - \upsilon^2 + 4\upsilon^3 + \upsilon^4\right) p_2 \\ &\quad + \left(-\upsilon^1 + \upsilon^2 + \upsilon^3 - 2\upsilon^4\right) p_3 + \left(\upsilon^2 + \upsilon^3 + 3\upsilon^4\right) p_4\end{aligned} \tag{3.2.17}$$

for all vectors $\mathbf{v} = \upsilon^1\mathbf{e}_1 + \upsilon^2\mathbf{e}_2 + \upsilon^3\mathbf{e}_3 + \upsilon^4\mathbf{e}_4 \in \mathscr{V}$, where $\left\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\right\}$ is a basis for $\mathscr{V}$ and $\left\{p_1, p_2, p_3, p_4\right\}$ is a basis for $\mathscr{P}_3$. Determine the matrix of $\mathbf{A}:\mathscr{V} \to \mathscr{P}_3$ with respect to the bases $\left\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\right\}$ and $\left\{p_1, p_2, p_3, p_4\right\}$.

Section 3.3. Properties of a Linear Transformation.

As with matrices, the *image space* of a linear transformation is defined to be set of all values of the function. More formally, the image space is the set

$$R(\mathbf{A}) = \{\mathbf{Ax} \mid \mathbf{x} \in \mathscr{V}\} \tag{3.3.1}$$

Utilizing the usual kind of proof, it is easy to show that the subset $R(\mathbf{A})$ of $\mathscr{U}$ is actually a subspace of $\mathscr{U}$. As such, we know from Theorem 2.5.2 that

$$\dim R(A) \le \dim \mathscr{U} \tag{3.3.2}$$

In Section 2.7, we introduced the rank of a matrix. Likewise, the *rank* of a linear transformation is $\dim R(\mathbf{A})$. If, the linear transformation $\mathbf{A}: \mathscr{V} \to \mathscr{U}$ has the property that

$$R(\mathbf{A}) = \mathscr{U} \tag{3.3.3}$$

it is said to be *onto*. If a linear transformation $\mathbf{A}: \mathscr{V} \to \mathscr{U}$ has the property that

$$\mathbf{Av}_1 = \mathbf{Av}_2 \quad \text{implies} \quad \mathbf{v}_1 = \mathbf{v}_2 \quad \text{for every } \mathbf{v}_1, \mathbf{v}_2 \in \mathscr{V} \tag{3.3.4}$$

then it is *one to one*. Linear transformations that are one to one are also called *regular* linear transformations.

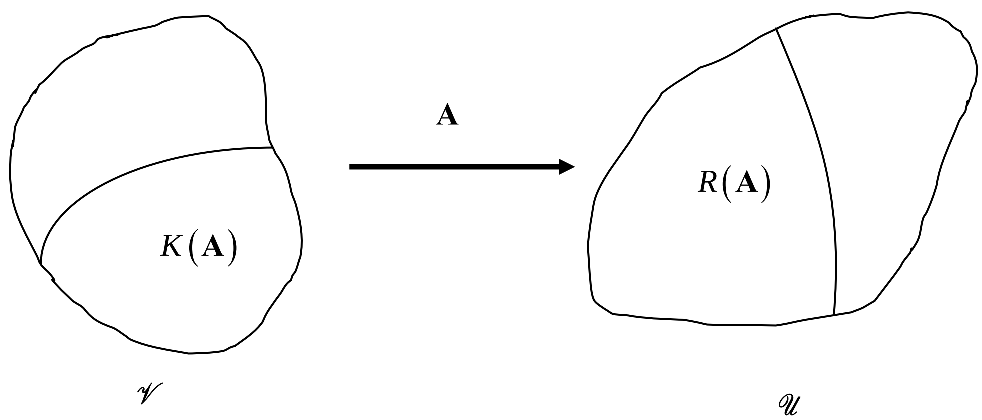
In Section 2.7, we introduced the kernel of a matrix. For linear transformations, the related idea is also called the *kernel*. The kernel of a linear transformation $\mathbf{A}: \mathscr{V} \to \mathscr{U}$ is the subset of $\mathscr{V}$ defined by

$$K(\mathbf{A}) = \{\mathbf{v} \mid \mathbf{Av} = \mathbf{0}\} \tag{3.3.5}$$

As with matrices, the *nullity* of a linear transformation is the dimension of the kernel, i.e., $\dim K(\mathbf{A})$. It is not difficult to establish that $K(\mathbf{A})$ is a subspace of $\mathscr{V}$. As such, it is true that

$$\dim K(\mathbf{A}) \le \dim \mathscr{V} \tag{3.3.6}$$

The following figure should be helpful as one tries to conceptualize the subspaces $K(\mathbf{A})$ and $R(\mathbf{A})$.



Theorem 3.3.1: A linear transformation $\mathbf{A}:\mathscr{V}\rightarrow\mathscr{U}$ is one to one, i.e., regular, if and only if $K(\mathbf{A})=\{\mathbf{0}\}$.

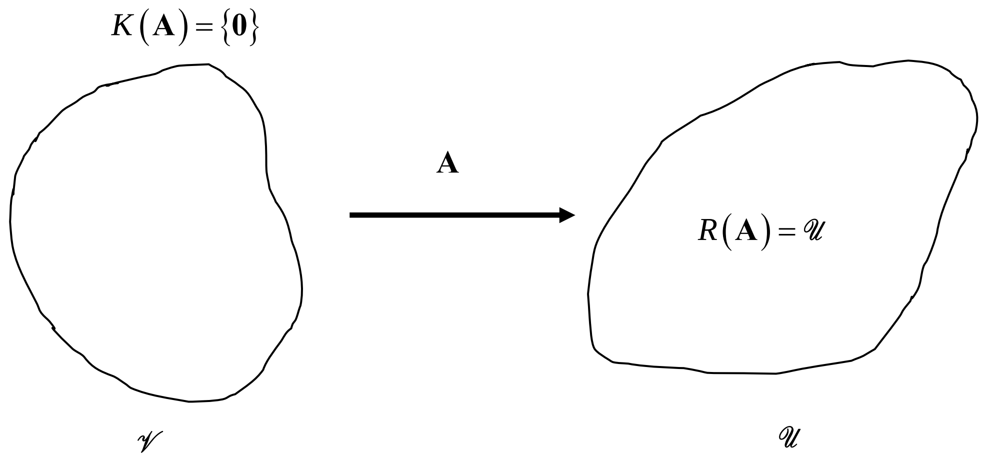
Proof: As with the corresponding proof for matrices, the key to the proof is the relationship $\mathbf{A}\mathbf{v}_1=\mathbf{A}\mathbf{v}_2$, which by linearity can be written $\mathbf{A}(\mathbf{v}_1-\mathbf{v}_2)=\mathbf{0}$. Thus, if $K(\mathbf{A})=\{\mathbf{0}\}$, then $\mathbf{A}\mathbf{v}_1=\mathbf{A}\mathbf{v}_2$ implies $\mathbf{v}_1=\mathbf{v}_2$. Conversely, assume $\mathbf{A}$ is one to one. Since $K(\mathbf{A})$ is a subspace it must contain the $\mathbf{0}\in\mathscr{V}$. Therefore, $\mathbf{A0}=\mathbf{0}$. If $K(\mathbf{A})$ contains any other element $\mathbf{v}$, as an element of the kernel it would be true that $\mathbf{A}\mathbf{v}=\mathbf{0}$. This formula contradicts the fact that $\mathbf{A}$ is one to one.

In Section 2.7, when discussing matrix equations, we established a uniqueness theorem. The linear transformation version of this theorem is

Theorem 3.3.2: If $\mathbf{b}\in R(\mathbf{A})$, then the solution to $\mathbf{A}\mathbf{x}=\mathbf{b}$ is unique if and only if $K(\mathbf{A})=\{\mathbf{0}\}$.

The proof of this theorem is formally the same as Theorem 2.7.4 for matrix equations.

The special case where $\mathbf{A}:\mathscr{V}\rightarrow\mathscr{U}$ is both one to one *and* onto would correspond to the figure



Theorem 3.3.3: If $\{\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_R\}$ is a linearly *dependent* set in $\mathscr{V}$ and if $\mathbf{A}:\mathscr{V}\to\mathscr{U}$ is a linear transformation, then $\{\mathbf{Av}_1,\mathbf{Av}_2,\ldots,\mathbf{Av}_R\}$ is a linearly dependent set in $\mathscr{U}$.

Proof Since the vectors $\{\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_R\}$ is a linearly dependent set, we can write

$$\sum_{j=1}^{R}\alpha_j\mathbf{v}_j=\mathbf{0} \tag{3.3.7}$$

where at least one coefficient is not zero. Therefore

$$\mathbf{A}\left(\sum_{j=1}^{R}\alpha_j\mathbf{v}_j\right)=\sum_{j=1}^{R}\alpha_j\mathbf{Av}_j=\mathbf{0} \tag{3.3.8}$$

Because the coefficients are not all zero, (3.3.8) establishes the result.

If the set $\{\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_R\}$ is *linearly independent*, then their image set $\{\mathbf{Av}_1,\mathbf{Av}_2,\ldots,\mathbf{Av}_R\}$ may or may not be linearly independent. For example, $\mathbf{A}$ might be a linear transformation that maps all vectors into $\mathbf{0}$.

The following theorem gives another condition for such linear transformations.

Theorem 3.3.4: A linear transformation $\mathbf{A}:\mathscr{V}\to\mathscr{U}$ is one to one, i.e., regular if and only if it maps linearly independent sets in $\mathscr{V}$ to linearly independent sets in $\mathscr{U}$.

Proof. We shall first prove that if $\mathbf{A}:\mathscr{V}\to\mathscr{U}$ is one to one then it maps linearly independent sets in $\mathscr{V}$ to linearly independent sets in $\mathscr{U}$. Let $\{\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_R\}$ be a linearly independent set in $\mathscr{V}$ and $\mathbf{A}:\mathscr{V}\to\mathscr{U}$ be a regular linear transformation. Consider the sum

$$\sum_{j=1}^{R}\alpha_j\mathbf{A}\mathbf{v}_j=\mathbf{0} \tag{3.3.9}$$

Equation (3.3.9) is equivalent to

$$\mathbf{A}\left(\sum_{j=1}^{R}\alpha_j\mathbf{v}_j\right)=\mathbf{0} \tag{3.3.10}$$

Theorem 3.1.1 tells us that $K(\mathbf{A})=\{\mathbf{0}\}$, therefore (3.3.10) yields

$$\sum_{j=1}^{R}\alpha_j\mathbf{v}_j=\mathbf{0} \tag{3.3.11}$$

Because $\{\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_R\}$ is a linearly independent set, (3.3.11) shows that $\alpha_1=\alpha_2=\cdots=\alpha_R=0$, which, from (3.3.9) implies that the set $\{\mathbf{A}\mathbf{v}_1,\ldots,\mathbf{A}\mathbf{v}_R\}$ is linearly independent. Next, we must prove that if $\mathbf{A}:\mathscr{V}\to\mathscr{U}$ maps linearly independent sets in $\mathscr{V}$ to linearly independent sets in $\mathscr{U}$, then it is one to one. The assumption that $\mathbf{A}$ preserves linear independence implies, in particular, that $\mathbf{A}\mathbf{v}\neq\mathbf{0}$ for every nonzero vector $\mathbf{v}\in\mathscr{V}$ since such a vector forms a linearly independent set. Therefore, $K(\mathbf{A})$ consists of the zero vector only, and thus $\mathbf{A}$ is one to one.

Theorem 3.3.5:

$$\underbrace{\dim\mathscr{V}}_{N}=\underbrace{\dim R(\mathbf{A})}_{\text{Rank}=R}+\underbrace{\dim K(\mathbf{A})}_{\text{Nullity}=P} \tag{3.3.12}$$

This theorem was proven for matrices in Section 2.7. It is the rank-nullity theorem for linear transformations.

Proof: For notational convenience, we have written in (3.3.12)

$$R=\dim R(\mathbf{A}) \quad\text{and}\qquad P=\dim K(\mathbf{A}) \tag{3.3.13}$$

We wish to show that $N=R+P$. Let $\{\mathbf{e}_1,\mathbf{e}_2,\ldots,\mathbf{e}_P,\mathbf{e}_{P+1},\mathbf{e}_{P+2},\ldots,\mathbf{e}_N\}$ be a basis for $\mathscr{V}$. We shall select these vectors such that the subset $\{\mathbf{e}_1,\mathbf{e}_2,\ldots,\mathbf{e}_P\}$ is a basis for $K(\mathbf{A})$. Our task is to show that

$N-P$ is the rank of $\mathbf{A}$. Our first step in the proof is to form the set $\{\mathbf{Ae}_1,\mathbf{Ae}_2,...,\mathbf{Ae}_P,\mathbf{Ae}_{P+1},\mathbf{Ae}_{P+2},...,\mathbf{Ae}_N\}$. This set spans $R(\mathbf{A})$ because $\{\mathbf{e}_1,\mathbf{e}_2,...,\mathbf{e}_P,\mathbf{e}_{P+1},\mathbf{e}_{P+2},...,\mathbf{e}_N\}$ is a basis for $\mathscr{V}$. This fact is formally expressed by the equation,

$$R(\mathbf{A})=\operatorname{Span}(\mathbf{Ae}_1,\mathbf{Ae}_2,...,\mathbf{Ae}_P,\mathbf{Ae}_{P+1},\mathbf{Ae}_{P+2},...,\mathbf{Ae}_N) \tag{3.3.14}$$

The next step utilizes the fact that, as defined, that the set $\{\mathbf{e}_1,\mathbf{e}_2,...,\mathbf{e}_P\}$ is a basis for $K(\mathbf{A})$. As such, it is true that $\mathbf{Ae}_1=\mathbf{Ae}_2=\cdots=\mathbf{Ae}_P=\mathbf{0}$. Therefore,

$$R(\mathbf{A})=\operatorname{Span}(\mathbf{Ae}_{P+1},\mathbf{Ae}_{P+2},...,\mathbf{Ae}_N) \tag{3.3.15}$$

If we can conclude that the set of $N-P$ vectors are linearly independent, then we can conclude that $R\equiv\dim R(\mathbf{A})=N-P$ and the proof is complete. The test for linear independence, as is always, is to analyze the equation

$$\alpha_1\mathbf{Ae}_{P+1}+\alpha_2\mathbf{Ae}_{P+2}+\cdots+\alpha_{N-P}\mathbf{Ae}_N=\mathbf{0} \tag{3.3.16}$$

Our challenge is to establish that the coefficients $\alpha_1,\alpha_2,...,\alpha_{N-P}$ are zero. The first step is to rewrite (3.3.16) as

$$\mathbf{A}(\alpha_1\mathbf{e}_{P+1}+\alpha_2\mathbf{e}_{P+2}+\cdots+\alpha_{N-P}\mathbf{e}_N)=\mathbf{0} \tag{3.3.17}$$

Therefore, $\alpha_1\mathbf{e}_{P+1}+\alpha_2\mathbf{e}_{P+2}+\cdots+\alpha_{N-P}\mathbf{e}_N\in K(\mathbf{A})$. However, we selected the vectors $\{\mathbf{e}_1,\mathbf{e}_2,...,\mathbf{e}_P,\mathbf{e}_{P+1},\mathbf{e}_{P+2},...,\mathbf{e}_N\}$ to be a basis for $\mathscr{V}$ such that the subset $\{\mathbf{e}_1,\mathbf{e}_2,...,\mathbf{e}_P\}$ is a basis for $K(\mathbf{A})$. The only vector of the form $\alpha_1\mathbf{e}_{P+1}+\alpha_2\mathbf{e}_{P+2}+\cdots+\alpha_{N-P}\mathbf{e}_N$, i.e., a linear combination of vectors not in $K(\mathbf{A})$ that is in $K(\mathbf{A})$, is the zero vector, $\mathbf{0}$. As a result,

$$\alpha_1\mathbf{e}_{P+1}+\alpha_2\mathbf{e}_{P+2}+\cdots+\alpha_{N-P}\mathbf{e}_N=\mathbf{0} \tag{3.3.18}$$

The sum (3.3.18) forces $\alpha_1=\alpha_2=\cdots=\alpha_{N-P}=0$ because of the linear independence of the set $\{\mathbf{e}_{P+1},\mathbf{e}_{P+2},...,\mathbf{e}_N\}$. This completes the proof.

Other relationships involving the dimensions of $\mathscr{V},\mathscr{U},R(\mathbf{A})$ and $K(\mathbf{A})$ are

$$\dim\mathscr{V}=\dim R(\mathbf{A})+\dim K(\mathbf{A}) \tag{3.3.19}$$

and

$$\dim R(\mathbf{A}) \le \min(\dim \mathscr{V}, \dim \mathscr{U}) \tag{3.3.20}$$

The proofs of these results are essentially the same as the proofs used in Section 2.7 for matrices. Theorem 2.5.2 told us that a subspace of a vector space equals the vector space if and only if the subspace and the vector space have the same dimension. This theorem and the fact that $R(\mathbf{A})$ is a subspace of $\mathscr{U}$ gives the following result

Theorem 3.3.6: $\dim R(\mathbf{A}) = \dim \mathscr{U}$ if and only if $\mathbf{A}$ is onto.

Another important result for one to one and onto linear transformations is the result

Theorem 3.3.7: If $\mathbf{A}: \mathscr{V} \to \mathscr{U}$ is one to one and onto, then

$$\dim \mathscr{V} = \dim \mathscr{U} \tag{3.3.21}$$

Proof: This result follows from Theorems 3.3.1, 3.3.4 and 3.3.5.

In the special case when $\dim \mathscr{V} = \dim \mathscr{U}$, it is possible to state the following important theorem:

Theorem 3.3.8: If $\mathbf{A}: \mathscr{V} \to \mathscr{U}$ is a linear transformation and if $\dim \mathscr{V} = \dim \mathscr{U}$, then $\mathbf{A}$ is a linear transformation *onto* $\mathscr{U}$ if and only if $\mathbf{A}$ is one to one.

Proof. Assume that $\mathbf{A}: \mathscr{V} \to \mathscr{U}$ is onto $\mathscr{U}$, then $\dim \mathscr{U} = \dim R(\mathbf{A}) = \dim \mathscr{V}$. Therefore, the equation $\dim \mathscr{V} = \dim R(\mathbf{A}) + \dim K(\mathbf{A})$ forces $\dim K(\mathbf{A}) = 0$ and thus $K(\mathbf{A}) = \{\mathbf{0}\}$ and $\mathbf{A}$ is one to one. Next assume that $\mathbf{A}$ is one-to-one. By Theorem 3.1.1, $K(\mathbf{A}) = \{\mathbf{0}\}$ and thus $\dim K(\mathbf{A}) = 0$. As a result, the equation $\dim \mathscr{V} = \dim R(\mathbf{A}) + \dim K(\mathbf{A})$ shows that

$$\dim \mathscr{V} = \dim \mathbf{A}(\mathscr{V}) = \dim \mathscr{U} \tag{3.3.22}$$

Therefore, by Theorem 3.3.4,

$$\mathbf{A}(\mathscr{V}) = \mathscr{U} \tag{3.3.23}$$

and $\mathbf{A}$ is onto.

In the case where $\mathbf{A}: \mathscr{V} \to \mathscr{U}$ is onto and one to one, the two vector spaces $\mathscr{V}$ and $\mathscr{U}$ are in one to one correspondence. As such, there is an *inverse function* $\mathbf{A}^{-1}: \mathscr{U} \to \mathscr{V}$. One of our objectives later will be to show that this inverse function, when it exists, is actually a linear transformation. However, before we can properly introduce this linear transformation, we need to define what is meant by the product of two linear transformations. This topic, among others, will be introduced in the next section.

In Section 1.8, we stated and proved Theorem 1.8.1, the consistency theorem for linear systems. That discussion concerned solutions of the matrix equation $A\mathbf{x} = \mathbf{b}$, and said that the system has a solution if and only if $\mathbf{b} \in R(A)$. When the problem to be solved is stated in terms of linear transformations instead of matrices, the theorem is readily generalized to be that the vector equation $\mathbf{Ax} = \mathbf{b}$ has a solution if and only if $\mathbf{b} \in R(\mathbf{A})$.

Section 3.4. Sums and Products of Linear Transformations

In this section, we shall assign meaning to the operations of addition and scalar multiplication for linear transformations. In addition, we shall explain how one forms the product of two linear transformations. For finite dimensional vector spaces, you will not be surprised to learn that these operations are generalizations of familiar matrix operations. First, we shall define the operations of addition of two linear transformations and multiplication of a linear transformation by a scalar.

Definition: If $\mathbf{A}$ and $\mathbf{B}$ are linear transformations $\mathscr{V} \to \mathscr{U}$, then their *sum* $\mathbf{A}+\mathbf{B}$ is a linear transformation $\mathscr{V} \to \mathscr{U}$ defined by

$$(\mathbf{A}+\mathbf{B})\mathbf{v} = \mathbf{A}\mathbf{v} + \mathbf{B}\mathbf{v} \tag{3.4.1}$$

for all $\mathbf{v} \in \mathscr{V}$. If $\lambda \in \mathscr{C}$, then $\lambda\mathbf{A}$ is a linear transformation $\mathscr{V} \to \mathscr{U}$ defined by

$$(\lambda\mathbf{A})\mathbf{v} = \lambda(\mathbf{A}\mathbf{v}) \tag{3.4.2}$$

for all $\mathbf{v} \in \mathscr{V}$.

If we write $\mathscr{L}(\mathscr{V};\mathscr{U})$ for the set of linear transformations from $\mathscr{V}$ to $\mathscr{U}$, then (3.4.1) and (3.4.2) make $\mathscr{L}(\mathscr{V};\mathscr{U})$ a *vector space*. The zero element in $\mathscr{L}(\mathscr{V};\mathscr{U})$ is the linear transformation $\mathbf{0}$ defined by

$$\mathbf{0}\mathbf{v} = \mathbf{0} \tag{3.4.3}$$

for all $\mathbf{v} \in \mathscr{V}$. The *negative* of $\mathbf{A} \in \mathscr{L}(\mathscr{V};\mathscr{U})$ is a linear transformation $-\mathbf{A} \in \mathscr{L}(\mathscr{V};\mathscr{U})$ defined by

$$-\mathbf{A} = -1\mathbf{A} \tag{3.4.4}$$

It follows from (3.4.4) that $-\mathbf{A}$ is the additive inverse of $\mathbf{A} \in \mathscr{L}(\mathscr{V};\mathscr{U})$. This assertion follows from

$$\mathbf{A}+(-\mathbf{A}) = \mathbf{A}+(-1\mathbf{A}) = 1\mathbf{A}+(-1\mathbf{A}) = (1-1)\mathbf{A} = 0\mathbf{A} = \mathbf{0} \tag{3.4.5}$$

where (3.4.1) and (3.4.2) have been used. Consistent with our notation introduced in equation (2.1.3), we shall write $\mathbf{A}-\mathbf{B}$ for the sum $\mathbf{A}+(-\mathbf{B})$ formed from the linear transformations $\mathbf{A}$ and $\mathbf{B}$. The formal proof that $\mathscr{L}(\mathscr{V};\mathscr{U})$ is a vector space is left as an exercise.

Theorem 3.4.1.

$$\dim \mathscr{L}(\mathscr{V};\mathscr{U}) = \dim \mathscr{V} \dim \mathscr{U} \tag{3.4.6}$$

Proof. Let $\{\mathbf{e}_1,...,\mathbf{e}_N\}$ be a basis for $\mathscr{V}$ and $\{\mathbf{b}_1,...,\mathbf{b}_M\}$ be a basis for $\mathscr{U}$. Define NM linear transformations $\mathbf{E}^k{}_j : \mathscr{V} \to \mathscr{U}$ by

$$\begin{aligned} \mathbf{E}^k{}_j \mathbf{e}_k &= \mathbf{b}_j, \quad k = 1,\dots,N; \quad j = 1,\dots,M \\ \mathbf{E}^k{}_j \mathbf{e}_p &= \mathbf{0}, \quad k \neq p \end{aligned} \tag{3.4.7}$$

If $\mathbf{A}$ is an arbitrary member of $\mathscr{L}(\mathscr{V};\mathscr{U})$, then $\mathbf{A}\mathbf{e}_k \in \mathscr{U}$, and thus can be expanded in the basis $\{\mathbf{b}_1,...,\mathbf{b}_M\}$. We shall write this expansion as we did in equation (3.2.2)

$$\mathbf{A}\mathbf{e}_k = \sum_{j=1}^{M} A^j{}_k \mathbf{b}_j \quad j = 1,2,...,M \text{ and } k = 1,2,...,N \tag{3.4.8}$$

Based upon the properties of $\mathbf{E}^k{}_j$ in (3.4.7), we can write (3.4.8) as

$$\mathbf{A}\mathbf{e}_k = \sum_{j=1}^{M} A^j{}_k \mathbf{E}^k{}_j \mathbf{e}_k = \sum_{j=1}^{M}\sum_{s=1}^{N} A^j{}_s \mathbf{E}^s{}_j \mathbf{e}_k \tag{3.4.9}$$

A simple rearrangement of (3.4.9) yields

$$\left(\mathbf{A} - \sum_{j=1}^{M}\sum_{s=1}^{N} A^j{}_s \mathbf{E}^s{}_j \right) \mathbf{e}_k = \mathbf{0} \tag{3.4.10}$$

Since the set of vectors $\{\mathbf{e}_1,...,\mathbf{e}_N\}$ form a basis of $\mathscr{V}$, an arbitrary vector $\mathbf{v} \in \mathscr{V}$ has the representation

$$\mathbf{v} = \sum_{k=1}^{N} \upsilon^k \mathbf{e}_k \tag{3.4.11}$$

Equation (3.4.11) allows (3.4.10) to be written

$$\left(\mathbf{A} - \sum_{j=1}^{M} \sum_{s=1}^{N} A^j{}_s \mathbf{E}^s{}_j \right) \mathbf{v} = \mathbf{0} \tag{3.4.12}$$

for all vectors $\mathbf{v} \in \mathscr{V}$. Thus, from (3.4.3), $\mathbf{A} - \sum_{j=1}^{M} \sum_{s=1}^{N} A^j{}_s \mathbf{E}^s{}_j$ is the zero liner transformation. As a result

$$\mathbf{A} = \sum_{j=1}^{M} \sum_{s=1}^{N} A^j{}_k \mathbf{E}^s{}_j \tag{3.4.13}$$

Equation (3.4.13)means that the set of MN linear transformations $\left\{ \mathbf{E}^s{}_j \text{ for } s = 1,...,N \text{ and } j = 1,...,M \right\}$ span $\mathscr{L}\left(\mathscr{V};\mathscr{U}\right)$. If we can prove that this set is linearly independent, then the proof of the theorem is complete. To this end, set

$$\sum_{j=1}^{M} \sum_{s=1}^{N} A^j{}_s \mathbf{E}^s{}_j = \mathbf{0} \tag{3.4.14}$$

Then, from (3.4.7),

$$\sum_{j=1}^{M} \sum_{s=1}^{N} A^j{}_s \mathbf{E}^s{}_j \left(\mathbf{e}_p \right) = \sum_{j=1}^{M} A^j{}_p \mathbf{b}_j = \mathbf{0} \tag{3.4.15}$$

Because the set $\left\{ \mathbf{b}_1,...,\mathbf{b}_M \right\}$ is linearly independent in $\mathscr{U}$, (3.4.15) yields $A^j{}_p = 0$. Hence the set $\left\{ \mathbf{E}^s{}_j \right\}$ is a basis of $\mathscr{L}\left(\mathscr{V};\mathscr{U}\right)$. As a result, we have

$$\dim \mathscr{L}\left(\mathscr{V};\mathscr{U}\right) = MN = \dim \mathscr{U} \dim \mathscr{V} \tag{3.4.16}$$

At the end of Section 3.3, we mentioned the need to define the *product* of two linear transformations. The formal definition is as follows:

Definition: If $\mathbf{A}: \mathscr{V} \to \mathscr{U}$ and $\mathbf{B}: \mathscr{U} \to \mathscr{W}$ are linear transformations, their *product* is a linear transformation $\mathscr{V} \to \mathscr{W}$, written $\mathbf{BA}$, defined by

$$\mathbf{BAv} = \mathbf{B}\left(\mathbf{Av}\right) \tag{3.4.17}$$

for all $\mathbf{v} \in \mathscr{V}$.

The properties of the product operation are summarized in the following theorem.

Theorem 3.4.2.

$$\begin{aligned}\mathbf{C}(\mathbf{BA}) &= (\mathbf{CB})\mathbf{A} \\ (\lambda\mathbf{A}+\mu\mathbf{B})\mathbf{C} &= \lambda\mathbf{AC}+\mu\mathbf{BC} \\ \mathbf{C}(\lambda\mathbf{A}+\mu\mathbf{B}) &= \lambda\mathbf{CA}+\mu\mathbf{CB}\end{aligned} \tag{3.4.18}$$

for all $\lambda,\mu\in\mathscr{C}$ and where it is understood that $\mathbf{A},\mathbf{B},\text{and}\,\mathbf{C}$ are defined on the proper vector spaces so as to make the indicated products defined.

The definition (3.4.17), which is given without the introduction of bases for the three vector spaces $\mathscr{V},\mathscr{U},\mathscr{W}$, actually implies the matrix multiplication formula

$$M\left(\mathbf{BA},\mathbf{e}_k,\mathbf{d}_q\right)=M\left(\mathbf{B},\mathbf{b}_j,\mathbf{d}_q\right)M\left(\mathbf{A},\mathbf{e}_k,\mathbf{b}_j\right) \tag{3.4.19}$$

where $\left\{\mathbf{e}_1,\mathbf{e}_2,...,\mathbf{e}_N\right\}$ is the basis of $\mathscr{V}$, $\left\{\mathbf{b}_1,\mathbf{b}_2,...,\mathbf{b}_M\right\}$ is the basis of $\mathscr{U}$ and $\left\{\mathbf{d}_1,\mathbf{d}_2,...,\mathbf{d}_P\right\}$ is the basis of $\mathscr{W}$. The proof of this formula involves utilizing the many definitions we have accumulated. The connection between $\mathbf{A}:\mathscr{V}\rightarrow\mathscr{U}$ and its components is (3.2.2), repeated,

$$\mathbf{A}\mathbf{e}_k=\sum_{j=1}^{M}A^j{}_k\mathbf{b}_j \qquad \text{for} \qquad k=1,2,...,N \tag{3.4.20}$$

Likewise, the connection between $\mathbf{B}:\mathscr{U}\rightarrow\mathscr{W}$ and its components is

$$\mathbf{B}\mathbf{b}_j=\sum_{q=1}^{P}B^q{}_j\mathbf{d}_q \qquad \text{for} \qquad j=1,2,...,M \tag{3.4.21}$$

From the definition (3.4.17),

$$\begin{aligned}\mathbf{BA}\mathbf{e}_k=\mathbf{B}\left(\mathbf{A}\mathbf{e}_k\right)&=\mathbf{B}\left(\sum_{j=1}^{M}A^j{}_k\mathbf{b}_j\right)=\sum_{j=1}^{M}A^j{}_k\mathbf{B}\mathbf{b}_j \\ &=\sum_{j=1}^{M}A^j{}_k\left(\sum_{q=1}^{P}B^q{}_j\mathbf{d}_q\right)=\sum_{q=1}^{P}\left(\sum_{j=1}^{M}B^q{}_jA^j{}_k\right)\mathbf{d}_q\end{aligned} \tag{3.4.22}$$

If we now recognize that the relationship between $\mathbf{BA}:\mathscr{V}\rightarrow\mathscr{W}$ and its components is, from the definition, an equation like (3.4.22), we see that we have derived an expression for the components of the product $\mathbf{BA}$. If we write the product as

$$\mathbf{C} = \mathbf{B}\mathbf{A} \tag{3.4.23}$$

then, the components of $\mathbf{C}$, by definition, are given by

$$\mathbf{C}\mathbf{e}_k = \sum_{q=1}^{P} C^q{}_k \mathbf{d}_q \qquad \text{for} \qquad k = 1, 2, \ldots, N \tag{3.4.24}$$

Therefore, the three sets of components connect by the formula

$$C^q{}_k = \sum_{j=1}^{M} B^q{}_j A^j{}_k \tag{3.4.25}$$

Equation (3.4.25)is precisely the matrix equation (3.4.19) given above.

Exercises

3.4.1 Show that

$$M\left(\lambda\mathbf{A} + \mu\mathbf{B}, \mathbf{e}_k, \mathbf{b}_j\right) = \lambda M\left(\mathbf{A}, \mathbf{e}_k, \mathbf{b}_j\right) + \mu M\left(\mathbf{B}, \mathbf{e}_k, \mathbf{b}_j\right) \tag{3.4.26}$$

Viewed as a function $M : \mathscr{L}\left(\mathscr{V};\mathscr{U}\right) \to \mathscr{M}^{M\times N}$, equation (3.4.26) shows that M is a linear transformation. It is easily shown to be one to one, and by virtue the result (3.4.16) and Theorem 3.3.7, $M : \mathscr{L}\left(\mathscr{V};\mathscr{U}\right) \to \mathscr{M}^{M\times N}$ is an onto linear transformation. The one to one correspondence between linear transformations between finite dimensional vector spaces and matrices is a fundamental and useful result in linear algebra. The fact that this correspondence depends on the choices of bases is also a fundamental result that is important in the applications.

3.4.2 You are given linear transformations $\mathbf{A} : \mathscr{V} \to \mathscr{U}$ and $\mathbf{B} : \mathscr{U} \to \mathscr{W}$ defined by

$$\begin{aligned} \mathbf{A}\mathbf{v} = {} & \upsilon^1\left(\mathbf{b}_1 + 2\mathbf{b}_2 + \mathbf{b}_3\right) + \upsilon^2\left(4\mathbf{b}_1 + \mathbf{b}_2 - 2\mathbf{b}_3\right) \\ & + \upsilon^3\left(\mathbf{b}_1 + \mathbf{b}_2 + \mathbf{b}_3\right) + \upsilon^4\left(\mathbf{b}_2 + 2\mathbf{b}_3\right) \end{aligned} \tag{3.4.27}$$

for all $\mathbf{v} = \upsilon^1\mathbf{e}_1 + \upsilon^2\mathbf{e}_2 + \upsilon^3\mathbf{e}_3 + \upsilon^4\mathbf{e}_4 \in \mathscr{V}$, and

$$\mathbf{B}\mathbf{u} = u^1\left(\mathbf{d}_1 + 5\mathbf{d}_2 + \mathbf{d}_3\right) + u^2\left(2\mathbf{d}_1 - \mathbf{d}_3\right) + u^3\left(-3\mathbf{d}_1 + 2\mathbf{d}_2 + \mathbf{d}_3\right) \tag{3.4.28}$$

for all $\mathbf{u} = u^1\mathbf{b}_1 + u^2\mathbf{b}_2 + u^3\mathbf{b}_3 \in \mathscr{U}$. Calculate the linear transformation $\mathbf{B}\mathbf{A} : \mathscr{V} \to \mathscr{W}$.

Section 3.5. One to One Onto Linear Transformations

In this section, we shall record some important properties of one to one and onto linear transformations $\mathbf{A}:\mathscr{V}\to\mathscr{U}$. In Section 3.3, it was mentioned that when a linear transformation $\mathbf{A}:\mathscr{V}\to\mathscr{U}$ is both one to one and onto, the elements of $\mathscr{V}$ are in one to one correspondence with the elements of $\mathscr{U}$. One to one onto linear transformations are sometimes called *isomorphisms*. The one to one correspondence between $\mathscr{V}$ and $\mathscr{U}$ means there exists a function $\mathbf{f}:\mathscr{U}\to\mathscr{V}$ such that

$$\mathbf{f}(\mathbf{A}\mathbf{v})=\mathbf{v} \quad \text{for all } \mathbf{v}\in\mathscr{V} \tag{3.5.1}$$

The next result we wish to establish is that $\mathbf{f}$ is actually a *linear transformation*. This means that if we are given $\mathbf{v}_1,\mathbf{v}_2\in\mathscr{V}$ and if we label the corresponding elements in $\mathscr{U}$ by

$$\mathbf{u}_1=\mathbf{A}\mathbf{v}_1 \quad \text{and} \quad \mathbf{u}_2=\mathbf{A}\mathbf{v}_2 \tag{3.5.2}$$

then

$$\mathbf{f}(\lambda\mathbf{u}_1+\mu\mathbf{u}_2)=\lambda\mathbf{f}(\mathbf{u}_1)+\mu\mathbf{f}(\mathbf{u}_2) \tag{3.5.3}$$

for all $\lambda,\mu\in\mathscr{C}$ and all $\mathbf{u}_1,\mathbf{u}_2$ in $\mathscr{U}$.

The proof of (3.5.3) goes as follows. First, form the left side of (3.5.3) and use the properties of $\mathbf{A}$ as a linear transformation. The result is

$$\begin{aligned}\mathbf{f}(\lambda\mathbf{u}_1+\mu\mathbf{u}_2)&=\mathbf{f}(\lambda\mathbf{A}\mathbf{v}_1+\mu\mathbf{A}\mathbf{v}_2)\\&=\mathbf{f}\big(\mathbf{A}(\lambda\mathbf{v}_1+\mu\mathbf{v}_2)\big)\\&=\lambda\mathbf{v}_1+\mu\mathbf{v}_2\end{aligned} \tag{3.5.4}$$

where (3.5.1) has been used. Next, it follows from (3.5.1) and (3.5.2) that

$$\mathbf{v}_1=\mathbf{f}(\mathbf{A}\mathbf{v}_1)=\mathbf{f}(\mathbf{u}_1) \tag{3.5.5}$$

and

$$\mathbf{v}_2=\mathbf{f}(\mathbf{A}\mathbf{v}_2)=\mathbf{f}(\mathbf{u}_2) \tag{3.5.6}$$

Equations (3.5.5) and (3.5.6), when utilized in (3.5.4) yield the result (3.5.3).

As with matrices, it is customary to denote the inverse linear transformation of a one to one onto linear transformation $\mathbf{A}:\mathscr{V}\rightarrow\mathscr{U}$ by $\mathbf{A}^{-1}$. Also, the linear transformation $\mathbf{A}^{-1}:\mathscr{U}\rightarrow\mathscr{V}$ is one to one and onto whose inverse obeys

$$\left(\mathbf{A}^{-1}\right)^{-1}=\mathbf{A} \tag{3.5.7}$$

Theorem 3.5.1. If $\mathbf{A}:\mathscr{V}\rightarrow\mathscr{U}$ and $\mathbf{B}:\mathscr{U}\rightarrow\mathscr{W}$ are one to one linear transformations, then $\mathbf{BA}:\mathscr{V}\rightarrow\mathscr{W}$ is a one to one onto linear transformation whose inverse is computed by

$$\left(\mathbf{BA}\right)^{-1}=\mathbf{A}^{-1}\mathbf{B}^{-1} \tag{3.5.8}$$

Proof. The fact that $\mathbf{BA}$ is a one to one and onto follows directly from the corresponding properties of $\mathbf{A}$ and $\mathbf{B}$. The fact that the inverse of $\mathbf{BA}$ is computed by (3.5.8) follows directly because if

$$\mathbf{u}=\mathbf{Av} \qquad \text{and} \qquad \mathbf{w}=\mathbf{Bu} \tag{3.5.9}$$

then

$$\mathbf{v}=\mathbf{A}^{-1}\mathbf{u} \qquad \text{and} \qquad \mathbf{u}=\mathbf{B}^{-1}\mathbf{w} \tag{3.5.10}$$

It follows from (3.5.9) that

$$\mathbf{w}=\mathbf{Bu}=\mathbf{B}\left(\mathbf{Av}\right)=\mathbf{BAv} \tag{3.5.11}$$

which implies that

$$\mathbf{v}=\left(\mathbf{BA}\right)^{-1}\mathbf{w} \tag{3.5.12}$$

It follows from (3.5.10) that

$$\mathbf{v}=\mathbf{A}^{-1}\mathbf{u}=\mathbf{A}^{-1}\mathbf{B}^{-1}\mathbf{w} \tag{3.5.13}$$

Therefore,

$$\mathbf{v}=\left(\mathbf{BA}\right)^{-1}\mathbf{w}=\mathbf{A}^{-1}\mathbf{B}^{-1}\mathbf{w} \tag{3.5.14}$$

As a result of (3.5.14)

$$\left((\mathbf{BA})^{-1}-\mathbf{A}^{-1}\mathbf{B}^{-1}\right)\mathbf{w}=\mathbf{0} \quad \text{for all} \quad \mathbf{w}\in\mathscr{W} \tag{3.5.15}$$

which implies (3.5.8).

The notation $\mathbf{A}^{-1}$ for the inverse allows (3.5.1) to be written

$$\mathbf{A}^{-1}(\mathbf{Av})=\mathbf{v} \quad \text{for all } \mathbf{v}\in\mathscr{V} \tag{3.5.16}$$

The product definition (3.4.17) allows (3.5.16) to be written

$$\mathbf{A}^{-1}\mathbf{Av}=\mathbf{v} \quad \text{for all } \mathbf{v}\in\mathscr{V} \tag{3.5.17}$$

Likewise, for all $\mathbf{u}\in\mathscr{U}$, we can write

$$\mathbf{AA}^{-1}\mathbf{u}=\mathbf{u} \quad \text{for all } \mathbf{u}\in\mathscr{U} \tag{3.5.18}$$

The *identity* linear transformation $\mathbf{I}:\mathscr{V}\to\mathscr{V}$ was introduced in Example 3.1.1. Recall that it is defined by

$$\mathbf{Iv}=\mathbf{v} \tag{3.5.19}$$

for all $\mathbf{v}$ in $\mathscr{V}$. Often it is desirable to distinguish the identity linear transformations on different vector spaces. In these cases we shall denote the identity linear transformation by $\mathbf{I}_{\mathscr{V}}$. It follows from (3.5.17), (3.5.18) and (3.5.19) that

$$\mathbf{AA}^{-1}=\mathbf{I}_{\mathscr{U}} \qquad \text{and} \qquad \mathbf{A}^{-1}\mathbf{A}=\mathbf{I}_{\mathscr{V}} \tag{3.5.20}$$

Conversely, if $\mathbf{A}$ is a linear transformation from $\mathscr{V}$ to $\mathscr{U}$, and if there exists a linear transformation $\mathbf{B}:\mathscr{U}\to\mathscr{V}$ such that $\mathbf{AB}=\mathbf{I}_{\mathscr{U}}$ and $\mathbf{BA}=\mathbf{I}_{\mathscr{V}}$, then $\mathbf{A}$ is one to one and onto and $\mathbf{B}=\mathbf{A}^{-1}$. The proof of this assertion is left as an exercise. As with matrices, linear transformations that have inverses, i.e., one to one onto linear transformations, are referred to as *nonsingular*.

Exercises

3.5.1 Show that

$$M\left(\mathbf{A}^{-1},\mathbf{e}_j,\mathbf{e}_k\right)=M\left(\mathbf{A},\mathbf{e}_k,\mathbf{e}_j\right)^{-1} \tag{3.5.21}$$

for a nonsingular linear transformation $\mathbf{A}:\mathscr{V}\rightarrow\mathscr{V}$.

3.5.2 In Example 3.2.2, we were given the linear transformation $\mathbf{A}:\mathscr{V}\rightarrow\mathscr{U}$ defined by

$$\begin{aligned}\mathbf{A}\mathbf{v} &= \left(\upsilon^2+2\upsilon^3+3\upsilon^4\right)\mathbf{b}_1+\left(-\upsilon^1+\upsilon^2+4\upsilon^3+\upsilon^4\right)\mathbf{b}_2\\ &\quad+\left(-\upsilon^1+\upsilon^2+\upsilon^3-2\upsilon^4\right)\mathbf{b}_3+\left(\upsilon^1+\upsilon^2-2\upsilon^3+2\upsilon^4\right)\mathbf{b}_4\end{aligned} \tag{3.5.22}$$

for all $\mathbf{v}=\upsilon^1\mathbf{e}_1+\upsilon^2\mathbf{e}_2+\upsilon^3\mathbf{e}_3+\upsilon^4\mathbf{e}_4\in\mathscr{V}$. Show that $\mathbf{A}$ is nonsingular and that the inverse $\mathbf{A}^{-1}:\mathscr{U}\rightarrow\mathscr{V}$ is defined by

$$\begin{aligned}\mathbf{A}^{-1}\mathbf{u} &= \left(-7u_1+\frac{19}{3}u_2-\frac{10}{3}u_3+4u_4\right)\mathbf{e}_1\\ &\quad+\left(-u_1+u_2+u_4\right)\mathbf{e}_2\\ &\quad+\left(-2u_1+2u_2-u_3+u_4\right)\mathbf{e}_3\\ &\quad+\left(2u_1-\frac{5}{3}u_2+\frac{2}{3}u_3-u_4\right)\mathbf{e}_4\end{aligned} \tag{3.5.23}$$

for all $\mathbf{u}=u^1\mathbf{b}_1+u^2\mathbf{b}_2+u^3\mathbf{b}_3+u^4\mathbf{b}_4\in\mathscr{U}$.

Section 3.6. Change of Basis for Linear Transformations

In Section 2.6, we introduced the idea of a *change of basis* and were led to the idea of a *transition matrix*. For a finite dimensional vector space $\mathscr{V}$, we considered two bases $\{\mathbf{e}_1,\mathbf{e}_2,...,\mathbf{e}_N\}$ and $\{\hat{\mathbf{e}}_1,\hat{\mathbf{e}}_2,...,\hat{\mathbf{e}}_N\}$. Because each basis element of the first set can be written as a linear combination of the elements of the second set, we were able to write equation (2.6.3), repeated,

$$\hat{\mathbf{e}}_j = \sum_{k=1}^{N} T_j^k \mathbf{e}_k \quad \text{for } j=1,2,...,N \tag{3.6.1}$$

and its inverse, equation (2.6.4), repeated,

$$\mathbf{e}_k = \sum_{j=1}^{N} \hat{T}_k^j \hat{\mathbf{e}}_j \quad \text{for } k=1,2,...,N \tag{3.6.2}$$

We defined the transition matrix for the basis change $\{\mathbf{e}_1,\mathbf{e}_2,...,\mathbf{e}_N\}\rightarrow\{\hat{\mathbf{e}}_1,\hat{\mathbf{e}}_2,...,\hat{\mathbf{e}}_N\}$ to be the matrix $(2.6.9)_1$,repeated,

$$T=\left[T_j^k\right]=\begin{bmatrix} T_1^1 & T_2^1 & T_3^1 & \cdot & \cdot & T_N^1 \\ T_1^2 & T_2^2 & T_3^2 & & & T_N^2 \\ T_1^3 & T_2^3 & T_3^3 & & & \cdot \\ \cdot & & & \cdot & & \cdot \\ \cdot & & & & & \cdot \\ T_1^N & T_2^N & \cdot & \cdot & \cdot & T_N^N \end{bmatrix} \tag{3.6.3}$$

Also, in Section 2.6, we derived the transformation rules for the components of a vector $\mathbf{v}\in\mathscr{V}$. This calculation began with equations (2.6.26) and (2.6.27), repeated,

$$\mathbf{v} = \sum_{j=1}^{N} \upsilon^j \mathbf{e}_j = \sum_{j=1}^{N} \hat{\upsilon}^j \hat{\mathbf{e}}_j \tag{3.6.4}$$

The two sets of components $\{\upsilon^1,\upsilon^2,...,\upsilon^N\}$ and $\{\hat{\upsilon}^1,\hat{\upsilon}^2,...,\hat{\upsilon}^N\}$ are connected by the formulas (2.6.31) and (2.6.32), repeated,

$$\upsilon^k = \sum_{j=1}^{N} T_j^k \hat{\upsilon}^j \tag{3.6.5}$$

$$\hat{\upsilon}^k = \sum_{j=1}^{N} \hat{T}^k_j \upsilon^j \tag{3.6.6}$$

In this section, out task is to derive the formula which connects the components of a linear transformation $\mathbf{A}:\mathscr{V} \to \mathscr{U}$ when one changes bases in $\mathscr{V}$ *and* $\mathscr{U}$. The result, like the results (3.6.5) and (3.6.6), reveals the dependence of $\mathbf{A}:\mathscr{V} \to \mathscr{U}$ on the bases. The dependence will be displayed as a matrix equation which connects the matrix of $\mathbf{A}:\mathscr{V} \to \mathscr{U}$ to *two* transition matrices, one for $\mathscr{V}$ and one for $\mathscr{U}$. This formula, when it is derived, will allow us to define what is meant by *similar matrices*.

The derivation we are going to perform, like virtually all in a linear algebra course, simply follows by a consistent application of the definitions. We are given $\mathbf{A}:\mathscr{V} \to \mathscr{U}$, bases $\{\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_N\}$ and $\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, ..., \hat{\mathbf{e}}_N\}$ *for* $\mathscr{V}$ and bases $\{\mathbf{b}_1, \mathbf{b}_2, ..., \mathbf{b}_M\}$ and $\{\hat{\mathbf{b}}_1, \hat{\mathbf{b}}_2, ..., \hat{\mathbf{b}}_M\}$ *for* $\mathscr{U}$. The connection between the bases $\{\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_N\}$ and $\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, ..., \hat{\mathbf{e}}_N\}$ is given by (3.6.1) and (3.6.2). By exactly the same logic, the bases $\{\mathbf{b}_1, \mathbf{b}_2, ..., \mathbf{b}_M\}$ and $\{\hat{\mathbf{b}}_1, \hat{\mathbf{b}}_2, ..., \hat{\mathbf{b}}_M\}$ are given by

$$\hat{\mathbf{b}}_j = \sum_{k=1}^{M} U^k_j \mathbf{b}_k \quad \text{for } j = 1, 2, ..., M \tag{3.6.7}$$

and its inverse,

$$\mathbf{b}_k = \sum_{j=1}^{M} \hat{U}^j_k \hat{\mathbf{b}}_j \quad \text{for } k = 1, 2, ..., M \tag{3.6.8}$$

From (3.2.2), the components of $\mathbf{A}:\mathscr{V} \to \mathscr{U}$ are

$$\mathbf{A}\mathbf{e}_k = \sum_{j=1}^{M} A^j{}_k \mathbf{b}_j \qquad k = 1, 2, ..., N \tag{3.6.9}$$

when one selects the basis $\{\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_N\}$ for $\mathscr{V}$ and the basis $\{\mathbf{b}_1, \mathbf{b}_2, ..., \mathbf{b}_M\}$ for $\mathscr{U}$. However, if we select the other bases $\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, ..., \hat{\mathbf{e}}_N\}$ and $\{\hat{\mathbf{b}}_1, \hat{\mathbf{b}}_2, ..., \hat{\mathbf{b}}_M\}$, the components of $\mathbf{A}:\mathscr{V} \to \mathscr{U}$ are given by

$$\mathbf{A}\hat{\mathbf{e}}_k = \sum_{j=1}^{M} \tilde{A}^j{}_k \hat{\mathbf{b}}_j \qquad k = 1, 2, ..., N \tag{3.6.10}$$

The transformation rule we are trying to derive arises from the substitution of the equations which connect the bases for $\mathscr{V}$ and the bases for $\mathscr{U}$ into (3.6.9) and the result is written in the form of (3.6.10). The result we are going to derive is

$$\tilde{A}^j{}_k = \sum_{q=1}^{N}\sum_{s=1}^{M} \hat{U}_s^{\,j} A^s{}_q T_k^q \qquad j=1,\dots,M \quad \text{and} \quad k=1,2,\dots,N \tag{3.6.11}$$

The derivation of this result is as follows: First, substitute (3.6.1) into (3.6.10). The result of this substitution is

$$\sum_{j=1}^{M} \tilde{A}^j{}_k \hat{\mathbf{b}}_j = \mathbf{A}\hat{\mathbf{e}}_k = \sum_{q=1}^{N} T_k^q \mathbf{A}\mathbf{e}_q \qquad k=1,2,\dots,N \tag{3.6.12}$$

We next substitute equation (3.6.9) into the right side of (3.6.12) to obtain

$$\sum_{j=1}^{M} \tilde{A}^j{}_k \hat{\mathbf{b}}_j = \sum_{q=1}^{N} T_k^q \mathbf{A}\mathbf{e}_q = \sum_{q=1}^{N}\sum_{s=1}^{M} A^s{}_q T_k^q \mathbf{b}_s \qquad k=1,2,\dots,N \tag{3.6.13}$$

Finally, we use (3.6.8) to rewrite (3.6.13) as

$$\sum_{j=1}^{M} \tilde{A}^j{}_k \hat{\mathbf{b}}_j = \sum_{q=1}^{N}\sum_{s=1}^{M} A^s{}_q T_k^q \mathbf{b}_s = \sum_{j=1}^{M}\sum_{q=1}^{N}\sum_{s=1}^{M} A^s{}_q T_k^q \hat{U}_s^{\,j} \hat{\mathbf{b}}_j \qquad k=1,2,\dots,N \tag{3.6.14}$$

A rearrangement of (3.6.14) yields

$$\sum_{j=1}^{M} \left(\tilde{A}^j{}_k - \sum_{q=1}^{N}\sum_{s=1}^{M} \hat{U}_s^{\,j} A^s{}_q T_k^q \right) \hat{\mathbf{b}}_j = \qquad k=1,2,\dots,N \tag{3.6.15}$$

Because the set of vectors $\left\{ \hat{\mathbf{b}}_1, \hat{\mathbf{b}}_2, \dots, \hat{\mathbf{b}}_M \right\}$ is a basis and, thus, linearly independent, (3.6.15) yields

$$\tilde{A}^j{}_k = \sum_{q=1}^{N}\sum_{s=1}^{M} \hat{U}_s^{\,j} A^s{}_q T_k^q \qquad j=1,\dots,M \quad \text{and} \quad k=1,2,\dots,N$$

As a matrix equation, this set of equations is equivalent to

$$\tilde{A} = U^{-1} A T \tag{3.6.16}$$

If we adopt the notation introduced in equation (3.2.9), equation (3.6.16) can be written in the slightly more informative notation

$$M\left(\mathbf{A}, \hat{\mathbf{e}}_k, \hat{\mathbf{b}}_j\right) = U^{-1} M\left(\mathbf{A}, \mathbf{e}_q, \mathbf{b}_s\right) T \tag{3.6.17}$$

An important special case is for the linear transformation $\mathbf{A}:\mathscr{V}\to\mathscr{V}$, a linear transformation on $\mathscr{V}$ with values in the same vector space $\mathscr{V}$. In this case, the components of $\mathbf{A}$ are related by

$$M\left(\mathbf{A},\hat{\mathbf{e}}_k,\hat{\mathbf{e}}_j\right)=T^{-1}M\left(\mathbf{A},\mathbf{e}_q,\mathbf{e}_s\right)T \tag{3.6.18}$$

Equation (3.6.18) is an example of *similar matrices*. More formally, similar matrices are defined by

Definition: If two square matrices A and B are related by a nonsingular matrix S by the formula $B=S^{-1}AS$, then A and B are said to be *similar*.

The derivation of (3.6.18) shows that the matrix $M\left(\mathbf{A},\hat{\mathbf{e}}_k,\hat{\mathbf{e}}_j\right)$ is *similar* to the matrix $M\left(\mathbf{A},\mathbf{e}_q,\mathbf{e}_s\right)$. It is the transition matrix which plays the role of S in the definition.

The fact that the linear transformation $\mathbf{A}:\mathscr{V}\to\mathscr{V}$ represents a quantity that does not depend upon the choice of basis for $\mathscr{V}$ means we can represent it in components with respect to any basis. The choice of basis which produces the result most amenable to finding a solution to a physical problem is always a desirable choice. This choice usually results in the utilization of (3.6.18) to calculate the transition matrix from given information about $M\left(\mathbf{A},\hat{\mathbf{e}}_k,\hat{\mathbf{e}}_j\right)$ and $M\left(\mathbf{A},\mathbf{e}_q,\mathbf{e}_s\right)$. We shall see an example of this kind of calculation later in this textbook.

Similar matrices arise in a lot of applications and are worthy of a short discussion. Some of the easily established properties of similar matrices are as follows:

a) If B is similar to A, then A is similar to B. This is true because if $B=S^{-1}AS$, then $A=SBS^{-1}$
b) If B is similar to A and C is similar to B, then C is similar to A. This is true because

$$B=S^{-1}AS \qquad \text{and} \qquad C=W^{-1}BW \tag{3.6.19}$$

implies

$$C=W^{-1}S^{-1}ASW=\left(SW\right)^{-1}ASW \tag{3.6.20}$$

c) If B is similar to A, then B^T is similar to A^T. This is true because if $B=S^{-1}AS$, then $B^T=S^TA^T\left(S^T\right)^{-1}$.
d) If B is similar to A, then $\det B=\det A$. This is true because $\det B=\det\left(S^{-1}AS\right)=\det S^{-1}\det A\det S=\det\left(S^{-1}S\right)\det A=\det A$.
e) If B is similar to A, and A is nonsingular, then B is nonsingular. This follows from d) because $\det B=\det A\neq 0$.

f) If B is similar to A, and A is nonsingular, then A^{-1} is similar to B^{-1}. This follows because $B = S^{-1}AS$ implies $B^{-1} = S^{-1}A^{-1}S$ when A is nonsingular.

An example of the use of (3.6.17) is provided by the following:

Example 3.6.1: You are given a linear transformation $\mathbf{A} : \mathscr{V} \to \mathscr{V}$ whose matrix with respect to a basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is

$$M\left(\mathbf{A}, \mathbf{e}_j, \mathbf{e}_k\right) = \begin{bmatrix} 2 & 2 & 0 \\ 1 & 1 & 2 \\ 1 & 1 & 2 \end{bmatrix} \tag{3.6.21}$$

and a change of basis

$$\begin{aligned} \hat{\mathbf{e}}_1 &= \mathbf{e}_1 - \mathbf{e}_2 \\ \hat{\mathbf{e}}_2 &= -2\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 \\ \hat{\mathbf{e}}_3 &= \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 \end{aligned} \tag{3.6.22}$$

It follows from (3.6.1) that the transition matrix is given by

$$T = \begin{bmatrix} T^1{}_1 & T^1{}_2 & T^1{}_3 \\ T^2{}_1 & T^2{}_2 & T^2{}_3 \\ T^3{}_1 & T^3{}_2 & T^3{}_3 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 1 \\ -1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \tag{3.6.23}$$

The inverse of the matrix (3.6.23) is

$$\hat{T} = T^{-1} = \begin{bmatrix} 0 & -1 & 1 \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \tag{3.6.24}$$

Therefore,

$$M\left(\mathbf{A},\hat{\mathbf{e}}_r,\hat{\mathbf{e}}_s\right)=T^{-1}M\left(\mathbf{A},\mathbf{e}_j,\mathbf{e}_k\right)T=\begin{bmatrix}0 & -1 & 1\\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3}\\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3}\end{bmatrix}\begin{bmatrix}2 & 2 & 0\\ 1 & 1 & 2\\ 1 & 1 & 2\end{bmatrix}\begin{bmatrix}1 & -2 & 1\\ -1 & 1 & 1\\ 0 & 1 & 1\end{bmatrix}$$
$$=\begin{bmatrix}0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 4\end{bmatrix} \tag{3.6.25}$$

Example 3.6.2: You are given a linear transformation $\mathbf{A}:\mathscr{V}\rightarrow\mathscr{U}$ defined by

$$\mathbf{A}\mathbf{v}=\upsilon^1\left(9\mathbf{b}_1+6\mathbf{b}_2-5\mathbf{b}_3+4\mathbf{b}_4\right)+\upsilon^2\left(-\mathbf{b}_1-\mathbf{b}_2+\mathbf{b}_3\right)+\upsilon^3\left(8\mathbf{b}_1+5\mathbf{b}_2-4\mathbf{b}_3+5\mathbf{b}_4\right) \tag{3.6.26}$$

For all vectors $\mathbf{v}=\upsilon^1\mathbf{e}_1+\upsilon^2\mathbf{e}_2+\upsilon^3\mathbf{e}_3\in\mathscr{V}$. The matrix of $\mathbf{A}$ with respect to the bases $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ and $\{\mathbf{b}_1,\mathbf{b}_2,\mathbf{b}_3,\mathbf{b}_4\}$ is

$$M\left(\mathbf{A},\mathbf{e}_q,\mathbf{b}_s\right)=\begin{bmatrix}9 & -1 & 8\\ 6 & -1 & 5\\ -5 & 1 & -4\\ 4 & 0 & 5\end{bmatrix} \tag{3.6.27}$$

You are given bases $\{\hat{\mathbf{e}}_1,\hat{\mathbf{e}}_2,\hat{\mathbf{e}}_3\}$ and $\{\hat{\mathbf{b}}_1,\hat{\mathbf{b}}_2,\hat{\mathbf{b}}_3,\hat{\mathbf{b}}_4\}$ defined by

$$\begin{aligned}\hat{\mathbf{e}}_1&=\mathbf{e}_1-\mathbf{e}_2\\ \hat{\mathbf{e}}_2&=-2\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3\\ \hat{\mathbf{e}}_3&=\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3\end{aligned} \tag{3.6.28}$$

and

$$\begin{aligned}\hat{\mathbf{b}}_1&=2\mathbf{b}_1+3\mathbf{b}_2+3\mathbf{b}_3-2\mathbf{b}_4\\ \hat{\mathbf{b}}_2&=3\mathbf{b}_1-2\mathbf{b}_2+2\mathbf{b}_3+4\mathbf{b}_4\\ \hat{\mathbf{b}}_3&=-2\mathbf{b}_1+\mathbf{b}_2+3\mathbf{b}_3\\ \hat{\mathbf{b}}_4&=4\mathbf{b}_1+2\mathbf{b}_2+4\mathbf{b}_3+5\mathbf{b}_4\end{aligned} \tag{3.6.29}$$

The problem is to find the matrix $M\left(\mathbf{A},\hat{\mathbf{e}}_k,\hat{\mathbf{b}}_j\right)$. It follows from (3.6.28) that

$$T = \begin{bmatrix} T_1^1 & T_2^1 & T_3^1 \\ T_1^2 & T_2^2 & T_3^2 \\ T_1^3 & T_2^3 & T_3^3 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 1 \\ -1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \tag{3.6.30}$$

and from (3.6.29) that

$$U = \begin{bmatrix} U_1^1 & U_2^1 & U_3^1 & U_4^1 \\ U_1^2 & U_2^2 & U_3^2 & U_4^2 \\ U_1^3 & U_2^3 & U_3^3 & U_4^3 \\ U_1^4 & U_2^4 & U_3^4 & U_4^4 \end{bmatrix} = \begin{bmatrix} 2 & 3 & -2 & 4 \\ 3 & -2 & 1 & 2 \\ 3 & 2 & 3 & 4 \\ -2 & 4 & 0 & 5 \end{bmatrix} \tag{3.6.31}$$

Given (3.6.27), (3.6.30) and (3.6.31), we can use (3.6.17) and find

$$\begin{aligned} M\left(\mathbf{A},\hat{\mathbf{e}}_k,\hat{\mathbf{b}}_j\right) = U^{-1}M\left(\mathbf{A},\mathbf{e}_q,\mathbf{b}_s\right)T &= \begin{bmatrix} 2 & 3 & -2 & 4 \\ 3 & -2 & 1 & 2 \\ 3 & 2 & 3 & 4 \\ -2 & 4 & 0 & 5 \end{bmatrix}^{-1} \begin{bmatrix} 9 & -1 & 8 \\ 6 & -1 & 5 \\ -5 & 1 & -4 \\ 4 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & -2 & 1 \\ -1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \\ &= \frac{1}{286}\begin{bmatrix} 48 & -15 & 37 & -62 \\ 34 & -100 & 56 & -32 \\ -60 & -17 & 61 & 6 \\ -8 & 74 & -30 & 58 \end{bmatrix} \begin{bmatrix} 9 & -1 & 8 \\ 6 & -1 & 5 \\ -5 & 1 & -4 \\ 4 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & -2 & 1 \\ -1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \\ &= \frac{1}{286}\begin{bmatrix} -95 & 37 & -236 \\ -824 & 914 & -1192 \\ -1061 & 1205 & -1564 \\ 850 & -888 & 1374 \end{bmatrix} = \begin{bmatrix} -0.332 & 0.1294 & -0.8252 \\ -2.8811 & 3.1958 & -4.178 \\ -3.7098 & 4.2133 & -5.4685 \\ 2.9720 & -3.2049 & 4.8042 \end{bmatrix} \end{aligned} \tag{3.6.32}$$

If $\mathbf{A}$ is a linear transformation $\mathbf{A}:\mathscr{V}\rightarrow\mathscr{V}$, the transformation formula is again (3.6.18). We shall use (3.6.18) to motivate the concept of the *determinant* of $\mathbf{A}:\mathscr{V}\rightarrow\mathscr{V}$ and the trace of $\mathbf{A}:\mathscr{V}\rightarrow\mathscr{V}$. First, we shall discuss the determinant. The determinant of the square matrix $M\left(\mathbf{A},\mathbf{e}_q,\mathbf{e}_s\right)$, written $\det M\left(\mathbf{A},\mathbf{e}_q,\mathbf{e}_s\right)$, can be computed by the formulas in Section 1.10. It follows from (3.6.18) and equation (1.10.60) that

$$\begin{aligned} \det M\left(\mathbf{A},\mathbf{e}_q,\mathbf{e}_s\right) &= \left(\det T\right)\left(M\left(\mathbf{A},\hat{\mathbf{e}}_k,\hat{\mathbf{e}}_j\right)\right)\left(\det T^{-1}\right) \\ &= \det M(\mathbf{A},\hat{\mathbf{e}}_k,\hat{\mathbf{e}}_j) \end{aligned} \tag{3.6.33}$$

Thus, we obtain the important result that for a linear transformation $\mathbf{A}:\mathscr{V}\to\mathscr{V}$, $\det M(\mathbf{A},\mathbf{e}_q,\mathbf{e}_s)$ is *independent* of the choice of basis for $\mathscr{V}$. With this fact in mind, we *define* the determinant of a linear transformation $\mathbf{A}:\mathscr{V}\to\mathscr{V}$, written $\det\mathbf{A}$, by

$$\det\mathbf{A}=\det M(\mathbf{A},\mathbf{e}_k,\mathbf{e}_q) \tag{3.6.34}$$

By the above argument, we are assured that $\det\mathbf{A}$ is a property of $\mathbf{A}$ alone.

The definition of the *trace* of a linear transformation $\mathbf{A}:\mathscr{V}\to\mathscr{V}$ is motivated by the same kind of argument that was just used for the determinant. The trace of a matrix is defined by equation (1.1.6). Therefore, the trace of the square matrix $M\left(\mathbf{A},\mathbf{e}_q,\mathbf{e}_s\right)$, written $\operatorname{tr}M\left(\mathbf{A},\mathbf{e}_q,\mathbf{e}_s\right)$ is just the sum of the diagonal elements. Because of the transformation formula (3.6.18) and the special property of the trace operation, equation (1.1.26), it follows that

$$\begin{aligned}\operatorname{tr}M\left(\mathbf{A},\hat{\mathbf{e}}_k,\hat{\mathbf{e}}_j\right)&=\operatorname{tr}\left(T^{-1}M\left(\mathbf{A},\mathbf{e}_q,\mathbf{e}_s\right)T\right)\\&=\operatorname{tr}\left(TT^{-1}M\left(\mathbf{A},\mathbf{e}_q,\mathbf{e}_s\right)\right)\\&=\operatorname{tr}\left(IM\left(\mathbf{A},\mathbf{e}_q,\mathbf{e}_s\right)\right)\\&=\operatorname{tr}\left(M\left(\mathbf{A},\mathbf{e}_q,\mathbf{e}_s\right)\right)\end{aligned} \tag{3.6.35}$$

Therefore, the trace of a matrix is independent of the basis used to derive the matrix from the linear transformation. As a result, we *define* the trace of a linear transformation by the formula

$$\operatorname{tr}\mathbf{A}=\operatorname{tr}M(\mathbf{A},\mathbf{e}_k,\mathbf{e}_q) \tag{3.6.36}$$

Exercises:

3.6.1 You are given a linear transformation $\mathbf{A}:\mathscr{V}\to\mathscr{V}$ defined by

$$\mathbf{A}\mathbf{v}=\upsilon^1\left(12\mathbf{e}_1+3\sqrt{2}\mathbf{e}_2\right)+\upsilon^2\left(3\sqrt{2}\mathbf{e}_1+15\mathbf{e}_2\right)+\upsilon^3\mathbf{e}_3 \tag{3.6.37}$$

for all vectors $\mathbf{v}=\upsilon^1\mathbf{e}_1+\upsilon^2\mathbf{e}_2+\upsilon^3\mathbf{e}_3\in\mathscr{V}$ in a three dimensional vector space $\mathscr{V}$. You are given a basis $\left\{\hat{\mathbf{e}}_1,\hat{\mathbf{e}}_2,\hat{\mathbf{e}}_3\right\}$ defined by

$$\begin{aligned}\hat{\mathbf{e}}_1&=\frac{1}{\sqrt{3}}\mathbf{e}_1+\frac{\sqrt{2}}{\sqrt{3}}\mathbf{e}_2\\\hat{\mathbf{e}}_2&=-\frac{\sqrt{2}}{\sqrt{3}}\mathbf{e}_1+\frac{1}{\sqrt{3}}\mathbf{e}_2\\\hat{\mathbf{e}}_3&=\mathbf{e}_3\end{aligned} \tag{3.6.38}$$

Find the matrix $M\left(\mathbf{A},\hat{\mathbf{e}}_k,\hat{\mathbf{e}}_j\right)$. You should obtain

$$M\left(\mathbf{A},\hat{\mathbf{e}}_k,\hat{\mathbf{e}}_j\right)=\begin{bmatrix}18 & 0 & 0\\ 0 & 9 & 0\\ 0 & 0 & 1\end{bmatrix} \tag{3.6.39}$$

3.6.2 You are given a linear transformation $\mathbf{A}:\mathscr{V}\rightarrow\mathscr{U}$ defined by

$$\mathbf{A}\mathbf{v}=\upsilon^1\left(8\mathbf{b}_1-2\mathbf{b}_2\right)+\upsilon^2\left(6\mathbf{b}_1-\mathbf{b}_2\right)+\upsilon^3\left(-5\mathbf{b}_1+\mathbf{b}_2\right) \tag{3.6.40}$$

for all vectors $\mathbf{v}=\upsilon^1\mathbf{e}_1+\upsilon^2\mathbf{e}_2+\upsilon^3\mathbf{e}_3\in\mathscr{V}$. You are given bases $\{\hat{\mathbf{e}}_1,\hat{\mathbf{e}}_2,\hat{\mathbf{e}}_3\}$ and $\{\hat{\mathbf{b}}_1,\hat{\mathbf{b}}_2\}$ defined by

$$\begin{aligned}\hat{\mathbf{e}}_1&=\mathbf{e}_1+3\mathbf{e}_2-2\mathbf{e}_3\\ \hat{\mathbf{e}}_2&=3\mathbf{e}_1-2\mathbf{e}_2+\mathbf{e}_3\\ \hat{\mathbf{e}}_3&=3\mathbf{e}_1+2\mathbf{e}_2+3\mathbf{e}_3\end{aligned} \tag{3.6.41}$$

and

$$\begin{aligned}\hat{\mathbf{b}}_1&=\mathbf{b}_1+\mathbf{b}_2\\ \hat{\mathbf{b}}_2&=-2\mathbf{b}_1+\mathbf{b}_2\end{aligned} \tag{3.6.42}$$

Find the matrix $M\left(\mathbf{A},\hat{\mathbf{e}}_j,\hat{\mathbf{b}}_k\right)$. You should obtain

$$M\left(\mathbf{A},\hat{\mathbf{e}}_k,\hat{\mathbf{b}}_j\right)=\frac{1}{3}\begin{bmatrix}22 & 1 & 11\\ -43 & -10 & -26\end{bmatrix} \tag{3.6.43}$$

3.6.3 A certain linear transformation $\mathbf{A}:\mathscr{V}\rightarrow\mathscr{V}$ has the matrix

$$M\left(\mathbf{A},\mathbf{e}_j,\mathbf{e}_k\right)=\begin{bmatrix}1 & 0 & 2\\ 0 & 2 & 1\\ 0 & 1 & 2\end{bmatrix} \tag{3.6.44}$$

with respect to a basis $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$. You are given a change of basis defined by

$$\begin{aligned}\mathbf{e}_1 &= \frac{3}{2}\hat{\mathbf{e}}_1 + \frac{1}{2}\hat{\mathbf{e}}_2\\ \mathbf{e}_2 &= -\frac{1}{2}\hat{\mathbf{e}}_1 + \frac{3}{2}\hat{\mathbf{e}}_2\\ \mathbf{e}_3 &= \hat{\mathbf{e}}_3\end{aligned} \tag{3.6.45}$$

Calculate the matrix of $\mathbf{A}:\mathscr{V}\to\mathscr{V}$ with respect to the basis $\{\hat{\mathbf{e}}_1,\hat{\mathbf{e}}_2,\hat{\mathbf{e}}_3\}$. You should obtain

$$M\left(\mathbf{A},\hat{\mathbf{e}}_j,\hat{\mathbf{e}}_k\right)=\begin{bmatrix}\frac{11}{10} & -\frac{3}{10} & \frac{5}{2}\\ -\frac{3}{10} & \frac{19}{10} & \frac{5}{2}\\ -\frac{1}{5} & \frac{3}{5} & 2\end{bmatrix} \tag{3.6.46}$$

3.6.4 You are given a linear transformation $\mathbf{A}:\mathscr{V}\to\mathscr{U}$ defined by

$$\mathbf{A}\mathbf{v}=\upsilon_1\left(\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_4\right)+\upsilon_2\left(\mathbf{e}_1-3\mathbf{e}_2+2\mathbf{e}_3\right)+\upsilon_3\mathbf{e}_4+\upsilon_4\left(4\mathbf{e}_1-3\mathbf{e}_2\right) \tag{3.6.47}$$

for all vectors $\mathbf{v}=\upsilon^1\mathbf{e}_1+\upsilon^2\mathbf{e}_2+\upsilon^3\mathbf{e}_3+\upsilon^4\mathbf{e}_4\in\mathscr{V}$. You are given bases $\{\hat{\mathbf{e}}_1,\hat{\mathbf{e}}_2,\hat{\mathbf{e}}_3,\hat{\mathbf{e}}_4\}$ defined by

$$\begin{aligned}\mathbf{e}_1 &= \hat{\mathbf{e}}_2+\hat{\mathbf{e}}_3+3\hat{\mathbf{e}}_4\\ \mathbf{e}_2 &= \hat{\mathbf{e}}_1+\hat{\mathbf{e}}_2+\hat{\mathbf{e}}_3+2\hat{\mathbf{e}}_4\\ \mathbf{e}_3 &= \hat{\mathbf{e}}_3+\hat{\mathbf{e}}_4\\ \mathbf{e}_4 &= -\hat{\mathbf{e}}_1\end{aligned} \tag{3.6.48}$$

Find the matrix $M\left(\mathbf{A},\hat{\mathbf{e}}_k,\hat{\mathbf{e}}_j\right)$.

3.6.5 You are given a linear transformation $\mathbf{A}:\mathscr{V}\to\mathscr{U}$ defined by

$$\begin{aligned}\mathbf{A}\mathbf{v}=&\,\upsilon^1\left(-\mathbf{b}_2-\mathbf{b}_3+\mathbf{b}_4\right)+\upsilon^2\left(\mathbf{b}_1+\mathbf{b}_2+\mathbf{b}_3+\mathbf{b}_4\right)\\ &+\upsilon^3\left(2\mathbf{b}_1+4\mathbf{b}_2+\mathbf{b}_3-2\mathbf{b}_4\right)+\upsilon^4\left(3\mathbf{b}_1+\mathbf{b}_2-2\mathbf{b}_3+2\mathbf{b}_4\right)\end{aligned} \tag{3.6.49}$$

for all vectors $\mathbf{v}=\upsilon^1\mathbf{e}_1+\upsilon^2\mathbf{e}_2+\upsilon^3\mathbf{e}_3+\upsilon^4\mathbf{e}_4\in\mathscr{V}$. You are given bases $\{\hat{\mathbf{e}}_1,\hat{\mathbf{e}}_2,\hat{\mathbf{e}}_3,\hat{\mathbf{e}}_4\}$ and $\{\hat{\mathbf{b}}_1,\hat{\mathbf{b}}_2,\hat{\mathbf{b}}_3,\hat{\mathbf{b}}_4\}$ defined by

$$\begin{aligned}
\hat{\mathbf{e}}_1 &= \mathbf{e}_1 - \mathbf{e}_2 + \mathbf{e}_4 \\
\hat{\mathbf{e}}_2 &= -2\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 + 2\mathbf{e}_4 \\
\hat{\mathbf{e}}_3 &= \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4 \\
\hat{\mathbf{e}}_4 &= 2\mathbf{e}_1 + \mathbf{e}_2 + 3\mathbf{e}_3 + \mathbf{e}_4
\end{aligned} \tag{3.6.50}$$

and

$$\begin{aligned}
\hat{\mathbf{b}}_1 &= 2\mathbf{b}_1 + 3\mathbf{b}_2 + 3\mathbf{b}_3 - 2\mathbf{b}_4 \\
\hat{\mathbf{b}}_2 &= 3\mathbf{b}_1 - 2\mathbf{b}_2 + 2\mathbf{b}_3 + 4\mathbf{b}_4 \\
\hat{\mathbf{b}}_3 &= -2\mathbf{b}_1 + \mathbf{b}_2 + 3\mathbf{b}_3 \\
\hat{\mathbf{b}}_4 &= 4\mathbf{b}_1 + 2\mathbf{b}_2 + 4\mathbf{b}_3 + 5\mathbf{b}_4
\end{aligned} \tag{3.6.51}$$

Find the matrix $M\left(\mathbf{A}, \hat{\mathbf{e}}_k, \hat{\mathbf{b}}_j\right)$. You should obtain

$$M\left(\mathbf{A}, \hat{\mathbf{e}}_k, \hat{\mathbf{b}}_j\right) = \frac{1}{286}\begin{bmatrix} -161 & 235 & 52 & 362 \\ -120 & -626 & -416 & -828 \\ -335 & -687 & -494 & -810 \\ -146 & 652 & 468 & 750 \end{bmatrix} = \begin{bmatrix} -0.5629 & 0.8217 & 0.1818 & 1.2657 \\ -0.4196 & -2.1888 & -1.4545 & -2.8951 \\ -1.1713 & -2.4021 & -1.7273 & -2.8322 \\ 0.5105 & 2.2797 & 1.6364 & 2.6224 \end{bmatrix} \tag{3.6.52}$$

3.6.6 Given the linear transformation $\mathbf{A}: \mathscr{V} \rightarrow \mathscr{V}$ from Exercise 3.6.3, calculate the determinants $\det M\left(\mathbf{A}, \mathbf{e}_k, \mathbf{e}_j\right)$ and $\det M\left(\mathbf{A}, \hat{\mathbf{e}}_k, \hat{\mathbf{e}}_j\right)$. This calculation should illustrate the theoretical result (3.6.33) above.

Chapter 4

VECTOR SPACES WITH INNER PRODUCT

There is no concept of *length* or *magnitude* in the definition of a vector space. In this chapter, we first take a formal approach and add structure to that of a vector space so as to make it an *inner product space*. We shall then illustrate by examples how one makes many of the vector spaces we have encountered thus far into inner product spaces. From this discussion, we will be led naturally into ideas of *length* and *angle* that one can associate with inner product spaces. This discussion will add a familiar geometric vocabulary to that we have been using for vector spaces.

In the application of mathematics to solve problems, it is often the case that one cannot obtain an exact solution. One has to settle for an approximate solution of some type. The ability to speak sensibly about an approximation requires ideas of distance and closely related ideas of length and angle. The way one judges the validity of an approximation is to specify that it is close in some sense to the exact solution. The ability of assign a mathematical meaning to “close” depends upon the structure of the mathematical model used to model the problem being solved. For the vast majority of problems one encounters, the underlying mathematical model establishes relationships between elements of vector spaces. If these elements have the additional structure which allows one to sensibly talk about distance, then the door is open to discussions of approximate solutions and their closeness to exact solutions. It is the structure of an inner product space which will provide us the way to talk about concepts such as distance, length and angle. These are the concepts that allow one to discuss the idea of an approximate solution to a problem being close to its exact solution.

Section 4.1 Definition of an Inner Product Space

In all of our discussions in Chapters 1, 2 and 3, we have allowed the scalar field to be the set of complex numbers $\mathscr{C}$. Most of our examples have utilized the special case of real numbers $\mathscr{R}$. In this chapter we shall discuss a so called *eigenvalue* problem that arises in many areas of the applications. As we shall see, a discussion of this problems is best done in the context where the scalar field is allowed to be $\mathscr{C}$. It is for this reason, we shall continue to allow complex numbers for the scalar field. Thus, we shall first define an inner product in this more general case. We will often illustrate the concepts with examples where the scalar field is the set of real numbers.

It is useful to briefly remind ourselves of certain arithmetic manipulations with complex numbers. You will recall that complex numbers always have the representation

$$\lambda = a + ib \tag{4.1.1}$$

where a and b are real numbers and the symbol i obeys $i^2 = -1$. The *complex conjugate* of a complex number λ is the complex number

$$\bar{\lambda} = a - ib \tag{4.1.2}$$

It follows from (4.1.1) and (4.1.2) that

$$\lambda + \bar{\lambda} = a + ib + a - ib = 2a = 2\operatorname{Re}(\lambda) \tag{4.1.3}$$

where $\operatorname{Re}(\lambda)$ means *the real part* of the complex number λ. Also, if $\lambda = a + ib$, then

$$\bar{\lambda}\lambda = \lambda\bar{\lambda} = (a+ib)(a-ib) = a^2 + iab - iab - i^2 b^2 = a^2 + b^2 \tag{4.1.4}$$

is a real number. If one refers to the absolute value of a complex number it is a real number denoted by $|\lambda|$ and it is defined by

$$|\lambda| = \sqrt{\lambda\bar{\lambda}} = \sqrt{a^2 + b^2} \tag{4.1.5}$$

When dealing with complex numbers, it is convenient to useful to have a formal condition which reflects the special case when a complex number is, in fact, real. This condition is the equation $\lambda = \bar{\lambda}$. In order to see that this condition does imply that λ is real simply form the equality $\lambda = \bar{\lambda}$. The result is

$$\lambda = a + ib = \bar{\lambda} = a - ib \tag{4.1.6}$$

Equation (4.1.6) implies that $b = 0$, and thus the complex number λ is real.

If you are given a vector space with its long list of properties summarized in Section 2.1, an inner product space is a vector space with additional properties that we shall now summarize.

Definition: An *inner product* on a complex vector space $\mathscr{V}$ is a function $f : \mathscr{V} \times \mathscr{V} \rightarrow \mathscr{C}$ with the following properties:

(1) $f(\mathbf{u}, \mathbf{v}) = \overline{f(\mathbf{v}, \mathbf{u})}$

(2) $\lambda f(\mathbf{u}, \mathbf{v}) = f(\lambda\mathbf{u}, \mathbf{v})$

(3) $f(\mathbf{u} + \mathbf{w}, \mathbf{v}) = f(\mathbf{u}, \mathbf{v}) + f(\mathbf{w}, \mathbf{v})$

(4) $f(\mathbf{u}, \mathbf{u}) \geq 0$ and $f(\mathbf{u}, \mathbf{u}) = 0$ if and only if $\mathbf{u} = \mathbf{0}$

for all $\mathbf{u},\mathbf{v},\mathbf{w}\in\mathscr{V}$ and $\lambda\in\mathscr{C}$. In Property 1 the bar denotes the complex conjugate. Properties 2 and 3 require that f be *linear* in its first argument; i.e.,

$$f\left(\lambda\mathbf{u}+\mu\mathbf{v},\mathbf{w}\right)=\lambda f\left(\mathbf{u},\mathbf{w}\right)+\mu f\left(\mathbf{v},\mathbf{w}\right) \tag{4.1.7}$$

for all $\mathbf{u},\mathbf{v},\mathbf{w}\in\mathscr{V}$ and all $\lambda,\mu\in\mathscr{C}$. Property 1 and the linearity implied by Properties 2 and 3 insure that f is *conjugate linear* in its second argument; i.e.,

$$f\left(\mathbf{u},\lambda\mathbf{v}+\mu\mathbf{w}\right)=\bar{\lambda} f\left(\mathbf{u},\mathbf{v}\right)+\bar{\mu} f\left(\mathbf{u},\mathbf{w}\right) \tag{4.1.8}$$

for all $\mathbf{u},\mathbf{v},\mathbf{w}\in\mathscr{V}$ and all $\lambda,\mu\in\mathscr{C}$. Note that Property 1 shows that $f\left(\mathbf{u},\mathbf{u}\right)$ is real. Property 4 requires that f be *positive definite*.

There are many notations for the inner product. In cases where the vector space is a real one, the notation of the "dot product" is used. In this case, one would write the function $f:\mathscr{V}\times\mathscr{V}\rightarrow\mathscr{R}$ as

$$f\left(\mathbf{u},\mathbf{v}\right)=\mathbf{u}\cdot\mathbf{v} \tag{4.1.9}$$

In cases where the vector space is a complex one, it is useful to adopt the notation

$$f\left(\mathbf{u},\mathbf{v}\right)=\left\langle\mathbf{u},\mathbf{v}\right\rangle \tag{4.1.10}$$

for the function $f:\mathscr{V}\times\mathscr{V}\rightarrow\mathscr{C}$. In this work, we shall adopt the notation (4.1.10).

An *inner product space* is simply a vector space with an inner product. To emphasize the importance of this idea and to focus simultaneously all its details, we restate the definition as follows.

Definition. A *complex inner product space*, or simply an *inner product space*, is a set $\mathscr{V}$ and a field $\mathscr{C}$ such that:

(a) There exists a binary operation in $\mathscr{V}$ called addition and denoted by $+$ such that:

(1) $\left(\mathbf{u}+\mathbf{v}\right)+\mathbf{w}=\mathbf{u}+\left(\mathbf{v}+\mathbf{w}\right)$ for all $\mathbf{u},\mathbf{v},\mathbf{w}\in\mathscr{V}$

(2) $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$ for all $\mathbf{u},\mathbf{v}\in\mathscr{V}$

(3) There exists an element $\mathbf{0}\in\mathscr{V}$ such that $\mathbf{u}+\mathbf{0}=\mathbf{u}$ for all $\mathbf{u}\in\mathscr{V}$

(4) For every $\mathbf{u}\in\mathscr{V}$ there exists an element $-\mathbf{u}\in\mathscr{V}$ such that $\mathbf{u}+\left(-\mathbf{u}\right)=\mathbf{0}$

(b) There exists an operation called *scalar multiplication* in which every scalar $\lambda\in\mathscr{C}$ can be combined with every element $\mathbf{u}\in\mathscr{V}$ to give an element $\lambda\mathbf{u}\in\mathscr{V}$ such that:

(l) $\lambda\left(\mu\mathbf{u}\right)=\left(\lambda\mu\right)\mathbf{u}$

(2) $(\lambda+\mu)\mathbf{u}=\lambda\mathbf{u}+\mu\mathbf{u}$

(3) $\lambda(\mathbf{u}+\mathbf{v})=\lambda\mathbf{u}+\lambda\mathbf{v}$

(4) $1\mathbf{u}=\mathbf{u}$

for all $\lambda,\mu,\in\mathscr{C}$ and all $\mathbf{u},\mathbf{v}\in\mathscr{V}$

(c) There exists an operation called *inner product* by which any ordered pair of vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathscr{V}$ determines an element of $\mathscr{C}$ denoted by $\langle\mathbf{u},\mathbf{v}\rangle$ such that

(1) $\langle\mathbf{u},\mathbf{v}\rangle=\overline{\langle\mathbf{v},\mathbf{u}\rangle}$

(2) $\lambda\langle\mathbf{u},\mathbf{v}\rangle=\langle\lambda\mathbf{u},\mathbf{v}\rangle$

(3) $\langle\mathbf{u}+\mathbf{w},\mathbf{v}\rangle=\langle\mathbf{u},\mathbf{v}\rangle+\langle\mathbf{w},\mathbf{v}\rangle$

(4) $\langle\mathbf{u},\mathbf{u}\rangle\geq\mathbf{0}$ and $\langle\mathbf{u},\mathbf{u}\rangle=\mathbf{0}$ if and only if $\mathbf{u}=0$

for all $\mathbf{u},\mathbf{v},\mathbf{w}\in\mathscr{V}$ and $\lambda\in\mathscr{C}$

A *real* inner product space is defined similarly. One simple formally omits the appearance of complex conjugates in the definitions.

Example 4.1.1: The vector space $\mathscr{R}^N$ becomes an inner product space (a real inner product space) if, for any two vectors $\mathbf{u},\mathbf{v}\in\mathscr{R}^N$,where $\mathbf{u}=(u_1,u_2,...,u_N)$ and $\mathbf{v}=(\upsilon_1,\upsilon_2,...,\upsilon_N)$, we define the inner product of $\mathbf{u}$ and $\mathbf{v}$ by

$$\langle\mathbf{u},\mathbf{v}\rangle=\sum_{j=1}^{N}u_j\upsilon_j \tag{4.1.11}$$

Example 4.1.2: The vector space $\mathscr{C}^N$ becomes an inner product space if, for any two vectors $\mathbf{u},\mathbf{v}\in\mathscr{C}^N$. where $\mathbf{u}=(u_1,u_2,...,u_N)$ and $\mathbf{v}=(\upsilon_1,\upsilon_2,...,\upsilon_N)$, we define the inner product of $\mathbf{u}$ and $\mathbf{v}$ by

$$\langle\mathbf{u},\mathbf{v}\rangle=\sum_{j=1}^{N}u_j\bar{\upsilon}_j \tag{4.1.12}$$

Example 4.1.3: The vector space $\mathscr{M}^{N\times 1}$ becomes an inner product space if, for any two column vectors $\mathbf{u},\mathbf{v}\in\mathscr{M}^{N\times 1}$, we define the inner product of $\mathbf{u}$ and $\mathbf{v}$ by

$$\langle \mathbf{u},\mathbf{v}\rangle = \begin{bmatrix} u_1 & u_2 & u_3 & \cdot & \cdot & u_N \end{bmatrix} \begin{bmatrix} \bar{\upsilon}_1 \\ \bar{\upsilon}_2 \\ \bar{\upsilon}_3 \\ \cdot \\ \cdot \\ \bar{\upsilon}_N \end{bmatrix} = \sum_{j=1}^{N} u_j \bar{\upsilon}_j = \mathbf{u}^T \bar{\mathbf{v}} = \bar{\mathbf{v}}^T \mathbf{u} \tag{4.1.13}$$

The next example shows how one assigns an inner product to the vector space of $M \times N$ matrices, $\mathscr{M}^{M\times N}$. Before we state this definition, recall that in Section 1.2 we defined the *trace* of a square matrix. In particular, if A is a square matrix, say a matrix in $\mathscr{M}^{N\times N}$, the *trace* of A, written $\operatorname{tr} A$, is defined by equation (1.1.6), repeated,

$$\operatorname{tr} A = A_{11} + A_{22} + A_{33} + \cdots + A_{NN} = \sum_{k=1}^{N} A_{kk} \tag{4.1.14}$$

If $A \in \mathscr{M}^{M\times N}$ and $B \in \mathscr{M}^{N\times M}$, then the rule for multiplying matrices and the definition (4.1.14) of the trace yield

$$\operatorname{tr}(AB) = \sum_{j=1}^{M}\sum_{k=1}^{N} A_{jk} B_{kj} = \operatorname{tr}(BA) \tag{4.1.15}$$

Equation (4.1.15) is equation (1.1.26) repeated.

Example 4.1.4: The vector space $\mathscr{M}^{M\times N}$ becomes an inner product space if, for any two $M \times N$ matrices $A, B \in \mathscr{M}^{M\times N}$, we define the inner product of A and B by

$$\begin{aligned}\langle A,B\rangle = \operatorname{tr}\left(A\bar{B}^T\right) &= \operatorname{tr}\left(\begin{bmatrix} A_{11} & A_{12} & A_{13} & \cdot & \cdot & A_{1N} \\ A_{21} & A_{22} & A_{23} & & & A_{2N} \\ A_{31} & A_{32} & A_{33} & & & A_{3N} \\ \cdot & & & \cdot & & \cdot \\ \cdot & & & & \cdot & \cdot \\ A_{M1} & \cdot & \cdot & \cdot & \cdot & A_{MN} \end{bmatrix} \begin{bmatrix} \bar{B}_{11} & \bar{B}_{21} & \bar{B}_{31} & \cdot & \cdot & \bar{B}_{M1} \\ \bar{B}_{12} & \bar{B}_{22} & \bar{B}_{32} & & & \bar{B}_{M2} \\ \bar{B}_{13} & \bar{B}_{23} & \bar{B}_{33} & & & \bar{B}_{M3} \\ \cdot & & & \cdot & & \cdot \\ \cdot & & & & \cdot & \cdot \\ \bar{B}_{1N} & \cdot & \cdot & \cdot & \cdot & \bar{B}_{MN} \end{bmatrix} \right) \\ &= \sum_{j=1}^{M}\sum_{k=1}^{N} A_{jk}\bar{B}_{jk}\end{aligned} \tag{4.1.16}$$

Example 4.1.5: Recall in Example 2.5.5 we introduced the symbol $\mathscr{P}_\infty$ for the infinite dimensional real vector space of all polynomials. If we view these functions as defined over an interval $[a,b]$ and if $w \in C[a,b]$ is a positive valued continuous function on $[a,b]$, the real vector space $\mathscr{P}_\infty$

becomes a real inner produce space if for any two polynomials $p,q \in \mathscr{P}_\infty$, we define the inner produce of p and q by

$$\langle p,q\rangle = \int_a^b w(x)\,p(x)\,q(x)\,dx \tag{4.1.17}$$

The positive valued function $w \in C[a,b]$ is called a *weighting function*.

Example 4.1.6: Next consider the real vector space of polynomial of degree less than or equal to N. In Section 2.1, we gave this vector space the symbol $\mathscr{P}_N$. In Example 2.5.4 we showed that this vector space has dimension $\dim \mathscr{P}_N = N+1$. An inner product of two polynomials in $\mathscr{P}_N$ is defined as follows. If we are given K, where $K > N+1$, *distinct* values of x which we shall denote by $x_1, x_2, ..., x_K$, the inner product in $\mathscr{P}_N$ is defined by

$$\langle p,q\rangle = \sum_{\alpha=1}^{K} p(x_\alpha)\,q(x_\alpha) \tag{4.1.18}$$

Example 4.1.7: The vector space $C[a,b]$ becomes an inner produce space (a real inner product space) if for any two functions $f,g \in C[a,b]$, we define the inner product of f and g by

$$\langle f,g\rangle = \int_a^b f(x)g(x)dx \tag{4.1.19}$$

The actual proof that the inner product spaces introduced in the above seven examples requires that each definition be shown to obey the rules (c) above.

The structure of an inner product space makes it possible to introduce the idea of *length* or *norm* of a vector.

Definition: Given an inner product space $\mathscr{V}$, the *length* or *norm* of a vector is an operation, denoted by $\|\ \|$, that assigns to each nonzero vector $\mathbf{v} \in \mathscr{V}$ a positive real number defined by:

$$\|\mathbf{v}\| = \sqrt{\langle \mathbf{v},\mathbf{v}\rangle} \tag{4.1.20}$$

Of course, it follows from rule (c)(4) above that the length of the zero vector is zero. The definition represented by equation (4.1.20) is for an inner product space of N dimensions. It generalizes the concept of “length” or “magnitude” from elementary Euclidean plane geometry to *N*-dimensional spaces.

Example 4.1.8: For the vector space $\mathscr{R}^N$, the length of a vector $\mathbf{v} \in \mathscr{R}^N$ is

$$\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle} = \left(\sum_{j=1}^{N} \upsilon_j^2 \right)^{1/2} \tag{4.1.21}$$

Example 4.1.9: For the vector space $\mathscr{C}^N$, it follows from (4.1.13) and (4.1.20) that the length of a vector $\mathbf{v} \in \mathscr{C}^N$ is

$$\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle} = \left(\sum_{j=1}^{N} \bar{\upsilon}_j \upsilon_j \right)^{1/2} = \left(\sum_{j=1}^{N} |\upsilon_j|^2 \right)^{1/2} \tag{4.1.22}$$

Example 4.1.10: For the vector space $\mathscr{M}^{N\times 1}$, the length of a column vector $\mathbf{v} \in \mathscr{M}^{N\times 1}$ is

$$\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle} = \sqrt{\mathbf{v}^T \bar{\mathbf{v}}} = \left(\sum_{j=1}^{N} \upsilon_j \bar{\upsilon}_j \right)^{1/2} = \left(\sum_{j=1}^{N} |\upsilon_j|^2 \right)^{1/2} \tag{4.1.23}$$

Example 4.1.11: For the vector space $\mathscr{M}^{M\times N}$, the length of the matrix $A \in \mathscr{M}^{M\times N}$ is

$$\|A\| = \sqrt{\langle A, A \rangle} = \sqrt{\operatorname{tr}\left(A\bar{A}^T \right)} = \left(\sum_{j=1}^{M} \sum_{k=1}^{N} A_{jk} \bar{A}_{jk} \right)^{1/2} = \left(\sum_{j=1}^{M} \sum_{k=1}^{N} |A_{jk}|^2 \right)^{1/2} \tag{4.1.24}$$

Example 4.1.12: For the vector space $\mathscr{P}_\infty$, the length of the polynomial $p \in \mathscr{P}_\infty$ is

$$\|p\| = \sqrt{\langle p, p \rangle} = \left(\int_a^b w(x)\, p^2(x)\, dx \right)^{1/2} \tag{4.1.25}$$

Example 4.1.13: For the vector space $\mathscr{P}_N$, the length of the polynomial $p \in \mathscr{P}_N$ is

$$\|p\| = \sqrt{\langle p, p \rangle} = \left(\sum_{\alpha=1}^{K} p^2(x_\alpha) \right)^{1/2} \tag{4.1.26}$$

Example 4.1.14: For the vector space $C[a,b]$, the length of the function $f \in C[a,b]$ is

$$\| f \| = \sqrt{\langle f, f \rangle} = \left(\int_a^b \left(f(x) \right)^2 dx \right)^{1/2} \tag{4.1.27}$$

Exercises

4.1.1 In Example 4.1.3 an inner product was defined on the vector space $\mathscr{M}^{N\times 1}$. It is possible for the same vector space to have more than one inner product. Define an inner product different than the one in equation (4.1.13) by the rule

$$\begin{aligned}\langle \mathbf{u},\mathbf{v}\rangle &= \begin{bmatrix} u_1 & u_2 & u_3 & \cdot & \cdot & u_N \end{bmatrix} \begin{bmatrix} g_1 & 0 & 0 & \cdot & \cdot & 0 \\ 0 & g_2 & 0 & \cdot & \cdot & 0 \\ 0 & 0 & g_3 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & g_N \end{bmatrix} \begin{bmatrix} \bar{\upsilon}_1 \\ \bar{\upsilon}_2 \\ \bar{\upsilon}_3 \\ \cdot \\ \cdot \\ \bar{\upsilon}_N \end{bmatrix} \\ &= \sum_{j=1}^{N} g_j u_j \bar{\upsilon}_j \end{aligned} \tag{4.1.28}$$

where $g_j > 0$ for $j = 1, 2, ..., N$. Show that the definition (4.1.28) makes $\mathscr{M}^{N\times 1}$ an inner product space.

Section 4.2 Schwarz Inequality and Triangle Inequality

As we build upon the properties of an inner product space, there are two fundamental inequalities that allow us to associate, with inner product spaces in general, certain concepts familiar from elementary geometry. The first is an inequality called as the *Schwarz Inequality* and the second one is called the *Triangle Inequality*.

Theorem 4.2.1: The *Schwarz inequality*

$$\left|\left\langle \mathbf{u},\mathbf{v}\right\rangle\right| \le \|\mathbf{u}\|\|\mathbf{v}\| \tag{4.2.1}$$

is valid for any two vectors $\mathbf{u},\mathbf{v}$ in an inner product space.

Proof: The Schwarz inequality is easily seen to be trivially true when either $\mathbf{u}$ or $\mathbf{v}$ is the $\mathbf{0}$ vector, so we shall assume that neither $\mathbf{u}$ nor $\mathbf{v}$ is zero. Next, consider the vector $\left\langle \mathbf{u},\mathbf{u}\right\rangle \mathbf{v}-\left\langle \mathbf{v},\mathbf{u}\right\rangle \mathbf{u}$ and employ Property (c4), which requires that every vector have a nonnegative length, hence

$$\left\|\left\langle \mathbf{u},\mathbf{u}\right\rangle \mathbf{v}-\left\langle \mathbf{v},\mathbf{u}\right\rangle \mathbf{u}\right\|^2 = \left\langle \left\langle \mathbf{u},\mathbf{u}\right\rangle \mathbf{v}-\left\langle \mathbf{v},\mathbf{u}\right\rangle \mathbf{u}, \left\langle \mathbf{u},\mathbf{u}\right\rangle \mathbf{v}-\left\langle \mathbf{v},\mathbf{u}\right\rangle \mathbf{u}\right\rangle \ge 0 \tag{4.2.2}$$

We shall algebraically rearrange this equation and show that

$$\left\|\left\langle \mathbf{u},\mathbf{u}\right\rangle \mathbf{v}-\left\langle \mathbf{v},\mathbf{u}\right\rangle \mathbf{u}\right\|^2 = \left(\|\mathbf{u}\|^2\|\mathbf{v}\|^2-\left\langle \mathbf{u},\mathbf{v}\right\rangle\overline{\left\langle \mathbf{u},\mathbf{v}\right\rangle}\right)\|\mathbf{u}\|^2 \ge 0 \tag{4.2.3}$$

The algebra necessary to derive this result is

$$\begin{aligned}
&\left\|\left\langle \mathbf{u},\mathbf{u}\right\rangle \mathbf{v}-\left\langle \mathbf{v},\mathbf{u}\right\rangle \mathbf{u}\right\|^2 = \left\langle \left\langle \mathbf{u},\mathbf{u}\right\rangle \mathbf{v}-\left\langle \mathbf{v},\mathbf{u}\right\rangle \mathbf{u}, \left\langle \mathbf{u},\mathbf{u}\right\rangle \mathbf{v}-\left\langle \mathbf{v},\mathbf{u}\right\rangle \mathbf{u}\right\rangle \\
&= \underbrace{\left\langle \left\langle \mathbf{u},\mathbf{u}\right\rangle \mathbf{v}, \left\langle \mathbf{u},\mathbf{u}\right\rangle \mathbf{v}-\left\langle \mathbf{v},\mathbf{u}\right\rangle \mathbf{u}\right\rangle - \left\langle \left\langle \mathbf{v},\mathbf{u}\right\rangle \mathbf{u}, \left\langle \mathbf{u},\mathbf{u}\right\rangle \mathbf{v}-\left\langle \mathbf{v},\mathbf{u}\right\rangle \mathbf{u}\right\rangle}_{\text{Used linearity in the first argument of the inner product.}} \\
&= \underbrace{\left\langle \mathbf{u},\mathbf{u}\right\rangle\left\langle \mathbf{v}, \left\langle \mathbf{u},\mathbf{u}\right\rangle \mathbf{v}-\left\langle \mathbf{v},\mathbf{u}\right\rangle \mathbf{u}\right\rangle - \left\langle \mathbf{v},\mathbf{u}\right\rangle\left\langle \mathbf{u}, \left\langle \mathbf{u},\mathbf{u}\right\rangle \mathbf{v}-\left\langle \mathbf{v},\mathbf{u}\right\rangle \mathbf{u}\right\rangle}_{\text{Factor the scalars from the first argument of the inner product.}} \\
&= \underbrace{\left\langle \mathbf{u},\mathbf{u}\right\rangle\overline{\left\langle \mathbf{u},\mathbf{u}\right\rangle}\left\langle \mathbf{v},\mathbf{v}\right\rangle - \left\langle \mathbf{u},\mathbf{u}\right\rangle\overline{\left\langle \mathbf{v},\mathbf{u}\right\rangle}\left\langle \mathbf{v},\mathbf{u}\right\rangle \underbrace{- \left\langle \mathbf{v},\mathbf{u}\right\rangle\overline{\left\langle \mathbf{u},\mathbf{u}\right\rangle}\left\langle \mathbf{u},\mathbf{v}\right\rangle + \left\langle \mathbf{v},\mathbf{u}\right\rangle\overline{\left\langle \mathbf{v},\mathbf{u}\right\rangle}\left\langle \mathbf{u},\mathbf{u}\right\rangle}_{\text{These terms cancel each other.}}}_{\text{Used conjugate linearity in the second argument of the inner product}} \\
&= \|\mathbf{u}\|^4\|\mathbf{v}\|^2 - \left\langle \mathbf{v},\mathbf{u}\right\rangle\overline{\left\langle \mathbf{v},\mathbf{u}\right\rangle}\|\mathbf{u}\|^2 = \left(\|\mathbf{u}\|^2\|\mathbf{v}\|^2-\left\langle \mathbf{u},\mathbf{v}\right\rangle\overline{\left\langle \mathbf{u},\mathbf{v}\right\rangle}\right)\|\mathbf{u}\|^2 \ge 0
\end{aligned} \tag{4.2.4}$$

Returning to the proof of the theorem, since $\mathbf{u}$ must not be zero, it follows that $\|\mathbf{u}\| > 0$ and, as a result, it follows from (4.2.4) that

$$\|\mathbf{u}\|^2\|\mathbf{v}\|^2 \geq \langle \mathbf{u},\mathbf{v}\rangle\overline{\langle \mathbf{u},\mathbf{v}\rangle} = |\langle \mathbf{u},\mathbf{v}\rangle|^2 \tag{4.2.5}$$

where the equality $|\langle \mathbf{u},\mathbf{v}\rangle| = \sqrt{\langle \mathbf{u},\mathbf{v}\rangle\overline{\langle \mathbf{u},\mathbf{v}\rangle}}$ has been used. The positive square root of the last equation is Schwarz's inequality.

Theorem 4.2.2: The triangle inequality

$$\|\mathbf{u}+\mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\| \tag{4.2.6}$$

is valid for any two vectors $\mathbf{u},\mathbf{v}$ in an inner product space.

Proof: The squared length of $\mathbf{u}+\mathbf{v}$ can be written in the form

$$\begin{aligned}\|\mathbf{u}+\mathbf{v}\|^2 &= \langle \mathbf{u}+\mathbf{v},\mathbf{u}+\mathbf{v}\rangle = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + \langle \mathbf{u},\mathbf{v}\rangle + \langle \mathbf{v},\mathbf{u}\rangle \\ &= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + \langle \mathbf{u},\mathbf{v}\rangle + \overline{\langle \mathbf{u},\mathbf{v}\rangle} \\ &= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + 2\operatorname{Re}(\langle \mathbf{u},\mathbf{v}\rangle)\end{aligned} \tag{4.2.7}$$

where, as with (4.1.3), Re signifies the real part. The inequality $\operatorname{Re}(\langle \mathbf{u},\mathbf{v}\rangle) \leq |\langle \mathbf{u},\mathbf{v}\rangle|$ and (4.2.7) show that

$$\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + 2\operatorname{Re}(\langle \mathbf{u},\mathbf{v}\rangle) \leq \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + 2|\langle \mathbf{u},\mathbf{v}\rangle| \tag{4.2.8}$$

The Schwarz inequality (4.2.1) allows this inequality to be written

$$\|\mathbf{u}+\mathbf{v}\|^2 \leq \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + 2|\langle \mathbf{u},\mathbf{v}\rangle| \leq \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + 2\|\mathbf{u}\|\|\mathbf{v}\| = (\|\mathbf{u}\| + \|\mathbf{v}\|)^2 \tag{4.2.9}$$

The positive square root of (4.2.9) yields the triangle inequality (4.2.6).

For a *real* inner product space $\mathscr{V}$, the concept of *angle* is arises from the following definition:

Definition: The *angle* between two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathscr{V}$ is by θ and is defined by

$$\cos\theta = \frac{\langle \mathbf{u},\mathbf{v}\rangle}{\|\mathbf{u}\|\|\mathbf{v}\|} \tag{4.2.10}$$

This definition angle between two vectors in a real vector space is meaningful because the Schwarz inequality (4.2.1) shows that

$$-1 \le \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} \le +1 \tag{4.2.11}$$

For a real inner product space, the definition (4.2.10) can be used to write

$$\|\mathbf{u}+\mathbf{v}\|^2 = \langle \mathbf{u}+\mathbf{v}, \mathbf{u}+\mathbf{v}\rangle = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + 2\langle \mathbf{u},\mathbf{v}\rangle = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + 2\|\mathbf{u}\|\|\mathbf{v}\|\cos\theta \tag{4.2.12}$$

In elementary geometry, (4.2.12) is known as the "Law of Cosines."

As the material in this section shows, when the structure of an inner product is added to that of a vector space, we can define, in very general terms the geometric ideas of length and, in the real case, angle. This structure, for example, when applied to the vector space $\mathscr{M}^{M\times N}$, allows one to talk about the angle between two $M \times N$ matrices.

Example 4.2.1: Consider the following 2×3 matrices

$$A = \begin{bmatrix} 5 & 9 & 7 \\ 8 & 5 & 6 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 1 & 3 & 4 \\ 9 & 0 & 2 \end{bmatrix} \tag{4.2.13}$$

a) The length of A and B are

$$\begin{aligned} \|A\| = \sqrt{\operatorname{tr} AA^T} &= \left(\operatorname{tr}\left(\begin{bmatrix} 5 & 9 & 7 \\ 8 & 5 & 6 \end{bmatrix} \begin{bmatrix} 5 & 8 \\ 9 & 5 \\ 7 & 6 \end{bmatrix} \right)\right)^{1/2} \\ &= \left(\operatorname{tr}\begin{pmatrix} 155 & 127 \\ 127 & 125 \end{pmatrix} \right)^{1/2} = (155+125)^{1/2} = (280)^{1/2} = 2\sqrt{70} \end{aligned} \tag{4.2.14}$$

and

$$\begin{aligned} \|B\| = \sqrt{\operatorname{tr} BB^T} &= \left(\operatorname{tr}\left(\begin{bmatrix} 1 & 3 & 4 \\ 9 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 9 \\ 3 & 0 \\ 4 & 2 \end{bmatrix} \right)\right)^{1/2} \\ &= \left(\operatorname{tr}\left(\begin{bmatrix} 26 & 17 \\ 17 & 85 \end{bmatrix} \right)\right)^{1/2} = (26+85)^{1/2} = \sqrt{111} \end{aligned} \tag{4.2.15}$$

b) The cosine of the angle between A and B is

$$\cos\theta = \frac{\langle A,B\rangle}{\|A\|\|B\|} = \frac{\operatorname{tr}\left(AB^T\right)}{\|A\|\|B\|} = \frac{1}{2\sqrt{70(111)}}\operatorname{tr}\left(\begin{bmatrix}5 & 9 & 7\\ 8 & 5 & 6\end{bmatrix}\begin{bmatrix}1 & 9\\ 3 & 0\\ 4 & 2\end{bmatrix}\right)$$
$$= \frac{1}{2\sqrt{70(111)}}\operatorname{tr}\left(\begin{bmatrix}60 & 59\\ 47 & 84\end{bmatrix}\right) = \frac{1}{2\sqrt{70(111)}}\left(60+84\right) = \frac{72}{\sqrt{70(111)}} = .817 \tag{4.2.16}$$

Therefore, the angle between A and B is approximately 35^0.

There is another idea which is familiar from elementary geometry that has a more general counterpart for inner product spaces. It is the concept of *distance*. The formal definition is as follows:

Definition: Given an inner product space $\mathscr{V}$ the *distance* between vectors $\mathbf{u},\mathbf{v}\in\mathscr{V}$ is $\|\mathbf{u}-\mathbf{v}\|$.

In more abstract mathematical discussions, there is the idea of distance but it is not necessarily derived from the equation $\|\mathbf{u}-\mathbf{v}\|$. Such structures are called *metric spaces*. Also, one sometimes hears of *normed linear spaces*. These are vector spaces for which the idea of length is defined without the necessity of the full structure of an inner product.[1] In our discussion, these structures all collapse together when we introduce the idea of a vector space with inner product.

The inner product as well as the expressions (4.1.20) for the length of a vector can be expressed in terms of any basis and the components of the vectors relative to that basis. Let $\left\{\mathbf{e}_1,\mathbf{e}_2,...,\mathbf{e}_N\right\}$ be a basis of $\mathscr{V}$ and denote the inner product of any two base vectors by e_{jk}, i.e.,

$$e_{jk} = \left\langle \mathbf{e}_j,\mathbf{e}_k\right\rangle = \overline{\left\langle \mathbf{e}_k,\mathbf{e}_j\right\rangle} = \bar{e}_{kj} \tag{4.2.17}$$

Thus, if the vectors $\mathbf{u}$ and $\mathbf{v}$ have the representations

$$\mathbf{u} = \sum_{j=1}^{N} u^j\mathbf{e}_j, \qquad \mathbf{v} = \sum_{k=1}^{N} \upsilon^k\mathbf{e}_k \tag{4.2.18}$$

[1] An example of a norm that is not derived from an inner product is the so called Manhattan or taxicab norm defined for two vectors in a finite dimensional vector space by the sum of the absolute differences in their coordinates. A nice discussion of this norm can be found at http://en.wikipedia.org/wiki/Taxicab_geometry.

relative to the basis $\{\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_N\}$ then the inner product of $\mathbf{u}$ and $\mathbf{v}$ is given by

$$\langle \mathbf{u}, \mathbf{v} \rangle = \left\langle \sum_{j=1}^{N} u^j \mathbf{e}_j, \sum_{k=1}^{N} \upsilon^k \mathbf{e}_k \right\rangle = \sum_{j=1}^{N} \sum_{k=1}^{N} u^j \overline{\upsilon}^k \langle \mathbf{e}_j, \mathbf{e}_k \rangle = \sum_{j=1}^{N} \sum_{k=1}^{N} e_{jk} u^j \overline{\upsilon}^k \tag{4.2.19}$$

From the definition (4.1.20) for the length of a vector $\mathbf{v}$, and from (4.2.17) and $(4.2.18)_2$, we can write

$$\|\mathbf{v}\| = \left(\sum_{j=1}^{N} \sum_{k=1}^{N} e_{jk} \upsilon^j \overline{\upsilon}^k \right)^{1/2} \tag{4.2.20}$$

Equations (4.2.19) and (4.2.20) give the component representations of the inner product and of the length for the basis $\{\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_N\}$.

Exercises

4.2.1 Show that the length or norm $\| \ \|$ induced by an inner product according to the definition (4.1.20) must satisfy the following *parallelogram law*;

$$\|\mathbf{u} + \mathbf{v}\|^2 + \|\mathbf{u} - \mathbf{v}\|^2 = 2\|\mathbf{v}\|^2 + 2\|\mathbf{u}\|^2 \tag{4.2.21}$$

for all vectors $\mathbf{u}$ and $\mathbf{v}$.

4.2.2 Show that in an inner product space

$$2\langle \mathbf{u}, \mathbf{v} \rangle + 2\langle \mathbf{v}, \mathbf{u} \rangle = \|\mathbf{u} + \mathbf{v}\|^2 - \|\mathbf{u} - \mathbf{v}\|^2 \tag{4.2.22}$$

4.2.3 Show that for a complex inner product space

$$2\langle \mathbf{u}, \mathbf{v} \rangle - 2\langle \mathbf{v}, \mathbf{u} \rangle = i\|\mathbf{u} + i\mathbf{v}\|^2 - i\|\mathbf{u} - i\mathbf{v}\|^2 \tag{4.2.23}$$

4.2.4 Use (4.2.22) and (4.2.23) and show that

$$2\langle \mathbf{u}, \mathbf{v} \rangle = \|\mathbf{u} + \mathbf{v}\|^2 + i\|\mathbf{u} + i\mathbf{v}\|^2 - (1+i)\|\mathbf{u}\|^2 - (1+i)\|\mathbf{v}\|^2 \tag{4.2.24}$$

which expresses the inner product of two vectors in terms of the length or norm. This formula is known as the *polar identity*.[2]

4.2.5 Show that for *real* vector spaces the Schwarz and triangle inequalities become equalities if and only if the vectors concerned are linearly dependent.

4.2.6 Show that $\|\mathbf{u}-\mathbf{v}\| \geq \|\mathbf{u}\| - \|\mathbf{v}\|$ for all $\mathbf{u}, \mathbf{v}$ in an inner product space $\mathscr{V}$. Hint: Write $\mathbf{u} = \mathbf{v} + \mathbf{u}$ - $\mathbf{v}$ and apply the triangle inequality.

4.2.7 Prove that the $N \times N$ matrix $\left[e_{jk} \right]$ defined by (4.2.17) is nonsingular. Hint: Utilize the fact that $\left\{ \mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_N \right\}$ is a basis of $\mathscr{V}$ and prove that the rank of $\left[e_{jk} \right]$ is N.

4.2.8 You are given a real inner product space $\mathscr{V}$ of dimension 3. You are also given two vectors $\mathbf{v}$ and $\mathbf{u}$ in $\mathscr{V}$ that have the representations

$$\begin{aligned} \mathbf{v} &= 2\hat{\mathbf{e}}_1 + 2\hat{\mathbf{e}}_2 - 4\hat{\mathbf{e}}_3 \\ \mathbf{u} &= -\hat{\mathbf{e}}_1 - 5\hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3 \end{aligned} \tag{4.2.25}$$

with respect to a basis $\left\{ \hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3 \right\}$. You are also given a basis $\left\{ \mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3 \right\}$ which is related to the basis $\left\{ \hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3 \right\}$ by the equations

$$\begin{aligned} \hat{\mathbf{e}}_1 &= \mathbf{i}_1 + \mathbf{i}_2 + \mathbf{i}_3 \\ \hat{\mathbf{e}}_2 &= \mathbf{i}_1 - \mathbf{i}_2 - \mathbf{i}_3 \\ \hat{\mathbf{e}}_3 &= 2\mathbf{i}_1 + \mathbf{i}_2 - \mathbf{i}_3 \end{aligned} \tag{4.2.26}$$

where the basis $\left\{ \mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3 \right\}$ has the special property

$$\left\langle \mathbf{i}_j, \mathbf{i}_k \right\rangle = \delta_{jk} \tag{4.2.27}$$

Calculate $\cos\theta$, where θ is the angle between the vector $\mathbf{v}$ and the vector $\mathbf{u}$. The answer you should obtain is $\dfrac{2}{5\sqrt{6}}$

[2] In the case of a real vector space, equation (4.2.22) reduces to

$$\left\langle \mathbf{u}, \mathbf{v} \right\rangle = \frac{1}{4}\left(\|\mathbf{u}+\mathbf{v}\|^2 - \|\mathbf{u}-\mathbf{v}\|^2 \right)$$

This result is the polar identity for real inner product spaces. This result and the similar result for a complex inner product space, equation (4.2.24) show that given an inner product space, there is a norm function defined and conversely. However, there are other ways to define a norm on a vector space that is not derived from the inner product structure, i.e., from the definition (4.1.20).

4.2.9 You given a real inner product space $\mathscr{V}$ of dimension 3. You are also given vectors $\mathbf{v}$ and $\mathbf{u}$ in $\mathscr{V}$ that have the representations

$$\begin{aligned}\mathbf{v} &= 2\mathbf{e}_1 + \mathbf{e}_2 - 4\mathbf{e}_3 \\ \mathbf{u} &= \mathbf{e}_1 - 5\mathbf{e}_2 + \mathbf{e}_3\end{aligned} \tag{4.2.28}$$

with respect to a basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. You are also given the six inner products

$$\begin{aligned}&\langle \mathbf{e}_1, \mathbf{e}_1 \rangle = 3, \langle \mathbf{e}_1, \mathbf{e}_2 \rangle = -1, \langle \mathbf{e}_1, \mathbf{e}_3 \rangle = 2 \\ &\langle \mathbf{e}_2, \mathbf{e}_2 \rangle = 3, \langle \mathbf{e}_2, \mathbf{e}_3 \rangle = 2, \langle \mathbf{e}_3, \mathbf{e}_3 \rangle = 6\end{aligned} \tag{4.2.29}$$

Calculate $\cos\theta$, where θ is the angle between the vector $\mathbf{v}$ and the vector $\mathbf{u}$. The answer you should obtain is 0.2064

4.2.10 Consider the inner product space $\mathscr{M}^{3\times 1}$ as defined in Example 4.1.2. You are given a basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ defined by

$$\mathbf{e}_1 = \begin{bmatrix} -\dfrac{4i}{5} \\ \dfrac{3}{5} \\ 0 \end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix} -\dfrac{7i}{3} \\ \dfrac{7}{3} \\ 0 \end{bmatrix}, \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{4.2.30}$$

Calculate the six quantities $e_{jk} = \langle \mathbf{e}_j, \mathbf{e}_k \rangle$ for $j,k = 1,2,3$ and the length of the vector

$$\mathbf{v} = 2i\mathbf{e}_1 + 2\mathbf{e}_2 + 5i\mathbf{e}_3 \tag{4.2.31}$$

4.2.11 A three dimensional inner product space $\mathscr{V}$ has a basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. A vector $\mathbf{v} \in \mathscr{V}$ has the component representation

$$\mathbf{v} = 2\mathbf{e}_1 + 4i\mathbf{e}_2 + \mathbf{e}_3 \tag{4.2.32}$$

You are also given a change of basis to a new basis for $\mathscr{V}$, $\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3\}$, defined by

$$\begin{aligned}\hat{\mathbf{e}}_1 &= 3i\mathbf{e}_2 + \mathbf{e}_3 \\ \hat{\mathbf{e}}_2 &= \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 \\ \hat{\mathbf{e}}_3 &= i\mathbf{e}_3\end{aligned} \tag{4.2.33}$$

Determine the components of the vector $\mathbf{v} \in \mathscr{V}$ with respect to the basis $\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3\}$ of $\mathscr{V}$. If you are given the numbers for the inner products $e_{jk} = \langle \mathbf{e}_j, \mathbf{e}_k \rangle$ as follows:

$$\left[e_{jk} \right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4.2.34}$$

and a vector

$$\mathbf{u} = 2i\mathbf{e}_1 + 6\mathbf{e}_2 + i\mathbf{e}_3 \tag{4.2.35}$$

Determine the distance between $\mathbf{u}$ and $\mathbf{v}$.

4.2.12 You are given a complex inner product space $\mathscr{V}$ of dimension 3. $\mathbf{v}$ and $\mathbf{u}$ are two vectors in $\mathscr{V}$ that have the representations

$$\begin{aligned} \mathbf{v} &= 2i\hat{\mathbf{e}}_1 + 2\hat{\mathbf{e}}_2 - 3\hat{\mathbf{e}}_3 \\ \mathbf{u} &= -\hat{\mathbf{e}}_1 - 5\hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3 \end{aligned} \tag{4.2.36}$$

with respect to a basis $\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3\}$. You are also given a basis $\{\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3\}$ which is related to the basis $\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3\}$ by the equations

$$\begin{aligned} \hat{\mathbf{e}}_1 &= \mathbf{i}_1 + i\mathbf{i}_2 + \mathbf{i}_3 \\ \hat{\mathbf{e}}_2 &= \mathbf{i}_1 - \mathbf{i}_2 - \mathbf{i}_3 \\ \hat{\mathbf{e}}_3 &= 2\mathbf{i}_1 + \mathbf{i}_2 - \mathbf{i}_3 \end{aligned} \tag{4.2.37}$$

where the basis $\{\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3\}$ has the special property

$$\langle \mathbf{i}_j, \mathbf{i}_k \rangle = \delta_{jk} \tag{4.2.38}$$

Calculate the distance between the vector $\mathbf{v}$ and the vector $\mathbf{u}$. The answer is $\sqrt{182}$

4.2.13 Equation (4.2.19) arose from the choice of a basis $\{\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_N\}$ for the inner product space $\mathscr{V}$. Because the choice of basis was arbitrary, the resulting component expression is independent of the choice $\{\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_N\}$. However, it is instructive to start with (4.2.19) and show that its right hand side is unchanged when there is a basis transformation $\{\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_N\} \rightarrow \{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, ..., \hat{\mathbf{e}}_N\}$.

Section 4.3. Orthogonal Vectors and Orthonormal Bases

The structure of an inner product space allows for a meaning to be assigned to the idea that a pair of vectors to be orthogonal or perpendicular. Motivated by the definition (4.2.10) for real inner product spaces, for inner product spaces in general, two vectors $\mathbf{u},\mathbf{v}\in\mathscr{V}$ are *orthogonal* or, equivalently, *perpendicular*, if

$$\left\langle \mathbf{u},\mathbf{v}\right\rangle = 0 \tag{4.3.1}$$

A *set* of vectors in an inner product space $\mathscr{V}$ is said to be an *orthogonal set* if all the vectors in the set are mutually orthogonal.

Theorem 4.3.1: If $\left\{\mathbf{v}_1,\mathbf{v}_2,...,\mathbf{v}_N\right\}$ is an orthogonal set, then the set of vectors $\left\{\mathbf{v}_1,\mathbf{v}_2,...,\mathbf{v}_N\right\}$ is linearly independent.

Proof: As with virtually all proofs of linear independence, we first form the sum

$$\alpha_1\mathbf{v}_1+\alpha_2\mathbf{v}_2+\cdots+\alpha_N\mathbf{v}_N=\mathbf{0} \tag{4.3.2}$$

Next, calculate the inner product of the vector $\alpha_1\mathbf{v}_1+\alpha_2\mathbf{v}_2+\cdots+\alpha_N\mathbf{v}_N$ with one of the members of the set, say $\mathbf{v}_j$. The result is

$$\left\langle \alpha_1\mathbf{v}_1+\alpha_2\mathbf{v}_2+\cdots+\alpha_N\mathbf{v}_N,\mathbf{v}_j\right\rangle = 0 \tag{4.3.3}$$

By virtue of the fact that the inner product is linear in its first slot, this result becomes

$$\alpha_1\left\langle \mathbf{v}_1,\mathbf{v}_j\right\rangle+\alpha_2\left\langle \mathbf{v}_2,\mathbf{v}_j\right\rangle+\cdots+\alpha_N\left\langle \mathbf{v}_N,\mathbf{v}_j\right\rangle = 0 \tag{4.3.4}$$

Next, use the fact that the set $\left\{\mathbf{v}_1,\mathbf{v}_2,...,\mathbf{v}_N\right\}$ is orthogonal and you find

$$\alpha_j = 0 \quad \text{for } j=1,2,...,N \tag{4.3.5}$$

The result (4.3.5) establishes the liner independence of the set $\left\{\mathbf{v}_1,\mathbf{v}_2,...,\mathbf{v}_N\right\}$.

It is often beneficial to utilize bases that consist of mutually orthogonal elements. In addition, it is convenient to normalize the basis vectors so that they have unit length. Vectors with a length of 1 are called *unit vectors* or *normalized vectors*. An *orthonormal set* is an orthogonal set with the property that all of its elements are unit vectors. More formally, a set of vectors $\left\{\mathbf{i}_1,...,\mathbf{i}_M\right\}$ is an orthonormal set if

$$\langle \mathbf{i}_j, \mathbf{i}_k \rangle = \delta_{jk} = \begin{cases} 1 & \text{if} \quad j = k \\ 0 & \text{if} \quad j \neq k \end{cases} \tag{4.3.6}$$

for $j,k = 1,2,...,M$. Of course, the symbol δ_{jk} in (4.3.6) is the *Kronecker* delta introduced in Section 1.1.

If we are given a basis $\{\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_N\}$ for an inner product space $\mathscr{V}$, we shall next outline a procedure that will construct an *orthonormal basis* $\{\mathbf{i}_1, ..., \mathbf{i}_N\}$. This procedure is called the *Gram-Schmidt orthogonalization* process. More precisely, the Gram-Schmidt process is summarized in the following theorem:

Theorem 4.3.2. Given a basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_N\}$ of an inner product space $\mathscr{V}$, then there exists an orthonormal basis $\{\mathbf{i}_1, \ldots, \mathbf{i}_N\}$ such that, for $K = 1,2,...,N$, $\operatorname{Span}(\mathbf{e}_1 \ldots, \mathbf{e}_K) = \operatorname{Span}(\mathbf{i}_1, \ldots, \mathbf{i}_K)$.

Proof: The construction proceeds in two steps; first a set of orthogonal vectors is constructed, then this set is normalized. Let $\{\mathbf{d}_1, \mathbf{d}_2, ..., \mathbf{d}_N\}$ denote a set of orthogonal, but not unit, vectors. This set is constructed from $\{\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_N\}$ as follows: Let

$$\mathbf{d}_1 = \mathbf{e}_1 \tag{4.3.7}$$

and define $\mathbf{d}_2$ by the formula

$$\mathbf{d}_2 = \mathbf{e}_2 + \xi \mathbf{d}_1 \tag{4.3.8}$$

The unknown scalar ξ will be calculated from the requirement that $\mathbf{d}_2$ is orthogonal to $\mathbf{d}_1$. This orthogonality requires that

$$\langle \mathbf{d}_2, \mathbf{d}_1 \rangle = \langle \mathbf{e}_2, \mathbf{d}_1 \rangle + \xi \langle \mathbf{d}_1 \cdot \mathbf{d}_1 \rangle = 0 \tag{4.3.9}$$

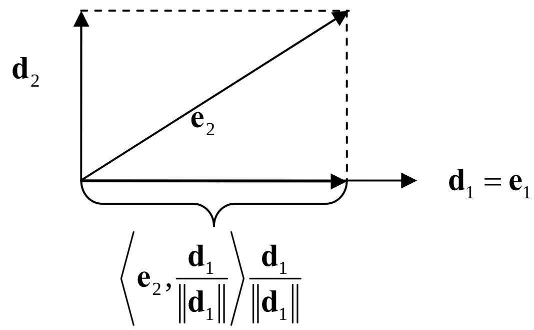
Therefore, the scalar ξ is determined by

$$\xi = -\frac{\langle \mathbf{e}_2, \mathbf{d}_1 \rangle}{\langle \mathbf{d}_1, \mathbf{d}_1 \rangle} \tag{4.3.10}$$

Of course, $\mathbf{d}_1 \cdot \mathbf{d}_1 = \mathbf{e}_1 \cdot \mathbf{e}_1 \neq 0$ since $\mathbf{e}_1 \neq 0$. Given (4.3.10), it follows from (4.3.8) that

$$\mathbf{d}_2 = \mathbf{e}_2 - \frac{\langle \mathbf{e}_2, \mathbf{d}_1 \rangle}{\langle \mathbf{d}_1, \mathbf{d}_1 \rangle} \mathbf{d}_1 = \mathbf{e}_2 - \left\langle \mathbf{e}_2, \frac{\mathbf{d}_1}{\|\mathbf{d}_1\|} \right\rangle \frac{\mathbf{d}_1}{\|\mathbf{d}_1\|} \tag{4.3.11}$$

The vector $\mathbf{d}_2$ is not zero, because $\mathbf{e}_1 = \mathbf{d}_1$ and $\mathbf{e}_2$ are linearly independent. The analytical expressions that define $\mathbf{d}_1$ and $\mathbf{d}_2$ are suggested by the following elementary figure:



While (4.3.11) is a general result for any inner product space, this figure, drawn in two dimensional space, illustrates that the calculation of $\mathbf{d}_2$ simply removes from $\mathbf{e}_2$ the projection of $\mathbf{e}_2$ in the direction of $\mathbf{d}_1 = \mathbf{e}_1$. The result is the vector $\mathbf{d}_2$ which is perpendicular to $\mathbf{d}_1 = \mathbf{e}_1$.

The vector $\mathbf{d}_3$ is defined by

$$\mathbf{d}_3 = \mathbf{e}_3 + \xi^2 \mathbf{d}_2 + \xi^1 \mathbf{d}_1 \tag{4.3.12}$$

The scalars ξ^2 and ξ^1 are determined by the requirement that $\mathbf{d}_3$ be orthogonal to both $\mathbf{d}_1$ and $\mathbf{d}_2$; thus

$$\langle \mathbf{d}_3, \mathbf{d}_1 \rangle = \langle \mathbf{e}_3, \mathbf{d}_1 \rangle + \xi^1 \langle \mathbf{d}_1, \mathbf{d}_1 \rangle = 0 \tag{4.3.13}$$

$$\langle \mathbf{d}_3, \mathbf{d}_2 \rangle = \langle \mathbf{e}_3, \mathbf{d}_2 \rangle + \xi^2 \langle \mathbf{d}_2, \mathbf{d}_2 \rangle = 0 \tag{4.3.14}$$

and, as a result,

$$\xi^1 = -\frac{\langle \mathbf{e}_3, \mathbf{d}_1 \rangle}{\langle \mathbf{d}_1, \mathbf{d}_1 \rangle} \tag{4.3.15}$$

$$\xi^2 = -\frac{\langle \mathbf{e}_3, \mathbf{d}_2 \rangle}{\langle \mathbf{d}_2, \mathbf{d}_2 \rangle} \tag{4.3.16}$$

Therefore, equation (4.3.12) becomes

$$\mathbf{d}_3 = \mathbf{e}_3 - \frac{\langle \mathbf{e}_3, \mathbf{d}_2 \rangle}{\langle \mathbf{d}_2, \mathbf{d}_2 \rangle}\mathbf{d}_2 - \frac{\langle \mathbf{e}_3, \mathbf{d}_1 \rangle}{\langle \mathbf{d}_1, \mathbf{d}_1 \rangle}\mathbf{d}_1 \tag{4.3.17}$$

The linear independence of $\mathbf{e}_1, \mathbf{e}_2$, and $\mathbf{e}_3$ requires that $\mathbf{d}_3$ be nonzero. Motivated by the discussion of the above figure, one can see that $\mathbf{d}_3$ is the vector resulting from the removal from $\mathbf{e}_3$ its projections in the two directions $\mathbf{d}_1$ and $\mathbf{d}_2$.

It should now be clear that the schemes used to calculate $\mathbf{d}_1, \mathbf{d}_2$ and $\mathbf{d}_3$ can be repeated until a set of K orthogonal vectors $\{\mathbf{d}_1, \mathbf{d}_2, \ldots, \mathbf{d}_K\}$ has been obtained. The orthonormal set is then obtained by defining

$$\mathbf{i}_k = \mathbf{d}_k / \|\mathbf{d}_k\|, \quad k = 1, 2, \ldots, K \tag{4.3.18}$$

It is easy to see from the above construction that $\{\mathbf{e}_1 \ldots, \mathbf{e}_K\}$, $\{\mathbf{d}_1, \ldots, \mathbf{d}_K\}$, and, hence, $\{\mathbf{i}_1, \ldots, \mathbf{i}_K\}$ generate the same subspace of $\mathscr{V}$ for each K.

It is convenient to collect the above formulas and eliminate the intermediate set of vectors $\{\mathbf{d}_1, \mathbf{d}_2, ..., \mathbf{d}_K\}$. It follows from (4.3.7) and (4.3.18) that

$$\mathbf{i}_1 = \frac{\mathbf{e}_1}{\|\mathbf{e}_1\|} \tag{4.3.19}$$

It follows from (4.3.11), (4.3.18) and (4.3.7) that

$$\mathbf{i}_2 = \frac{\mathbf{d}_2}{\|\mathbf{d}_2\|} = \frac{\mathbf{e}_2 - \langle \mathbf{e}_2, \mathbf{i}_1 \rangle \mathbf{i}_1}{\|\mathbf{e}_2 - \langle \mathbf{e}_2, \mathbf{i}_1 \rangle \mathbf{i}_1\|} \tag{4.3.20}$$

In a similar fashion

$$\mathbf{i}_3 = \frac{\mathbf{e}_3 - \langle \mathbf{e}_3, \mathbf{i}_2 \rangle \mathbf{i}_2 - \langle \mathbf{e}_3, \mathbf{i}_1 \rangle \mathbf{i}_1}{\|\mathbf{e}_3 - \langle \mathbf{e}_3, \mathbf{i}_2 \rangle \mathbf{i}_2 - \langle \mathbf{e}_3, \mathbf{i}_1 \rangle \mathbf{i}_1\|} \tag{4.3.21}$$

The pattern of these results reveals that

$$\mathbf{i}_k = \frac{\mathbf{e}_k - \sum_{j=1}^{k-1} \langle \mathbf{e}_k, \mathbf{i}_j \rangle \mathbf{i}_j}{\left\| \mathbf{e}_k - \sum_{j=1}^{k-1} \langle \mathbf{e}_k, \mathbf{i}_j \rangle \mathbf{i}_j \right\|} \quad \text{for} \quad k = 1, 2, ..., K \tag{4.3.22}$$

is the formula for the k^{th} unit vector in the orthonormal set $\{\mathbf{i}_1,\ldots,\mathbf{i}_K\}$.

Example 4.3.1: Given the following three column vectors in $\mathscr{M}^{4\times 1}$.

$$\mathbf{e}_1=\begin{bmatrix}2\\2\\2\\2\end{bmatrix},\mathbf{e}_2=\begin{bmatrix}-1\\4\\4\\-1\end{bmatrix},\mathbf{e}_3=\begin{bmatrix}4\\-2\\2\\0\end{bmatrix} \tag{4.3.23}$$

These vectors are linearly independent and span a certain subspace of $\mathscr{M}^{4\times 1}$. The objective is to use the Gram-Schmidt process to find three orthonormal vectors that span the same subspace of $\mathscr{M}^{4\times 1}$. Recall that the inner product of the vector space $\mathscr{M}^{4\times 1}$ is given by (4.1.13), repeated,

$$\langle \mathbf{u},\mathbf{v}\rangle = \mathbf{u}^T\overline{\mathbf{v}} = \overline{\mathbf{v}}^T\mathbf{u} \tag{4.3.24}$$

Step 1: The unit vector $\mathbf{i}_1$ is given by (4.3.19). Therefore,

$$\mathbf{i}_1=\frac{\mathbf{e}_1}{\|\mathbf{e}_1\|}=\frac{1}{\sqrt{\mathbf{e}_1^T\mathbf{e}_1}}\mathbf{e}_1=\frac{1}{4}\begin{bmatrix}2\\2\\2\\2\end{bmatrix}=\frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} \tag{4.3.25}$$

Step 2: The unit vector $\mathbf{i}_2$ is given by (4.3.20). It follows from (4.3.23) and (4.3.25) that

$$\mathbf{e}_2-\langle\mathbf{e}_2,\mathbf{i}_1\rangle\mathbf{i}_1=\begin{bmatrix}-1\\4\\4\\-1\end{bmatrix}-\left(\frac{1}{2}\begin{bmatrix}-1\\4\\4\\-1\end{bmatrix}^T\begin{bmatrix}1\\1\\1\\1\end{bmatrix}\right)\frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}=\begin{bmatrix}-1\\4\\4\\-1\end{bmatrix}-(3)\frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}=\frac{1}{2}\begin{bmatrix}-5\\5\\5\\-5\end{bmatrix} \tag{4.3.26}$$

The norm of the column matrix (4.3.26) is

$$\|\mathbf{e}_2-\langle\mathbf{e}_2,\mathbf{i}_1\rangle\mathbf{i}_1\|=\sqrt{\left(\frac{1}{2}\begin{bmatrix}-5\\5\\5\\-5\end{bmatrix}\right)^T\left(\frac{1}{2}\begin{bmatrix}-5\\5\\5\\-5\end{bmatrix}\right)}=\frac{1}{2}\sqrt{100}=5 \tag{4.3.27}$$

Therefore,

$$\mathbf{i}_2 = \frac{\mathbf{e}_2 - \langle \mathbf{e}_2, \mathbf{i}_1 \rangle \mathbf{i}_1}{\left\| \mathbf{e}_2 - \langle \mathbf{e}_2, \mathbf{i}_1 \rangle \mathbf{i}_1 \right\|} = \frac{1}{2}\begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix} \tag{4.3.28}$$

Step 3: The unit vector $\mathbf{i}_3$ is defined by (4.3.21). It follows from (4.3.23), (4.3.25) and (4.3.28) that

$$\begin{aligned} \mathbf{e}_3 - \langle \mathbf{e}_3, \mathbf{i}_2 \rangle \mathbf{i}_2 - \langle \mathbf{e}_3, \mathbf{i}_1 \rangle \mathbf{i}_1 &= \begin{bmatrix} 4 \\ -2 \\ 2 \\ 0 \end{bmatrix} - \frac{1}{2}\left(\frac{1}{2}\begin{bmatrix} 4 \\ -2 \\ 2 \\ 0 \end{bmatrix}^T \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix} \right) \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix} - \frac{1}{2}\left(\frac{1}{2}\begin{bmatrix} 4 \\ -2 \\ 2 \\ 0 \end{bmatrix}^T \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right) \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 4 \\ -2 \\ 2 \\ 0 \end{bmatrix} - \frac{1}{2}(-2)\begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix} - \frac{1}{2}(2)\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} -2 \\ 0 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 2 \\ -2 \end{bmatrix} \end{aligned} \tag{4.3.29}$$

The norm of the column matrix (4.3.29) is

$$\left\| \mathbf{e}_3 - \langle \mathbf{e}_3, \mathbf{i}_2 \rangle \mathbf{i}_2 - \langle \mathbf{e}_3, \mathbf{i}_1 \rangle \mathbf{i}_1 \right\| = \sqrt{\left(\begin{bmatrix} 2 \\ -2 \\ 2 \\ -2 \end{bmatrix} \right)^T \left(\begin{bmatrix} 2 \\ -2 \\ 2 \\ -2 \end{bmatrix} \right)} = 4 \tag{4.3.30}$$

Therefore,

$$\mathbf{i}_3 = \frac{\mathbf{e}_3 - \langle \mathbf{e}_3, \mathbf{i}_2 \rangle \mathbf{i}_2 - \langle \mathbf{e}_3, \mathbf{i}_1 \rangle \mathbf{i}_1}{\left\| \mathbf{e}_3 - \langle \mathbf{e}_3, \mathbf{i}_2 \rangle \mathbf{i}_2 - \langle \mathbf{e}_3, \mathbf{i}_1 \rangle \mathbf{i}_1 \right\|} = \frac{1}{4}\begin{bmatrix} 2 \\ -2 \\ 2 \\ -2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \tag{4.3.31}$$

Given the Gram-Schmidt construction process as described, the result is a basis for the subspace spanned by $\{\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_K\}$ and an orthonormal basis $\{\mathbf{i}_1, \mathbf{i}_2, ..., \mathbf{i}_K\}$ that spans the same subspace. These two bases are necessarily connected by the usual change of basis formula as discussed in Section 2.6. We shall write this formula as

$$\mathbf{e}_k = \sum_{q=1}^{K} R_k^q \mathbf{i}_q \qquad \text{for} \quad k = 1, ..., K \tag{4.3.32}$$

Because the set $\{\mathbf{i}_1, \mathbf{i}_2, ..., \mathbf{i}_K\}$ is orthonormal and, thus, obeys (4.3.6), the coefficients $R_k^q, q, k = 1, 2, ..., K$, are given by

$$R_k^q = \langle \mathbf{e}_k, \mathbf{i}_q \rangle \tag{4.3.33}$$

Example 4.3.2: For Example 4.3.1, it is possible to use

$$\mathbf{e}_1 = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix} -1 \\ 4 \\ 4 \\ -1 \end{bmatrix}, \mathbf{e}_3 = \begin{bmatrix} 4 \\ -2 \\ 2 \\ 0 \end{bmatrix}, \mathbf{i}_1 = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{i}_2 = \frac{1}{2}\begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \mathbf{i}_3 = \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \tag{4.3.34}$$

and calculate the coefficients $R_k^q, q, k = 1, 2, 3$. The results are

$$\begin{gathered}
R_1^1 = \langle \mathbf{e}_1, \mathbf{i}_1 \rangle = \frac{1}{2}\begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix}^T \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = 4 \quad R_2^1 = \langle \mathbf{e}_2, \mathbf{i}_1 \rangle = \frac{1}{2}\begin{bmatrix} -1 \\ 4 \\ 4 \\ -1 \end{bmatrix}^T \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = 3 \quad R_3^1 = \langle \mathbf{e}_3, \mathbf{i}_1 \rangle = \frac{1}{2}\begin{bmatrix} 4 \\ -2 \\ 2 \\ 0 \end{bmatrix}^T \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = 2 \\
R_1^2 = \langle \mathbf{e}_1, \mathbf{i}_2 \rangle = \frac{1}{2}\begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix}^T \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix} = 0 \quad R_2^2 = \langle \mathbf{e}_2, \mathbf{i}_2 \rangle = \frac{1}{2}\begin{bmatrix} -1 \\ 4 \\ 4 \\ -1 \end{bmatrix}^T \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix} = 5 \quad R_3^2 = \langle \mathbf{e}_3, \mathbf{i}_2 \rangle = \frac{1}{2}\begin{bmatrix} 4 \\ -2 \\ 2 \\ 0 \end{bmatrix}^T \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix} = -2 \\
R_1^3 = \langle \mathbf{e}_1, \mathbf{i}_3 \rangle = \frac{1}{2}\begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix}^T \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} = 0 \quad R_2^3 = \langle \mathbf{e}_2, \mathbf{i}_3 \rangle = \frac{1}{2}\begin{bmatrix} -1 \\ 4 \\ 4 \\ -1 \end{bmatrix}^T \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} = 0 \quad R_3^3 = \langle \mathbf{e}_3, \mathbf{i}_3 \rangle = \frac{1}{2}\begin{bmatrix} 4 \\ -2 \\ 2 \\ 0 \end{bmatrix}^T \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} = 4
\end{gathered} \tag{4.3.35}$$

Therefore, the matrix $R = \left[R_k^q \right]$ is the upper triangle matrix

$$R = \begin{bmatrix} R_1^1 & R_2^1 & R_3^1 \\ R_1^2 & R_2^2 & R_3^2 \\ R_1^3 & R_2^3 & R_3^3 \end{bmatrix} = \begin{bmatrix} 4 & 3 & 2 \\ 0 & 5 & -2 \\ 0 & 0 & 4 \end{bmatrix} \tag{4.3.36}$$

These results, reduce (4.3.32) to

$$\begin{aligned}\mathbf{e}_1 &= 4\mathbf{i}_1 \\ \mathbf{e}_2 &= 3\mathbf{i}_1 + 5\mathbf{i}_2 \\ \mathbf{e}_3 &= 2\mathbf{i}_1 - 2\mathbf{i}_2 + 4\mathbf{i}_3\end{aligned} \tag{4.3.37}$$

It is a general result that the Gram-Schmidt process yields an upper triangular matrix as illustrated by the example (4.3.36). This result is contained in the formula (4.3.22) which can be solved for $\mathbf{e}_k, k = 1,2,...,K$ to obtain

$$\mathbf{e}_k = \left\| \mathbf{e}_k - \sum_{j=1}^{k-1} \langle \mathbf{e}_k, \mathbf{i}_j \rangle \mathbf{i}_j \right\| \mathbf{i}_k + \sum_{j=1}^{k-1} \langle \mathbf{e}_k, \mathbf{i}_j \rangle \mathbf{i}_j \quad \text{for} \quad k = 1,2,...,K \tag{4.3.38}$$

Equations (4.3.38) and (4.3.32) identify the components of $R = \left[R_k^q \right]$. The results are

$$R_k^q = \begin{cases} \langle \mathbf{e}_k, \mathbf{i}_q \rangle & \text{for} \quad q = 1,2,...,k-1 \\ \left\| \mathbf{e}_k - \sum_{j=1}^{k-1} \langle \mathbf{e}_k, \mathbf{i}_j \rangle \mathbf{i}_j \right\| & \text{for} \quad q = k \\ 0 & \text{for} \quad q > k \end{cases} \tag{4.3.39}$$

An equivalent but possibly more informative display of the matrix R is

$$R = \begin{bmatrix} \|\mathbf{e}_1\| & \langle \mathbf{e}_2, \mathbf{i}_1 \rangle & \langle \mathbf{e}_3, \mathbf{i}_1 \rangle & \cdot & \cdot & \langle \mathbf{e}_N, \mathbf{i}_1 \rangle \\ 0 & \| \mathbf{e}_2 - \langle \mathbf{e}_2, \mathbf{i}_1 \rangle \mathbf{i}_1 \| & \langle \mathbf{e}_3, \mathbf{i}_2 \rangle & \cdot & \cdot & \langle \mathbf{e}_N, \mathbf{i}_2 \rangle \\ 0 & 0 & \| \mathbf{e}_3 - \langle \mathbf{e}_3, \mathbf{i}_2 \rangle \mathbf{i}_2 - \langle \mathbf{e}_3, \mathbf{i}_1 \rangle \mathbf{i}_1 \| & \cdot & \cdot & \langle \mathbf{e}_N, \mathbf{i}_3 \rangle \\ \cdot & \cdot & 0 & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \left\| \mathbf{e}_K - \sum_{j=1}^{K-1} \langle \mathbf{e}_K, \mathbf{i}_j \rangle \mathbf{i}_j \right\| \end{bmatrix} \tag{4.3.40}$$

Example 4.3.3: The Gram-Schmidt process is a good way to introduce the concept of a set of *orthogonal polynomials*. The study of orthogonal polynomials is a large and somewhat complicated undertaking. They arise as the solution to certain second order ordinary differential equations that occur in many applications. The orthogonal polynomials introduced in this example are called Legendre Polynomials.[3] In addition to arising during the study or ordinary differential equations, they have application in certain numerical integration applications known as the

[3] Information about the French mathematician Adrien-Marie Legendre can be found at http://en.wikipedia.org/wiki/Adrien-Marie_Legendre.

Newton-Cotes formulae.[4] In any case, a Legendre polynomial of degree k, where $k = 0,1,2,.....$, is a polynomial of degree k defined on the interval $[-1,1]$. It is given the symbol P_k. The first six Legendre polynomials turn out to be are

$$\begin{aligned}
P_0(x) &= 1 \\
P_1(x) &= x \\
P_2(x) &= \frac{1}{2}(3x^2 - 1) \\
P_3(x) &= \frac{1}{2}(5x^3 - 3x) \\
P_4(x) &= \frac{1}{8}(35x^4 - 30x^2 + 3) \\
P_5(x) &= \frac{1}{8}(63x^5 - 70x^3 + 15x) \\
P_6(x) &= \frac{1}{16}(231x^6 - 315x^4 + 105x^2 - 5)
\end{aligned} \tag{4.3.41}$$

They are normalized such that

$$P_k(1) = 1 \tag{4.3.42}$$

They also obey a recursion relationship

$$P_{k+1}(x) = \left(\frac{2k+1}{k+1}\right) x P_k(x) - \left(\frac{k}{k+1}\right) P_{k-1}(x) \quad \text{for} \quad k = 1,2,3,.... \tag{4.3.43}$$

Finally, they obey the integral condition

$$\int_{x=-1}^{x=1} P_k(x)^2 \, dx = \frac{2}{2k+1} \tag{4.3.44}$$

In Example 4.1.5, we assigned an inner product to the $N+1$ dimensional vector space of polynomials $\mathscr{P}_N$. The inner product we wish to assign is the special case of (4.1.17) defined by

$$\langle p, q \rangle = \int_{x=-1}^{x=1} p(x) q(x) \, dx \tag{4.3.45}$$

---

[4] A good summary of the Newton-Cotes formulae can be found at
http://en.wikipedia.org/wiki/Newton%E2%80%93Cotes_formulas

In Example 2.5.4, we showed that the set of polynomials $\left\{1, x, x^2, ..., x^N\right\}$ form a basis for $\mathscr{P}_N$. Next, we shall apply the Gram-Schmidt procedure to the set $\left\{1, x, x^2, ..., x^N\right\}$. We shall utilize the inner product (4.3.45). If, for the purposes of this example, we take $N = 6$, and use the symbol

$$p_k(x) = x^{k-1} \quad \text{for} \quad k = 1,2,3,4,5,6 \tag{4.3.46}$$

and $\left\{i_1, i_2, i_3, i_4, i_5, i_6\right\}$ to denote the orthonormal set that results from the Gram-Schmidt procedure. In order to implement the procedure, we need to adopt the general formula (4.3.22) for $k = 1, 2, ..., 6$. Of course, the inner product is the one defined by (4.3.45). As a result, the first three elements of the set $\left\{i_1, i_2, i_3, i_4, i_5, i_6\right\}$ are

for $k = 1$ and $p_1(x) = 1$

$$i_1(x) = \frac{p_1}{\sqrt{\langle p_1, p_1\rangle}} = \frac{1}{\left(\int_{x=-1}^{x=1} dx\right)^{1/2}} = \frac{1}{\sqrt{2}} \tag{4.3.47}$$

for $k = 2$ and $p_2(x) = x$

$$\begin{aligned} i_2(x) &= \frac{p_2 - \langle p_2, i_1\rangle i_1}{\left\| p_2 - \langle p_2, i_1\rangle i_1\right\|} = \frac{x - \left(\int_{-1}^{1} \frac{1}{\sqrt{2}} x dx\right)\frac{1}{\sqrt{2}}}{\sqrt{\int_{x=-1}^{x=1}\left(x - \left(\int_{-1}^{1}\frac{1}{\sqrt{2}} x dx\right)\frac{1}{\sqrt{2}}\right)^2 dx}} \\ &= \frac{x}{\sqrt{\int_{x=-1}^{x=1} x^2 dx}} = \sqrt{\frac{3}{2}} x \end{aligned} \tag{4.3.48}$$

for $k = 3$, and $p_3(x) = x^2$

$$
\begin{aligned}
i_3 &= \frac{p_3 - \langle p_3, i_1\rangle i_1 - \langle p_3, i_2\rangle i_2}{\left\| p_3 - \langle p_3, i_1\rangle i_1 - \langle p_3, i_2\rangle i_2 \right\|} \\
&= \frac{x^2 - \left(\int_{-1}^{1}\left(x^2\right)\left(\frac{1}{\sqrt{2}}\right)dx\right)\frac{1}{\sqrt{2}} - \left(\int_{-1}^{1}\left(x^2\right)\left(\sqrt{\frac{3}{2}}x\right)dx\right)\sqrt{\frac{3}{2}}x}{\sqrt{\int_{x=-1}^{x=1}\left(x^2 - \left(\int_{-1}^{1}\left(x^2\right)\left(\frac{1}{\sqrt{2}}\right)dx\right)\frac{1}{\sqrt{2}} - \left(\int_{-1}^{1}\left(x^2\right)\left(\sqrt{\frac{3}{2}}x\right)dx\right)\sqrt{\frac{3}{2}}x\right)^2 dx}} \\
&= \frac{x^2 - \left(\int_{-1}^{1}\left(x^2\right)\left(\frac{1}{\sqrt{2}}\right)dx\right)\frac{1}{\sqrt{2}}}{\sqrt{\int_{x=-1}^{x=1}\left(x^2 - \left(\int_{-1}^{1}\left(x^2\right)\left(\frac{1}{\sqrt{2}}\right)dx\right)\frac{1}{\sqrt{2}}\right)^2 dx}} = \frac{x^2 - \frac{1}{2}\left.\frac{x^3}{3}\right|_{x=-1}^{x=1}}{\sqrt{\int_{x=-1}^{x=1}\left(x^2 - \frac{1}{2}\left.\frac{x^3}{3}\right|_{x=-1}^{x=1}\right)^2 dx}} \\
&= \frac{x^2 - \frac{1}{3}}{\sqrt{\int_{x=-1}^{x=1}\left(x^2 - \frac{1}{3}\right)^2 dx}} = \frac{x^2 - \frac{1}{3}}{\sqrt{\int_{x=-1}^{x=1}\left(x^4 - \frac{2}{3}x^2 + \frac{1}{9}\right)dx}} = \frac{x^2 - \frac{1}{3}}{\sqrt{\frac{8}{45}}} = \sqrt{\frac{5}{8}}\left(3x^2 - 1\right)
\end{aligned}
\tag{4.3.49}
$$

Similar calculations yield

$$
\begin{aligned}
i_4(x) &= \sqrt{\frac{7}{8}}\left(5x^3 - 3x\right) \\
i_5(x) &= \sqrt{\frac{9}{128}}\left(35x^4 - 30x^2 + 3\right) \\
i_6(x) &= \sqrt{\frac{11}{128}}\left(63x^5 - 70x^3 + 15x\right)
\end{aligned}
\tag{4.3.50}
$$

Equations (4.3.47), (4.3.48), (4.3.49) and (4.3.50) show the relationships between the orthonormal polynomials produced by the Gram-Schmidt procedure and the Legendre polynomials defined by (4.3.41). The Legendre polynomials are normalized to obey (4.3.42) and the polynomials in $\{i_1, i_2, i_3, i_4, i_5, i_6\}$ are normalized to have unit length. The point of this example is that one can begin with the polynomials in the set $\{1, x, x^2, ..., x^N\}$, adopt a definition inner product as illustrated by (4.3.45) and generate a set of orthogonal polynomials that have importance in certain applications.

Other examples of orthogonal polynomials that arise in the applications go by the names of Chebyshev, Gegenbaur, Hermite, Jacobi and Leguerre.

Exercises

4.3.1: Determine an orthonormal basis for the basis $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ of $\mathscr{M}^{3\times 1}$ defined by

$$\mathbf{e}_1=\begin{bmatrix}1\\-1\\1\end{bmatrix},\mathbf{e}_2=\begin{bmatrix}1\\3\\-1\end{bmatrix},\mathbf{e}_3=\begin{bmatrix}4\\-2\\1\end{bmatrix} \tag{4.3.51}$$

The correct answer is

$$\mathbf{i}_1=\begin{bmatrix}\frac{1}{\sqrt{3}}\\-\frac{1}{\sqrt{3}}\\\frac{1}{\sqrt{3}}\end{bmatrix},\mathbf{i}_2=\begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\\0\end{bmatrix},\mathbf{i}_3=\begin{bmatrix}\frac{1}{\sqrt{6}}\\-\frac{1}{\sqrt{6}}\\-\frac{2}{\sqrt{6}}\end{bmatrix} \tag{4.3.52}$$

4.3.2: Determine an orthonormal basis for the basis $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ of $\mathscr{M}^{3\times 1}$ defined by

$$\mathbf{e}_1=\begin{bmatrix}0\\3\\3i\end{bmatrix},\mathbf{e}_2=\begin{bmatrix}i\\-2\\2\end{bmatrix},\mathbf{e}_3=\begin{bmatrix}i\\3i\\0\end{bmatrix} \tag{4.3.53}$$

The correct answer is

$$\mathbf{i}_1=\begin{bmatrix}0\\\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}i\end{bmatrix},\mathbf{i}_2=\frac{1}{\sqrt{5}}\begin{bmatrix}i\\-1+i\\1+i\end{bmatrix},\mathbf{i}_3=\frac{1}{5\sqrt{2}}\begin{bmatrix}1-3i\\-1+\frac{1}{2}i\\\frac{1}{2}+i\end{bmatrix} \tag{4.3.54}$$

4.3.3 Determine an orthonormal basis for the basis $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3,\mathbf{e}_4\}$ for the vector space $\mathscr{M}^{4\times 1}$ defined by

$$\mathbf{e}_1=\begin{bmatrix}2\\3\\3i\\-2i\end{bmatrix},\mathbf{e}_2=\begin{bmatrix}3i\\-2\\2\\4\end{bmatrix},\mathbf{e}_3=\begin{bmatrix}-2i\\1\\3i\\0\end{bmatrix},\mathbf{e}_4=\begin{bmatrix}4\\2i\\4\\5i\end{bmatrix} \tag{4.3.55}$$

The correct answer is

$$\mathbf{i}_1 = \begin{bmatrix} 0.3922 \\ 0.5883 \\ 0.5883i \\ -0.3922i \end{bmatrix}, \mathbf{i}_2 = \begin{bmatrix} 0.0855 + 0.4416i \\ -0.2422 - 0.1710i \\ 0.5414 + 0.1282i \\ 0.6268 - 0.0855i \end{bmatrix}, \mathbf{i}_3 = \begin{bmatrix} -0.0660 - 0.6967i \\ -0.3899 + 0.3374i \\ 0.0684 + 0.3850i \\ 0.2934 - 0.0733i \end{bmatrix}, \mathbf{i}_4 = \begin{bmatrix} 0.3925 + 0.0016i \\ 0.1884 + 0.5144i \\ 0.4331 - 0.0618i \\ -0.1235 + 0.5824i \end{bmatrix} \quad (4.3.56)$$

4.3.4 You are given a matrix $A \in \mathscr{M}^{3\times 2}$ defined by

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 4 & 5 \end{bmatrix} \quad (4.3.57)$$

Find an orthonormal basis for the image space $R(A)$ if A.

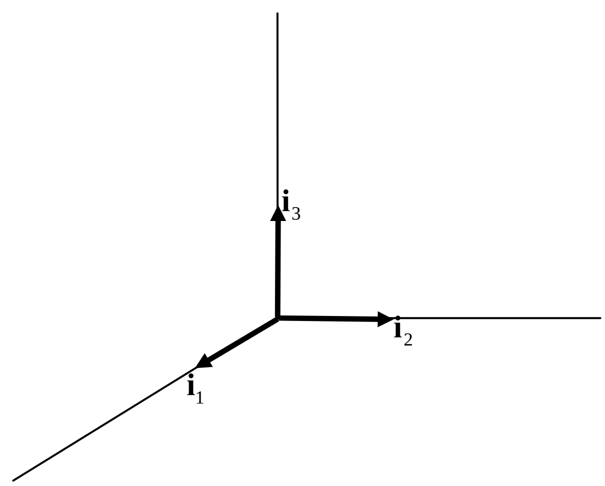
4.3.5 You are given a matrix $A \in \mathscr{M}^{3\times 2}$ defined by

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \\ i & 5 \end{bmatrix} \quad (4.3.58)$$

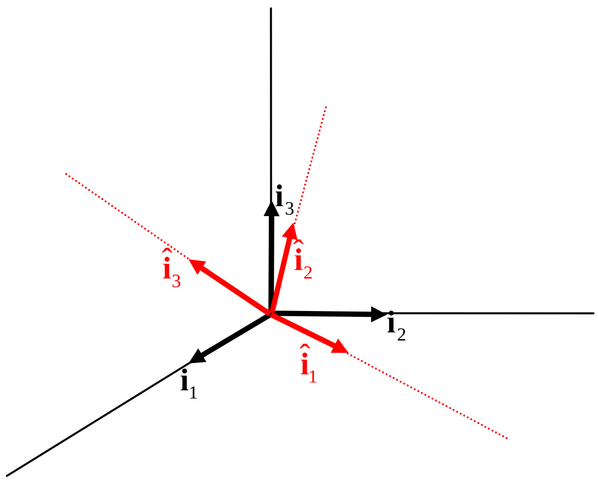
Find an orthonormal basis that spans $R(A)$.

Section 4.4. Orthonormal Bases in Three Dimensions

The real inner product space of dimension three is especially important in applied mathematics. It arises as the underlying mathematical structure of almost all application of the broad area known as applied mechanics. For this reason and others, in this section, we shall built upon the results of Section 4.3 and study this special case in greater detail. We begin with the assumption that we have an orthonormal basis for this real inner product space $\mathscr{V}$ that we will denote by $\{\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3\}$. Given any basis for $\mathscr{V}$, the Gram-Schmidt process discussed in Section 4.3 allows this orthonormal basis to be constructed. It is customary to illustrate this basis with a figure like the following



The first topic we wish to discuss is the basis change from the orthonormal basis $\{\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3\}$ to a *second* orthonormal basis $\{\hat{\mathbf{i}}_1, \hat{\mathbf{i}}_2, \hat{\mathbf{i}}_3\}$. Geometrically, we can illustrate a possible second basis on the above figure as follows:



As follows from (4.3.6), the fact that the two bases are orthonormal is reflected in the requirements

$$\left\langle \mathbf{i}_s, \mathbf{i}_k \right\rangle = \delta_{sk} \qquad s,k = 1,2,3 \tag{4.4.1}$$

and

$$\left\langle \hat{\mathbf{i}}_j, \hat{\mathbf{i}}_q \right\rangle = \delta_{jq} \qquad j,k = 1,2,3 \tag{4.4.2}$$

As we have discussed several times and discussed in detail in Section 2.6, the two sets of bases are related by an expression of the form

$$\hat{\mathbf{i}}_j = \sum_{k=1}^{3} Q_{kj} \mathbf{i}_k \quad \text{for} \quad j = 1,2,3 \tag{4.4.3}$$

It follows from (4.4.3) and (4.4.1) that the coefficients of the transition matrix are given by

$$\left\langle \mathbf{i}_s, \hat{\mathbf{i}}_j \right\rangle = \left\langle \mathbf{i}_s, \sum_{k=1}^{3} Q_{kj} \mathbf{i}_k \right\rangle = \sum_{k=1}^{3} Q_{kj} \left\langle \mathbf{i}_s, \mathbf{i}_k \right\rangle = Q_{sj} \tag{4.4.4}$$

Because we are dealing with a real vector space, the result (4.4.4) combined with the definition (4.2.10) tells us that $Q_{sj} = \left\langle \mathbf{i}_s, \hat{\mathbf{i}}_j \right\rangle$ is the *cosine of the angle between the vectors* $\mathbf{i}_s$ *and* $\hat{\mathbf{i}}_j$. These cosines are the usual *direction cosines* that are familiar from elementary geometry.

The coefficients of the transition matrix

$$Q = \left[Q_{kj} \right] = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{bmatrix} \tag{4.4.5}$$

are restricted by the two requirements (4.4.1) and (4.4.2). The nature of the restriction is revealed if we substitute (4.4.3) into (4.4.2) to obtain

$$\left\langle \hat{\mathbf{i}}_j, \hat{\mathbf{i}}_q \right\rangle = \left\langle \sum_{s=1}^{3} Q_{sj} \mathbf{i}_s, \sum_{k=1}^{3} Q_{kq} \mathbf{i}_k \right\rangle = \sum_{s=1}^{3} \sum_{k=1}^{3} Q_{sj} Q_{kq} \left\langle \mathbf{i}_s, \mathbf{i}_k \right\rangle = \delta_{jq} \tag{4.4.6}$$

If we now utilize (4.4.1), (4.4.6) reduces to

$$\sum_{s=1}^{3} \sum_{k=1}^{3} Q_{sj} Q_{kq} \delta_{sk} = \sum_{k=1}^{3} Q_{kj} Q_{kq} = \delta_{jq} \tag{4.4.7}$$

The matrix form of $(4.4.7)_2$ is

$$Q^T Q = I \tag{4.4.8}$$

Because the transition matrix is nonsingular, it follows from (4.4.8) that

$$Q^{-1} = Q^T \tag{4.4.9}$$

Therefore, *the inverse of the transition matrix between two orthonormal bases is equal to its transpose*. The special result (4.4.9) also implies

$$QQ^T = I \tag{4.4.10}$$

in addition to (4.4.8). Also, because of (1.10.21), it follows from (4.4.8) that

$$\det\left(QQ^T\right) = \det Q \det Q^T = \left(\det Q\right)^2 = 1 \tag{4.4.11}$$

and, thus,

$$\det Q = \pm 1 \tag{4.4.12}$$

The transition matrix Q is an example of what is known as an *orthogonal matrix*. It has the property, as reflected in the construction above, of preserving lengths and angles.

If we view the transition matrix as consisting of 3 column vectors $\mathbf{q}_1, \mathbf{q}_2$ and $\mathbf{q}_3$ defined by

$$\mathbf{q}_j = \begin{bmatrix} Q_{1j} \\ Q_{2j} \\ Q_{3j} \end{bmatrix} \qquad \text{for} \quad j = 1,2,3 \tag{4.4.13}$$

then it is easy to restate the orthogonality condition (4.4.8) as a condition of orthogonality on the column vectors (4.4.13). The formal condition that reflects this fact is

$$\mathbf{q}_j^T \mathbf{q}_k = \delta_{jk} \tag{4.4.14}$$

Example 4.4.1: An elementary example of the basis change described above is one where Q takes the simple form

$$Q = \left[Q_{kj} \right] = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{21} & Q_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4.4.15}$$

With this choice for the transition matrix, the basis change defined by (4.4.3) reduces to

$$\begin{aligned} \hat{\mathbf{i}}_1 &= Q_{11}\mathbf{i}_1 + Q_{21}\mathbf{i}_2 \\ \hat{\mathbf{i}}_2 &= Q_{12}\mathbf{i}_1 + Q_{22}\mathbf{i}_2 \\ \hat{\mathbf{i}}_3 &= \mathbf{i}_3 \end{aligned} \tag{4.4.16}$$

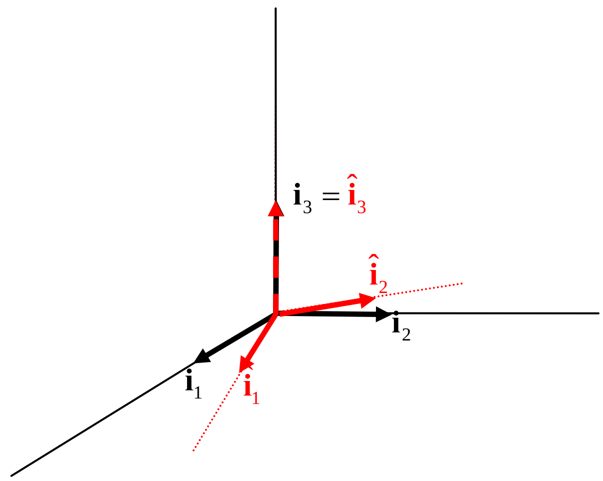
where, from (4.4.8)

$$\begin{aligned} \left(Q_{11}\right)^2 + \left(Q_{21}\right)^2 &= 1 \\ \left(Q_{12}\right)^2 + \left(Q_{22}\right)^2 &= 1 \\ Q_{11}Q_{12} + Q_{21}Q_{22} &= 0 \end{aligned} \tag{4.4.17}$$

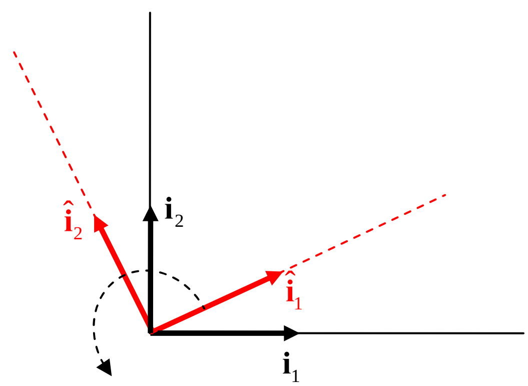
and, from(4.4.10),

$$\begin{aligned} \left(Q_{11}\right)^2 + \left(Q_{12}\right)^2 &= 1 \\ \left(Q_{21}\right)^2 + \left(Q_{22}\right)^2 &= 1 \\ Q_{11}Q_{21} + Q_{12}Q_{22} &= 0 \end{aligned} \tag{4.4.18}$$

For the basis change defined by (4.4.16), the above figure is replaced by



Thus, the choice (4.4.16) corresponds to some type of a rotation about the 3 axis. If we view the above figure from the prospective of a rotation in the plane, the result is



where the 3 axis can be viewed as pointing out of the page. The six equations (4.4.17) and (4.4.18) have certain obvious implications. First, $(4.4.17)_1$ and $(4.4.18)_1$ imply

$$Q_{21} = \pm Q_{12} \tag{4.4.19}$$

The result (4.4.19) reduce $(4.4.17)_2$ and $(4.4.18)_2$ to the same equation. Also, given (4.4.19), it follows from $(4.4.17)_1$ and $(4.4.17)_2$ that

$$Q_{22} = \pm Q_{11} \tag{4.4.20}$$

Given (4.4.19) and (4.4.20), equation $(4.4.17)_3$ or, equivalently, $(4.4.18)_3$ yield

$$Q_{11}Q_{21} + Q_{12}Q_{22} = Q_{11}\left(\pm Q_{12}\right) + Q_{12}\left(\pm Q_{11}\right) = 0 \tag{4.4.21}$$

If we exclude the trivial case where Q_{11} or Q_{12} are zero, (4.4.21) says that the multiplicity reflected in (4.4.19) and (4.4.20) must be paired such that if

$$Q_{22} = \pm Q_{11} \tag{4.4.22}$$

then

$$Q_{21} = \mp Q_{12} \tag{4.4.23}$$

Given (4.4.22) and (4.4.23), we see that the first column of (4.4.15) determines the two possible choices for the second column of (4.4.15). The remaining condition on the elements of the first column is $(4.4.18)_1$, repeated,

$$\left(Q_{11}\right)^2 + \left(Q_{12}\right)^2 = 1 \tag{4.4.24}$$

or, equivalently,

$$Q_{12} = \pm\sqrt{1 - \left(Q_{11}\right)^2} \tag{4.4.25}$$

Equations (4.4.22) through (4.4.25) reduce the transition matrix (4.4.15) to *eight* possible choices. Four of these are as follows:

Case 1:

$$Q = \left[Q_{kj}\right] = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{21} & Q_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} Q_{11} & -\sqrt{1-\left(Q_{11}\right)^2} & 0 \\ \sqrt{1-\left(Q_{11}\right)^2} & Q_{11} & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4.4.26}$$

Case 2:

$$Q = \left[Q_{kj} \right] = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{21} & Q_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} Q_{11} & \sqrt{1-(Q_{11})^2} & 0 \\ -\sqrt{1-(Q_{11})^2} & Q_{11} & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4.4.27}$$

Case 3:

$$Q = \left[Q_{kj} \right] = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{21} & Q_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} Q_{11} & -\sqrt{1-(Q_{11})^2} & 0 \\ -\sqrt{1-(Q_{11})^2} & -Q_{11} & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4.4.28}$$

and, finally,

Case 4:

$$Q = \left[Q_{kj} \right] = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{21} & Q_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} Q_{11} & \sqrt{1-(Q_{11})^2} & 0 \\ \sqrt{1-(Q_{11})^2} & -Q_{11} & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4.4.29}$$

The other four cases result from formally replacing Q_{11} with $-Q_{11}$ in the above four. Returning to the four cases listed, the details become a little more transparent if we introduce an angle θ that makes $\cos\theta$ the direction cosine between $\mathbf{i}_1$ and $\hat{\mathbf{i}}_1$. Given this interpretation, we can use (4.4.4) to write

$$Q_{11} = \left\langle \mathbf{i}_1, \hat{\mathbf{i}}_1 \right\rangle = \cos\theta \tag{4.4.30}$$

which reduces the above four cases to

Case 1:

$$Q = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4.4.31}$$

Case 2:

$$Q = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4.4.32}$$

Case 3:

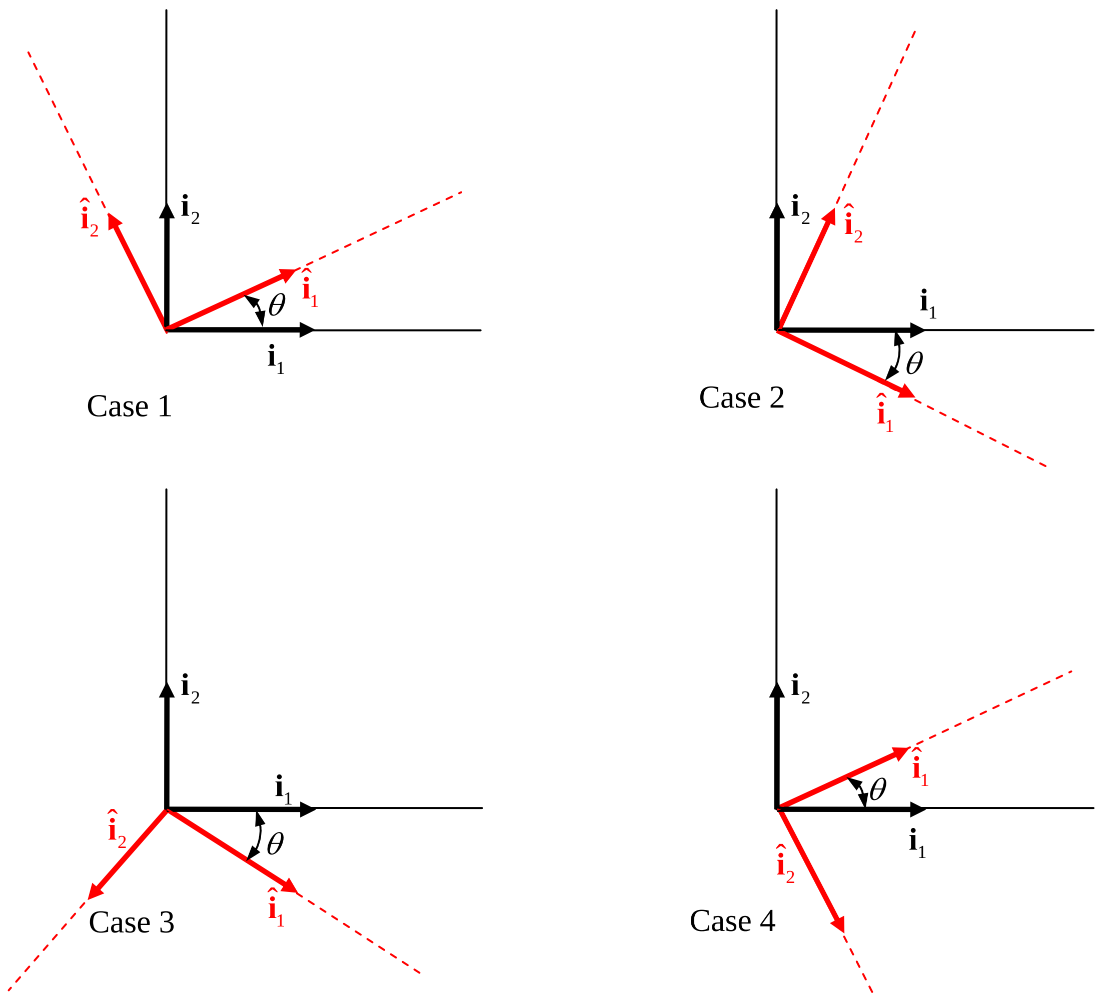
$$Q = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ \sin\theta & -\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4.4.33}$$

and

Case 4:

$$Q = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ -\sin\theta & -\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4.4.34}$$

The first two cases correspond to the situation where $\det Q = 1$ and the second two cases correspond to the case where $\det Q = -1$. Also, Case 2 differs from Case 1 and case 4 from case 3 by the sign of the angle. If Cases 1 and 3 represent some sort of rotation by an amount θ, then Cases 2 and 4 represent the same type of rotation by an amount $-\theta$. The following four figures display geometrically these four cases:



In a sense that we will describe, Case 1 is the fundamental case illustrated by this example. Cases 2 and 3 are similar to 1 and 4, respectively. They simply represent rotations by a negative angle. Case 3 can be thought of as being Case 2 followed by another basis change $\hat{\mathbf{i}}_1 \rightarrow \hat{\mathbf{i}}_1$ and $\hat{\mathbf{i}}_2 \rightarrow -\hat{\mathbf{i}}_2$. Likewise Case 4 can be thought of as being Case 1 followed by another basis change $\hat{\mathbf{i}}_1 \rightarrow \hat{\mathbf{i}}_1$ and $\hat{\mathbf{i}}_2 \rightarrow -\hat{\mathbf{i}}_2$. The bottom line of all of this is that Case 1 represents a basic rotation. Cases 2,3 and 4 represent a negative rotation in one case and a rotation followed by a second basis change which simply flips one of the axes. Cases 3 and 4, as mentioned, have the property that $\det Q = -1$. Rotations with this property involve the kind of axis inversion illustrated by Cases 3 and 4. They are sometimes called *improper rotations*. If we rule out improper rotations and recognize that Case 2 is a special case of Case 1, the transition matrix that characterizes a rotation about the $\mathbf{i}_3$ axis is (4.4.31), repeated,

$$Q = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4.4.35}$$

It should be evident that the transition matrix for rotations about the other axes are

$$Q=\begin{bmatrix}\cos\theta & 0 & -\sin\theta\\ 0 & 1 & 0\\ \sin\theta & 0 & \cos\theta\end{bmatrix} \tag{4.4.36}$$

for a rotation about the $\mathbf{i}_2$ axis and

$$Q=\begin{bmatrix}1 & 0 & 0\\ 0 & \cos\theta & -\sin\theta\\ 0 & \sin\theta & \cos\theta\end{bmatrix} \tag{4.4.37}$$

for a rotation about the $\mathbf{i}_1$ axis.

Exercises

4.4.1 In the applications it is often the case where the angle θ depends upon a parameter t such as the time. The result is, for example, that the orthogonal matrix Q depends upon t. Given $Q(t)$ defined by

$$Q(t)=\begin{bmatrix}\cos\theta(t) & -\sin\theta(t) & 0\\ \sin\theta(t) & \cos\theta(t) & 0\\ 0 & 0 & 1\end{bmatrix} \tag{4.4.38}$$

Show that

$$\frac{dQ(t)}{dt}=Q(t)\begin{bmatrix}0 & -\dfrac{d\theta(t)}{dt} & 0\\ \dfrac{d\theta(t)}{dt} & 0 & 0\\ 0 & 0 & 0\end{bmatrix} \tag{4.4.39}$$

Therefore, the skew symmetric matrix $\begin{bmatrix} 0 & -\frac{d\theta(t)}{dt} & 0 \\ \frac{d\theta(t)}{dt} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, which is determined by $\frac{d\theta(t)}{dt}$, determines the angular velocity for the rotation (4.4.35)

4.4.2 Generalize the results of Exercise 4.4.1 for an arbitrary orthogonal matrix and show that

$$\frac{dQ(t)}{dt} = Q(t)Z(t) \tag{4.4.40}$$

where $Z(t)$ is a skew symmetric matrix, i.e., where $Z(t) = -Z(t)^T$. The result (4.4.40) generalizes the special result (4.4.39) and shows that the angular velocity of the basis $\{\hat{\mathbf{i}}_1, \hat{\mathbf{i}}_2, \hat{\mathbf{i}}_3\}$ with respect to the basis $\{\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3\}$ is determined by $Z(t)$. As a skew symmetric matrix in three dimensions, $Z(t)$ can have only three nonzero components. It is customary to use these three components to define a three dimensional vector which is known as the *angular velocity vector* of the basis $\{\hat{\mathbf{i}}_1, \hat{\mathbf{i}}_2, \hat{\mathbf{i}}_3\}$ with respect to the basis $\{\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3\}$. If $\boldsymbol{\omega}(t)$ is the angular velocity vector, and its components with respect to the basis $\{\hat{\mathbf{i}}_1, \hat{\mathbf{i}}_2, \hat{\mathbf{i}}_3\}$ are related to the components of $Z(t)$ by the simple relationship

$$Z(t) = \begin{bmatrix} 0 & Z_{12} & Z_{13} \\ -Z_{12} & 0 & Z_{23} \\ -Z_{13} & -Z_{23} & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \tag{4.4.41}$$

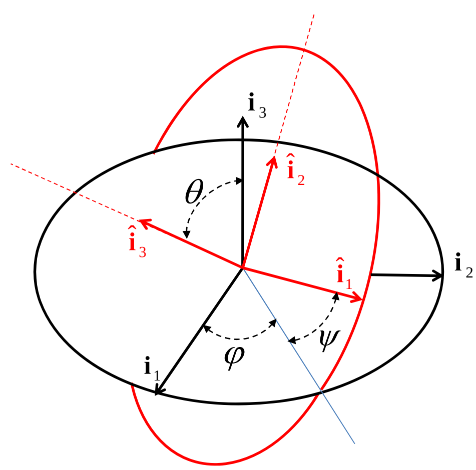
where

$$\boldsymbol{\omega}(t) = \omega_1(t)\hat{\mathbf{i}}_1 + \omega_2(t)\hat{\mathbf{i}}_2 + \omega_3(t)\hat{\mathbf{i}}_3 \tag{4.4.42}$$

Section 4.5 Euler Angles[5]

Geometric constructions like the one discussed in Section 4.4 arise in a lot of applications. There is an entire branch of mechanics where one studies the motion of rigid bodies like, for example, gyroscopes, where the basis $\{\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3\}$ is a reference or fixed orientation in space and the basis $\{\hat{\mathbf{i}}_1, \hat{\mathbf{i}}_2, \hat{\mathbf{i}}_3\}$ is fixed to the rigid body and, thus, defines the position of the rigid body relative to the fixed orientation. Another application is when one thinks of the basis $\{\hat{\mathbf{i}}_1, \hat{\mathbf{i}}_2, \hat{\mathbf{i}}_3\}$ as fixed to the body of an aircraft and its orientation relative to $\{\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3\}$ gives the orientation of the aircraft. These applications get rather complicated as one tries to characterize the position of, for example, a rigid body as a consequence of a general rotation. The usual approach is to view the final position as the result of a sequence of three rotations of the form of (4.4.35) through (4.4.37). In aerodynamics the three rotations are known by the names of *roll*, *pitch* and *yaw*. In a more general context of rigid body dynamics they are known as the *Euler angles*.[6]

The usual way these three rotations are represented is shown in the following figure:



The sequence of rotations are:

1. Rotate about the $\mathbf{i}_3$ axis by an angle φ.

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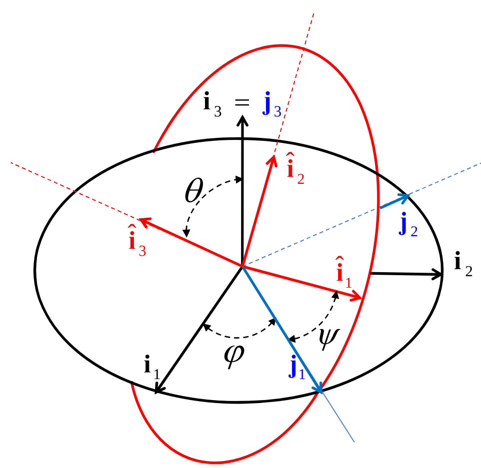
[5] Information about Leonhard Paul Euler can be found at http://en.wikipedia.org/wiki/Leonhard_Euler.
[6] There is an excellent discussion of Euler angles at http://en.wikipedia.org/wiki/Euler_angles.

2. Rotate about the "rotated" $\mathbf{i}_1$ axis by an angle θ which aligns the "rotated" $\mathbf{i}_3$ with $\hat{\mathbf{i}}_3$.
3. Rotate about the $\hat{\mathbf{i}}_3$ axis by an angle ψ to align with $\hat{\mathbf{i}}_1$ and $\hat{\mathbf{i}}_2$

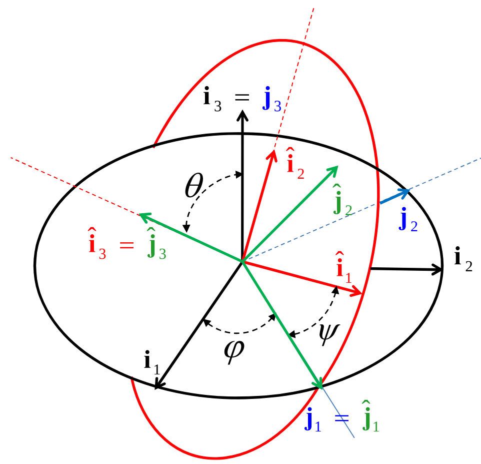
The details to construct these bases changes are complicated. We really need four bases to fully characterize the three rotations listed. Two of them are the first basis, $\{\mathbf{i}_1,\mathbf{i}_2,\mathbf{i}_3\}$, and the final basis, $\{\hat{\mathbf{i}}_1,\hat{\mathbf{i}}_2,\hat{\mathbf{i}}_3\}$. Unfortunately, we need to introduce more symbols. The four bases are labeled as follows:

$$\{\mathbf{i}_1,\mathbf{i}_2,\mathbf{i}_3\}\xrightarrow[\varphi]{}\{\mathbf{j}_1,\mathbf{j}_2,\mathbf{j}_3\}\xrightarrow[\theta]{}\{\hat{\mathbf{j}}_1,\hat{\mathbf{j}}_2,\hat{\mathbf{j}}_3\}\xrightarrow[\psi]{}\{\hat{\mathbf{i}}_1,\hat{\mathbf{i}}_2,\hat{\mathbf{i}}_3\}$$

The basis change from $\{\mathbf{i}_1,\mathbf{i}_2,\mathbf{i}_3\}$ to $\{\mathbf{j}_1,\mathbf{j}_2,\mathbf{j}_3\}$ is shown in the following figure:



The basis change from $\{\mathbf{i}_1,\mathbf{i}_2,\mathbf{i}_3\}$ to $\{\mathbf{j}_1,\mathbf{j}_2,\mathbf{j}_3\}$ from $\{\mathbf{j}_1,\mathbf{j}_2,\mathbf{j}_3\}$ to $\{\hat{\mathbf{j}}_1,\hat{\mathbf{j}}_2,\hat{\mathbf{j}}_3\}$ and $\{\hat{\mathbf{j}}_1,\hat{\mathbf{j}}_2,\hat{\mathbf{j}}_3\}$ to $\{\hat{\mathbf{i}}_1,\hat{\mathbf{i}}_2,\hat{\mathbf{i}}_3\}$ are shown in the following figure:



The three transition matrices that characterize the three rotations just described are

$$\{\mathbf{i}_1,\mathbf{i}_2,\mathbf{i}_3\}\xrightarrow[\varphi]{}\{\mathbf{j}_1,\mathbf{j}_2,\mathbf{j}_3\}$$

$$Q_\varphi=\begin{bmatrix}\cos\varphi & -\sin\varphi & 0\\ \sin\varphi & \cos\varphi & 0\\ 0 & 0 & 1\end{bmatrix} \tag{4.5.1}$$

$$\{\mathbf{j}_1,\mathbf{j}_2,\mathbf{j}_3\}\xrightarrow[\theta]{}\{\hat{\mathbf{j}}_1,\hat{\mathbf{j}}_2,\hat{\mathbf{j}}_3\}$$

$$Q_\theta=\begin{bmatrix}1 & 0 & 0\\ 0 & \cos\theta & -\sin\theta\\ 0 & \sin\theta & \cos\theta\end{bmatrix} \tag{4.5.2}$$

and

$$\{\hat{\mathbf{j}}_1,\hat{\mathbf{j}}_2,\hat{\mathbf{j}}_3\}\xrightarrow[\psi]{}\{\hat{\mathbf{i}}_1,\hat{\mathbf{i}}_2,\hat{\mathbf{i}}_3\}$$

$$Q_\psi=\begin{bmatrix}\cos\psi & -\sin\psi & 0\\ \sin\psi & \cos\psi & 0\\ 0 & 0 & 1\end{bmatrix} \tag{4.5.3}$$

The transition matrix for the rotation $\{\mathbf{i}_1,\mathbf{i}_2,\mathbf{i}_3\}\to\{\hat{\mathbf{i}}_1,\hat{\mathbf{i}}_2,\hat{\mathbf{i}}_3\}$ is the product[7]

$$Q = Q_\varphi Q_\theta Q_\psi \tag{4.5.4}$$

Equation (4.5.4) gives the transition matrix for a complicated rotation in three dimensions in terms of three elementary rotations. The utility of the Euler angles is the ability to break down complicated rotations in this fashion. The mathematical feature that makes a representation like (4.5.4) possible is a special property of the set of orthogonal matrices. The set of orthogonal matrices is an example of what is known as a *group*. For our purposes, we do not need to examine this concept in any detail. It suffices to point out that the identity matrix is orthogonal and for each orthogonal matrix, its inverse is also orthogonal. In addition, orthogonal matrices have the features that if two orthogonal matrices are multiplied together, the result is an orthogonal matrix and they obey the usual matrix multiplication property of associativity.

Exercises:

4.5.1 Carry out the multiplication in (4.5.4) and show that

$$Q=\begin{bmatrix} \cos\varphi\cos\psi-\sin\varphi\cos\theta\sin\psi & -\sin\psi\cos\varphi-\cos\psi\cos\theta\sin\varphi & \sin\theta\sin\varphi \\ \cos\psi\sin\varphi+\sin\psi\cos\theta\cos\varphi & \cos\varphi\cos\theta\cos\psi-\sin\varphi\sin\psi & -\sin\theta\cos\varphi \\ \sin\theta\sin\psi & \cos\psi\sin\theta & \cos\theta \end{bmatrix} \tag{4.5.5}$$

If one were given the components of the orthogonal matrix Q, equation (4.5.5) can be used to calculate the three angles φ,θ and ψ. This information then determines the three elementary rotations represented by the matrices (4.5.1), (4.5.2) and (4.5.3). The key formulas that follow from (4.5.5) are

$$Q_{33}=\cos\theta \tag{4.5.6}$$

$$\left.\begin{aligned} Q_{31}&=\sin\theta\sin\psi \\ Q_{32}&=\sin\theta\cos\psi \end{aligned}\right\}\Rightarrow \tan\psi=\frac{Q_{31}}{Q_{32}} \qquad \text{if} \quad \sin\theta\neq 0 \tag{4.5.7}$$

and

$$\left.\begin{aligned} Q_{13}&=\sin\theta\sin\varphi \\ Q_{23}&=-\sin\theta\cos\varphi \end{aligned}\right\}\Rightarrow \tan\varphi=-\frac{Q_{13}}{Q_{23}} \quad \text{if} \quad \sin\theta\neq 0 \tag{4.5.8}$$

[7] Equation (4.5.4) follows from the application of (3.6.18) to each of the three basis transformations represented by Q_φ, Q_θ and Q_ψ.

Because of the multiplicity of the inverse trigometric functions, the solution obtained from (4.5.6), (4.5.7) and (4.5.8) will not be unique.

4.5.2 Recall from Exercise 4.4.2 that the angular velocity associated with an orthogonal matrix $Q(t)$ is determined by a skew symmetric matrix $Z(t)$ defined by

$$Z(t) = Q(t)^T \frac{dQ(t)}{dt} \tag{4.5.9}$$

Utilize (4.5.4) and show that the skew symmetric matrix $Z(t)$ can be expressed in terms of the corresponding angular velocities of the rotations Q_ψ, Q_θ and Q_φ by the formula

$$Z(t) = Z_\psi(t) + Q_\psi{}^T Z_\theta Q_\psi + Q_\psi{}^T Q_\theta{}^T Z_\varphi Q_\theta Q_\psi \tag{4.5.10}$$

The components of equation (4.5.10) give the angular velocity of the basis $\{\hat{\mathbf{i}}_1, \hat{\mathbf{i}}_2, \hat{\mathbf{i}}_3\}$ with respect to the basis $\{\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3\}$. Its components are the components of a linear transformation projected into the basis $\{\hat{\mathbf{i}}_1, \hat{\mathbf{i}}_2, \hat{\mathbf{i}}_3\}$.

4.5.3 Utilize the formulas for the various matrices in (4.5.10) and show that

$$Z_\varphi(t) = \begin{bmatrix} 0 & -\dot{\varphi}(t) & 0 \\ \dot{\varphi}(t) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{4.5.11}$$

$$Z_\theta(t) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\dot{\theta}(t) \\ 0 & \dot{\theta}(t) & 0 \end{bmatrix} \tag{4.5.12}$$

$$Z_\psi(t) = \begin{bmatrix} 0 & -\dot{\psi}(t) & 0 \\ \dot{\psi}(t) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{4.5.13}$$

and

$$Z(t)=\begin{bmatrix} 0 & -\dot{\psi}(t)-\dot{\varphi}(t)\cos\theta & \dot{\varphi}(t)\cos\psi\sin\theta-\dot{\theta}(t)\sin\psi \\ \dot{\psi}(t)+\dot{\varphi}(t)\cos\theta & 0 & -\dot{\theta}(t)\cos\psi-\dot{\varphi}(t)\sin\psi\sin\theta \\ -\dot{\varphi}(t)\cos\psi\sin\theta+\dot{\theta}(t)\sin\psi & \dot{\theta}(t)\cos\psi+\dot{\varphi}(t)\sin\psi\sin\theta & 0 \end{bmatrix} \tag{4.5.14}$$

The components of the matrix (4.5.14) define the components of the angular velocity vector with respect to the basis $\{\hat{\mathbf{i}}_1,\hat{\mathbf{i}}_2,\hat{\mathbf{i}}_3\}$ according to the correspondence (4.4.41). Therefore,

$$\begin{aligned}\boldsymbol{\omega}(t)&=\omega_{\hat{x}_1}\hat{\mathbf{i}}_1+\omega_{\hat{x}_2}\hat{\mathbf{i}}_2+\omega_{\hat{x}_3}\hat{\mathbf{i}}_3\\ &=\left(\dot{\theta}(t)\cos\psi+\dot{\varphi}(t)\sin\psi\sin\theta\right)\hat{\mathbf{i}}_1+\left(\dot{\varphi}(t)\cos\psi\sin\theta-\dot{\theta}(t)\sin\psi\right)\hat{\mathbf{i}}_2+\left(\dot{\psi}(t)+\dot{\varphi}(t)\cos\theta\right)\hat{\mathbf{i}}_3\end{aligned} \tag{4.5.15}$$

4.5.4 As explained above, equation (4.5.14) represents the components of a linear transformation project into the basis $\{\hat{\mathbf{i}}_1,\hat{\mathbf{i}}_2,\hat{\mathbf{i}}_3\}$. That same linear transformation projected into the basis $\{\mathbf{i}_1,\mathbf{i}_2,\mathbf{i}_3\}$ is related to the matrix (4.5.14) by the change of basis formula (3.6.18). In this case, the transition matrix is (4.5.4). Show that the resulting matrix is

$$QZ(t)Q^T=\begin{bmatrix} 0 & -\dot{\varphi}(t)-\dot{\psi}(t)\cos\theta & \dot{\theta}(t)\sin\varphi-\dot{\psi}(t)\cos\varphi\sin\theta \\ \dot{\varphi}(t)+\dot{\psi}(t)\cos\theta & 0 & -\dot{\theta}(t)\cos\varphi-\dot{\psi}(t)\sin\varphi\sin\theta \\ -\dot{\theta}\sin\varphi+\dot{\psi}(t)\cos\varphi\sin\theta & \dot{\theta}(t)\cos\varphi+\dot{\psi}(t)\sin\varphi\sin\theta & 0 \end{bmatrix} \tag{4.5.16}$$

The components of the matrix (4.5.16) define the components of the angular velocity vector with respect to the basis $\{\mathbf{i}_1,\mathbf{i}_2,\mathbf{i}_3\}$ according to the correspondence (4.4.41). Therefore,

$$\begin{aligned}\boldsymbol{\omega}(t)&=\omega_{x_1}\mathbf{i}_1+\omega_{x_2}\mathbf{i}_2+\omega_{x_3}\mathbf{i}_3\\ &=\left(\dot{\theta}\cos\varphi+\dot{\psi}(t)\sin\varphi\sin\theta\right)\mathbf{i}_1+\left(\dot{\theta}\sin\varphi-\dot{\psi}(t)\cos\varphi\sin\theta\right)\mathbf{i}_2+\left(\dot{\varphi}(t)+\dot{\psi}(t)\cos\theta\right)\mathbf{i}_3\end{aligned} \tag{4.5.17}$$

Section 4.6. Cross Products on Three Dimensional Inner Product Spaces

In Sections 4.4 and 4.5 involved certain ideas that are special to three dimensional real inner product spaces. This section concerns another topic that is also special to these spaces. While portions of the topics in Sections 4.4 and 4.5 can be generalized to inner product spaces of arbitrary dimension, the topic of this section is for all practical purposes a unique one for three dimensional vector spaces. The topic we shall briefly discuss is how to assign the operation of a *cross product* to a real inner product space of dimension three.

Definition: A *cross product* in a three dimensional real inner product space $\mathscr{V}$ is a function, written $\mathbf{u}\times\mathbf{v}$, from $\mathscr{V}\times\mathscr{V}\to\mathscr{V}$ such that

(1) $\mathbf{u}\times\mathbf{v}=-\mathbf{v}\times\mathbf{u}$

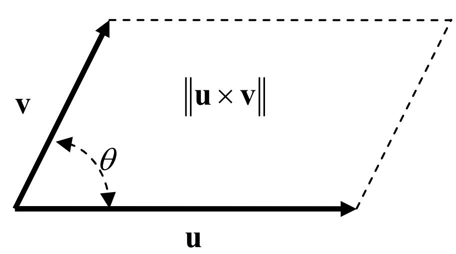
(2) $\mathbf{w}\times(\mathbf{u}+\mathbf{v})=\mathbf{w}\times\mathbf{u}+\mathbf{w}\times\mathbf{v}$

(3) $\mu(\mathbf{u}\times\mathbf{v})=(\mu\mathbf{u})\times\mathbf{v}$ for all $\mathbf{u},\mathbf{v},\mathbf{w}\in\mathscr{V}$ and $\mu\in\mathscr{R}$

(4) $\langle\mathbf{u},\mathbf{u}\times\mathbf{v}\rangle=\mathbf{0}$

(5) $\|\mathbf{u}\times\mathbf{v}\|=\|\mathbf{u}\|\|\mathbf{v}\|\sin\theta$ where $\cos\theta=\dfrac{\langle\mathbf{u},\mathbf{v}\rangle}{\|\mathbf{u}\|\|\mathbf{v}\|}$ and $0\le\theta\le 180^o$.

Geometrically, $\|\mathbf{u}\times\mathbf{v}\|$ is the area of the parallelogram with sides $\|\mathbf{u}\|$ and $\|\mathbf{v}\|$ intersecting at an angle θ.



It follows from properties (1) and (2) that

$$(\mathbf{u}+\mathbf{v})\times\mathbf{w}=\mathbf{u}\times\mathbf{w}+\mathbf{v}\times\mathbf{w} \tag{4.6.1}$$

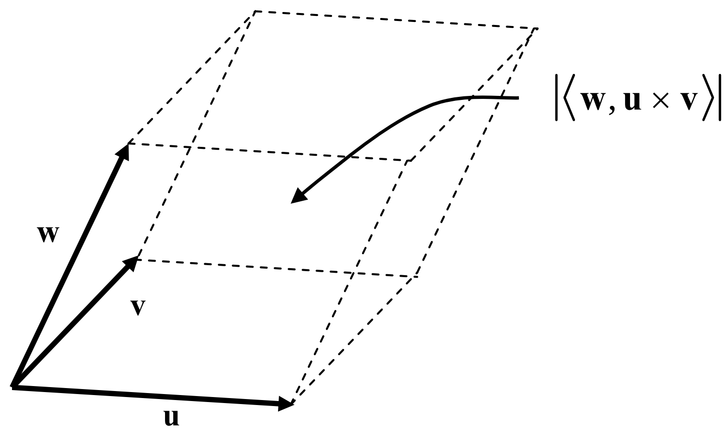
and from properties (1) and (3)

$$\mu(\mathbf{u}\times\mathbf{v}) = \mathbf{u}\times(\mu\mathbf{v}) \tag{4.6.2}$$

Also, from property (1) that

$$\mathbf{u}\times\mathbf{u} = \mathbf{0} \tag{4.6.3}$$

The *scalar triple product* of three vectors $\mathbf{u},\mathbf{v}$ and $\mathbf{w}$ in $\mathscr{V}$ is $\langle\mathbf{w},\mathbf{u}\times\mathbf{v}\rangle$. Geometrically, $|\langle\mathbf{w},\mathbf{u}\times\mathbf{v}\rangle|$ is the volume of the parallelepiped formed by the co-terminus sides $\mathbf{u},\mathbf{v}$ and $\mathbf{w}$.



If $\{\mathbf{i}_1,\mathbf{i}_2,\mathbf{i}_3\}$ is an orthonormal basis for $\mathscr{V}$. Then the cross product connects the basis members by the formulas

$$\begin{aligned}\mathbf{i}_1\times\mathbf{i}_2 &= \pm\mathbf{i}_3\\ \mathbf{i}_3\times\mathbf{i}_1 &= \pm\mathbf{i}_2\\ \mathbf{i}_2\times\mathbf{i}_3 &= \pm\mathbf{i}_1\end{aligned} \tag{4.6.4}$$

The proof of (4.6.4) involves the kinds of arguments we have used many times. Since $\mathbf{i}_1\times\mathbf{i}_2\in\mathscr{V}$, this vector can be expanded in the basis $\{\mathbf{i}_1,\mathbf{i}_2,\mathbf{i}_3\}$ and written

$$\mathbf{i}_1\times\mathbf{i}_2 = \mu\mathbf{i}_1+\lambda\mathbf{i}_2+\xi\mathbf{i}_3 \tag{4.6.5}$$

According property (4), $\langle\mathbf{i}_1,\mathbf{i}_1\times\mathbf{i}_2\rangle=0$ and $\langle\mathbf{i}_2,\mathbf{i}_1\times\mathbf{i}_2\rangle=0$. Therefore, $\mu=\lambda=0$ and (4.6.5) reduces to

$$\mathbf{i}_1\times\mathbf{i}_2 = \xi\mathbf{i}_3 \tag{4.6.6}$$

It follows from property (5) that $\left\|\mathbf{i}_1 \times \mathbf{i}_2\right\| = 1$. This fact and (4.6.6) yield $\xi = \pm 1$ and the result $(4.6.4)_1$ is obtained. The other two results follow by an identical argument.

Given the choices (4.6.4), the component formula for the cross product for vectors

$$\mathbf{u} = \sum_{j=1}^{3} u_j \mathbf{i}_j \tag{4.6.7}$$

and

$$\mathbf{v} = \sum_{k=1}^{3} \upsilon_k \mathbf{i}_k \tag{4.6.8}$$

can be written

$$\begin{aligned} \mathbf{u} \times \mathbf{v} &= \sum_{j=1}^{3} u_j \mathbf{i}_j \times \sum_{k=1}^{3} \upsilon_k \mathbf{i}_k = \sum_{j=1}^{3}\sum_{k=1}^{3} u_j \upsilon_k \mathbf{i}_j \times \mathbf{i}_k \\ &= \pm(u_2\upsilon_3 - u_3\upsilon_2)\mathbf{i}_1 \pm (u_3\upsilon_1 - u_1\upsilon_3)\mathbf{i}_2 \pm (u_1\upsilon_2 - u_2\upsilon_1)\mathbf{i}_3 \end{aligned} \tag{4.6.9}$$

Equation (4.6.9), which has its origin in (4.6.4) shows that there are *two* cross products in the three dimensional inner product space $\mathscr{V}$. These two cases are characterized as follows:

Definition: A vector space $\mathscr{V}$ with the cross product

$$\mathbf{u} \times \mathbf{v} = (u_2\upsilon_3 - u_3\upsilon_2)\mathbf{i}_1 + (u_3\upsilon_1 - u_1\upsilon_3)\mathbf{i}_2 + (u_1\upsilon_2 - u_2\upsilon_1)\mathbf{i}_3 \tag{4.6.10}$$

is said to have *positive orientation*.

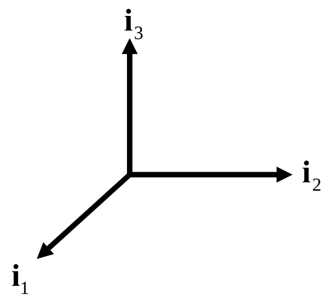
Definition: A vector space $\mathscr{V}$ with the cross product

$$\mathbf{u} \times \mathbf{v} = -(u_2\upsilon_3 - u_3\upsilon_2)\mathbf{i}_1 - (u_3\upsilon_1 - u_1\upsilon_3)\mathbf{i}_2 - (u_1\upsilon_2 - u_2\upsilon_1)\mathbf{i}_3 \tag{4.6.11}$$

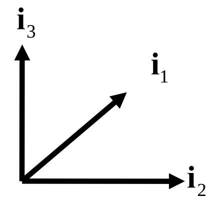
is said to have *negative orientation*.

Vector spaces with positive orientation are sometimes called *right handed*, and vector spaces with negative orientation are sometimes call *left handed*.

The following two figures illustrate the bases for positive oriented and negative oriented vector spaces:



Positive Orientation



Negative Orientation

Given the component formula (4.6.9), the scalar triple product $\langle \mathbf{w}, \mathbf{u}\times\mathbf{v}\rangle$ of three vectors $\mathbf{u},\mathbf{v}$ and $\mathbf{w}$ in $\mathscr{V}$ is

$$\begin{aligned}\langle \mathbf{w}, \mathbf{u}\times\mathbf{v}\rangle &= \left\langle \mathbf{w}, \sum_{j=1}^{3}\sum_{k=1}^{3} u_j \upsilon_k \mathbf{i}_j \times \mathbf{i}_k \right\rangle \\ &= \pm(u_2\upsilon_3 - u_3\upsilon_2)\langle \mathbf{w}, \mathbf{i}_1\rangle \pm (u_3\upsilon_1 - u_1\upsilon_3)\langle \mathbf{w}, \mathbf{i}_2\rangle \pm (u_1\upsilon_2 - u_2\upsilon_1)\langle \mathbf{w}, \mathbf{i}_3\rangle \\ &= \pm w_1(u_2\upsilon_3 - u_3\upsilon_2) \pm w_2(u_3\upsilon_1 - u_1\upsilon_3) \pm w_3(u_1\upsilon_2 - u_2\upsilon_1)\end{aligned} \tag{4.6.12}$$

Often the result (4.6.12) is written

$$\langle \mathbf{w}, \mathbf{u}\times\mathbf{v}\rangle = \pm\begin{vmatrix} w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \\ \upsilon_1 & \upsilon_2 & \upsilon_3 \end{vmatrix} \tag{4.6.13}$$

One of the results illustrated by (4.6.12) or, equivalently, (4.6.13) is

$$\begin{aligned}\langle \mathbf{w}, \mathbf{u}\times\mathbf{v}\rangle = \langle \mathbf{v}, \mathbf{w}\times\mathbf{u}\rangle = \langle \mathbf{u}, \mathbf{v}\times\mathbf{w}\rangle &= -\langle \mathbf{w}, \mathbf{v}\times\mathbf{u}\rangle \\ &= -\langle \mathbf{v}, \mathbf{u}\times\mathbf{w}\rangle \\ &= -\langle \mathbf{u}, \mathbf{w}\times\mathbf{v}\rangle\end{aligned} \tag{4.6.14}$$

The results (4.6.14) also follow from a geometric argument based upon the interpretation of $\left|\langle \mathbf{w}, \mathbf{u}\times\mathbf{v}\rangle\right|$ as the volume shown above.

Exercises

4.6.1 Derive the vector identity

$$\mathbf{u}\times\left(\mathbf{v}\times\mathbf{w}\right)=\left(\mathbf{w}\times\mathbf{v}\right)\times\mathbf{u}=\mathbf{v}\left\langle\mathbf{u},\mathbf{w}\right\rangle-\mathbf{w}\left\langle\mathbf{u},\mathbf{v}\right\rangle \tag{4.6.15}$$

4.6.2 Derive the vector identity

$$\left\langle\mathbf{s}\times\mathbf{u},\mathbf{v}\times\mathbf{w}\right\rangle=\left\langle\mathbf{s},\mathbf{v}\right\rangle\left\langle\mathbf{u},\mathbf{w}\right\rangle-\left\langle\mathbf{s},\mathbf{w}\right\rangle\left\langle\mathbf{u},\mathbf{v}\right\rangle \tag{4.6.16}$$

Section 4.7. Reciprocal Bases

The results of Sections 4.4, 4.5 and 4.6 are based upon the convenience of orthonormal bases. The results of these sections were also restricted to the special case of a *real* inner product space of *three* dimensions. These results are useful in a large variety of applications of linear algebra. Unfortunately, there are important physical problems for which an orthonormal basis is not the best choice and, in addition, there are important applications where the vector space is not real and it is not of dimension three. Therefore, in this section, we shall look at bases that capture some but not all of the convenience of orthonormal bases called *reciprocal bases*. In addition, we shall not assume the vector space is real or that it is three dimensional.

Let $\{\mathbf{e}_1,\mathbf{e}_2,...,\mathbf{e}_N\}$ be a basis for an N dimensional vector space $\mathscr{V}$. In equation (4.2.17) of Section 4.2, we defined the symbols e_{jk}, for $j,k=1,2,...,N$, by

$$e_{jk}=\langle \mathbf{e}_j,\mathbf{e}_k\rangle=\overline{\langle \mathbf{e}_k,\mathbf{e}_j\rangle}=\overline{e_{kj}} \tag{4.7.1}$$

Because $\{\mathbf{e}_1,\mathbf{e}_2,...,\mathbf{e}_N\}$ is a basis, the $N\times N$ matrix

$$\left[e_{jk}\right]=\begin{bmatrix} \langle \mathbf{e}_1,\mathbf{e}_1\rangle & \langle \mathbf{e}_1,\mathbf{e}_2\rangle & \langle \mathbf{e}_1,\mathbf{e}_3\rangle & \cdot & \cdot & \langle \mathbf{e}_1,\mathbf{e}_N\rangle \\ \langle \mathbf{e}_2,\mathbf{e}_1\rangle & \langle \mathbf{e}_2,\mathbf{e}_2\rangle & \langle \mathbf{e}_2,\mathbf{e}_3\rangle & & & \langle \mathbf{e}_2,\mathbf{e}_N\rangle \\ \langle \mathbf{e}_3,\mathbf{e}_1\rangle & \langle \mathbf{e}_3,\mathbf{e}_1\rangle & \langle \mathbf{e}_3,\mathbf{e}_1\rangle & & & \cdot \\ \cdot & & & \cdot & & \cdot \\ \cdot & & & & \cdot & \cdot \\ \langle \mathbf{e}_N,\mathbf{e}_1\rangle & \langle \mathbf{e}_N,\mathbf{e}_1\rangle & \cdot & \cdot & \cdot & \langle \mathbf{e}_N,\mathbf{e}_N\rangle \end{bmatrix} \tag{4.7.2}$$

has rank N and is thus nonsingular. To confirm this assertion about the rank, we can examine the linear independence of the columns by the usual test, namely, examine whether or not the equation

$$\alpha_1\begin{bmatrix} \langle \mathbf{e}_1,\mathbf{e}_1\rangle \\ \langle \mathbf{e}_2,\mathbf{e}_1\rangle \\ \langle \mathbf{e}_3,\mathbf{e}_1\rangle \\ \cdot \\ \cdot \\ \langle \mathbf{e}_N,\mathbf{e}_1\rangle \end{bmatrix}+\alpha_2\begin{bmatrix} \langle \mathbf{e}_1,\mathbf{e}_2\rangle \\ \langle \mathbf{e}_2,\mathbf{e}_2\rangle \\ \langle \mathbf{e}_3,\mathbf{e}_2\rangle \\ \cdot \\ \cdot \\ \langle \mathbf{e}_N,\mathbf{e}_2\rangle \end{bmatrix}+\cdots+\alpha_N\begin{bmatrix} \langle \mathbf{e}_1,\mathbf{e}_N\rangle \\ \langle \mathbf{e}_2,\mathbf{e}_N\rangle \\ \langle \mathbf{e}_3,\mathbf{e}_N\rangle \\ \cdot \\ \cdot \\ \langle \mathbf{e}_N,\mathbf{e}_N\rangle \end{bmatrix}=0 \tag{4.7.3}$$

forces the coefficients $\alpha_1,\alpha_2,...,\alpha_N$ to be zero. If we use the linearity of the inner product in its second slot, (4.7.3) can be rewritten

$$\left\langle \mathbf{e}_j, \bar{\alpha}_1\mathbf{e}_1 + \bar{\alpha}_2\mathbf{e}_2 + \cdots + \bar{\alpha}_N\mathbf{e}_N \right\rangle = 0 \qquad \text{for} \quad j = 1,2,\ldots,N \tag{4.7.4}$$

Because $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_N\}$ is a basis, (4.7.4) implies

$$\left\langle \mathbf{v}, \bar{\alpha}_1\mathbf{e}_1 + \bar{\alpha}_2\mathbf{e}_2 + \cdots + \bar{\alpha}_N\mathbf{e}_N \right\rangle = 0 \qquad \text{for all} \quad \mathbf{v} \in \mathscr{V} \tag{4.7.5}$$

Property (4) of the defining properties of an inner product space listed in Section 4.1 tells us that (4.7.5) implies

$$\bar{\alpha}_1\mathbf{e}_1 + \bar{\alpha}_2\mathbf{e}_2 + \cdots + \bar{\alpha}_N\mathbf{e}_N = \mathbf{0} \tag{4.7.6}$$

If we again use the fact that $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_N\}$ is a basis, we can conclude that $\alpha_1 = \alpha_2 = \cdots = \alpha_N = 0$.

Given the basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_N\}$ of $\mathscr{V}$, we shall next define what is known as the reciprocal basis as follows:

Definition: The *reciprocal basis* to a basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_N\}$ of $\mathscr{V}$ is a basis $\{\mathbf{e}^1, \mathbf{e}^2, \ldots, \mathbf{e}^N\}$ of $\mathscr{V}$ defined by

$$\left\langle \mathbf{e}^k, \mathbf{e}_j \right\rangle = \delta_j^k \qquad \text{for} \quad j,k = 1,2,\ldots,N \tag{4.7.7}$$

Geometrically, the definition (4.7.7) says that the reciprocal basis vector $\mathbf{e}^k$ is a vector *perpendicular* to the $N-1$ basis vectors $\mathbf{e}_1, \ldots, \mathbf{e}_{k-1}, \mathbf{e}_{k+1}, \ldots, \mathbf{e}_N$.

An obvious example of a reciprocal basis is the orthonormal basis $\{\mathbf{i}_1, \mathbf{i}_2, \ldots, \mathbf{i}_N\}$ which by (4.3.6) is its own reciprocal basis. Thus, the distinction between basis and reciprocal basis vanishes when one begins with an orthonormal basis.

Given the two basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_N\}$ and $\{\mathbf{e}^1, \mathbf{e}^2, \ldots, \mathbf{e}^N\}$ for $\mathscr{V}$, they must be connected by the usual formulas like (2.6.3). If we rewrite (2.6.3) to fit the notation used in this section, the two bases $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_N\}$ and $\{\mathbf{e}^1, \mathbf{e}^2, \ldots, \mathbf{e}^N\}$ must be related, for example, by

$$\mathbf{e}_j = \sum_{q=1}^{N} T_{qj}\mathbf{e}^q \tag{4.7.8}$$

It immediately follows from (4.7.1) and (4.7.7) that the components of the transition matrix are $T_{kj} = e_{jk}$, and, as a result, the basis change (4.7.8) is

$$\mathbf{e}_j = \sum_{q=1}^{N} e_{jq}\mathbf{e}^q \quad \text{for} \quad j = 1,2,...,N \tag{4.7.9}$$

The basis change $\{\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_N\} \rightarrow \{\mathbf{e}^1, \mathbf{e}^2, ..., \mathbf{e}^N\}$ is

$$\mathbf{e}^j = \sum_{q=1}^{N} e^{jq}\mathbf{e}_q \quad \text{for} \quad j = 1,2,...,N \tag{4.7.10}$$

where the matrix $\left[e^{jq}\right]$ is the inverse matrix to the matrix $\left[e_{jk}\right]$. It is useful to note that (4.7.7) and (4.7.10) show that

$$e^{jk} = \left\langle \mathbf{e}^j, \mathbf{e}^k \right\rangle = \overline{\left\langle \mathbf{e}^j, \mathbf{e}^k \right\rangle} = \overline{e^{kj}} \tag{4.7.11}$$

Example 4.7.1: Consider the following elementary construction for the two dimensional inner product space $\mathscr{R}^2$. The base vectors $\{\mathbf{e}_1, \mathbf{e}_2\}$ consist of two vectors 45^o apart, the first one, $\mathbf{e}_1$, two units long and the second one, $\mathbf{e}_2$, one unit long. If we calculate the inner products in (4.7.1), the given information yields

$$\left[e_{jk}\right] = \begin{bmatrix} \langle \mathbf{e}_1, \mathbf{e}_1 \rangle & \langle \mathbf{e}_1, \mathbf{e}_2 \rangle \\ \langle \mathbf{e}_2, \mathbf{e}_1 \rangle & \langle \mathbf{e}_2, \mathbf{e}_2 \rangle \end{bmatrix} = \begin{bmatrix} 4 & \sqrt{2} \\ \sqrt{2} & 1 \end{bmatrix} \tag{4.7.12}$$

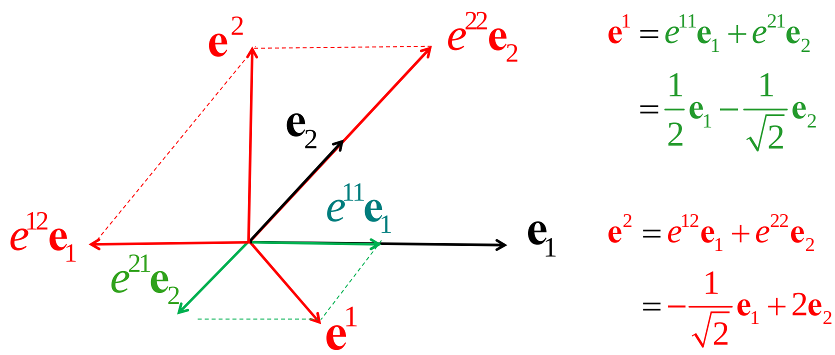
The inverse of this matrix is easily seen to be

$$\left[e^{jk}\right] = \begin{bmatrix} \dfrac{1}{2} & -\dfrac{1}{\sqrt{2}} \\ -\dfrac{1}{\sqrt{2}} & 2 \end{bmatrix} \tag{4.7.13}$$

Therefore, from (4.7.10),

$$\begin{aligned} \mathbf{e}^1 &= e^{11}\mathbf{e}_1 + e^{21}\mathbf{e}_2 = \frac{1}{2}\mathbf{e}_1 - \frac{1}{\sqrt{2}}\mathbf{e}_2 \\ \mathbf{e}^2 &= e^{21}\mathbf{e}_1 + e^{22}\mathbf{e}_2 = -\frac{1}{\sqrt{2}}\mathbf{e}_1 + 2\mathbf{e}_2 \end{aligned} \tag{4.7.14}$$

The following figure shows the basis $\{\mathbf{e}_1, \mathbf{e}_2\}$, its reciprocal $\{\mathbf{e}^1, \mathbf{e}^2\}$ and the graphical representation of (4.7.14).



Example 4.7.2: You are given a basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ of $\mathscr{M}^{3\times 1}$ defined by

$$\mathbf{e}_1 = \begin{bmatrix} 0.9659 \\ 0.2588 \\ 0 \end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix} 0.2588 \\ 0.9659 \\ 0 \end{bmatrix}, \mathbf{e}_3 = \begin{bmatrix} 0.2500 \\ 0.0670 \\ 0.9659 \end{bmatrix} \tag{4.7.15}$$

The problem is to find its reciprocal basis $\{\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3\}$. The inner product for $\mathscr{M}^{3\times 1}$ is, of course, given by (4.1.13), repeated,

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^T \overline{\mathbf{v}} = \overline{\mathbf{v}}^T \mathbf{u} \tag{4.7.16}$$

This definition allows us to utilize (4.7.15) and (4.7.2) to obtain

$$\left[e_{jk} \right] = \begin{bmatrix} \langle \mathbf{e}_1, \mathbf{e}_1 \rangle & \langle \mathbf{e}_1, \mathbf{e}_2 \rangle & \langle \mathbf{e}_1, \mathbf{e}_3 \rangle \\ \langle \mathbf{e}_2, \mathbf{e}_1 \rangle & \langle \mathbf{e}_2, \mathbf{e}_2 \rangle & \langle \mathbf{e}_2, \mathbf{e}_3 \rangle \\ \langle \mathbf{e}_3, \mathbf{e}_1 \rangle & \langle \mathbf{e}_3, \mathbf{e}_2 \rangle & \langle \mathbf{e}_3, \mathbf{e}_3 \rangle \end{bmatrix} = \begin{bmatrix} 1.0000 & 0.5000 & 0.2588 \\ 0.5000 & 1.0000 & 0.1294 \\ 0.2588 & 0.1294 & 1.0000 \end{bmatrix} \tag{4.7.17}$$

The inverse of this matrix can be shown to be

$$\left[e^{jq} \right] = \begin{bmatrix} 1.4051 & -0.6667 & -0.2774 \\ -0.6667 & 1.333 & 0 \\ -0.2774 & 0 & 1.0718 \end{bmatrix} \tag{4.7.18}$$

Therefore, from (4.7.10),

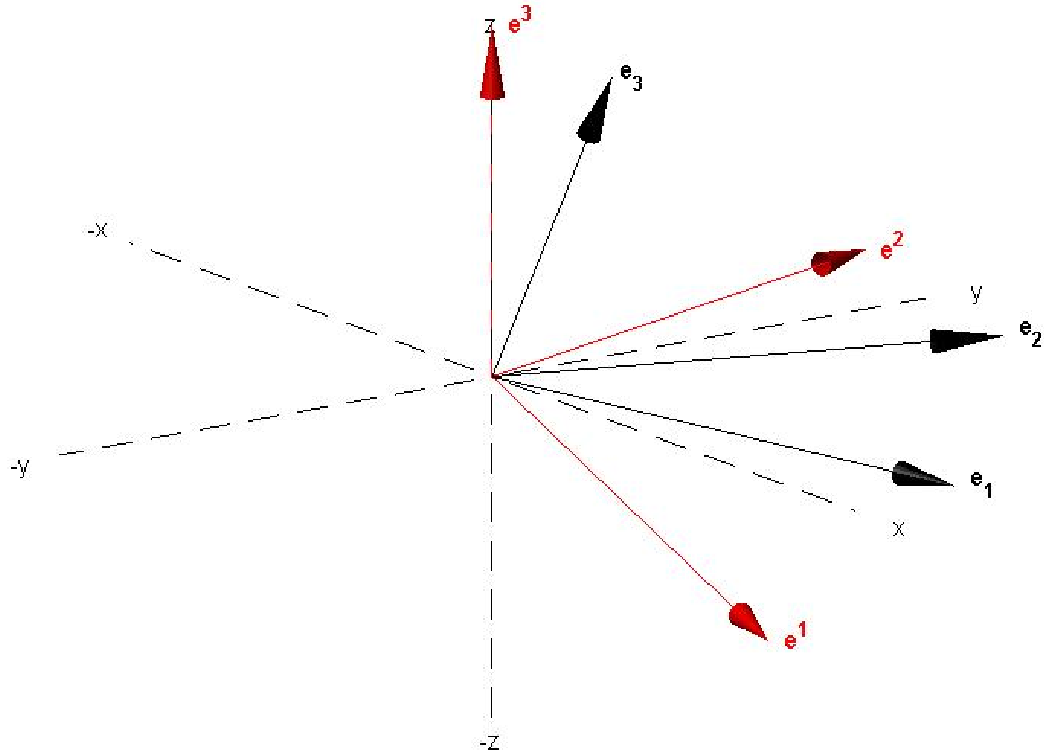
$$\begin{aligned}\mathbf{e}^1 &= e^{11}\mathbf{e}_1 + e^{12}\mathbf{e}_2 + e^{13}\mathbf{e}_3 \\ &= 1.4051\begin{bmatrix}0.9659\\0.2588\\0\end{bmatrix} - 0.667\begin{bmatrix}0.2588\\0.9659\\0\end{bmatrix} - 0.2774\begin{bmatrix}0.2500\\0.0670\\0.9659\end{bmatrix} = \begin{bmatrix}1.1154\\-0.2989\\-0.2679\end{bmatrix}\end{aligned} \tag{4.7.19}$$

$$\mathbf{e}^2 = e^{21}\mathbf{e}_1 + e^{22}\mathbf{e}_2 + e^{23}\mathbf{e}_3 = -0.667\begin{bmatrix}0.9659\\0.2588\\0\end{bmatrix} + 1.333\begin{bmatrix}0.2588\\0.9659\\0\end{bmatrix} = \begin{bmatrix}-0.2989\\1.1154\\0\end{bmatrix} \tag{4.7.20}$$

and

$$\mathbf{e}^3 = e^{31}\mathbf{e}_1 + e^{32}\mathbf{e}_2 + e^{33}\mathbf{e}_3 = -0.2774\begin{bmatrix}0.9659\\0.2588\\0\end{bmatrix} + 1.0718\begin{bmatrix}0.2500\\0.0670\\0.9659\end{bmatrix} = \begin{bmatrix}0\\0\\1.0353\end{bmatrix} \tag{4.7.21}$$

The following figure illustrates these vectors for this example:



If one has constructed a reciprocal basis $\left\{\mathbf{e}^1,\mathbf{e}^2,...,\mathbf{e}^N\right\}$, it is possible to expand a vector $\mathbf{v}\in\mathscr{V}$ in that basis as follows:

$$\mathbf{v}=\sum_{j=1}^{N}\upsilon_j\mathbf{e}^j \tag{4.7.22}$$

Equation (2.6.26), repeated, is

$$\mathbf{v}=\sum_{k=1}^{N}\upsilon^k\mathbf{e}_k \tag{4.7.23}$$

which represents $\mathbf{v}\in\mathscr{V}$ with respect to the basis $\left\{\mathbf{e}_1,\mathbf{e}_2,...,\mathbf{e}_N\right\}$. The two sets of components $\left\{\upsilon^1,\upsilon^2,...,\upsilon^N\right\}$ and $\left\{\upsilon_1,\upsilon_2,...,\upsilon_N\right\}$ are connected by the usual change of basis formulas discussed in Section 2.6. Given the special notation being used in this section, it is convenient to rewrite these formulas in this notation. Two important formulas that follow from (4.7.22) and (4.7.23) are

$$\left\langle\mathbf{v},\mathbf{e}_k\right\rangle=\left\langle\sum_{j=1}^{N}\upsilon_j\mathbf{e}^j,\mathbf{e}_k\right\rangle=\sum_{j=1}^{N}\upsilon_j\left\langle\mathbf{e}^j,\mathbf{e}_k\right\rangle=\sum_{j=1}^{N}\upsilon_j\delta_k^j=\upsilon_k \tag{4.7.24}$$

and

$$\left\langle\mathbf{v},\mathbf{e}^j\right\rangle=\left\langle\sum_{k=1}^{N}\upsilon^k\mathbf{e}_k,\mathbf{e}^j\right\rangle=\sum_{k=1}^{N}\upsilon^k\left\langle\mathbf{e}_k,\mathbf{e}^j\right\rangle=\sum_{j=1}^{N}\upsilon^k\delta_k^j=\upsilon^j \tag{4.7.25}$$

where the defining condition (4.7.7) has been used. These formulas simply say that the components $\left\{\upsilon^1,\upsilon^2,...,\upsilon^N\right\}$ are the *projections* of the vector $\mathbf{v}\in\mathscr{V}$ in the directions of the reciprocal basis $\left\{\mathbf{e}^1,\mathbf{e}^2,...,\mathbf{e}^N\right\}$, and the components $\left\{\upsilon_1,\upsilon_2,...,\upsilon_N\right\}$ are the projects of the vector $\mathbf{v}\in\mathscr{V}$ in the directions of the basis $\left\{\mathbf{e}_1,\mathbf{e}_2,...,\mathbf{e}_N\right\}$. It follows from (4.7.1), (4.7.23) (4.7.24) that the two sets of components are connected by

$$\upsilon_k=\left\langle\mathbf{v},\mathbf{e}_k\right\rangle=\left\langle\sum_{j=1}^{N}\upsilon^j\mathbf{e}_j,\mathbf{e}_k\right\rangle=\sum_{j=1}^{N}\upsilon^j\left\langle\mathbf{e}_j,\mathbf{e}_k\right\rangle=\sum_{j=1}^{N}e_{jk}\upsilon^j \tag{4.7.26}$$

or, by the reciprocal relationship, utilizing (4.7.11), (4.7.22) and (4.7.25)

$$\upsilon^j=\left\langle\mathbf{v},\mathbf{e}^j\right\rangle=\left\langle\sum_{k=1}^{N}\upsilon_k\mathbf{e}^k,\mathbf{e}^j\right\rangle=\sum_{k=1}^{N}\upsilon_k\left\langle\mathbf{e}^k,\mathbf{e}^j\right\rangle=\sum_{k=1}^{N}e^{kj}\upsilon_k \tag{4.7.27}$$

It is customary to refer to the components $\{\upsilon^1, \upsilon^2, ..., \upsilon^N\}$ as the *contravariant* components of $\mathbf{v} \in \mathscr{V}$ and the components $\{\upsilon_1, \upsilon_2, ..., \upsilon_N\}$ as the *covariant* components of $\mathbf{v} \in \mathscr{V}$.

As an example of the covariant and the contravarient components of a vector, consider the geometric construction introduced in Example 4.7.1 above.

Example 4.7.3: In this example the base vectors $\{\mathbf{e}_1, \mathbf{e}_2\}$ consist of two vectors 45^o apart, the first one, $\mathbf{e}_1$, two units long and the second one, $\mathbf{e}_2$, one unit long. You are given a vector $\mathbf{v}$ defined by

$$\mathbf{v} = \frac{1}{2}\mathbf{e}_1 + 2\mathbf{e}_2 \tag{4.7.28}$$

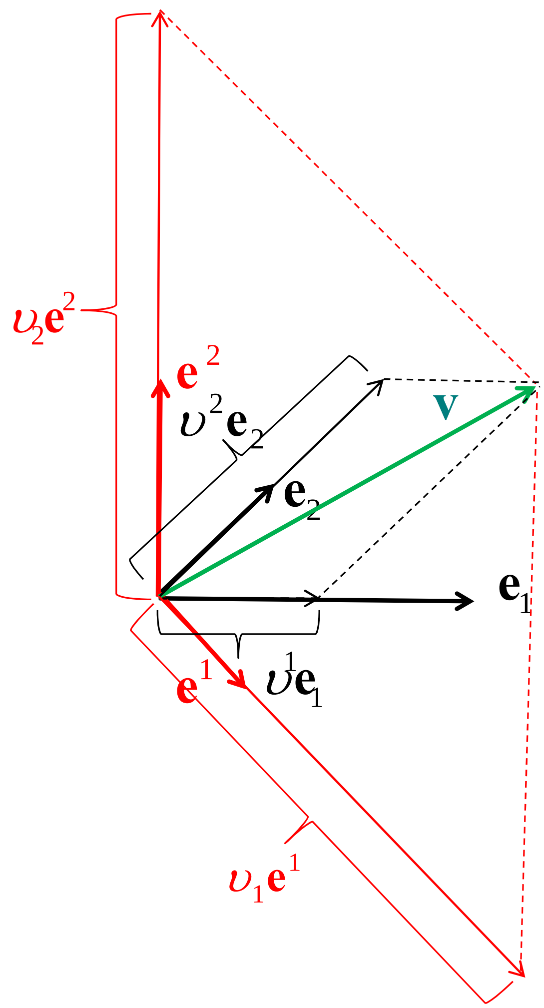
Therefore, for this example, the contravariant components of $\mathbf{v}$ are

$$\upsilon^1 = \frac{1}{2} \quad \text{and} \quad \upsilon^2 = 2 \tag{4.7.29}$$

Equations (4.7.12), (4.7.26) and (4.7.29) yield the following covariant components:

$$\begin{aligned} \upsilon_1 &= e_{11}\upsilon^1 + e_{21}\upsilon^2 = 4\left(\frac{1}{2}\right) + \sqrt{2}(2) = 2 + 2\sqrt{2} \\ \upsilon_2 &= e_{12}\upsilon^1 + e_{22}\upsilon^2 = \sqrt{2}\left(\frac{1}{2}\right) + 2(1) = \frac{1}{\sqrt{2}} + 2 \end{aligned} \tag{4.7.30}$$

The figure below shows the reciprocal basis and the covariant and contravarient components of $\mathbf{v}$.



Example 4.7.4: You are given the basis and reciprocal basis calculated in Example 4.7.2 above. You are also given a $\mathbf{v} \in \mathscr{M}^{3\times 1}$ defined with respect to the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ by

$$\begin{aligned}\mathbf{v} &= \upsilon^1\mathbf{e}_1 + \upsilon^2\mathbf{e}_2 + \upsilon^3\mathbf{e}_3 \\ &= 0.3167\mathbf{e}_1 + 0.4714\mathbf{e}_2 + 0.5799\mathbf{e}_3 \\ &= \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}\end{aligned} \tag{4.7.31}$$

The covariant components of $\mathbf{v} \in \mathscr{M}^{3\times 1}$ are given by equation (4.7.27) where the numerical values of the e_{kj} for $k, j = 1,2,3$ are given by (4.7.17). Therefore,

$$\begin{aligned}\upsilon_1 &= e_{11}\upsilon^1 + e_{12}\upsilon^2 + e_{13}\upsilon^3 \\ &= (1.0000)(0.3167) + (0.5000)(0.4714) + (0.2588)(0.5799) = 0.7071 \\ \upsilon_2 &= e_{21}\upsilon^1 + e_{22}\upsilon^2 + e_{23}\upsilon^3 \\ &= (0.5000)(0.3167) + (1.0000)(0.4714) + (0.1294)(0.5799) = 0.7071 \\ \upsilon_3 &= e_{31}\upsilon^1 + e_{32}\upsilon^2 + e_{33}\upsilon^3 \\ &= (0.2588)(0.3167) + (0.1294)(0.4714) + (1.000)(0.5799) = 0.7407\end{aligned} \tag{4.7.32}$$

and, with respect to $\{\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3\}$, the vector $\mathbf{v} \in \mathscr{M}^{3\times 1}$ is given by

$$\begin{aligned}\mathbf{v} &= \upsilon_1\mathbf{e}^1 + \upsilon_2\mathbf{e}^2 + \upsilon_3\mathbf{e}^3 \\ &= 0.7071\mathbf{e}^1 + 0.7071\mathbf{e}^2 + 0.7407\mathbf{e}^3 \\ &= \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}\end{aligned} \tag{4.7.33}$$

Exercises

4.7.1 You are given a basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ of $\mathscr{M}^{3\times 1}$ defined by

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}, \mathbf{e}_3 = \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix} \tag{4.7.34}$$

Find the reciprocal basis $\{\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3\}$. The answer is

$$\mathbf{e}^1 = \begin{bmatrix} -\frac{1}{8} \\ \frac{5}{8} \\ \frac{7}{4} \end{bmatrix}, \mathbf{e}^2 = \begin{bmatrix} \frac{1}{8} \\ \frac{3}{8} \\ \frac{1}{4} \end{bmatrix}, \mathbf{e}^3 = \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{4} \\ -\frac{1}{2} \end{bmatrix} \tag{4.7.35}$$

4.7.2 If we are given the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ for the vector space $\mathscr{M}^{4\times 1}$ defined by

$$\mathbf{e}_1 = \begin{bmatrix} 2 \\ 3 \\ 3i \\ -2i \end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix} 3i \\ -2 \\ 2 \\ 4 \end{bmatrix}, \mathbf{e}_3 = \begin{bmatrix} -2i \\ 1 \\ 3i \\ 0 \end{bmatrix}, \mathbf{e}_4 = \begin{bmatrix} 4 \\ 2i \\ 4 \\ 5i \end{bmatrix} \tag{4.7.36}$$

Find the reciprocal basis to $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$. The answer is

$$\begin{aligned} \mathbf{e}^1 &= \begin{bmatrix} 0.0631+0.1516i \\ 0.2733 - 0.1085\mathrm{i} \\ 0.0059 + 0.0100\mathrm{i} \\ 0.0200 - 0.0119\mathrm{i} \end{bmatrix}, \mathbf{e}^2 = \begin{bmatrix} 0.0756 + 0.0362\mathrm{i} \\ -0.1430 - 0.1042\mathrm{i} \\ 0.0157 + 0.0718\mathrm{i} \\ 0.1436 - 0.0313\mathrm{i} \end{bmatrix}, \\ \mathbf{e}^3 &= \begin{bmatrix} -0.0387 - 0.2743\mathrm{i} \\ -0.2481 + 0.1691\mathrm{i} \\ 0.0305 + 0.2332\mathrm{i} \\ 0.0664 - 0.0611\mathrm{i} \end{bmatrix}, \mathbf{e}^4 = \begin{bmatrix} 0.0542 + 0.0002\mathrm{i} \\ 0.0260 + 0.0710\mathrm{i} \\ 0.0598 - 0.0085\mathrm{i} \\ -0.0171 + 0.0804\mathrm{i} \end{bmatrix} \end{aligned} \tag{4.7.37}$$

Section 4.8. Reciprocal Bases and Linear Transformations

In Section 3.2, we introduced the components of a linear transformation $\mathbf{A}:\mathscr{V}\rightarrow\mathscr{U}$ with respect to a basis $\{\mathbf{e}_1,\mathbf{e}_2,...,\mathbf{e}_N\}$ for $\mathscr{V}$ and a basis $\{\mathbf{b}_1,\mathbf{b}_2,...,\mathbf{b}_M\}$ for $\mathscr{U}$. The fundamental formula that defined the components is equation (3.2.2), repeated,

$$\mathbf{A}\mathbf{e}_k=\sum_{j=1}^{M}A^j{}_k\mathbf{b}_j \qquad k=1,2,...,N \tag{4.8.1}$$

The matrix of the linear transformation was defined by $\mathbf{A}:\mathscr{V}\rightarrow\mathscr{U}$ by equation (3.2.9), repeated,

$$M\left(\mathbf{A},\mathbf{e}_k,\mathbf{b}_j\right)=\begin{bmatrix} A^1{}_1 & A^1{}_2 & \cdot & \cdot & \cdot & A^1{}_N \\ A^2{}_1 & A^2{}_2 & & & & A^2{}_N \\ A^3{}_1 & & A^3{}_3 & & & \cdot \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ A^M{}_1 & A^M{}_2 & \cdot & \cdot & \cdot & A^M{}_N \end{bmatrix}=\left[A^j{}_k\right] \tag{4.8.2}$$

In Section 3.6, we examined how the matrix (4.8.2) is altered when there are basis changes $\{\mathbf{e}_1,\mathbf{e}_2,...,\mathbf{e}_N\}\rightarrow\{\hat{\mathbf{e}}_1,\hat{\mathbf{e}}_2,...,\hat{\mathbf{e}}_N\}$ and $\{\mathbf{b}_1,\mathbf{b}_2,...,\mathbf{b}_M\}\rightarrow\{\hat{\mathbf{b}}_1,\hat{\mathbf{b}}_2,...,\hat{\mathbf{b}}_M\}$. The result was equation (3.6.17), repeated,

$$M\left(\mathbf{A},\hat{\mathbf{e}}_k,\hat{\mathbf{b}}_j\right)=U^{-1}M\left(\mathbf{A},\mathbf{e}_q,\mathbf{b}_s\right)T \tag{4.8.3}$$

where T is the transition matrix for $\{\mathbf{e}_1,\mathbf{e}_2,...,\mathbf{e}_N\}\rightarrow\{\hat{\mathbf{e}}_1,\hat{\mathbf{e}}_2,...,\hat{\mathbf{e}}_N\}$ and U is the transition matrix for $\{\mathbf{b}_1,\mathbf{b}_2,...,\mathbf{b}_M\}\rightarrow\{\hat{\mathbf{b}}_1,\hat{\mathbf{b}}_2,...,\hat{\mathbf{b}}_M\}$.

Now that we have complicated the discussion by the introduction of the reciprocal basis $\{\mathbf{e}^1,\mathbf{e}^2,...,\mathbf{e}^N\}$ and $\{\mathbf{b}^1,\mathbf{b}^2,...,\mathbf{b}^M\}$, and introduced the basis change $\{\mathbf{e}_1,\mathbf{e}_2,...,\mathbf{e}_N\}\rightarrow\{\mathbf{e}^1,\mathbf{e}^2,...,\mathbf{e}^N\}$ and $\{\mathbf{b}_1,\mathbf{b}_2,...,\mathbf{b}_M\}\rightarrow\{\mathbf{b}^1,\mathbf{b}^2,...,\mathbf{b}^M\}$, we have three other ways to create matrices from $\mathbf{A}:\mathscr{V}\rightarrow\mathscr{U}$. Repeating (4.8.1) and adding the other three, the four sets of components of $\mathbf{A}:\mathscr{V}\rightarrow\mathscr{U}$ are defined by

$$
\begin{aligned}
\mathbf{A}\mathbf{e}_k &= \sum_{j=1}^{M} A^j{}_k \mathbf{b}_j \\
\mathbf{A}\mathbf{e}_k &= \sum_{j=1}^{M} A_{jk} \mathbf{b}^j \\
\mathbf{A}\mathbf{e}^k &= \sum_{j=1}^{M} A^{jk} \mathbf{b}_j \\
\mathbf{A}\mathbf{e}^k &= \sum_{j=1}^{M} A_j{}^k \mathbf{b}^j
\end{aligned}
\tag{4.8.4}
$$

for $k = 1, 2, ..., N$. The corresponding matrices are

$$
\begin{aligned}
M\left(\mathbf{A}, \mathbf{e}_k, \mathbf{b}_j\right) &= \left[A^j{}_k\right] \\
M\left(\mathbf{A}, \mathbf{e}_k, \mathbf{b}^j\right) &= \left[A_{jk}\right] \\
M\left(\mathbf{A}, \mathbf{e}^k, \mathbf{b}_j\right) &= \left[A^{jk}\right] \\
M\left(\mathbf{A}, \mathbf{e}^k, \mathbf{b}^j\right) &= \left[A_j{}^k\right]
\end{aligned}
\tag{4.8.5}
$$

Consistent with (4.8.1), the convention in building the various matrices is that the first index, whether a subscript or a subscript, denotes the row of the matrix and the second index the column. Now that our vector spaces are inner product spaces, we can derive from (4.8.4) the following formulas for the components.

$$
\begin{aligned}
A^j{}_k &= \left\langle \mathbf{A}\mathbf{e}_k, \mathbf{b}^j \right\rangle \\
A_{jk} &= \left\langle \mathbf{A}\mathbf{e}_k, \mathbf{b}_j \right\rangle \\
A^{jk} &= \left\langle \mathbf{A}\mathbf{e}^k, \mathbf{b}^j \right\rangle \\
A_j{}^k &= \left\langle \mathbf{A}\mathbf{e}^k, \mathbf{b}_j \right\rangle
\end{aligned}
\tag{4.8.6}
$$

These formulas arise by forming the respective right hand sides, substituting from (4.8.4) in terms of the components and then making use of the definition of the reciprocal basis, equation (4.7.7). The four sets of components of $\mathbf{A}: \mathscr{V} \rightarrow \mathscr{U}$ in (4.8.6) are the *mixed contravariant-covariant* components, the *covariant components*, the *contravariant* components and the *mixed covariant-contravariant* components, respectively.

As with any basis change, the various sets of components are connected by a basis change formula identical to (4.8.3). The problem is that an already complicated notation gets even more complicated. It is probably more direct to derive the formulas that connect the components in (4.8.6) in each case rather than try to adapt (4.8.3). The bases $\left\{\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_N\right\}$ and $\left\{\mathbf{e}^1, \mathbf{e}^2, ..., \mathbf{e}^N\right\}$ are connected by(4.7.9) and (4.7.10), repeated.

$$\mathbf{e}_j = \sum_{q=1}^{N} e_{jq}\mathbf{e}^q \quad \text{for} \quad j = 1,2,...,N \tag{4.8.7}$$

$$\mathbf{e}^j = \sum_{q=1}^{N} e^{jq}\mathbf{e}_q \quad \text{for} \quad j = 1,2,...,N \tag{4.8.8}$$

Likewise, the bases $\{\mathbf{b}_1, \mathbf{b}_2, ..., \mathbf{b}_M\}$ and $\{\mathbf{b}^1, \mathbf{b}^2, ..., \mathbf{b}^M\}$ are connected by

$$\mathbf{b}_j = \sum_{q=1}^{M} b_{jq}\mathbf{b}^q \quad \text{for} \quad j = 1,2,...,M \tag{4.8.9}$$

$$\mathbf{b}^j = \sum_{q=1}^{M} b^{jq}\mathbf{b}_q \quad \text{for} \quad j = 1,2,...,M \tag{4.8.10}$$

The approach we shall take is to derive formulas that relate the components in $(4.8.6)_1$ to each set of components in the remaining three equations. Given $(4.8.6)_1$, it follows from $(4.8.6)_2$ that

$$A_{jk} = \left\langle \mathbf{A}\mathbf{e}_k, \mathbf{b}_j \right\rangle = \left\langle \mathbf{A}\mathbf{e}_k, \sum_{q=1}^{M} b_{jq}\mathbf{b}^q \right\rangle = \sum_{q}^{M} \overline{b_{jq}} \left\langle \mathbf{A}\mathbf{e}_k, \mathbf{b}^q \right\rangle = \sum_{q}^{M} \overline{b_{jq}} A^q{}_k \tag{4.8.11}$$

where (4.8.10) and $(4.8.6)_1$ have been used. If we next use (4.7.1) applied to the basis $\{\mathbf{b}_1, \mathbf{b}_2, ..., \mathbf{b}_M\}$, the transformation formula (4.8.11) simplifies to

$$A_{jk} = \sum_{q}^{M} b_{qj} A^q{}_k \tag{4.8.12}$$

Identical calculations starting from $(4.8.6)_3$ yields

$$A^{jk} = \left\langle \mathbf{A}\mathbf{e}^k, \mathbf{b}^j \right\rangle = \sum_{p=1}^{N} A^j{}_p e^{kp} \tag{4.8.13}$$

Likewise, when one starts from (4.8.6) it follows that

$$A_j{}^k = \left\langle \mathbf{A}\mathbf{e}^k, \mathbf{b}_j \right\rangle = \sum_{p=1}^{N} \sum_{q=1}^{M} b_{qj} A^q{}_p e^{kp} \tag{4.8.14}$$

At the risk of introducing more notation, if we introduce the matrices

$$E = \left[e^{jq} \right] \tag{4.8.15}$$

and

$$B = \left[b^{jq} \right] \tag{4.8.16}$$

Then the matrix versions of (4.8.12), (4.8.13) and (4.8.14) are, respectively,

$$M\left(\mathbf{A}, \mathbf{e}_k, \mathbf{b}^j\right) = \overline{B^{-1}} M\left(\mathbf{A}, \mathbf{e}_k, \mathbf{b}_j\right) \tag{4.8.17}$$

$$M\left(\mathbf{A}, \mathbf{e}^k, \mathbf{b}_j\right) = M\left(\mathbf{A}, \mathbf{e}_k, \mathbf{b}_j\right)\overline{E} \tag{4.8.18}$$

and

$$M\left(\mathbf{A}, \mathbf{e}^k, \mathbf{b}^j\right) = \overline{B^{-1}} M\left(\mathbf{A}, \mathbf{e}_k, \mathbf{b}_j\right)\overline{E} \tag{4.8.19}$$

Equation (4.8.19) can also be read off from (4.8.3). The transition matrix for the transformation $\left\{\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_N\right\} \to \left\{\mathbf{e}^1, \mathbf{e}^2, ..., \mathbf{e}^N\right\}$ turns out to be the complex conjugate of the matrix (4.8.15), and the transition matrix for the transformation $\left\{\mathbf{b}_1, \mathbf{b}_2, ..., \mathbf{b}_M\right\} \to \left\{\mathbf{b}^1, \mathbf{b}^2, ..., \mathbf{b}^M\right\}$ the transition matrix is the complex conjugate of the matrix (4.8.16).

In the special case where the linear transformation is one from the vector space $\mathscr{V}$ to itself, i.e., when $\mathbf{A}: \mathscr{V} \to \mathscr{V}$, the transition matrices E and B become the same. Also, operations like the determinant and trace of a linear transformation can be performed on $\mathbf{A}: \mathscr{V} \to \mathscr{V}$. In Section 3.6, we defined the determinant of a linear transformation $\mathbf{A}: \mathscr{V} \to \mathscr{V}$ by equation (3.6.34), repeated,

$$\det \mathbf{A} = \det M(\mathbf{A}, \mathbf{e}_k, \mathbf{e}_j) \tag{4.8.20}$$

It is important to recall that the matrix $\det M(\mathbf{A}, \mathbf{e}_j, \mathbf{e}_k)$ consists of elements defined by the equation

$$\mathbf{A}\mathbf{e}_k = \sum_{j=1}^{N} A^j{}_k \mathbf{e}_j \tag{4.8.21}$$

Equation (4.8.21) is equation $(4.8.4)_1$ specialized to the case of a linear transformation $\mathbf{A}: \mathscr{V} \to \mathscr{V}$.

In Section 3.6, we also illustrated how the determinant does not depend upon the choice of basis. The proof of this result, as shown in Section 3.6, arose from the fact that the matrix $\det M(\mathbf{A}, \mathbf{e}_k, \mathbf{e}_j)$ is similar to the matrix $M\left(\mathbf{A}, \hat{\mathbf{e}}_k, \hat{\mathbf{e}}_j\right)$ resulting from a basis change $\left\{\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_N\right\} \to \left\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, ..., \hat{\mathbf{e}}_N\right\}$. The question naturally arises as to what is the relationship between the determinant of $\mathbf{A}$ defined by (4.8.20) and the three matrices $M\left(\mathbf{A}, \mathbf{e}_k, \mathbf{b}^j\right), M\left(\mathbf{A}, \mathbf{e}^k, \mathbf{b}_j\right)$ and

$M\left(\mathbf{A},\mathbf{e}^k,\mathbf{b}^j\right)$. The answer is provided by the transformation formulas (4.8.17), (4.8.18) and (4.8.19) applied in the case $B = E$. The results are summarized as follows:

$$\begin{aligned}\det \mathbf{A} &= \det M(\mathbf{A},\mathbf{e}_k,\mathbf{e}_j) = \det M\left(\mathbf{A},\mathbf{e}^k,\mathbf{e}^j\right) \\ &= \det\left(M\left(\mathbf{A},\mathbf{e}^k,\mathbf{b}_j\right)\overline{E^{-1}}\right) = \det\left(\overline{B}M\left(\mathbf{A},\mathbf{e}_k,\mathbf{b}^j\right)\right)\end{aligned} \tag{4.8.22}$$

For the case under discussion, equation (4.8.19) shows that the matrix $M(\mathbf{A},\mathbf{e}_k,\mathbf{e}_j)$ is similar to the matrix $M\left(\mathbf{A},\mathbf{e}^k,\mathbf{e}^j\right)$. Equation $(4.8.22)_2$ reflects the result mentioned in Section 3.6 that similar matrices have the same determinant.

A similar calculation as that leading to (4.8.22) shows that the trace of $\mathbf{A}:\mathscr{V}\rightarrow\mathscr{V}$ is related to the trace of the four matrices by

$$\begin{aligned}\operatorname{tr}\mathbf{A} &= \operatorname{tr} M(\mathbf{A},\mathbf{e}_k,\mathbf{e}_j) = \operatorname{tr} M\left(\mathbf{A},\mathbf{e}^k,\mathbf{e}^j\right) \\ &= \operatorname{tr}\left(\overline{B}M\left(\mathbf{A},\mathbf{e}_k,\mathbf{b}^j\right)\right) = \operatorname{tr}\left(M\left(\mathbf{A},\mathbf{e}^k,\mathbf{b}_j\right)\overline{E^{-1}}\right)\end{aligned} \tag{4.8.23}$$

Exercises

4.8.1 You are given a linear transformation $\mathbf{A}:\mathscr{V}\rightarrow\mathscr{U}$ defined by

$$\mathbf{A}\mathbf{v} = \upsilon^1\left(9\mathbf{b}_1 + 6\mathbf{b}_2 - 5\mathbf{b}_3 + 4\mathbf{b}_4\right) + \upsilon^2\left(-\mathbf{b}_1 - \mathbf{b}_2 + \mathbf{b}_3\right) + \upsilon^3\left(8\mathbf{b}_1 + 5\mathbf{b}_2 - 4\mathbf{b}_3 + 5\mathbf{b}_4\right) \tag{4.8.24}$$

For all vectors $\mathbf{v} = \upsilon^1\mathbf{e}_1 + \upsilon^2\mathbf{e}_2 + \upsilon^3\mathbf{e}_3 \in \mathscr{V}$, where $\mathscr{V}$ is a real inner product space. The matrix of $\mathbf{A}$ with respect to the bases $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ and $\{\mathbf{b}_1,\mathbf{b}_2,\mathbf{b}_3,\mathbf{b}_4\}$ is

$$M\left(\mathbf{A},\mathbf{e}_k,\mathbf{b}_j\right) = \begin{bmatrix} 9 & -1 & 8 \\ 6 & -1 & 5 \\ -5 & 1 & -4 \\ 4 & 0 & 5 \end{bmatrix} \tag{4.8.25}$$

You are given that the matrix of inner products (4.7.1) is

$$\left[e_{jk}\right] = \begin{bmatrix} 3 & -3 & 7 \\ -3 & 11 & -3 \\ 7 & -3 & 21 \end{bmatrix} \tag{4.8.26}$$

$$\left[b_{jk}\right]=\begin{bmatrix} 1 & -1 & -3 & 1 \\ -1 & 3 & 2 & -4 \\ -3 & 2 & 2 & -2 \\ 1 & -4 & -2 & 1 \end{bmatrix} \tag{4.8.27}$$

Calculate the three sets of components for the linear transformation, $\mathbf{A}:\mathscr{V}\to\mathscr{V}$, $M\left(\mathbf{A},\mathbf{e}_k,\mathbf{b}^j\right), M\left(\mathbf{A},\mathbf{e}^k,\mathbf{b}_j\right)$ and $M\left(\mathbf{A}^*,\mathbf{e}^k,\mathbf{e}_j\right)$. The answers are

$$M\left(\mathbf{A},\mathbf{e}_k,\mathbf{b}^j\right)=\left[A_{jk}\right]=\begin{bmatrix} -8 & 3 & -4 \\ -17 & 0 & -21 \\ -33 & 3 & -33 \\ -1 & 1 & 1 \end{bmatrix} \tag{4.8.28}$$

$$M\left(\mathbf{A},\mathbf{e}^k,\mathbf{b}_j\right)=\left[A^{jk}\right]=\frac{1}{16}\begin{bmatrix} 353 & 67 & -102 \\ 475 & 89 & -69 \\ -199 & -37 & 58 \\ 137 & 27 & -38 \end{bmatrix} \tag{4.8.29}$$

and

$$M\left(\mathbf{A},\mathbf{e}^k,\mathbf{b}^j\right)=\left[A_j^{\ k}\right]=\begin{bmatrix} -\frac{689}{32} & -\frac{123}{32} & \frac{103}{16} \\ -\frac{1173}{32} & -\frac{231}{32} & \frac{163}{16} \\ -\frac{157}{2} & -15 & \frac{45}{2} \\ -\frac{31}{8} & -\frac{5}{8} & \frac{5}{4} \end{bmatrix} \tag{4.8.30}$$

Section 4.9. The Adjoint Linear Transformation

In this section, we shall briefly examine certain special concepts involving linear transformations defined on inner product spaces. In particular, we shall introduce what is known as the *adjoint linear transformation*. This linear transformation generalizes the transpose of a linear transpose in those cases where the inner product space is not real. As we shall learn, with the proper choice of bases, the adjoint of a linear transformation or the transpose of a linear transformation is directly related to the corresponding concept for matrices.

We begin our discussion by consideration of a linear transformation $\mathbf{A}:\mathscr{V}\to\mathscr{U}$ defined on two inner product spaces $\mathscr{V}$ and $\mathscr{U}$. At this point in the discussion, we shall allow $\mathscr{V}$ and $\mathscr{U}$ to be complex inner product spaces. The adjoint linear transformation to $\mathbf{A}:\mathscr{V}\to\mathscr{U}$ is defined as follows:

Definition: Given a linear transformation $\mathbf{A}:\mathscr{V}\to\mathscr{U}$, the *adjoint linear transformation* to $\mathbf{A}$ is a linear transformation $\mathbf{A}^*:\mathscr{U}\to\mathscr{V}$ defined by

$$\left\langle \mathbf{u},\mathbf{A}\mathbf{v}\right\rangle=\left\langle \mathbf{A}^*\mathbf{u},\mathbf{v}\right\rangle \qquad \text{for all} \quad \mathbf{v}\in\mathscr{V} \quad \text{and} \quad \mathbf{u}\in\mathscr{U} \tag{4.9.1}$$

Note that the inner product on the left side of (4.9.1) is the one in $\mathscr{U}$ and the one on the right side of (4.9.1) is the one in $\mathscr{V}$. Also, note that the property $\left\langle \mathbf{u},\mathbf{v}\right\rangle=\overline{\left\langle \mathbf{v},\mathbf{u}\right\rangle}$ of an inner product allows us to replace the definition (4.9.1) by the equivalent definition

$$\left\langle \mathbf{A}\mathbf{v},\mathbf{u}\right\rangle=\left\langle \mathbf{v},\mathbf{A}^*\mathbf{u}\right\rangle \qquad \text{for all} \quad \mathbf{v}\in\mathscr{V} \quad \text{and} \quad \mathbf{u}\in\mathscr{U} \tag{4.9.2}$$

In the special case where the inner produce spaces $\mathscr{V}$ and $\mathscr{U}$ are real, the adjoint is called the *transpose* and the definition (4.9.1) is written

$$\left\langle \mathbf{u},\mathbf{A}\mathbf{v}\right\rangle=\left\langle \mathbf{A}^T\mathbf{u},\mathbf{v}\right\rangle \qquad \text{for all} \quad \mathbf{v}\in\mathscr{V} \quad \text{and} \quad \mathbf{u}\in\mathscr{U} \tag{4.9.3}$$

Equation (4.9.1) and the special case (4.9.3) are *component free* definitions of the adjoint and the transpose, respectively. This is an important feature of the definition. Physical quantities represented by geometric objects like vectors and linear transformations do not depend upon the special basis that might be selected in a particular application. Thus, it is important to understand when a quantity does or does not depend upon the basis.

Properties of the adjoint that follow from the definition (4.9.1) are summarized in the following theorem.

Theorem 4.9.1:

$$\left(\mathbf{A}+\mathbf{B}\right)^{*}=\mathbf{A}^{*}+\mathbf{B}^{*} \tag{4.9.4}$$

$$\left(\mathbf{AB}\right)^{*}=\mathbf{B}^{*}\mathbf{A}^{*} \tag{4.9.5}$$

$$\left(\lambda\mathbf{A}\right)^{*}=\bar{\lambda}\mathbf{A}^{*} \tag{4.9.6}$$

$$\mathbf{I}^{*}=\mathbf{I} \tag{4.9.7}$$

$$\mathbf{0}^{*}=\mathbf{0} \tag{4.9.8}$$

$$\left(\mathbf{A}^{*}\right)^{*}=\mathbf{A} \tag{4.9.9}$$

and, if $\mathbf{A}$ is nonsingular, so is $\mathbf{A}^{*}$ and, in addition,

$$\left(\mathbf{A}^{*}\right)^{-1}=\left(\mathbf{A}^{-1}\right)^{*} \tag{4.9.10}$$

The proof of the above theorem is straightforward and is left as an exercise.

Equation (4.9.2) defines the adjoint in a component free or basis free fashion. In other words, the definition does not involve choices of the bases in $\mathscr{V}$ or $\mathscr{U}$. Some of our operations on linear transformations, such as the determinant and trace operations introduced in Section 3.6, were introduced by defining certain operations on the components with respect to specific basis choices. We then showed that the results, in this case the determinant and the trace, were actually independent of the choice of bases. The important conclusion was that the determinant and the trace of the matrix was a basis free quantity. We could have followed a similar approach with the definition of the adjoint but the component free definition is preferred when possible. A component free definition of the determinant is possible but not without a diversion into topics we do not wish to discuss at this point.

Having stressed the benefits of a component free definition of the adjoint, it is useful and important to select bases for $\mathscr{V}$ and $\mathscr{U}$ and examine how the adjoint and, for the real case, the transpose are related to their matrix. First, we shall select bases for $\mathscr{V}$ and $\mathscr{U}$. As in Section 4.8, the situation is complicated by the number of choices. If we denote the basis for $\mathscr{V}$ by $\left\{\mathbf{e}_1,\mathbf{e}_2,...,\mathbf{e}_N\right\}$, we have a reciprocal basis $\left\{\mathbf{e}^1,\mathbf{e}^2,...,\mathbf{e}^N\right\}$. If we denote the basis for $\mathscr{U}$ by $\left\{\mathbf{b}_1,\mathbf{b}_2,...,\mathbf{b}_M\right\}$, we have a reciprocal basis $\left\{\mathbf{b}^1,\mathbf{b}^2,...,\mathbf{b}^M\right\}$. The four sets of components corresponding to these choices are listed in equation (4.8.4). A similar set of formulas hold for the components of the adjoint linear transformation $\mathbf{A}^{*}:\mathscr{U}\rightarrow\mathscr{V}$. We shall write these formulas as

$$
\begin{aligned}
\mathbf{A}^*\mathbf{b}_j &= \sum_{k=1}^{N} A^{*k}{}_j \mathbf{e}_k \\
\mathbf{A}^*\mathbf{b}_j &= \sum_{k=1}^{N} A^*{}_{kj} \mathbf{e}^k \\
\mathbf{A}^*\mathbf{b}^j &= \sum_{k=1}^{N} A^{*kj} \mathbf{e}_k \\
\mathbf{A}^*\mathbf{b}^j &= \sum_{k=1}^{N} A^*{}_k{}^j \mathbf{e}^k
\end{aligned}
\tag{4.9.11}
$$

for $j = 1, 2, ..., M$.

The basic question is how are the coefficients $A^{*k}{}_j$, $A^*{}_{kj}$, A^{*kj} and $A^*{}_k{}^j$ in (4.9.11) related to the components of the linear transformation $\mathbf{A}$ in (4.8.4). Unfortunately, the relationships can be complicated. Equations (4.8.6) give the various components of $\mathbf{A}$ in terms of $\mathbf{A}$ and the inner product with the basis vectors. For the adjoint, $\mathbf{A}^*$,similar formulas are

$$
\begin{aligned}
A^{*k}{}_j &= \left\langle \mathbf{A}^*\mathbf{b}_j, \mathbf{e}^k \right\rangle \\
A^*{}_{kj} &= \left\langle \mathbf{A}^*\mathbf{b}_j, \mathbf{e}_k \right\rangle \\
A^{*kj} &= \left\langle \mathbf{A}^*\mathbf{b}^j, \mathbf{e}^k \right\rangle \\
A^*{}_k{}^j &= \left\langle \mathbf{A}^*\mathbf{b}^j, \mathbf{e}_k \right\rangle
\end{aligned}
\tag{4.9.12}
$$

The four sets of components of $\mathbf{A}^* : \mathscr{U} \rightarrow \mathscr{V}$ in (4.9.12) are the *mixed contravariant-covariant* components, the *covariant* components, the *contravariant* and the mixed *covariant-contravariant* components, respectively.

Our goal is to determine how the various sets of components are related. The four results we do wish to record are

$$A^{*k}{}_j = \overline{A_j{}^k} \tag{4.9.13}$$

$$A^*{}_{kj} = \overline{A_{jk}} \tag{4.9.14}$$

$$A^{*kj} = \overline{A^{jk}} \tag{4.9.15}$$

and

$$A^*{}_k{}^j = \overline{A^j{}_k} \tag{4.9.16}$$

Equations (4.9.13) through (4.9.16) simply state that the components of the adjoint are obtained by transposing and forming the complex conjugates of the appropriate components of the linear transformation $\mathbf{A}:\mathscr{V}\to\mathscr{U}$.

The derivation, for example, of (4.9.13) begins with the result $(4.9.12)_1$ and the definition (4.9.1). The sequence of calculations is

$$A^{*k}{}_{j}=\left\langle \mathbf{A}^{*}\mathbf{b}_{j},\mathbf{e}^{k}\right\rangle=\left\langle \mathbf{b}_{j},\mathbf{A}\mathbf{e}^{k}\right\rangle=\overline{\left\langle \mathbf{A}\mathbf{e}^{k},\mathbf{b}_{j}\right\rangle} \tag{4.9.17}$$

The result (4.9.13) follows by application of the result $(4.8.6)_4$ to (4.9.17). The results (4.9.14), (4.9.15) and (4.9.16) follow by identical arguments.

If we were to decide, for example, to derive a formula that connected the components $A^{*k}{}_{j}$, for $k=1,2,...,N$ and $j=1,2,...,M$ to the components $A^{j}{}_{k}$, for $j=1,2,...,M$ and $k=1,2,...,N$, the results are much more complicated. The derivation of the kind of formula is not difficult but the details are more involved. The starting place is the formula (4.9.17). This formula and (4.8.14) combine to yield

$$A^{*k}{}_{j}=\overline{\left\langle \mathbf{A}\mathbf{e}^{k},\mathbf{b}_{j}\right\rangle}=\overline{\sum_{p=1}^{N}\sum_{q=1}^{M}b_{qj}A^{q}{}_{p}e^{kp}}=\sum_{p=1}^{N}\sum_{q=1}^{M}b_{jq}\overline{A^{q}{}_{p}}e^{pk} \tag{4.9.18}$$

The complexity of the relationship between the coefficients $A^{q}{}_{p}$, for $p=1,2,...,N$ and $q=1,2,...,M$ and $A^{*k}{}_{j}$ for $k=1,2,...,N$ and $j=1,2,...,M$ in equation (4.9.18) obscures the more elementary result (4.9.13).

It is instructive to write the results (4.9.13) through (4.9.16) as matrix equations. From the definition of the matrix of a linear transformation given in Section 3.2, it is true that

$$\begin{aligned}
M\left(\mathbf{A}^{*},\mathbf{b}_{j},\mathbf{e}_{k}\right)&=\left[A^{*k}{}_{j}\right]\\
M\left(\mathbf{A}^{*},\mathbf{b}_{j},\mathbf{e}^{k}\right)&=\left[A^{*}{}_{kj}\right]\\
M\left(\mathbf{A}^{*},\mathbf{b}^{j},\mathbf{e}_{k}\right)&=\left[A^{*kj}\right]\\
M\left(\mathbf{A}^{*},\mathbf{b}^{j},\mathbf{e}^{k}\right)&=\left[A^{*}{}_{k}{}^{j}\right]
\end{aligned} \tag{4.9.19}$$

With these expressions and the corresponding definitions for the various matrices of $\mathbf{A}$ given in (4.8.5), equations (4.9.13) through (4.9.16) can be written

$$M\left(\mathbf{A}^{*},\mathbf{b}_{j},\mathbf{e}_{k}\right)=\overline{M\left(\mathbf{A},\mathbf{e}^{k},\mathbf{b}^{j}\right)^{T}} \tag{4.9.20}$$

$$M\left(\mathbf{A}^{*},\mathbf{b}_{j},\mathbf{e}^{k}\right)=\overline{M\left(\mathbf{A},\mathbf{e}_{k},\mathbf{b}^{j}\right)^{T}} \tag{4.9.21}$$

$$M\left(\mathbf{A}^*, \mathbf{b}^j, \mathbf{e}_k\right) = \overline{M\left(\mathbf{A}, \mathbf{e}^k, \mathbf{b}_j\right)^T} \tag{4.9.22}$$

and

$$M\left(\mathbf{A}^*, \mathbf{b}^j, \mathbf{e}^k\right) = \overline{M\left(\mathbf{A}, \mathbf{e}_k, \mathbf{b}_j\right)^T} \tag{4.9.23}$$

The more complicated formula (4.9.18) is equivalent to the matrix equation

$$M\left(\mathbf{A}^*, \mathbf{b}_j, \mathbf{e}_k\right) = \overline{E}\,\overline{M\left(\mathbf{A}, \mathbf{e}_k, \mathbf{b}_j\right)^T}\,\overline{B^{-1}} \tag{4.9.24}$$

where the transition matrices are defined by (4.8.15) and (4.8.16). One way to derive (4.9.24) is to substitute (4.8.19) into (4.9.20).

The results (4.9.20) through (4.9.23) reveal the fact that, with the proper choice of basis, the matrix of $\mathbf{A}^*$ is the transposed complex conjugate of the matrix of $\mathbf{A}$. The phrase "proper choice of basis" is fundamental. Equation (4.9.24) shows that the matrix $M\left(\mathbf{A}^*, \mathbf{b}_j, \mathbf{e}_k\right)$ *is not* the transposed complex conjugate of the matrix $M\left(\mathbf{A}, \mathbf{e}_k, \mathbf{b}_j\right)$.

For the special case where the adjoint linear transformation is one from the vector space $\mathscr{V}$ to itself, i.e., when $\mathbf{A}^* : \mathscr{V} \to \mathscr{V}$, an application of the definition of the determinant, equation (3.6.34) yields the result

$$\det \mathbf{A}^* = \det M(\mathbf{A}^*, \mathbf{e}_j, \mathbf{e}_k) \tag{4.9.25}$$

where the matrix $\det M(\mathbf{A}^*, \mathbf{e}_j, \mathbf{e}_k)$ consists of elements defined by the equation

$$\mathbf{A}^*\mathbf{e}_j = \sum_{k=1}^{N} A^{*k}{}_j \mathbf{e}_k \tag{4.9.26}$$

Equation (4.9.26) is equation $(4.9.11)_1$ specialized to the case of a linear transformation $\mathbf{A}^* : \mathscr{V} \to \mathscr{V}$. Our interest is the relationship between the determinant of $\mathbf{A} : \mathscr{V} \to \mathscr{V}$ and the determinant of $\mathbf{A}^* : \mathscr{V} \to \mathscr{V}$. The answer is given by the definition (4.9.25) and equation (4.9.24) in the special case we are discussing, namely, where $B = E$. These two equations yield

$$\begin{aligned}\det \mathbf{A}^* = \det M(\mathbf{A}^*, \mathbf{e}_j, \mathbf{e}_k) &= \det\left(E^{-1}\overline{M\left(\mathbf{A}, \mathbf{e}_k, \mathbf{e}_j\right)^T} E\right)\\ &= \det E^{-1} \det E \det\left(\overline{M\left(\mathbf{A}, \mathbf{e}_k, \mathbf{e}_j\right)^T}\right)\\ &= \overline{\det\left(M\left(\mathbf{A}, \mathbf{e}_k, \mathbf{e}_j\right)^T\right)} = \overline{\det\left(M\left(\mathbf{A}, \mathbf{e}_k, \mathbf{e}_j\right)\right)}\end{aligned} \tag{4.9.27}$$

The definition of the determinant of $\mathbf{A}$, equation (3.6.34), reduces (4.9.27) to the simple result

$$\det \mathbf{A}^* = \overline{\det \mathbf{A}} \tag{4.9.28}$$

Also in Section 3.6, we defined the trace of a linear transformation $\mathbf{A} : \mathscr{V} \to \mathscr{V}$ by (3.6.36), repeated,

$$\operatorname{tr} \mathbf{A} = \operatorname{tr} M(\mathbf{A}, \mathbf{e}_k, \mathbf{e}_j) \tag{4.9.29}$$

Likewise, the trace of the adjoint is defined by

$$\operatorname{tr} \mathbf{A}^* = \operatorname{tr} M(\mathbf{A}^*, \mathbf{e}_k, \mathbf{e}_j) \tag{4.9.30}$$

If one starts with (4.9.24), of course with $B = E$, it is easily shown that

$$\operatorname{tr} \mathbf{A}^* = \overline{\operatorname{tr} \mathbf{A}} \tag{4.9.31}$$

The point of this discussion that needs to be seen through the equations and their many subscripts and superscripts is that the matrix of the adjoint linear transformation is not simply the matrix created by transposing and taking the complex conjugate of the matrix of the original linear transformation. It takes a special choice of the bases for simple formulas such as (4.9.20) through (4.9.23) to be valid. The basis dependence of these relationships stands in contrast to relationships like (3.5.18), repeated,

$$M\left(\mathbf{A}^{-1}, \mathbf{e}_j, \mathbf{e}_k\right) = M\left(\mathbf{A}, \mathbf{e}_k, \mathbf{e}_j\right)^{-1} \tag{4.9.32}$$

In this case, one starts with a basis, creates the matrix, inverts the matrix and then, *utilizing the same basis*, creates the linear transformation that is the inverse. In the case of the adjoint, the matrix operation, i.e., transposing and forming the complex conjugate, does not produce the adjoint linear transformation unless the proper bases are adopted.

Another important result to extract from the above detail is that if $\left\{\mathbf{i}_1, \mathbf{i}_2, ..., \mathbf{i}_N\right\}$ is an orthonormal basis for $\mathscr{V}$ and $\left\{\mathbf{j}_1, \mathbf{j}_2, ..., \mathbf{j}_M\right\}$ is the orthonormal basis for $\mathscr{U}$, the matrix relationships between a linear transformation and its adjoint take the simple form

$$M\left(\mathbf{A}^*,\mathbf{j}_q,\mathbf{i}_k\right)=\overline{M\left(\mathbf{A},\mathbf{i}_k,\mathbf{j}_q\right)}^T \tag{4.9.33}$$

Most of our discussions that require that we utilize the components of a linear transformation will allow the use of orthonormal bases. Thus, the complexity of the component representations of the adjoint will not create difficulties. The fact that most of our fundamental discussions can be conducted in a component free fashion is also a benefit as we try to keep our discussions simple from a notation standpoint.

As we have seen in this section, the inner product structure on the vector spaces $\mathscr{V}$ and $\mathscr{U}$ allows for the introduction of the adjoint linear transformation to a linear transformation $\mathbf{A}:\mathscr{V}\to\mathscr{U}$. Given the idea of an adjoint linear transformation, we can define an inner product structure for the set of linear transformations $\mathscr{L}\left(\mathscr{V};\mathscr{U}\right)$ in a fashion that generalizes the inner product we introduced on $\mathscr{M}^{M\times N}$ in Example 4.1.4. The formal definition is

$$\left\langle\mathbf{A},\mathbf{C}\right\rangle=\operatorname{tr}\left(\mathbf{AC}^*\right) \tag{4.9.34}$$

for all linear transformations $\mathbf{A},\mathbf{C}\in\mathscr{L}\left(\mathscr{V};\mathscr{U}\right)$. It is elementary to show that the definition (4.9.34) obeys the four properties of an inner product given in Section 4.1. The first three of these properties are more or less obvious. The fourth one, namely, that $\left\langle\mathbf{A},\mathbf{A}\right\rangle\geq 0$ and $\left\langle\mathbf{A},\mathbf{A}\right\rangle=0$ if and only if $\mathbf{A}=\mathbf{0}$ can be seen to be true by simply adopting an orthonormal basis for $\mathscr{V}$ and $\mathscr{U}$ and utilize the matrix result (4.1.16). The fact that the definition (4.9.34) does not depend upon the choice of the bases for $\mathscr{V}$ and $\mathscr{U}$ insures that the fourth property of an inner product space is obeyed.

If desired, one can express (4.9.34) in terms of the various components introduced in this section and obtain the following component represention of the inner product defined by (4.9.34):

$$\left\langle\mathbf{A},\mathbf{C}\right\rangle=\operatorname{tr}\left(\mathbf{AC}^*\right)=\sum_{k=1}^{N}\sum_{j=1}^{M}A^j{}_k\overline{C}_j{}^k=\sum_{k=1}^{N}\sum_{p=1}^{N}\sum_{j=1}^{M}\sum_{q=1}^{M}b_{jq}e^{pk}A^j{}_k\overline{C^q{}_p} \tag{4.9.35}$$

The matrix form of the complicated result (4.9.35) is

$$\begin{aligned}\left\langle\mathbf{A},\mathbf{C}\right\rangle&=\operatorname{tr}\left(\mathbf{AC}^*\right)=\operatorname{tr}\left(M\left(\mathbf{A},\mathbf{e}_k,\mathbf{b}_j\right)M\left(\mathbf{C}^*,\mathbf{b}_j,\mathbf{e}_k\right)\right)\\&=\operatorname{tr}\left(M\left(\mathbf{A},\mathbf{e}_k,\mathbf{b}_j\right)\overline{E}\,\overline{M\left(\mathbf{C},\mathbf{e}_k,\mathbf{b}_j\right)}^T\overline{B^{-1}}\right)\end{aligned} \tag{4.9.36}$$

where $(4.9.19)_1$ and (4.9.24) have been used.

In closing this section, it is important to call attention to the case where the inner product space is real. In this case, the adjoint reduces to the transpose, that we shall write $\mathbf{A}^T$. The above equations all remain valid with a simple elimination of the various complex conjugates that appear.

Exercises

4.9.1 Confirm the formula (4.9.10). This problem, as one would expect, uses the definition (4.9.1) and the defining property $\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$ for the inverse.

4.9.2 You are given a linear transformation $\mathbf{A}:\mathscr{V}\rightarrow\mathscr{V}$, defined by

$$\mathbf{A}\mathbf{v} = \left(\upsilon^1 + 4\upsilon^2 + \upsilon^3\right)\mathbf{e}_1 + \left(\upsilon^1 + \upsilon^2 - 2\upsilon^3\right)\mathbf{e}_2 + \left(\upsilon^1 - 3\upsilon^2 + 2\upsilon^3\right)\mathbf{e}_3 \tag{4.9.37}$$

for all $\mathbf{v} = \upsilon^1\mathbf{e}_1 + \upsilon^2\mathbf{e}_2 + \upsilon^3\mathbf{e}_2 \in \mathscr{V}$, where $\mathscr{V}$ is a real inner product space. It follows from (4.9.37) that the matrix of $\mathbf{A}$ with respect to the basis $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ is

$$M\left(\mathbf{A},\mathbf{e}_k,\mathbf{e}_j\right) = \begin{bmatrix} 1 & 4 & 1 \\ 1 & 1 & -2 \\ 1 & -3 & 2 \end{bmatrix} \tag{4.9.38}$$

You are given that the matrix of inner products (4.7.1) is

$$\left[e_{jk}\right] = \begin{bmatrix} 3 & -3 & 7 \\ -3 & 11 & -3 \\ 7 & -3 & 21 \end{bmatrix} \tag{4.9.39}$$

Calculate the four sets of components for the adjoint linear transformation, $\mathbf{A}^*:\mathscr{V}\rightarrow\mathscr{V}$, $M\left(\mathbf{A}^*,\mathbf{e}_j,\mathbf{e}_k\right) = \left[A^{*k}{}_j\right], M\left(\mathbf{A}^*,\mathbf{e}_j,\mathbf{e}^k\right) = \left[A^*{}_{kj}\right], M\left(\mathbf{A}^*,\mathbf{e}^j,\mathbf{e}_k\right) = \left[A^{*kj}\right]$ and $M\left(\mathbf{A}^*,\mathbf{e}^j,\mathbf{e}^k\right) = \left[A^*{}_k{}^j\right]$. The answers are

$$M\left(\mathbf{A}^*,\mathbf{e}_j,\mathbf{e}_k\right) = \left[A^{*k}{}_j\right] = \overline{M\left(\mathbf{A},\mathbf{e}^k,\mathbf{e}^j\right)^T} = \begin{bmatrix} -\frac{257}{32} & \frac{1777}{32} & \frac{107}{32} \\ -\frac{75}{32} & \frac{347}{32} & -\frac{71}{32} \\ \frac{110}{32} & -\frac{590}{32} & \frac{38}{32} \end{bmatrix} \tag{4.9.40}$$

$$M\left(\mathbf{A}^*,\mathbf{e}_j,\mathbf{e}^k\right) = \left[A^*{}_{kj}\right] = \overline{M\left(\mathbf{A},\mathbf{e}_k,\mathbf{e}^j\right)^T} = \begin{bmatrix} 7 & 5 & 25 \\ -12 & 8 & -38 \\ 23 & -31 & 55 \end{bmatrix} \tag{4.9.41}$$

$$M\left(\mathbf{A}^{*},\mathbf{e}^{j},\mathbf{e}_{k}\right)=\left[A^{*kj}\right]=\overline{M\left(\mathbf{A},\mathbf{e}^{k},\mathbf{e}_{j}\right)}^{T}=\begin{bmatrix}\frac{161}{32} & \frac{25}{4} & -\frac{5}{8}\\ \frac{43}{32} & \frac{5}{4} & -\frac{3}{8}\\ -\frac{23}{16} & -2 & \frac{1}{4}\end{bmatrix} \tag{4.9.42}$$

and

$$M\left(\mathbf{A}^{*},\mathbf{e}^{j},\mathbf{e}^{k}\right)=\left[A^{*}{}_{k}{}^{j}\right]=\overline{M\left(\mathbf{A},\mathbf{e}_{k},\mathbf{e}_{j}\right)}^{T}=\begin{bmatrix}1 & 1 & 1\\ 4 & 1 & -3\\ 1 & -2 & 2\end{bmatrix} \tag{4.9.43}$$

4.9.3 Utilize the results in Exercise 4.9.2 and calculate $\left\langle \mathbf{A},\mathbf{A}\right\rangle$. The answer is

$$\left\langle \mathbf{A},\mathbf{A}\right\rangle=\operatorname{tr}\left(\mathbf{A}\mathbf{A}^{*}\right)=\operatorname{tr}\left(\begin{bmatrix}1 & 4 & 1\\ 1 & 1 & -2\\ 1 & -3 & 2\end{bmatrix}\begin{bmatrix}-\frac{257}{32} & \frac{1777}{32} & \frac{107}{32}\\ -\frac{75}{32} & \frac{347}{32} & -\frac{71}{32}\\ \frac{110}{32} & -\frac{590}{32} & \frac{38}{32}\end{bmatrix}\right)=\frac{3253}{32} \tag{4.9.44}$$

4.9.4 You are given a linear transformation $A:\mathscr{V}\to\mathscr{U}$ defined, for all vectors $\mathbf{v}\in\mathscr{V}$, by

$$\begin{aligned}\mathbf{A}\mathbf{v}&=\left(\upsilon^{1}-i\upsilon^{2}+7\upsilon^{3}\right)\mathbf{b}_{1}+\left(-8i\upsilon^{1}+9\upsilon^{2}+8i\upsilon^{3}\right)\mathbf{b}_{2}\\ &\quad+\left(4\upsilon^{1}-9\upsilon^{2}+15i\upsilon^{3}\right)\mathbf{b}_{3}+\left(7i\upsilon^{1}+4i\upsilon^{2}\right)\mathbf{b}_{4}\end{aligned} \tag{4.9.45}$$

for all vectors $\mathbf{v}=\upsilon^{1}\mathbf{e}_{1}+\upsilon^{2}\mathbf{e}_{2}+\upsilon^{3}\mathbf{e}_{3}\in\mathscr{V}$, where $\left\{\mathbf{e}_{1},\mathbf{e}_{2},\mathbf{e}_{3}\right\}$ is a basis for $\mathscr{V}$ and $\left\{\mathbf{b}_{1},\mathbf{b}_{2},\mathbf{b}_{3},\mathbf{b}_{4}\right\}$ is a basis for $\mathscr{U}$. The linear transformation defined by (4.9.45) was introduced earlier in Exercise 3.3.3. It follows from (4.9.45) that the matrix of $\mathbf{A}$ with respect to the basis $\left\{\mathbf{e}_{1},\mathbf{e}_{2},\mathbf{e}_{3}\right\}$ is

$$M\left(\mathbf{A},\mathbf{e}_{j},\mathbf{b}_{k}\right)=\begin{bmatrix}1 & -i & 7\\ -8i & 9 & 8i\\ 4 & -9 & 15i\\ 7i & 4i & 0\end{bmatrix} \tag{4.9.46}$$

You are given that the matrix of inner products (4.7.1) is

$$\left[e_{jk}\right]=\begin{bmatrix} 3 & 1+2i & 1+2i \\ 1-2i & 11 & -6+5i \\ 1-2i & -6-5i & 21 \end{bmatrix} \tag{4.9.47}$$

Likewise, for the basis $\left\{\mathbf{b}_1,\mathbf{b}_2,\mathbf{b}_3,\mathbf{b}_4\right\}$, the matrix of inner products (4.7.1) is

$$\left[b_{jk}\right]=\begin{bmatrix} 26 & -6+8i & 12-4i & -2-6i \\ -6-8i & 33 & -8+6i & 8+4i \\ 12+4i & -8-6i & 14 & -2i \\ -2+6i & 8-4i & 2i & 61 \end{bmatrix} \tag{4.9.48}$$

Calculate the four sets of components for the adjoint linear transformation, $\mathbf{A}^*:\mathscr{V}\to\mathscr{U}$, $M\left(\mathbf{A}^*,\mathbf{b}_j,\mathbf{e}_k\right)=\left[A^{*k}{}_j\right]$, $M\left(\mathbf{A}^*,\mathbf{b}_j,\mathbf{e}^k\right)=\left[A^*{}_{kj}\right]$, $M\left(\mathbf{A}^*,\mathbf{b}^j,\mathbf{e}_k\right)=\left[A^{*kj}\right]$ and $M\left(\mathbf{A}^*,\mathbf{b}^j,\mathbf{e}^k\right)=\left[A^*{}_k{}^j\right]$. The answers are

$$\begin{aligned} M\left(\mathbf{A}^*,\mathbf{b}_j,\mathbf{e}_k\right)&=\left[A^{*k}{}_j\right]=\overline{M\left(\mathbf{A},\mathbf{e}^k,\mathbf{b}^j\right)^T} \\ &=\begin{bmatrix} \frac{1185-1293i}{29} & \frac{1551+69623i}{290} & \frac{12919-8823i}{145} & \frac{11374-25103i}{145} \\ \frac{-3468+5364i}{145} & \frac{20142-6217i}{145} & \frac{-7428-4804i}{145} & \frac{-17311-13983i}{290} \\ \frac{1237+2319i}{145} & \frac{15769-19369i}{290} & \frac{-4473-1299i}{145} & \frac{-14361+47i}{290} \end{bmatrix} \end{aligned} \tag{4.9.49}$$

$$\begin{aligned} M\left(\mathbf{A}^*,\mathbf{b}_j,\mathbf{e}^k\right)&=\left[A^*{}_{kj}\right]=\overline{M\left(\mathbf{A},\mathbf{e}_k,\mathbf{b}^j\right)^T} \\ &=\begin{bmatrix} -32-50i & -10+224i & 102-60i & 30-349i \\ -186+142i & 393-92i & -210-42i & 66-300i \\ 186-132i & 48-200i & 36-118i & -16-22i \end{bmatrix} \end{aligned} \tag{4.9.50}$$

$$M\left(\mathbf{A}^*,\mathbf{b}^j,\mathbf{e}_k\right)=\left[A^{*kj}\right]=\overline{M\left(\mathbf{A},\mathbf{e}^k,\mathbf{b}_j\right)}^T$$
$$=\begin{bmatrix} \frac{-39+93i}{145} & \frac{79+2170i}{290} & \frac{634-63i}{145} & \frac{118-561i}{145} \\ \frac{72+52i}{145} & \frac{557}{145}-\frac{32i}{29} & -\frac{73}{58}-\frac{581i}{290} & \frac{-413-113i}{290} \\ \frac{92-3i}{145} & \frac{439}{290}-\frac{115i}{58} & -\frac{63}{58}-\frac{401i}{290} & \frac{-263+97i}{290} \end{bmatrix} \tag{4.9.51}$$

and

$$M\left(\mathbf{A}^*,\mathbf{b}^j,\mathbf{e}^k\right)=\left[A^{*}{}_{k}{}^{j}\right]=\overline{M\left(\mathbf{A},\mathbf{e}_k,\mathbf{b}_j\right)}^T=\begin{bmatrix} 1 & 8i & 4 & -7i \\ i & 9 & -9 & -4i \\ 7 & -8i & -15i & 0 \end{bmatrix} \tag{4.9.52}$$

4.9.5 Utilize the results in Exercise 4.9.4 and calculate $\left\langle \mathbf{A},\mathbf{A}\right\rangle$. The answer is

$$\left\langle \mathbf{A},\mathbf{A}\right\rangle=\operatorname{tr}\left(\mathbf{A}\mathbf{A}^*\right)$$
$$=\operatorname{tr}\left(\begin{bmatrix} 1 & -i & 7 \\ -8i & 9 & 8i \\ 4 & -9 & 15i \\ 7i & 4i & 0 \end{bmatrix}\begin{bmatrix} \frac{1185-1293i}{29} & \frac{1551+69623i}{290} & \frac{12919-8823i}{145} & \frac{11374-25103i}{145} \\ \frac{-3468+5364i}{145} & \frac{20142-6217i}{145} & \frac{-7428-4804i}{145} & \frac{-17311-13983i}{290} \\ \frac{1237+2319i}{145} & \frac{15769-19369i}{290} & \frac{-4473-1299i}{145} & \frac{-14361+47i}{290} \end{bmatrix}\right) \tag{4.9.53}$$
$$=\frac{885994}{145}$$

Section 4.10. Norm of a Linear Transformation

In Section 4.1, with equation (4.1.20), we introduced the idea of length or norm of a vector. While vector spaces can have norms that are not defined in terms of an inner product, thus far all of our examples have this property. In particular, we gave several examples in Section 4.1 for different types of vector spaces. In Section 4.9, we introduced the idea of an inner product in the vector space of linear transformations $\mathscr{L}\left(\mathscr{V};\mathscr{U}\right)$ by the definition (4.9.34). Given this definition, we can continue to calculate the norm from the inner product by the formula

$$\left\|\mathbf{A}\right\| = \sqrt{\left\langle \mathbf{A},\mathbf{A}\right\rangle} = \sqrt{\operatorname{tr}\left(\mathbf{A}\mathbf{A}^{*}\right)} \tag{4.10.1}$$

for a linear transformation $\mathbf{A}:\mathscr{V}\to\mathscr{U}$. The component representation of (4.10.1) can be read off from (4.9.35).

If we are given a linear transformation $\mathbf{A}:\mathscr{V}\to\mathscr{U}$, then for every vector $\mathbf{v}\in\mathscr{V}$, the vector $\mathbf{A}\mathbf{v}\in\mathscr{U}$ is defined and has the component representation

$$\mathbf{A}\mathbf{v} = \sum_{j=1}^{M}\sum_{k=1}^{N} A^{j}{}_{k}\upsilon^{k}\mathbf{b}_{j} \tag{4.10.2}$$

Therefore, from the definition of the norm of vectors in $\mathscr{U}$,

$$\begin{aligned}\left\|\mathbf{A}\mathbf{v}\right\| = \sqrt{\left\langle \mathbf{A}\mathbf{v},\mathbf{A}\mathbf{v}\right\rangle} &= \sqrt{\left\langle \sum_{j=1}^{M}\sum_{k=1}^{N} A^{j}{}_{k}\upsilon^{k}\mathbf{b}_{j}, \sum_{q=1}^{M}\sum_{s=1}^{N} A^{q}{}_{s}\upsilon^{s}\mathbf{b}_{q}\right\rangle} \\ &= \sqrt{\sum_{q=1}^{M}\sum_{s=1}^{N}\sum_{j=1}^{M}\sum_{k=1}^{N} A^{j}{}_{k}\overline{A}^{q}_{s}\upsilon^{k}\overline{\upsilon}^{s}b_{jq}}\end{aligned} \tag{4.10.3}$$

Because of the special rules for multiplying linear transformations in finite dimensional vector spaces, i.e., the matrix rules, equation (4.10.3) can be used to establish a relationship between the norm of a vector and the norm of a linear transformation. The particular result is

$$\left\|\mathbf{A}\mathbf{v}\right\| \le \left\|\mathbf{A}\right\|\left\|\mathbf{v}\right\| \tag{4.10.4}$$

For norms in general, the special relationship (4.10.4) is *not necessarily true*. In our case it is true. When it is true, the various norms involved are sometimes referred to as being *compatible*.

The derivation of (4.10.4) follows from (4.10.3) and the component version of (4.10.1). Because our equations are basis independent, there is no loss of generality to simply take all bases to be *orthonormal*. In this case, the component version of (4.10.1), as follows from (4.9.35), is

$$\|\mathbf{A}\| = \sqrt{\sum_{j=1}^{M}\sum_{k=1}^{N} A_{jk}\bar{A}_{jk}} = \sqrt{\sum_{j=1}^{M}\sum_{k=1}^{N} \left\|A_{jk}\right\|^2} \tag{4.10.5}$$

Likewise, (4.10.3) reduces to

$$\|\mathbf{Av}\| = \sqrt{\sum_{j=1}^{M}\sum_{s=1}^{N}\sum_{k=1}^{N} A_{jk}\bar{A}_{js}\upsilon_k\bar{\upsilon}_s} \tag{4.10.6}$$

Because each component of the vector $\mathbf{v}$ obeys $\upsilon_k = \langle \mathbf{v}, \mathbf{e}_k \rangle$, equation (4.10.6) can be written

$$\begin{aligned}\|\mathbf{Av}\| &= \sqrt{\sum_{j=1}^{M}\sum_{s=1}^{N}\sum_{k=1}^{N} A_{jk}\bar{A}_{js}\upsilon_k\bar{\upsilon}_s} = \sqrt{\sum_{j=1}^{M}\sum_{s=1}^{N}\sum_{k=1}^{N} \left\langle \mathbf{v}, \bar{A}_{jk}\mathbf{e}_k \right\rangle \overline{\left\langle \mathbf{v}, \bar{A}_{js}\mathbf{e}_s \right\rangle}} \\ &= \sqrt{\sum_{j=1}^{M} \left\langle \mathbf{v}, \sum_{k=1}^{N} \bar{A}_{jk}\mathbf{e}_k \right\rangle \overline{\left\langle \mathbf{v}, \sum_{s=1}^{N} \bar{A}_{js}\mathbf{e}_s \right\rangle}} = \sqrt{\sum_{j=1}^{M} \left| \left\langle \mathbf{v}, \sum_{k=1}^{N} \bar{A}_{jk}\mathbf{e}_k \right\rangle \right|^2}\end{aligned} \tag{4.10.7}$$

Next, we apply the Schwartz Inequality to each term in the sum in (4.10.7) to obtain

$$\|\mathbf{Av}\| = \sqrt{\sum_{j=1}^{M} \left| \left\langle \mathbf{v}, \sum_{k=1}^{N} \bar{A}_{jk}\mathbf{e}_k \right\rangle \right|^2} \le \sqrt{\sum_{j=1}^{M} \|\mathbf{v}\|^2 \left\| \sum_{k=1}^{N} \bar{A}_{jk}\mathbf{e}_k \right\|^2} = \|\mathbf{v}\| \sqrt{\sum_{j=1}^{M} \left\| \sum_{k=1}^{N} \bar{A}_{jk}\mathbf{e}_k \right\|^2} \tag{4.10.8}$$

The final manipulation involves rewriting the term $\left\| \sum_{k=1}^{N} \bar{A}_{jk}\mathbf{e}_k \right\|^2$ as

$$\left\| \sum_{k=1}^{N} \bar{A}_{jk}\mathbf{e}_k \right\|^2 = \left\langle \sum_{k=1}^{N} \bar{A}_{jk}\mathbf{e}_k, \sum_{s=1}^{N} \bar{A}_{js}\mathbf{e}_s \right\rangle = \sum_{k=1}^{N}\sum_{s=1}^{N} \bar{A}_{jk}A_{js} \left\langle \mathbf{e}_k, \mathbf{e}_s \right\rangle = \sum_{k=1}^{N} \bar{A}_{jk}A_{jk} = \sum_{k=1}^{N} \left|A_{jk}\right|^2 \tag{4.10.9}$$

where we have used our assumption that the basis $\{\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_N\}$ is orthonormal. When we substitute (4.10.9) into (4.10.8) and use (4.10.5) we obtain the result (4.10.4).[8]

Exercises

4.10.1 Utilize the results of Exercises 4.9.2 and 4.9.3 and illustrate the validity of (4.10.4) for the linear transformation defined by (4.9.37) and the particular vector

[8] An important and interesting result that we shall not pursue here is a theorem that establishes that in a finite dimensional vector space all norms are equivalent. The equivalence is based upon a formal definition. It is such that the various norms produce the same topological structure.

$$\mathbf{v} = \mathbf{e}_1 + 2\mathbf{e}_2 + 3\mathbf{e}_3 \tag{4.10.10}$$

The results for the factors in (4.10.4) turn out to be

$$\begin{aligned} \|\mathbf{Av}\| &= 3\sqrt{106} = 30.89 \\ \|\mathbf{A}\|\|\mathbf{v}\| &= \frac{1}{4}\sqrt{374095} = 152.91 \end{aligned} \tag{4.10.11}$$

4.10.2 Utilize the results of Exercises 4.9.4 and 4.9.5 and illustrate the validity of (4.10.4) for the linear transformation defined by (4.9.45) and the particular vector

$$\mathbf{v} = \mathbf{e}_1 + 2\mathbf{e}_2 + 3\mathbf{e}_3 \tag{4.10.12}$$

The results for the factors in (4.10.4) turn out to be

$$\begin{aligned} \|\mathbf{Av}\| &= 3\sqrt{5471} = 221.90 \\ \|\mathbf{A}\|\|\mathbf{v}\| &= \frac{2}{5}\sqrt{6644955} = 1031.11 \end{aligned} \tag{4.10.13}$$

Section 4.11. More About Linear Transformations on Inner Product Spaces

In this section, we shall briefly add to the information provided in Section 4.8 by studying certain other properties of linear transformations defined on complex inner product spaces. In Section 1.9, while discussing the idea of the transpose of a matrix, we introduced the idea of a matrix being symmetric and a matrix being skew-symmetric. The corresponding ideas for linear transformations defined on complex inner product spaces are Hermitian and skew-Hermitian. The formal definition is as follows:

Definition: A linear transformation $\mathbf{A}:\mathscr{V}\to\mathscr{V}$ is called *Hermitian* if $\mathbf{A}=\mathbf{A}^*$ and *skew-Hermitian* if $\mathbf{A}=-\mathbf{A}^*$.

If the underlying inner product space is real, the designations *symmetric* and *skew-symmetric* are often used instead of Hermitian and skew-Hermitian. The following theorem, which follows directly from the definition (4.9.1), characterizes Hermitian and skew-Hermitian linear transformations.

Theorem 4.11.1: A linear transformation $\mathbf{A}$ is Hermitian if and only if

$$\left\langle \mathbf{v}_1,\mathbf{A}\mathbf{v}_2\right\rangle=\left\langle \mathbf{A}\mathbf{v}_1,\mathbf{v}_2\right\rangle \tag{4.11.1}$$

for all $\mathbf{v}_1,\mathbf{v}_2\in\mathscr{V}$, and it is skew-Hermitian if and only if

$$\left\langle \mathbf{v}_1,\mathbf{A}\mathbf{v}_2\right\rangle=-\left\langle \mathbf{A}\mathbf{v}_1,\mathbf{v}_2\right\rangle \tag{4.11.2}$$

for all $\mathbf{v}_1,\mathbf{v}_2\in\mathscr{V}$.

As with matrices, see equation (1.9.22), a linear transformation $\mathbf{A}:\mathscr{V}\to\mathscr{V}$ can always be written

$$\mathbf{A}=\frac{1}{2}\left(\mathbf{A}+\mathbf{A}^*\right)+\frac{1}{2}\left(\mathbf{A}-\mathbf{A}^*\right) \tag{4.11.3}$$

which decomposes $\mathbf{A}$ into a Hermitian part and a skew-Hermitian part. In Section 3.4, we introduced the notation $\mathscr{L}(\mathscr{V};\mathscr{U})$ for the vector space of linear transformations from $\mathscr{V}$ to $\mathscr{U}$. In the current circumstance, where we are looking at a linear transformation $\mathbf{A}:\mathscr{V}\to\mathscr{V}$, we shall denote by $\mathscr{S}\left(\mathscr{V};\mathscr{V}\right)$ and $\mathscr{A}\left(\mathscr{V};\mathscr{V}\right)$ the *subsets* of $\mathscr{L}\left(\mathscr{V};\mathscr{V}\right)$ defined by

$$\mathscr{S}\left(\mathscr{V};\mathscr{V}\right)=\left\{\mathbf{A}\,\middle|\,\mathbf{A}\in\mathscr{L}\left(\mathscr{V};\mathscr{V}\right)\text{ and }\mathbf{A}=\mathbf{A}^*\right\} \tag{4.11.4}$$

and

$$\mathscr{A}(\mathscr{V};\mathscr{V}) = \left\{ \mathbf{A} \middle| \mathbf{A} \in \mathscr{L}(\mathscr{V};\mathscr{V}) \text{ and } \mathbf{A} = -\mathbf{A}^* \right\} \tag{4.11.5}$$

In the special case of a *real* inner product space, it is easy to show that the subsets $\mathscr{S}(\mathscr{V};\mathscr{V})$ and $\mathscr{A}(\mathscr{V};\mathscr{V})$ are both *subspaces* of $\mathscr{L}(\mathscr{V};\mathscr{V})$.[9] In particular, $\mathscr{L}(\mathscr{V};\mathscr{V})$ has the following decomposition:

Theorem 4.11.2: For a *real* inner product space $\mathscr{V}$, every linear transformation $\mathbf{A} \in \mathscr{L}(\mathscr{V};\mathscr{V})$ has the unique decomposition

$$\mathbf{A} = \mathbf{D} + \mathbf{W} \tag{4.11.6}$$

where $\mathbf{D} \in \mathscr{S}(\mathscr{V};\mathscr{V})$ and $\mathbf{W} \in \mathscr{A}(\mathscr{V};\mathscr{V})$.

Proof. If we utilize (4.11.3) in the case of a real vector space, the decomposition (4.11.6) follows from the definitions

$$\mathbf{D} = \frac{1}{2}\left(\mathbf{A} + \mathbf{A}^T\right) \tag{4.11.7}$$

and

$$\mathbf{W} = \frac{1}{2}\left(\mathbf{A} - \mathbf{A}^T\right) \tag{4.11.8}$$

Equations (4.11.7) and (4.11.8) establish the existence of the representation (4.11.6). The uniqueness of this representation follows by the following argument. We shall assume the representation (4.11.6) is not unique and write

$$\mathbf{A} = \mathbf{D}_1 + \mathbf{W}_1 = \mathbf{D}_2 + \mathbf{W}_2 \tag{4.11.9}$$

It follows from (4.11.9) that

$$\mathbf{D}_1 - \mathbf{D}_2 = \mathbf{W}_2 - \mathbf{W}_1 \tag{4.11.10}$$

Equation (4.11.10) forces the symmetric linear transformation $\mathbf{D}_1 - \mathbf{D}_2$ to equal the skew-symmetric linear transformation $\mathbf{W}_2 - \mathbf{W}_1$. Because the only linear transformation that is both symmetric and skew-symmetric is the zero linear transformation, $\mathbf{D}_1 = \mathbf{D}_2$ and $\mathbf{W}_2 = \mathbf{W}_1$ which establishes uniqueness.

[9] In the case of complex inner product spaces the subsets $\mathscr{S}(\mathscr{V};\mathscr{V})$ and $\mathscr{A}(\mathscr{V};\mathscr{V})$ are *not* subspaces.

In Section 4.4, we encountered the idea of an orthogonal matrix in $\mathscr{M}^{3\times 3}$. A generalization of this concept to linear transformations $\mathbf{A}:\mathscr{V}\rightarrow\mathscr{U}$ defined on complex inner product spaces is the idea of a *unitary* linear transformation.

Definition: A linear transformation $\mathbf{Q}:\mathscr{V}\rightarrow\mathscr{U}$ is *unitary* if

$$\left\langle \mathbf{Qv}_2,\mathbf{Qv}_1\right\rangle=\left\langle \mathbf{v}_2,\mathbf{v}_1\right\rangle \tag{4.11.11}$$

for all $\mathbf{v}_1,\mathbf{v}_2\in\mathscr{V}$.

For a real inner product space $\mathscr{V}$, equation (4.11.11) defines an *orthogonal* linear transformation. In the definition (4.11.11), the inner product on the left side of the equation is the one in $\mathscr{U}$ while the one on the right side of the equation is the inner product in $\mathscr{V}$. Geometrically, the definition (4.11.11) asserts that unitary (or orthogonal) linear transformations preserve the inner products. In geometric terms, they preserve lengths and angles.

Theorem 4.11.3: If $\mathbf{Q}$ is unitary, then it is one to one.

Proof. Take $\mathbf{v}_1=\mathbf{v}_2=\mathbf{v}$ in (4.11.11), and use the definition of norm (4.1.20). The result is

$$\left\|\mathbf{Qv}\right\|=\left\|\mathbf{v}\right\| \tag{4.11.12}$$

Thus, if $\mathbf{Qv}=\mathbf{0}$, then $\mathbf{v}=\mathbf{0}$. Therefore, the kernel of $\mathbf{Q}$, $K\left(\mathbf{Q}\right)$ only contains the zero vector. If we now use Theorem 3.3.1, this theorem is proven.

If we combine the definition of a unitary linear transformation (4.11.11), with the definition of an adjoint linear transformation, equation (4.9.1), then

$$\left\langle \mathbf{Qv}_2,\mathbf{Qv}_1\right\rangle=\left\langle \mathbf{v}_2,\mathbf{Q}^*\mathbf{Qv}_1\right\rangle=\left\langle \mathbf{v}_2,\mathbf{v}_1\right\rangle \tag{4.11.13}$$

Therefore, a unitary linear transformation $\mathbf{Q}:\mathscr{V}\rightarrow\mathscr{U}$ obeys

$$\mathbf{Q}^*\mathbf{Q}=\mathbf{I}_{\mathscr{V}} \tag{4.11.14}$$

If we require $\mathscr{V}$ and $\mathscr{U}$ to have the same dimension, then Theorems 4.11.3 and 3.3.5 ensure that a unitary transformation $\mathbf{Q}$ is a one to one and onto, thus, a nonsingular linear transformation. In the case we can use (4.11.14) and conclude

$$\mathbf{Q}^{-1}=\mathbf{Q}^* \tag{4.11.15}$$

Equation (4.11.15), in turn, implies

$$\mathbf{Q}\mathbf{Q}^* = \mathbf{I}_{\mathscr{U}} \tag{4.11.16}$$

Recall from Theorem 3.3.4 that a one to one linear transformation maps linearly independent vectors into linearly independent vectors. Therefore if $\{\mathbf{e}_1,\ldots,\mathbf{e}_N\}$ is a basis for $\mathscr{V}$ and $\mathbf{A}:\mathscr{V}\to\mathscr{U}$ is one to one, then $\{\mathbf{A}\mathbf{e}_1,\ldots,\mathbf{A}\mathbf{e}_N\}$ is basis for $R(\mathbf{A})$ which is a subspace in $\mathscr{U}$. If, in addition, $\{\mathbf{e}_1,\ldots,\mathbf{e}_N\}$ is orthonormal and $\mathbf{A}$ is unitary, it easily follows from the definition (4.11.11)that $\{\mathbf{A}\mathbf{e}_1,\ldots,\mathbf{A}\mathbf{e}_N\}$ is also orthonormal. Thus *the image of an orthonormal basis under a unitary transformation is also an orthonormal basis*. Conversely, one can show that a linear transformation which sends an orthonormal basis of $\mathscr{V}$ into an orthonormal basis of $R(\mathbf{A})$ must be unitary.

If we specialize our discussion to a three dimensional real vector space $\mathscr{V}$ as discussed in Sections 4.4 and 4.5, we can quickly extract some interesting results that have application in Mechanics. Among the topics we discussed in Section 4.4 was a change of basis from an orthonormal basis $\{\mathbf{i}_1,\mathbf{i}_2,\mathbf{i}_3\}$ to a *second* orthonormal basis $\{\hat{\mathbf{i}}_1,\hat{\mathbf{i}}_2,\hat{\mathbf{i}}_3\}$. Recall from equation (4.4.3), repeated,

$$\hat{\mathbf{i}}_j = \sum_{k=1}^{3} Q_{kj}\mathbf{i}_k \quad \text{for} \quad j=1,2,3 \tag{4.11.17}$$

that the two bases are connected by an orthogonal matrix

$$Q = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{bmatrix} \tag{4.11.18}$$

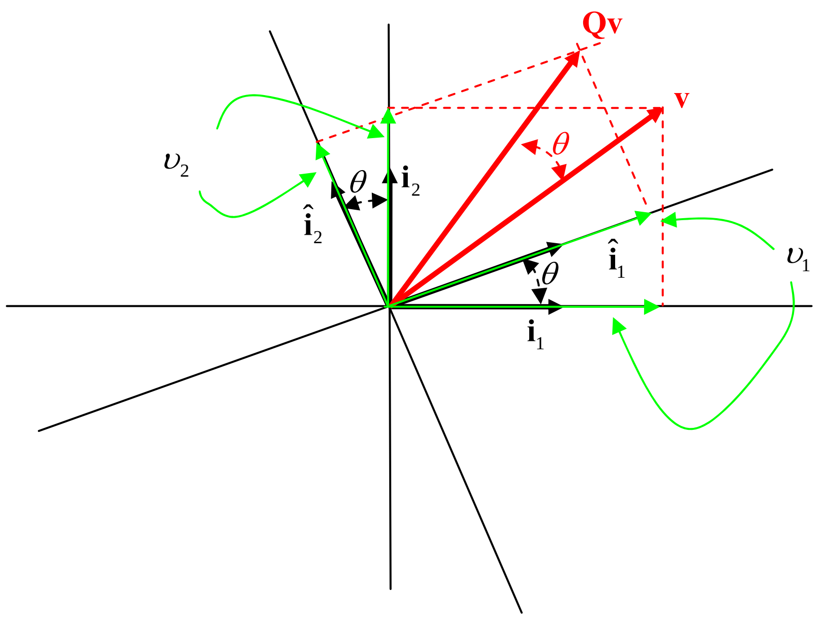
If we define an *orthogonal linear transformation* $\mathbf{Q}:\mathscr{V}\to\mathscr{V}$, by

$$\mathbf{Q}\mathbf{v} = \sum_{j=1}^{3}\sum_{k=1}^{3} Q_{kj}\upsilon_j\mathbf{i}_k \tag{4.11.19}$$

for all vectors $\mathbf{v} = \sum_{j=1}^{3}\upsilon_j\mathbf{i}_j \in \mathscr{V}$, then

$$\hat{\mathbf{i}}_j = \mathbf{Q}\mathbf{i}_j = \sum_{k=1}^{3} Q_{kj}\mathbf{i}_k \quad \text{for} \quad j=1,2,3 \tag{4.11.20}$$

In the special case where the rotation caused by (4.11.17) is in the plane, the result of the orthogonal linear transformation $\mathbf{Q}$ is such that it takes an arbitrary vector $\mathbf{v}$ and creates another one with the same components as $\mathbf{v}$ has with respect to the basis $\{\mathbf{i}_1,\mathbf{i}_2\}$ but points in the direction defined by $\{\hat{\mathbf{i}}_1,\hat{\mathbf{i}}_2\}$. It is perhaps easier to see the effect of $\mathbf{Q}$ by an examination of the following figure:



As displayed in the figure, the orthogonal linear transformation $\mathbf{Q}$ takes a vector $\mathbf{v}$ and rotates it by the angle θ while preserving its length. It also preserves the relative position of $\mathbf{Qv}$ to $\{\hat{\mathbf{i}}_1,\hat{\mathbf{i}}_2\}$ as that of $\mathbf{v}$ to $\{\mathbf{i}_1,\mathbf{i}_2\}$.

The study of the dynamics of systems of particles or the dynamics of rigid bodies necessitates a discussion of the so called kinematics of motion. This subject involves such things as viewing the motion of a particle from a basis that is rotating relative to one fixed in space. This view is modeled by allowing the orthogonal linear transformation $\mathbf{Q}$ to depend upon the time t. A time dependent vector $\mathbf{v}$ can be viewed instantaneously from the fixed basis or the rotating basis. The two component representations are given by

$$\mathbf{v}(t)=\sum_{j=1}^{3}\upsilon_j(t)\mathbf{i}_j=\sum_{j=1}^{3}\hat{\upsilon}_j(t)\hat{\mathbf{i}}_j(t) \tag{4.11.21}$$

It follows from (4.11.21) that

$$\frac{d\mathbf{v}(t)}{dt}=\sum_{j=1}^{3}\frac{d\upsilon_j(t)}{dt}\mathbf{i}_j=\sum_{j=1}^{3}\frac{d\hat{\upsilon}_j(t)}{dt}\hat{\mathbf{i}}_j(t)+\sum_{j=1}^{3}\hat{\upsilon}_j(t)\frac{d\hat{\mathbf{i}}_j(t)}{dt} \tag{4.11.22}$$

The term $\sum_{j=1}^{3}\frac{d\upsilon_j(t)}{dt}\mathbf{i}_j$ represents the time rate of change of $\mathbf{v}(t)$ seen by an observer fixed in the $\{\mathbf{i}_1,\mathbf{i}_2,\mathbf{i}_3\}$ basis. The term $\sum_{j=1}^{3}\frac{d\hat{\upsilon}_j(t)}{dt}\hat{\mathbf{i}}_j(t)$ represents the time rate of change of $\mathbf{v}(t)$ seen by an observer fixed in the $\{\hat{\mathbf{i}}_1,\hat{\mathbf{i}}_2,\hat{\mathbf{i}}_3\}$ basis. The term $\sum_{j=1}^{3}\hat{\upsilon}_j(t)\frac{d\hat{\mathbf{i}}_j(t)}{dt}$ is the time rate of change of $\mathbf{v}(t)$ caused by the rotation of the second basis relative to the first. It is this term that we shall now examine. It follows from (4.11.20) that

$$\frac{d\hat{\mathbf{i}}_j(t)}{dt}=\frac{d\mathbf{Q}(t)}{dt}\mathbf{i}_j \tag{4.11.23}$$

The derivative $\frac{d\mathbf{Q}(t)}{dt}$ is constrained by the formula

$$\mathbf{Q}(t)\mathbf{Q}(t)^T=\mathbf{I} \tag{4.11.24}$$

Equation (4.11.24) is (4.11.16) applied to the case of a real vector space. The derivative of (4.11.24) is

$$\frac{d\mathbf{Q}(t)}{dt}\mathbf{Q}(t)^T+\mathbf{Q}(t)\left(\frac{d\mathbf{Q}(t)}{dt}\right)^T=\mathbf{0} \tag{4.11.25}$$

Next, we observe that is we define a linear transformation $\mathbf{Z}$ by

$$\mathbf{Z}(t)=\frac{d\mathbf{Q}(t)}{dt}\mathbf{Q}(t)^T \tag{4.11.26}$$

then, by (4.11.25),

$$\mathbf{Z}(t)=-\mathbf{Z}(t)^T \tag{4.11.27}$$

In other words, the linear transformation defined by (4.11.26) is skew symmetric. Given the definition (4.11.26) and the equation (4.11.15), it follows that

$$\frac{d\mathbf{Q}(t)}{dt}=\mathbf{Z}(t)\mathbf{Q}(t) \tag{4.11.28}$$

This formula and (4.11.20) allow (4.11.23) to be written

$$\frac{d\hat{\mathbf{i}}_j(t)}{dt} = \frac{d\mathbf{Q}(t)}{dt}\mathbf{i}_j = \mathbf{Z}(t)\mathbf{Q}(t)\mathbf{i}_j = \mathbf{Z}(t)\hat{\mathbf{i}}_j(t) \tag{4.11.29}$$

Equation (4.11.29) shows that $\mathbf{Z}(t)$ measures the angular velocity of the basis $\{\hat{\mathbf{i}}_1, \hat{\mathbf{i}}_2, \hat{\mathbf{i}}_3\}$ with respect to the basis $\{\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3\}$. Since $\mathbf{Z}(t)$ is skew symmetric, its matrix with respect to, for example, the basis $\{\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3\}$ will be of the form

$$M\left(\mathbf{Z}(t), \mathbf{i}_j, \mathbf{i}_k\right) = \begin{bmatrix} 0 & Z_{12}(t) & Z_{13}(t) \\ -Z_{12}(t) & 0 & Z_{23}(t) \\ -Z_{13}(t) & -Z_{23}(t) & 0 \end{bmatrix} \tag{4.11.30}$$

Thus, the nine components of $\mathbf{Z}(t)$ are actually determined by three quantities. It is common practice to identify a three dimensional vector that has these components and write (4.11.29) in terms of that vector. The formal step is to define a vector $\boldsymbol{\omega}(t)$ by the requirement

$$\mathbf{Z}(t)\mathbf{v} = \boldsymbol{\omega}(t) \times \mathbf{v} \qquad \text{for all} \quad \mathbf{v} \in \mathscr{V} \tag{4.11.31}$$

where the cross product is defined in Section 4.6. While we do not need to introduce components, it is perhaps useful to point out that (4.11.31) can be used to write (4.11.30) as

$$M\left(\mathbf{Z}(t), \mathbf{i}_j, \mathbf{i}_k\right) = \begin{bmatrix} 0 & -\omega_3(t) & \omega_2(t) \\ \omega_3(t) & 0 & -\omega_1(t) \\ -\omega_2(t) & \omega_1(t) & 0 \end{bmatrix} \tag{4.11.32}$$

for the cross product associated with a positively oriented system. In dynamics, the vector $\boldsymbol{\omega}(t)$ is the *angular velocity*. The definition (4.11.31) allows (4.11.29) to be written in the possibly more familiar form

$$\frac{d\hat{\mathbf{i}}_j(t)}{dt} = \boldsymbol{\omega}(t) \times \hat{\mathbf{i}}_j(t) \tag{4.11.33}$$

and (4.11.22) to be written

$$\frac{d\mathbf{v}(t)}{dt} = \sum_{j=1}^{3} \frac{d\hat{\upsilon}_j(t)}{dt}\hat{\mathbf{i}}_j(t) + \boldsymbol{\omega}(t) \times \mathbf{v}(t) \tag{4.11.34}$$

It is customary to give the first term on the right side of (4.11.34) a different symbol such as $\dfrac{\delta \mathbf{v}(t)}{\delta t}$ to signify that it represents the rate of change as seen from the rotating basis.

Exercises:

4.11.1 If you are given a linear transformation $\mathbf{A}:\mathscr{V}\to\mathscr{U}$ show that the linear transformation $\mathbf{A}\mathbf{A}^{*}:\mathscr{U}\to\mathscr{U}$ and the linear transformation $\mathbf{A}^{*}\mathbf{A}:\mathscr{V}\to\mathscr{V}$ are Hermitian.

4.11.2 Use the definition of the scalar triple product introduced in Section 4.6 and show that the determinant of a linear transformation $\mathbf{A}:\mathscr{V}\to\mathscr{V}$, where $\mathscr{V}$ is a three dimensional inner product space, is given by the component free expression

$$\det \mathbf{A}\langle \mathbf{u},\mathbf{v}\times\mathbf{w}\rangle = \langle \mathbf{A}\mathbf{u},\mathbf{A}\mathbf{v}\times\mathbf{A}\mathbf{w}\rangle \qquad \text{for all} \quad \mathbf{u},\mathbf{v},\mathbf{w}\in\mathscr{V} \tag{4.11.35}$$

Equation (4.11.35) defines the determinant of a linear transformation in a completely component free fashion. It is limited to the case of three dimensions but, perhaps, gives some insight in how the determinant of a linear transformation could be defined when the vector space has dimension greater than three.

4.11.3 Given a linear transformation $\mathbf{A}:\mathscr{V}\to\mathscr{V}$, where $\mathscr{V}$ is a three dimensional inner product space, define a linear transformation $\mathbf{K}_{\mathbf{A}}:\mathscr{V}\to\mathscr{V}$ by

$$\langle \mathbf{K}_{\mathbf{A}}\mathbf{u},\mathbf{v}\times\mathbf{w}\rangle = \langle \mathbf{u},\mathbf{A}\mathbf{v}\times\mathbf{A}\mathbf{w}\rangle \quad \text{for all} \quad \mathbf{u},\mathbf{v},\mathbf{w}\in\mathscr{V} \tag{4.11.36}$$

Show that

$$\mathbf{K}_{\mathbf{A}}\mathbf{A} = \mathbf{A}\mathbf{K}_{\mathbf{A}} = (\det \mathbf{A})\mathbf{I} \tag{4.11.37}$$

Therefore, from (1.10.50) and (1.10.51), $\mathbf{K}_{\mathbf{A}}$ is the linear transformation whose matrix is the adjugate matrix of the matrix of $\mathbf{A}$. In the following we shall write $\operatorname{adj}\mathbf{A}$ to denote the linear transformation defined by (4.11.36).

4.11.4 Show that the trace of a linear transformation $\mathbf{A}:\mathscr{V}\to\mathscr{V}$, where $\mathscr{V}$ is a three dimensional inner product space, is given by the component free expression

$$\operatorname{tr}\mathbf{A}\langle \mathbf{u},\mathbf{v}\times\mathbf{w}\rangle = \langle \mathbf{A}\mathbf{u},\mathbf{v}\times\mathbf{w}\rangle + \langle \mathbf{u},\mathbf{A}\mathbf{v}\times\mathbf{w}\rangle + \langle \mathbf{u},\mathbf{v}\times\mathbf{A}\mathbf{w}\rangle \qquad \text{for all} \quad \mathbf{u},\mathbf{v},\mathbf{w}\in\mathscr{V} \tag{4.11.38}$$

Equation (4.11.38) defines the trace of a linear transformation in a component free fashion. Like the definition of the determinant given in Exercise 4.9.1, the definition (4.11.38) is limited to the three dimensional case.

4.11.5 Certain applications require that a determinant be differentiated. If $\mathbf{A} = \mathbf{A}(t)$ is a differentiable linear transformation, use the results of Exercises 4.11.1 and 4.11.2 and show that

$$\frac{d\left(\det\left(\mathbf{A}(t)\right)\right)}{dt} = \operatorname{tr}\left(\left(\operatorname{adj}\mathbf{A}(t)\right)\frac{d\mathbf{A}(t)}{dt}\right) \tag{4.11.39}$$

where $\operatorname{adj}\mathbf{A}$ is the linear transformation denoted by $\mathbf{K}_{\mathbf{A}}$ in Exercise 4.11.3. While we have not proven it here, equation (4.11.39) holds for linear transformations defined on vector spaces of arbitrary finite dimension.

Section 4.12. Fundamental Subspaces Theorem

In this section, we shall return to the discussion of linear transformations given in Section 3.3 and augment those concepts in the case where the underlying vector spaces are inner product spaces. We shall show how to interpret some of our earlier results in an interesting geometric fashion. The main result of this section is known as the *Fundamental Subspaces Theorem*.

We begin this discussion by again looking at a linear transformation $\mathbf{A}:\mathscr{V}\to\mathscr{U}$. Unlike the discussion in Section 3.3, we now allow $\mathscr{V}$ and $\mathscr{U}$ to be inner product spaces. We shall allow them to be complex inner product spaces. In Section 3.3, we defined the subspace of $\mathscr{V}$ known as the kernel of $\mathbf{A}:\mathscr{V}\to\mathscr{U}$. This subspace was defined by equation (3.3.5), repeated,

$$K(\mathbf{A})=\left\{\mathbf{v}\middle|\mathbf{A}\mathbf{v}=\mathbf{0}\right\} \tag{4.12.1}$$

We also defined the subspace of $\mathscr{U}$ known as the range. This subspace was defined by equation (3.3.1), repeated,

$$R(\mathbf{A})=\left\{\mathbf{A}\mathbf{v}\middle|\ \mathbf{v}\in\mathscr{V}\right\} \tag{4.12.2}$$

For the inner product spaces we are considering, the corresponding concepts for the adjoint $\mathbf{A}^*:\mathscr{U}\to\mathscr{V}$ are

$$K(\mathbf{A}^*)=\left\{\mathbf{u}\middle|\mathbf{A}^*\mathbf{u}=\mathbf{0}\right\} \tag{4.12.3}$$

for the kernel and

$$R(\mathbf{A}^*)=\left\{\mathbf{A}^*\mathbf{u}\middle|\ \mathbf{u}\in\mathscr{U}\right\} \tag{4.12.4}$$

The subspace $K(\mathbf{A})$ of $\mathscr{V}$ and the subspace $R(\mathbf{A}^*)$ also of $\mathscr{V}$ have a special geometric relationship that we now wish to characterize. In order to characterize this relationship, we make the following definitions:

Definition: A subspace $\mathscr{V}_1$ and a subspace $\mathscr{V}_2$ of the inner product space $\mathscr{V}$ are *orthogonal* if $\langle\mathbf{v}_1,\mathbf{v}_2\rangle=0$ for every $\mathbf{v}_1\in\mathscr{V}_1$ and $\mathbf{v}_2\in\mathscr{V}_2$.

If a subspace $\mathscr{V}_1$ and a subspace $\mathscr{V}_2$ of the inner product space $\mathscr{V}$ are orthogonal it is customary to write $\mathscr{V}_1\perp\mathscr{V}_2$. If $\mathscr{V}_1\perp\mathscr{V}_2$ the only element the two subspaces have in common is the zero vector, i.e. $\mathscr{V}_1\cap\mathscr{V}_2=\{\mathbf{0}\}$. A related geometric concept is the orthogonal complement of a subspace. This definition is as follows:

Definition: Let $\mathscr{V}_1$ be a subspace of an inner product space $\mathscr{V}$, the *orthogonal complement* of $\mathscr{V}_1$, written $\mathscr{V}_1^\perp$, is a subspace defined by

$$\mathscr{V}_1^\perp = \left\{ \mathbf{v} \in \mathscr{V} \middle| \left\langle \mathbf{v}, \mathbf{v}_1 \right\rangle = 0 \quad \text{for all } \mathbf{v}_1 \in \mathscr{V}_1 \right\} \tag{4.12.5}$$

In simple geometric terms the orthogonal complement consists of all of those vectors in the inner product space $\mathscr{V}$ that are perpendicular to those in $\mathscr{V}_1$.

The first part of the fundamental subspaces theorem is that the subspace $K(\mathbf{A})$ is the orthogonal compliment of $R(\mathbf{A}^*)$. In the notation we have just introduced, the assertion is

$$K(\mathbf{A}) = R(\mathbf{A}^*)^\perp \tag{4.12.6}$$

The proof of this result follows from the definition of the adjoint linear transformation, equation (4.9.1), repeated,

$$\left\langle \mathbf{u}, \mathbf{A}\mathbf{v} \right\rangle = \left\langle \mathbf{A}^*\mathbf{u}, \mathbf{v} \right\rangle \tag{4.12.7}$$

For a vector $\mathbf{v} \in K(\mathbf{A})$, since $\mathbf{A}\mathbf{v} = 0$, equation (4.12.7) yields $\left\langle \mathbf{A}^*\mathbf{u}, \mathbf{v} \right\rangle = 0$. Thus, the vector $\mathbf{A}^*\mathbf{u} \in R(\mathbf{A}^*)$ is orthogonal to $\mathbf{v} \in K(\mathbf{A})$. The second part of the fundamental subspaces theorem is

$$K(\mathbf{A}^*) = R(\mathbf{A})^\perp \tag{4.12.8}$$

The proof of this result is essentially the same as the proof of (4.12.6). One simply interchanges the roles of $\mathbf{A}$ and $\mathbf{A}^*$.

Example 4.12.1: You are given a matrix $A \in M^{4\times 3}$ defined by

$$A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 4 & 0 \\ 2 & 4 & 0 \\ 1 & 3 & 1 \end{bmatrix} \tag{4.12.9}$$

Viewed as a linear transformation $A : \mathscr{M}^{3\times 1} \to \mathscr{M}^{4\times 1}$, one can easily show that

$$R(A) = \text{span}\left(\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right) \tag{4.12.10}$$

and

$$K(A) = \text{span}\left(\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}\right) \tag{4.12.11}$$

Since,

$$A^T = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 3 & 4 & 4 & 3 \\ 1 & 0 & 0 & 1 \end{bmatrix} \tag{4.12.12}$$

a simple calculation shows that

$$R(A^T) = \text{span}\left(\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}\right) \tag{4.12.13}$$

$$K(A^T) = \text{span}\left(\begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right) \tag{4.12.14}$$

The fundamental subspaces theorem asserts that $K(A) = R(A^T)^{\perp}$ and $K(A^T) = R(A)^{\perp}$. These results are validated for this example because

For $K(A) = R(A^T)^{\perp}$

$$\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}^T \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}^T \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix} = 0 \tag{4.12.15}$$

and, for $K\left(A^T\right)=R\left(A\right)^\perp$

$$\begin{bmatrix}0\\-1\\1\\0\end{bmatrix}^T\begin{bmatrix}0\\1\\1\\0\end{bmatrix}=\begin{bmatrix}0\\-1\\1\\0\end{bmatrix}^T\begin{bmatrix}1\\0\\0\\1\end{bmatrix}=0$$
$$\begin{bmatrix}-1\\0\\0\\1\end{bmatrix}^T\begin{bmatrix}0\\1\\1\\0\end{bmatrix}=\begin{bmatrix}-1\\0\\0\\1\end{bmatrix}^T\begin{bmatrix}1\\0\\0\\1\end{bmatrix}=0 \tag{4.12.16}$$

Next, we shall introduce the ideas of the sum and the direct sum of two subspaces. The idea is useful in the context of the results (4.12.6) and (4.12.7).

Definition: If $\mathscr{V}_1$ and $\mathscr{V}_2$ are subspaces of a vector space $\mathscr{V}$, the *sum* of $\mathscr{V}_1$ and $\mathscr{V}_2$ is written $\mathscr{V}_1+\mathscr{V}_2$ and is the subspace of $\mathscr{V}$ defined by

$$\mathscr{V}_1+\mathscr{V}_2=\left\{\mathbf{v}\middle|\mathbf{v}=\mathbf{v}_1+\mathbf{v}_2 \text{ where } \mathbf{v}_1\in\mathscr{V}_1 \text{ and } \mathbf{v}_2\in\mathscr{V}_2\right\} \tag{4.12.17}$$

If *every* vector in $\mathscr{V}$ can be expressed in the form $\mathbf{v}=\mathbf{v}_1+\mathbf{v}_2$ where $\mathbf{v}_1\in\mathscr{V}_1$ and $\mathbf{v}_2\in\mathscr{V}_2$, then $\mathscr{V}=\mathscr{V}_1+\mathscr{V}_2$.

Definition: If $\mathscr{V}_1$ and $\mathscr{V}_2$ are subspaces of a vector space $\mathscr{V}$ such that the only vector they have in common is the zero vector, i.e., if $\mathscr{V}_1\cap\mathscr{V}_2=\{\mathbf{0}\}$, them the sum $\mathscr{V}_1+\mathscr{V}_2$ is called the *direct sum* and is written $\mathscr{V}_1\oplus\mathscr{V}_2$.

Theorem 4.12.1: If $\mathscr{V}_1$ and $\mathscr{V}_2$ are subspaces of a vector space $\mathscr{V}$, the representation $\mathbf{v}=\mathbf{v}_1+\mathbf{v}_2$ of a vector $\mathbf{v}\in\mathscr{V}_1+\mathscr{V}_2$ is unique if and only if $\mathscr{V}_1\cap\mathscr{V}_2=\{\mathbf{0}\}$.

Proof: Every vector in $\mathbf{v}\in\mathscr{V}_1+\mathscr{V}_2$ has the representation

$$\mathbf{v}=\underbrace{\mathbf{v}_1}_{\text{In } \mathscr{V}_1}+\underbrace{\mathbf{v}_2}_{\text{In } \mathscr{V}_2} \tag{4.12.18}$$

If $\mathscr{V}_1\cap\mathscr{V}_2=\{\mathbf{0}\}$, we need to prove that the representation (4.12.18) is unique. As with all uniqueness proofs, we assume the representation is not unique. In particular assume $\mathbf{v}\in\mathscr{V}_1+\mathscr{V}_2$ has two representations

$$\mathbf{v} = \underbrace{\mathbf{v}_1}_{\text{In } \mathscr{V}_1} + \underbrace{\mathbf{v}_2}_{\text{In } \mathscr{V}_2} = \underbrace{\mathbf{v}_1'}_{\text{In } \mathscr{V}_1} + \underbrace{\mathbf{v}_2'}_{\text{In } \mathscr{V}_2} \tag{4.12.19}$$

Rearrangement yields

$$\underbrace{\mathbf{v}_1}_{\text{In } \mathscr{V}_1} - \underbrace{\mathbf{v}_1'}_{\text{In } \mathscr{V}_1} = \underbrace{\mathbf{v}_2'}_{\text{In } \mathscr{V}_2} - \underbrace{\mathbf{v}_2}_{\text{In } \mathscr{V}_2} \tag{4.12.20}$$

Because the only vector in $\mathscr{V}_1$ that is also in $\mathscr{V}_2$ is $\mathbf{0}$, equation (4.12.20) yields $\mathbf{v}_1 = \mathbf{v}_1'$ and $\mathbf{v}_2' = \mathbf{v}_2$. Therefore the representation is unique.

Conversely, if we have uniqueness, it must be true that $\mathscr{V}_1 \cap \mathscr{V}_2 = \{\mathbf{0}\}$. Because, if this were not the case, any nonzero vector $\mathbf{w} \in \mathscr{V}_1 \cap \mathscr{V}_2$ could violate the assumed uniqueness by writing $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2 = \underbrace{\mathbf{v}_1 + \mathbf{w}}_{\text{In } \mathscr{V}_1} + \underbrace{\mathbf{v}_2 - \mathbf{w}}_{\text{In } \mathscr{V}_2}$.

Theorem 4.12.2: If $\mathscr{V}_1$ is a subspace of $\mathscr{V}$, then

$$\mathscr{V} = \mathscr{V}_1 \oplus \mathscr{V}_1^{\perp} \tag{4.12.21}$$

Proof: Let $\mathbf{v}_1$ be an arbitrary vector in $\mathscr{V}_1$. Without loss of generality, we can select $\mathbf{v}_1$ such that $\|\mathbf{v}_1\| = 1$. For example, if one selected a $\mathbf{v}_1$ which did not have unit length, the vector $\mathbf{v}_1 / \|\mathbf{v}_1\|$ would have unit length. For any vector $\mathbf{v} \in \mathscr{V}$, we can write the identity

$$\mathbf{v} = \underbrace{\mathbf{v} - \langle \mathbf{v}, \mathbf{v}_1 \rangle \mathbf{v}_1}_{\text{In } \mathscr{V}_1^{\perp}} + \underbrace{\langle \mathbf{v}, \mathbf{v}_1 \rangle \mathbf{v}_1}_{\text{In } \mathscr{V}_1} \tag{4.12.22}$$

The first term, $\mathbf{v} - \langle \mathbf{v}, \mathbf{v}_1 \rangle \mathbf{v}_1$, is in $\mathscr{V}_1^{\perp}$ because,

$$\begin{aligned} \langle \mathbf{v} - \langle \mathbf{v}, \mathbf{v}_1 \rangle \mathbf{v}_1, \mathbf{v}_1 \rangle &= \langle \mathbf{v}, \mathbf{v}_1 \rangle - \langle \mathbf{v}, \mathbf{v}_1 \rangle \langle \mathbf{v}_1, \mathbf{v}_1 \rangle \\ &= \langle \mathbf{v}, \mathbf{v}_1 \rangle - \langle \mathbf{v}, \mathbf{v}_1 \rangle \|\mathbf{v}_1\|^2 = \langle \mathbf{v}, \mathbf{v}_1 \rangle - \langle \mathbf{v}, \mathbf{v}_1 \rangle = 0 \end{aligned} \tag{4.12.23}$$

We have established that any vector in $\mathbf{v} \in \mathscr{V}$ has the decomposition

$$\mathbf{v} = \underbrace{\mathbf{v}_1}_{\text{In } \mathscr{V}_1} + \underbrace{\mathbf{v}_2}_{\text{In } \mathscr{V}_1^{\perp}} \tag{4.12.24}$$

To complete the proof of the theorem, we need to prove that this decomposition is unique. Because the only vector in $\mathscr{V}_1 \cap \mathscr{V}_1^{\perp}$ is $\mathbf{0}$, Theorem 4.8.1 tells us the decomposition is unique.

Corollary:

$$\dim \mathscr{V} = \dim \mathscr{V}_1 + \dim \mathscr{V}_1^{\perp} \tag{4.12.25}$$

Proof: One simply writes $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$, and expresses each vector in terms of a basis for each subspace. The union of these two bases is a basis for $\mathscr{V}$.

Given $\mathbf{A}: \mathscr{V} \to \mathscr{U}$, we know that the kernel, $K(\mathbf{A})$, is a subspace of $\mathscr{V}$. Theorem 4.8.1 applied to this subspace yields

$$\mathscr{V} = K(\mathbf{A}) \oplus K(\mathbf{A})^{\perp} \tag{4.12.26}$$

Equation (4.12.6) allows (4.12.26) to be written

$$\mathscr{V} = K(\mathbf{A}) \oplus R(\mathbf{A}^*) \tag{4.12.27}$$

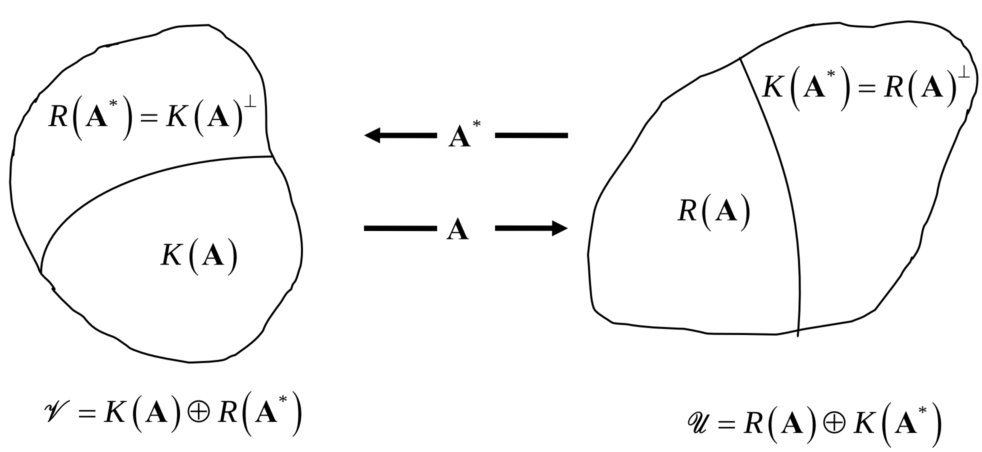
In a similar fashion, the fact that $R(\mathbf{A})$ is a subspace of $\mathscr{U}$ allows us to write

$$\mathscr{U} = R(\mathbf{A}) \oplus R(\mathbf{A})^{\perp} \tag{4.12.28}$$

which by (4.12.8)

$$\mathscr{U} = R(\mathbf{A}) \oplus K(\mathbf{A}^*) \tag{4.12.29}$$

The following figure summarizes the various geometric relationships reflected in equations (4.12.6) , (4.12.8), (4.12.27) and (4.12.29).



A final result that can be identified from the above geometric construction is that

$$\dim R(\mathbf{A}) = \dim R(\mathbf{A}^*) \tag{4.12.30}$$

This result is a direct consequence of (3.3.13) and (4.12.25) applied to equation (4.12.27). In the case of a matrix $A: \mathscr{M}^{N\times 1} \to \mathscr{M}^{M\times 1}$ with real elements, equation (4.12.30) reduces to equation (2.8.3). It was this earlier result that gave us the important fact that the row rank and the column rank of a matrix are the same.

Exercises

4.12.1 you were given the following system of equations:

$$\begin{aligned} x_1 + x_2 + 2x_3 + x_4 &= 5 \\ 2x_1 + 3x_2 - x_3 - 2x_4 &= 2 \\ 4x_1 + 5x_2 + 3x_3 &= 7 \end{aligned} \tag{4.12.31}$$

a) Show that the kernel of the matrix of coefficients is

$$K(A) = \operatorname{span}\left(\begin{bmatrix} -7 \\ 5 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 4 \\ 0 \\ 1 \end{bmatrix} \right) \tag{4.12.32}$$

b) Utilize (4.12.6) and show that

$$R(A^T) = \operatorname{span}\left(\begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ -1 \\ -2 \end{bmatrix} \right) \tag{4.12.33}$$

Section 4.13. Least Squares Problem

In the introduction to this Chapter, it was explained that the inner product structure makes it possible to construct approximations based upon the idea of elements of vector spaces being close in some sense. In this section, we shall illustrate an approximation procedure that occurs in many applications. It is based upon the idea of minimizing the distance between two vectors. The resulting idea is known as a least squares problem. The details of this idea will be presented in this section. In the next section, we shall apply the idea to the problem of fitting a curve to a set of data.

We begin by the assumption that we are given a *real* vector space $\mathscr{V}$ of dimension N. The problem we shall investigate is how to approximate a vector $\mathbf{v} \in \mathscr{V}$ by a vector $\mathbf{u}$ in a subspace of $\mathscr{V}$, we shall call $\mathscr{V}_1$. In a way to be described next, we shall define a quantity that measures the approximation.

Given $\mathbf{v} \in \mathscr{V}$, the approximation to $\mathbf{v}$, which we have called $\mathbf{u} \in \mathscr{V}_1$, is measured by the distance between $\mathbf{v}$ and $\mathbf{u}$. The structure of the inner product space provides a measure of this distance. Our first step is to define the *residual* by

$$\mathbf{r}(\mathbf{u}) = \mathbf{v} - \mathbf{u} \tag{4.13.1}$$

and measure the departure of $\mathbf{v}$ from $\mathbf{u}$ by the *squared norm*

$$\|\mathbf{r}(\mathbf{u})\|^2 = \|\mathbf{v} - \mathbf{u}\|^2 \tag{4.13.2}$$

The problem we are examining is, given $\mathbf{v} \in \mathscr{V}$, how do we calculate $\mathbf{u}$ in order to minimize the squared norm $\|\mathbf{r}(\mathbf{u})\|^2 = \|\mathbf{v} - \mathbf{u}\|^2$.

As a function of the $\mathbf{u}$, we shall minimize the squared norm (4.13.2). The result of this minimization problem is straight forward if we remember how the derivative of a function with respect to a vector is defined. If $f : \mathscr{V}_1 \rightarrow \mathscr{R}$ is a real valued function defined on an open set $\mathscr{V}_1$ of a vector space $\mathscr{V}$, then the *gradient* of f is a vector valued function on $\mathscr{V}_1$ defined by

$$\left.\frac{df(\mathbf{x} + \lambda\mathbf{c})}{d\lambda}\right|_{\lambda=0} = \langle \operatorname{grad} f(\mathbf{x}), \mathbf{c} \rangle \tag{4.13.3}$$

for all vectors $\mathbf{c} \in \mathscr{V}_1$. The minimum state we seek is the vector $\mathbf{u} \in \mathscr{V}_1$ which satisfies the condition

$$\left.\frac{d}{d\lambda}\left(\left\|\mathbf{r}\left(\mathbf{u}+\lambda\mathbf{c}\right)\right\|^{2}\right)\right|_{\lambda=0}=\left\langle\operatorname{grad}\left\|\mathbf{r}\left(\mathbf{u}\right)\right\|^{2},\mathbf{c}\right\rangle=0 \tag{4.13.4}$$

for all vectors $\mathbf{c}\in\mathscr{V}_1$. From (4.13.2) and $(4.13.4)_1$, at the point of minimum, it is necessary that

$$\left.\frac{d}{d\lambda}\left(\left\|\mathbf{v}-\left(\mathbf{u}+\lambda\mathbf{c}\right)\right\|^{2}\right)\right|_{\lambda=0}=0 \quad \text{for all} \quad \mathbf{c}\in\mathscr{V}_1 \tag{4.13.5}$$

Equation $(4.13.4)_2$ is equivalent to the M equations

$$\left\langle\operatorname{grad}\left\|\mathbf{v}-\mathbf{u}\right\|^{2},\mathbf{c}\right\rangle=0 \quad \text{for all} \quad \mathbf{c}\in\mathscr{V}_1 \tag{4.13.6}$$

The manipulations are slightly easier if we perform the calculation based upon (4.13.5). The result is

$$\begin{aligned}\left.\frac{d}{d\lambda}\left(\left\|\mathbf{v}-\left(\mathbf{u}+\lambda\mathbf{c}\right)\right\|^{2}\right)\right|_{\lambda=0}&=\left.\frac{d}{d\lambda}\left(\left\langle\mathbf{v}-\left(\mathbf{u}+\lambda\mathbf{c}\right),\mathbf{v}-\left(\mathbf{u}+\lambda\mathbf{c}\right)\right\rangle\right)\right|_{\lambda=0}\\&=-\left\langle\mathbf{v}-\mathbf{u},\mathbf{c}\right\rangle-\left\langle\mathbf{c},\mathbf{v}-\mathbf{u}\right\rangle\\&=-2\left\langle\mathbf{v}-\mathbf{u},\mathbf{c}\right\rangle=0 \qquad \text{for all} \quad \mathbf{c}\in\mathscr{V}_1\end{aligned} \tag{4.13.7}$$

If (4.13.7) is rewritten, the minimization condition is defined by

$$\left\langle\mathbf{v}-\mathbf{u},\mathbf{c}\right\rangle=0 \quad \text{for all} \quad \mathbf{c}\in\mathscr{V}_1 \tag{4.13.8}$$

Thus, we obtain the elementary result that the residual, as defined by (4.13.1) must be orthogonal to every vector in the subspace $\mathscr{V}_1$. Another way of stating this result is that the *minimum residual is a vector in the orthogonal complement of the subspace* $\mathscr{V}_1$.

Thus, in its most elementary form, if we wish to approximate a given vector $\mathbf{v}\in\mathscr{V}$ by a vector $\mathbf{u}$ in the subspace $\mathscr{V}_1$, we simply calculate the vector $\mathbf{u}$ that obeys (4.13.8). If we select a basis $\left\{\mathbf{e}_1,\mathbf{e}_2,...,\mathbf{e}_M\right\}$ for $\mathscr{V}_1$, and express $\mathbf{u}$ in components by the usual formula

$$\mathbf{u}=\sum_{j=1}^{M}u^j\mathbf{e}_j \tag{4.13.9}$$

it follows from (4.13.8) that

$$u^j=\left\langle\mathbf{u},\mathbf{e}^j\right\rangle=\left\langle\mathbf{v},\mathbf{e}^j\right\rangle \quad \text{for} \quad j=1,2,...,M \tag{4.13.10}$$

where $\{\mathbf{e}^1, \mathbf{e}^2, ..., \mathbf{e}^M\}$ is the reciprocal basis to $\{\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_M\}$. If $(4.13.10)_2$ is substituted into (4.13.9), the answer to our least squares problem is

$$\mathbf{u} = \sum_{j=1}^{M} \left\langle \mathbf{v}, \mathbf{e}^j \right\rangle \mathbf{e}_j \tag{4.13.11}$$

Equation (4.13.11) says we simply calculate the reciprocal basis to $\{\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_M\}$, project the given vector $\mathbf{v} \in \mathscr{V}$ in these M directions and substitute the result into (4.13.11).

Example 4.13.1 In Section 2.1 we gave a number of examples of vector spaces. One of these was the vector space $C[a,b]$ consisting of those continuous functions defined on every open subinterval of $[a,b]$. In this example, we shall select the interval to be $[0,b]$ and the associated vector space is written $C[0,b]$. A subspace of this vector space is the subspace created by the span of the set $\{f_1, f_2, f_3, f_4, f_5\}$ where

$$\begin{aligned} f_1(x) &= \frac{1}{\sqrt{b}} \\ f_j(x) &= \sqrt{\frac{2}{b}} \cos\frac{(j-1)\pi x}{b} \qquad \text{for} \quad j = 2,3,4,5 \end{aligned} \tag{4.13.12}$$

In the notation used in this section, $\mathscr{V}$ is the vector space $C[0,b]$ and $\mathscr{V}_1 = \operatorname{span}(f_1, f_2, f_3, f_4, f_5)$. The vector space becomes an inner product space with the definition (4.1.19), repeated,

$$\langle f, g \rangle = \int_0^b f(x) g(x) dx \tag{4.13.13}$$

for all vectors $f, g \in C[0,b]$. You are given that the set $\{f_1, f_2, f_3, f_4, f_5\}$ is orthonormal. This assertion can be established by utilization of (4.13.12) and the definition (4.13.13) to confirm that

$$\langle f_j, f_k \rangle = \delta_{jk} \quad \text{for} \quad j,k = 1,2,3,4,5 \tag{4.13.14}$$

In order to illustrate the approximation (4.13.11), we shall make the choice

$$g(x) = 1 - \frac{x}{b} \tag{4.13.15}$$

In the notation being utilized in this example, (4.13.11) takes the form

$$g(x) = 1 - \frac{x}{b} = \sum_{j=1}^{5} \langle g, f_j \rangle f_j \tag{4.13.16}$$

where the fact that $\{f_1, f_2, f_3, f_4, f_5\}$ is orthonormal has been used. The next step is to make explicit the approximation by evaluating the inner products in (4.13.16). The results turn out to be

$$\begin{aligned}\langle g, f_1\rangle &= \int_0^b \left(1-\frac{x}{b}\right)\left(\frac{1}{\sqrt{b}}\right)dx = \frac{\sqrt{b}}{2}\\ \langle g, f_j\rangle &= \int_0^b \left(1-\frac{x}{b}\right)\left(\sqrt{\frac{2}{b}}\cos\frac{(j-1)\pi x}{b}\right)dx \\ &= \sqrt{\frac{2}{b}}\left(\frac{b}{(j-1)^2\pi^2}\left(1-(-1)^{j-1}\right)\right) \qquad \text{for} \quad j=2,3,4,5\end{aligned} \tag{4.13.17}$$

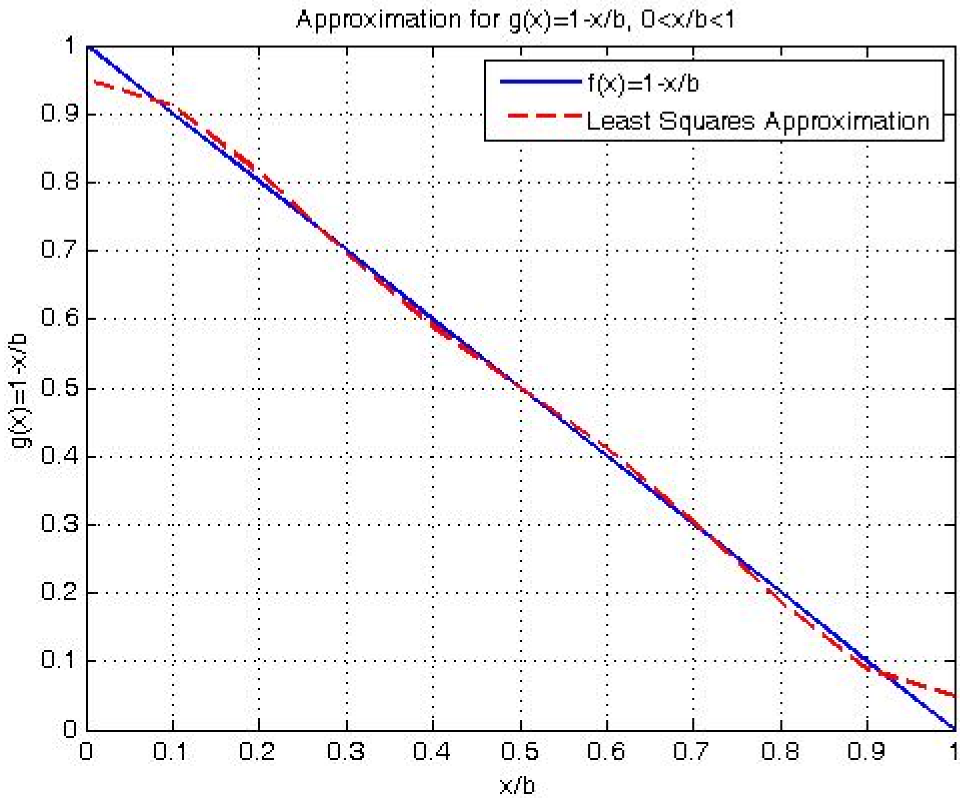
More explicitly,

$$\begin{aligned}\langle g, f_2\rangle &= \frac{\sqrt{8b}}{\pi^2}\\ \langle g, f_3\rangle &= 0\\ \langle g, f_4\rangle &= \frac{\sqrt{8b}}{9\pi^2}\\ \langle g, f_5\rangle &= 0\end{aligned} \tag{4.13.18}$$

If these results are utilized in (4.13.16), the resulting approximation is

$$g(x) = 1-\frac{x}{b} \cong \frac{1}{2} + \frac{4}{\pi^2}\cos\frac{\pi x}{b} + \frac{4}{9\pi^2}\cos\frac{3\pi x}{b} \tag{4.13.19}$$

The quality of the approximation (4.13.19) is illustrated in the following plot



Example 4.13.1 is an example of a *Fourier Cosine* series expansion for the function $g(x)=1-\dfrac{x}{b}$.

Section 4.14. Least Squares Problems and Overdetermined Systems

In this section, we shall formulate another solution procedure based upon the concept of least squares. Essentially, we shall examine a problem that does not have a solution and attempt to find an approximate solution that is close in some sense to the original problem. It will turn out that the approximate solution found is equivalent to the results in Section 4.13 above.

The basic idea leading to the approximate solution is as follows. Recall that when we are given a linear transformation $\mathbf{A}:\mathscr{V}\rightarrow\mathscr{U}$ and a vector $\mathbf{b}\in\mathscr{U}$, the consistency theorem for linear systems says that the system

$$\mathbf{Ax}=\mathbf{b} \tag{4.14.1}$$

has a solution if and only if $\mathbf{b}\in R(\mathbf{A})$. This theorem was discussed in Sections 1.8 and 2.7 for matrix systems. It was mentioned again in Section 3.3.

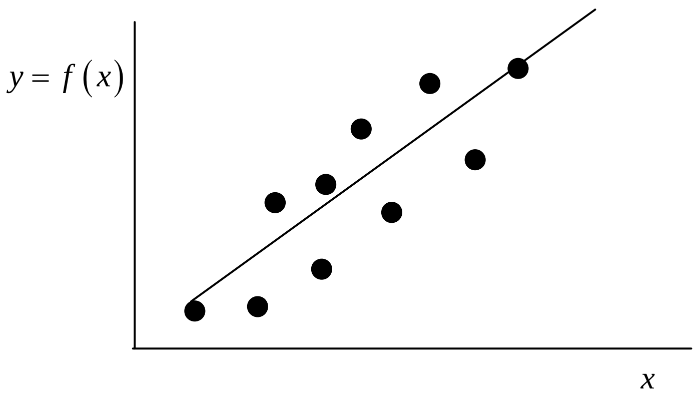
Interestingly, there are problems which arise in the applications that lead to systems like (4.14.1) for which $\mathbf{b}\notin R(\mathbf{A})$. Our theory tells us that these problems do not have a solution. The question still remains whether or not there is an *approximate solution* that is useful. A major application of this kind of problem is *curve fitting*. This is when one is trying to fit a curve to a set of data. A typical problem in the applications is when you are given a discrete table of data

y_1	y_2	y_3	.	.	.	y_K
x_1	x_2	x_3	.	.	.	x_K

and your application requires that you have values of y for values of x intermediate to values in the table. There are two *curve fitting* approaches to this problem that we shall be discussing.

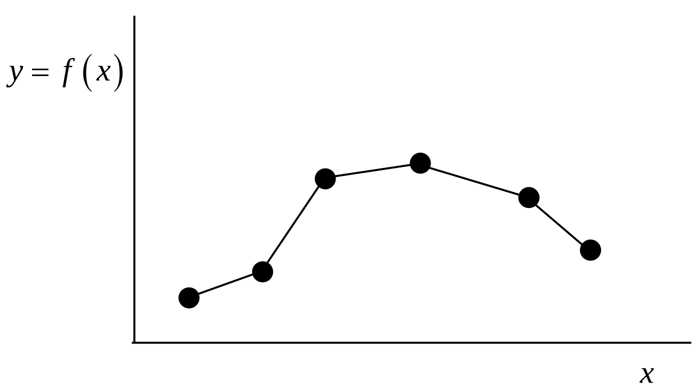
1. The data exhibits a significant degree of error or scatter as shown in the following figure.

   a. The approach is to derive a single curve that represents general trend of the data.
   b. You make no effort to find a curve that intersects the given points.
   c. The curve is designed to follow the pattern of points taken as a group.
   d. The approach is to try pass a curve through the data that *minimizes error* in some fashion.
   e. This kind of problem is known as a *regression* problem.

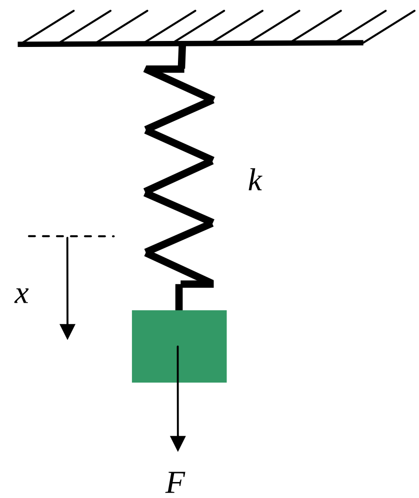


2. The data is known to be precise.

    a. The approach is to fit a curve or series of curves that pass directly through each of the points.
    b. The estimation of values between well known discrete points is called *interpolation*.



Consider the following problem as an example regression example. The example is an application involving spring constants. It is derived from an elementary idea in mechanics known as Hooke's Law. It is named after the 17th century British physicists Robert Hooke.[10] This so called law is an effort to *model*, in a crude way, the elongation of a spring resulting from an applied force:



You are given a force-elongation table, in some consistent set of units, is as follows:

Force	Elongation
0	0
2	2
4	3
7	4
11	5

A table such as this would arise from a series of experiments involving loading different weights to a spring. The problem is that Hooke's Law says that

$$F = kx \tag{4.14.2}$$

where k is a property of the spring, a constant known as the spring constant. The model presumes that there is one spring contant that defines the spring. Unfortunately, the experiment yields four different spring constants! They are $1. \frac{4}{3}, \frac{7}{4}, \frac{11}{5}$! Viewed as a system of equations in the matrix form $A\mathbf{x} = \mathbf{b}$, we have a system

[10] http://en.wikipedia.org/wiki/Robert_Hooke

$$\underbrace{\begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}}_{4\times 1}\underbrace{\begin{bmatrix} k \end{bmatrix}}_{1\times 1} = \underbrace{\begin{bmatrix} 2 \\ 4 \\ 7 \\ 11 \end{bmatrix}}_{4\times 1} \tag{4.14.3}$$

This is an *overdetermined* system of four equations in one unknown. It should be clear that the rank of the matrix of coefficients is one and the rank of the augmented matrix is two. Thus, when Theorem 2.7.2 is utilized, we see that the system (4.14.3) does not have a solution. In the language we have used when discussing systems of equations, our example is one where the column matrix

$$\mathbf{b} = \begin{bmatrix} 2 \\ 4 \\ 7 \\ 11 \end{bmatrix} \tag{4.14.4}$$

is not in the image space of the matrix

$$A = \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} \tag{4.14.5}$$

The matrix A is one to one but it is not onto. For this problem there are a couple of approaches one might utilize in an effort to build a model based upon the given data. One approach is simply to stop and declare that the physical problem is not modeled by Hooke's Law. In other words, there is a more complex physical model governing the spring. This is sometimes the best approach. However, you lose the benefit of simple (linear) mathematical equations that are more readily solved. The second approach is to find a value of k that produces a spring constant corresponding to a "best approximation" or "best fit" of the experimental data to the hypothetical physical law, i.e. Hookes Law.

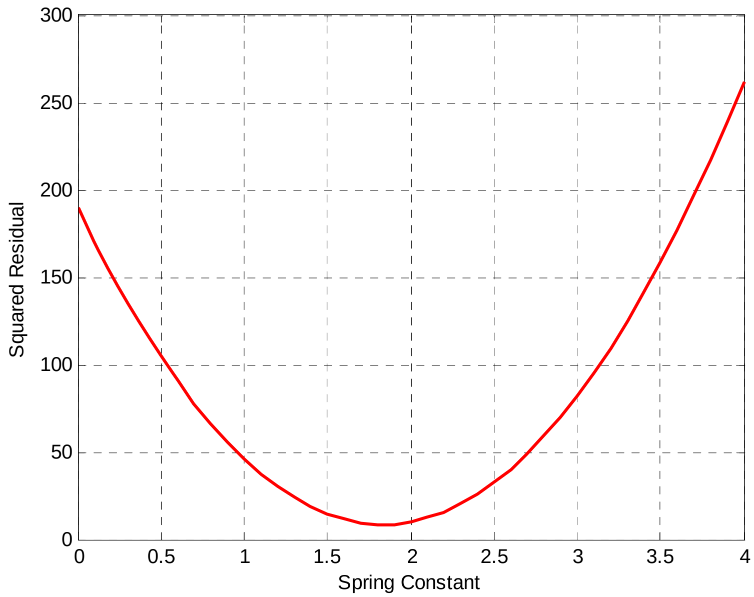
An approach which has proven effective is to define an *error* or *residual* for each experiment and to minimize the error in some fashion. The residuals are as follows:

$$\begin{aligned} r_1 &= 2 - 2k \\ r_2 &= 4 - 3k \\ r_3 &= 7 - 4k \\ r_3 &= 11 - 5k \end{aligned} \tag{4.14.6}$$

The least squares method is a method to find the value of k which causes the sum of the squared residuals to be a minimum. This means that we want to find k such that

$$r_1^2 + r_2^2 + r_3^2 + r_4^2 = (2-2k)^2 + (4-3k)^2 + (7-4k)^2 + (11-5k)^2 \tag{4.14.7}$$

is a *minimum*. The plot of this equation is



Therefore, the squared residual is never zero, but it can be minimized. At the minimum point, it would necessarily be true that

$$\frac{d\left(r_1^2 + r_2^2 + r_3^2 + r_4^2\right)}{dk} = 0 \tag{4.14.8}$$

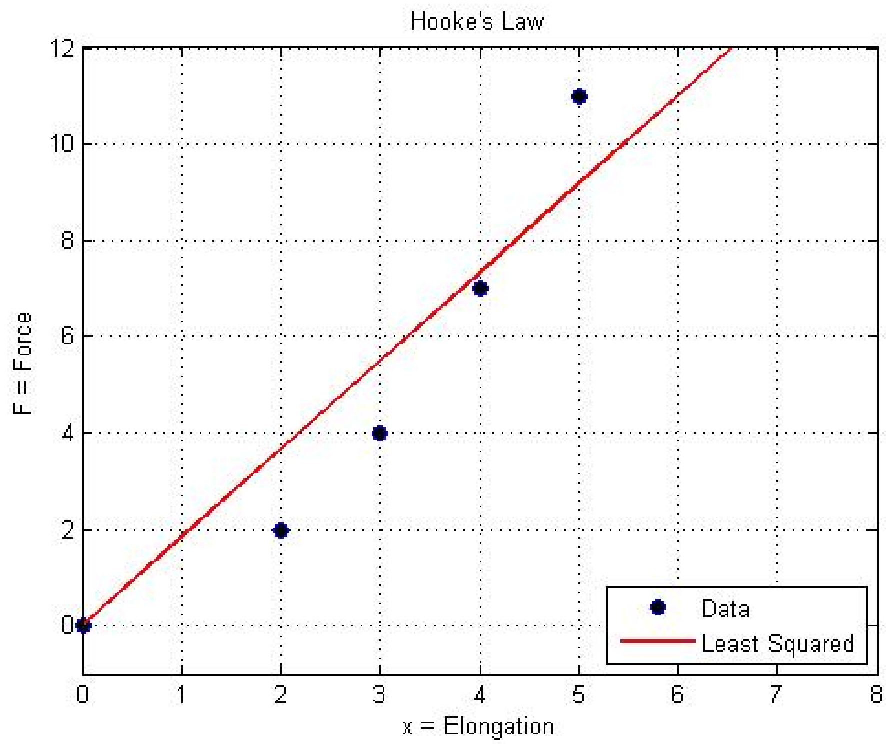
If this derivative is calculated and placed to zero, we get

$$-4(2-2k) - 6(4-3k) - 8(7-3k) - 10(11-5k) = 0 \tag{4.14.9}$$

or, after simplification,

$$k = \frac{99}{54} = 1.8333 \tag{4.14.10}$$

This number represents an approximation, derived from the set of four experiments, built around the decision to accept the least squares of the residuals as the definition of an approximate solution. The figure illustrates the result of the above calculation.



Example 4.14.1 It is instructive to cast the calculation leading to (4.14.10) in the notation used in Section 4.13. The vector space $\mathscr{V}$ corresponds to the four dimensional vector space $\mathscr{M}^{4\times 1}$. The given vector we denoted by $\mathbf{v}$ in Section 4.13 is the column matrix $\mathbf{b}$ in (4.14.4), repeated,

$$\mathbf{v} = \mathbf{b} = \begin{bmatrix} 2 \\ 4 \\ 7 \\ 11 \end{bmatrix} \tag{4.14.11}$$

The subspace of $\mathscr{V}$ we have denoted by $\mathscr{V}_1$ is the one dimensional subspace spanned by (4.14.5), repeated,

$$A = \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} \tag{4.14.12}$$

This problem is an application of the ideas in Section 4.13 but it is special in the sense that the subspace $\mathscr{V}_1$ is defined by the problem, i.e., by the matrix (4.14.12) rather than the result of a choice made in advance. In the notation of Section 4.13, we would write the vector (4.14.12) as

$$\mathbf{e}_1 = \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} \tag{4.14.13}$$

and $\mathscr{V}_1$ is spanned by the set containing the single vector $\mathbf{e}_1$. Given the above identifications, the general solution in the form (4.13.11) specializes to

$$\mathbf{u} = \left\langle \mathbf{v}, \mathbf{e}^1 \right\rangle \mathbf{e}_1 \tag{4.14.14}$$

In order to utilize (4.14.14) we need the vector $\mathbf{e}^1 \in \mathscr{V}_1$ that is reciprocal to $\mathbf{e}_1$ as is defined by (4.14.13). As with all reciprocal bases, we need to utilize (4.7.1) and (4.7.10). Because the subspace $\mathscr{V}_1$ one dimensional, the calculations are elementary. It follows from (4.14.13) that

$$e_{11} = \left\langle \mathbf{e}_1, \mathbf{e}_1 \right\rangle = \begin{bmatrix} 2 & 3 & 4 & 5 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} = 54 \tag{4.14.15}$$

and

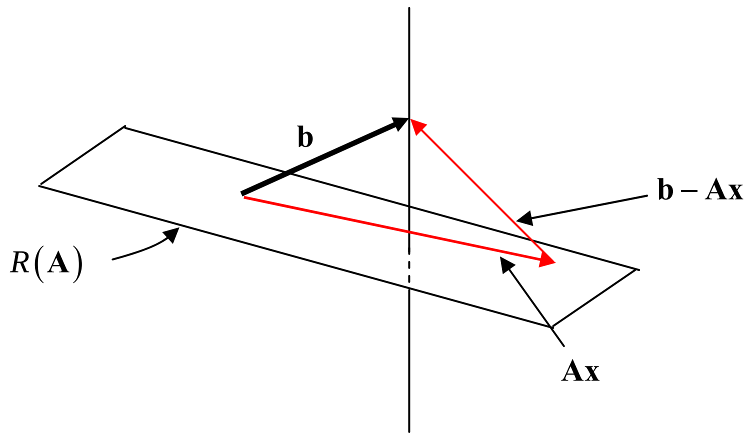
$$\mathbf{e}^1 = e^{11} \mathbf{e}_1 = \frac{1}{e_{11}} \mathbf{e}_1 = \frac{1}{54} \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} \tag{4.14.16}$$

Given (4.14.11), (4.14.16) and (4.14.13), the answer (4.14.14) is

$$\mathbf{u} = \left\langle \mathbf{v}, \mathbf{e}^1 \right\rangle \mathbf{e}_1 = \left(\frac{1}{54} \begin{bmatrix} 2 & 4 & 7 & 11 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} \right) \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} = \frac{99}{54} \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} \tag{4.14.17}$$

Therefore, the contravariant component of the approximate solution in the one dimensional direction $\mathbf{e}_1 = \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$ is $\frac{99}{44}$. Of course, the answer $\frac{99}{44}$ is the result (4.14.10) obtained earlier.

Given the motivation provided by the example leading to (4.14.10) and the same example worked in Example 4.14.1, we now turn to the following more general problem. We are given a linear transformation $\mathbf{A}:\mathscr{V}\to\mathscr{U}$ and a vector $\mathbf{b}\in\mathscr{U}$, where $\mathscr{V}$ and $\mathscr{U}$ are *real* inner product spaces. We are also given that $\mathbf{b}\notin R(\mathbf{A})$. Therefore, a system $\mathbf{Ax}=\mathbf{b}$ does not have a solution. At this point in the discussion, $\mathbf{A}:\mathscr{V}\to\mathscr{U}$ is perfectly general. We have said it is not onto and we do not require that it be one to one. For a single three dimensional vector space, the following figure suggests the geometric arrangement we are discussing



The plane shown is the image space of the linear transformation. The given vector $\mathbf{b}$ is not in the image space. The vector $\mathbf{b}-\mathbf{Ax}$ in some sense measures the inconsistency in the system. If we fit the notation used in the above figure with that used in Section 4.13, the vector space $\mathscr{U}$ which contains the vector $\mathbf{b}$ corresponds to the $\mathscr{V}$ of Section 4.13. The image space $R(\mathbf{A})$, as a subspace of $\mathscr{U}$, corresponds to the subspace $\mathscr{V}_1$ of $\mathscr{V}$. The product $\mathbf{Ax}$ corresponds to what we called $\mathbf{u}$ in Section 4.13. As with Example 4.14.1, the subspace $R(\mathbf{A})$ is not arbitrary. It is determined from the properties of $\mathbf{A}:\mathscr{V}\to\mathscr{U}$.

As with the general discussion in Section 4.13, the least squared method defines, for each vector $\mathbf{x}\in\mathscr{V}$, the *residual* to be the vector

$$\mathbf{r}(\mathbf{x})=\mathbf{b}-\mathbf{Ax} \tag{4.14.18}$$

Because our vector spaces have inner products, we can again measure the error by the length or norm squared of the residual (4.14.18). The length squared of this residual is

$$\left\|\mathbf{r}(\mathbf{x})\right\|^2=\langle\mathbf{b}-\mathbf{Ax},\mathbf{b}-\mathbf{Ax}\rangle \tag{4.14.19}$$

where the inner product is the one for $\mathscr{U}$. The fact that it is a dependence on $\mathbf{x}\in\mathscr{V}$ rather than a depence on $\mathbf{Ax}\in R(\mathbf{A})$ that makes the formulation in this section slightly different than the one in

Section 4.13. However, they are entirely equivalent. In any case, our problem is to find the $\mathbf{x} \in \mathscr{V}$, which makes the squared residual, $\left\|\mathbf{r}(\mathbf{x})\right\|^2$, a minimum. As with the case in Section 4.13 and as with the elementary example above, we will differentiate (4.14.19) and place the result to zero.

The result of this minimization will be an equation which will determine the particular $\mathbf{x} \in \mathscr{V}$ which minimizes the squared residual. The equation that we shall obtain is known as the *normal equation*. It says that $\mathbf{x}$ must obey the system of $N = \dim \mathscr{V}$ equations

$$\mathbf{A}^T \mathbf{A}\mathbf{x} = \mathbf{A}^T \mathbf{b} \tag{4.14.20}$$

The minimization process follows the same procedure used in Section 4.13. The minimum state we seek is the vector $\mathbf{x} \in \mathscr{V}$ which satisfies the condition

$$\left.\frac{d\left\|\mathbf{r}(\mathbf{x}+\lambda\mathbf{a})\right\|^2}{d\lambda}\right|_{\lambda=0} = \left\langle \operatorname{grad}\left\|\mathbf{r}(\mathbf{x})\right\|^2, \mathbf{a}\right\rangle = 0 \tag{4.14.21}$$

for all vectors $\mathbf{a} \in \mathscr{V}$. It is easier, for the simple function $\left\|\mathbf{r}(\mathbf{x})\right\|^2$, to form the derivative on the left side of (4.14.21) than it is to worry about how one calculates the gradient on the right. From the definition of the function $\left\|\mathbf{r}(\mathbf{x})\right\|^2$, equation (4.14.19), it follows that

$$\left\|\mathbf{r}(\mathbf{x}+\lambda\mathbf{a})\right\|^2 = \left\langle \mathbf{b} - \mathbf{A}(\mathbf{x}+\lambda\mathbf{a}), \mathbf{b} - \mathbf{A}(\mathbf{x}+\lambda\mathbf{a})\right\rangle \tag{4.14.22}$$

Therefore,

$$\left.\frac{d\left\|\mathbf{r}(\mathbf{x}+\lambda\mathbf{a})\right\|^2}{d\lambda}\right|_{\lambda=0} = \left\langle \mathbf{b}-\mathbf{A}\mathbf{x}, -\mathbf{A}\mathbf{a}\right\rangle + \left\langle -\mathbf{A}\mathbf{a}, \mathbf{b}-\mathbf{A}\mathbf{x}\right\rangle = -2\left\langle \mathbf{b}-\mathbf{A}\mathbf{x}, \mathbf{A}\mathbf{a}\right\rangle = 0 \tag{4.14.23}$$

Equation (4.14.23) corresponds to (4.13.7) in the formulation of Section 4.13. The next step is slightly different than used in Section 4.13. The inner product in (4.14.23) is the one in $\mathscr{U}$. The inner product in (4.14.23) vanishes for all vectors in $R(A)$, a subspace of $\mathscr{U}$ defined by $\mathbf{A}$. A more useful result is obtained if we use the definition of the transpose, equation (4.9.3). This definition allows (4.14.23) to be written

$$\left\langle \mathbf{A}^T(\mathbf{b}-\mathbf{A}\mathbf{x}), \mathbf{a}\right\rangle = 0 \quad \text{for all vectors } \mathbf{a} \in \mathscr{V} \tag{4.14.24}$$

where the inner product in (4.14.24) is the one in $\mathscr{V}$. Geometrically, (4.14.23) says that the residual $\mathbf{b} - \mathbf{A}\mathbf{x}$ is orthogonal to the subspace $R(A)$. The equivalent result, equation (4.14.24) says that $\mathbf{A}^T(\mathbf{b}-\mathbf{A}\mathbf{x})$ is orthogonal to all vectors in $\mathscr{V}$. Therefore,

$$\mathbf{A}^T\left(\mathbf{b}-\mathbf{Ax}\right)=\mathbf{0} \tag{4.14.25}$$

which is the result (4.14.20). As mentioned above, equation (4.14.20) is called the *normal equation*. A geometric interpretation of (4.14.25) is that $\mathbf{b}-\mathbf{Ax}$ is in $K\left(A^T\right)$, which from (4.12.8) equals $R\left(A\right)^\perp$. Therefore it is a consequence of (4.14.25) that

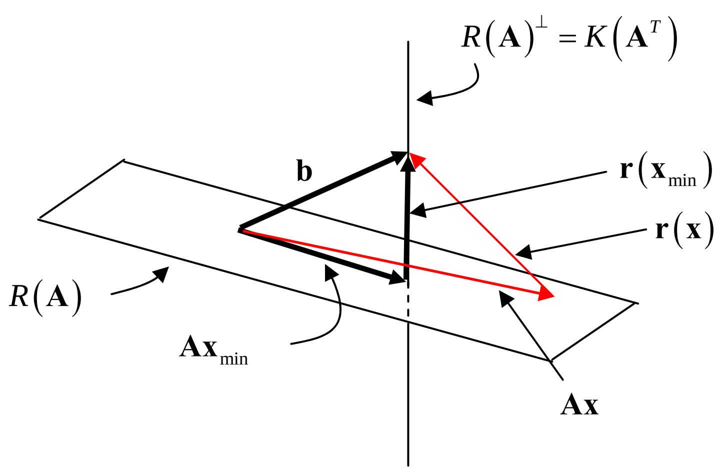
$$\mathbf{Ax}-\mathbf{b}\in K\left(\mathbf{A}^T\right)=R\left(\mathbf{A}\right)^\perp \tag{4.14.26}$$

Equation (4.14.20) and the earlier result, (4.13.11) are entirely equivalent. If we apply (4.13.11) to the case at hand, then it yields

$$\mathbf{u}=\mathbf{Ax}=\sum_{j=1}^{M}\left\langle \mathbf{v},\mathbf{e}^j\right\rangle \mathbf{e}_j \tag{4.14.27}$$

As a practical matter, (4.14.20) is more convenient. With (4.14.27), one must analyze the linear transformation $\mathbf{A}$ to determine a basis and reciprocal basis for its range. With (4.14.20), this calculation need not be performed. It is implicit in the normal equation (4.14.20).

It is helpful when trying to understand solutions to the normal equation to utilize the figure above augmented with the geometric result (4.14.26). If we adopt the notation $\mathbf{x}_{\min}$ for any solution to the normal equation (4.14.20), and display how it arises as a special choice of all possible vectors $\mathbf{x}\in\mathscr{V}$. The following figure is the result.



As the figure suggests, a solution $\mathbf{x}_{\min}$ is the one that produces a residual vector $\mathbf{r}(\mathbf{x}_{\min}) = \mathbf{b} - \mathbf{A}\mathbf{x}_{\min}$ orthogonal to the image space $R(\mathbf{A})$. In other words, $\mathbf{r}(\mathbf{x}_{\min})$ is the projection of $\mathbf{r}(\mathbf{x})$ into the orthogonal compliment of $R(\mathbf{A})$.

The last figure is a good motivation for the inequality

$$\|\mathbf{b} - A\mathbf{x}\| \geq \|\mathbf{b} - A\mathbf{x}_{\min}\| \tag{4.14.28}$$

Another way to write this inequality is $\|\mathbf{r}(\mathbf{x})\| \geq \|\mathbf{r}(\mathbf{x}_{\min})\|$. This fact, while obvious from the simple geometric construction above, needs to be established in general in order to be sure that we have actually found a minimum point when we find solutions of the normal equation. The derivation of (4.14.28) follows by the calculation

$$\begin{aligned}
\|\mathbf{b} - \mathbf{A}\mathbf{x}\|^2 &= \Bigg\| \underbrace{(\mathbf{b} - \mathbf{A}\mathbf{x}_{\min})}_{\text{In } K(\mathbf{A}^T) = R(\mathbf{A})^\perp} + \underbrace{(\mathbf{A}\mathbf{x}_{\min} - \mathbf{A}\mathbf{x})}_{\text{In } R(\mathbf{A})} \Bigg\|^2 \\
&= \left\langle (\mathbf{b} - \mathbf{A}\mathbf{x}_{\min}) + (\mathbf{A}\mathbf{x}_{\min} - \mathbf{A}\mathbf{x}), (\mathbf{b} - \mathbf{A}\mathbf{x}_{\min}) + (\mathbf{A}\mathbf{x}_{\min} - \mathbf{A}\mathbf{x}) \right\rangle \\
&= \|\mathbf{b} - \mathbf{A}\mathbf{x}_{\min}\|^2 + \|\mathbf{A}\mathbf{x}_{\min} - \mathbf{A}\mathbf{x}\|^2 + 2\underbrace{\langle \mathbf{b} - \mathbf{A}\mathbf{x}_{\min}, \mathbf{A}\mathbf{x}_{\min} - \mathbf{A}\mathbf{x} \rangle}_{\text{Equal to 0 because } \perp} \\
&= \|\mathbf{b} - \mathbf{A}\mathbf{x}_{\min}\|^2 + \|\mathbf{A}\mathbf{x}_{\min} - \mathbf{A}\mathbf{x}\|^2 \\
&\geq \|\mathbf{b} - \mathbf{A}\mathbf{x}_{\min}\|^2
\end{aligned} \tag{4.14.29}$$

The positive square root of the last inequality gives the result asserted.

Further properties of the normal equation (4.14.20) are as follows:

1. The normal equation (4.14.20) was derived to be useful for cases where $\mathbf{b} \notin R(\mathbf{A})$. If $\mathbf{b}$ is , in fact, in $R(\mathbf{A})$, then (4.14.20) written in the form (4.14.25) shows that $\mathbf{b} - \mathbf{A}\mathbf{x}$, which is a vector in $R(\mathbf{A})$, is also in $K(\mathbf{A}^T)$. Because from (4.12.8), $K(\mathbf{A}^T) = R(\mathbf{A})^\perp$. It is necessarily true that

$$\mathbf{A}\mathbf{x} = \mathbf{b} \tag{4.14.30}$$

The conclusion is that when $\mathbf{b} \in R(\mathbf{A})$, the normal equation (4.14.20) and (4.14.30) have the same solution.

2. The linear transformation $\mathbf{A}^T\mathbf{A} : \mathscr{V} \to \mathscr{V}$ is symmetric. This result simply follows from the calculation $(\mathbf{A}^T\mathbf{A})^T = \mathbf{A}^T(\mathbf{A}^T)^T = \mathbf{A}^T\mathbf{A}$.

3. The kernel of $\mathbf{A}^T\mathbf{A}:\mathscr{V}\to\mathscr{V}$ equals the kernel of $\mathbf{A}:\mathscr{V}\to\mathscr{U}$. In equation form this assertion is

$$K\left(\mathbf{A}^T\mathbf{A}\right)=K\left(\mathbf{A}\right) \tag{4.14.31}$$

The proof of (4.14.31) goes as follows: Let $\mathbf{v}$ be a vector in $K\left(\mathbf{A}\right)$. By the definition of the kernel, it is true that $\mathbf{Av}=\mathbf{0}$. Thus, it is true that $\mathbf{A}^T\mathbf{Av}=\mathbf{0}$ and, as a result, $\mathbf{v}\in K\left(\mathbf{A}^T\mathbf{A}\right)$. Conversely, let $\mathbf{v}$ be a vector in $K\left(\mathbf{A}^T\mathbf{A}\right)$. It follows that $\mathbf{A}^T\mathbf{Av}=\mathbf{0}$ and, as a result, the vector $\mathbf{Av}$ which is in $R\left(\mathbf{A}\right)$, is in $K\left(\mathbf{A}^T\right)$. Because from (4.12.8), $K\left(\mathbf{A}^T\right)=R\left(\mathbf{A}\right)^\perp$ and that the only vector that is both in $R\left(\mathbf{A}\right)$ and $K\left(\mathbf{A}^T\right)=R\left(\mathbf{A}\right)^\perp$ is the zero vector, we can conclude that $\mathbf{Av}=\mathbf{0}$ which forces $\mathbf{v}\in K\left(\mathbf{A}\right)$

4. The rank of the symmetric linear transformation $\mathbf{A}^T\mathbf{A}:\mathscr{V}\to\mathscr{V}$ and the linear transformation $\mathbf{A}:\mathscr{V}\to\mathscr{U}$ are the same. In other words

$$\dim R\left(\mathbf{A}^T\mathbf{A}\right)=\dim R\left(\mathbf{A}\right) \tag{4.14.32}$$

This result follows from (4.14.31) and the rank-nullity theorem, equation (3.3.12), applied to the linear transformations $\mathbf{A}^T\mathbf{A}$ and $\mathbf{A}$.

5. The solution of the normal equation (4.14.20) is unique if and only if $K\left(\mathbf{A}\right)=\left\{\mathbf{0}\right\}$. This assertion follows from (4.14.31) and Theorem 3.3.2.
6. In the case where the solution of the normal equation (4.14.20) is not unique, the residuals for the solutions are the same. More explicitly, if $\mathbf{x}_1$ and $\mathbf{x}_2$ are two solutions of the normal equation (4.14.20), then

$$A\mathbf{x}_1=A\mathbf{x}_2 \tag{4.14.33}$$

This equation follows from (4.14.31) which shows that when the solution is not unique $\mathbf{x}_1-\mathbf{x}_2$ must be in $K\left(\mathbf{A}^T\mathbf{A}\right)=K\left(\mathbf{A}\right)$. Since the least squares solution minimizes $\mathbf{b}-\mathbf{Ax}$, the two solutions $\mathbf{x}_1$ and $\mathbf{x}_2$ yield the same vector in $R\left(A\right)$ and that vector is the closest in the least squared sense to the vector $\mathbf{b}$.

In the special case where $K\left(\mathbf{A}\right)=\left\{\mathbf{0}\right\}$, i.e., when $\mathbf{A}:\mathscr{V}\to\mathscr{U}$ is one to one, we know from (3.3.12) that the rank of $\mathbf{A}$ and $\mathbf{A}^T\mathbf{A}$ is equal to $\dim\mathscr{V}$. As a result, the symmetric linear transformation $\mathbf{A}^T\mathbf{A}:\mathscr{V}\to\mathscr{V}$ is nonsingular even though the linear transformation $\mathbf{A}:\mathscr{V}\to\mathscr{U}$ is not. The unique solution of the normal equation (4.14.20) is then

$$\mathbf{x}=\left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{A}^T\mathbf{b} \tag{4.14.34}$$

In general, when one is given a linear transformation $\mathbf{A}:\mathscr{V}\to\mathscr{U}$ with the property that $\dim R(\mathbf{A})=\dim\mathscr{V}$, the combination $\left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{A}^T$ is known as the *left pseudo inverse* of $\mathbf{A}$. It is the left pseudo inverse that gives the solution to the least squared problem. If the linear transformation $\mathbf{A}$ is nonsingular, it is evident that $\left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{A}^T=\mathbf{A}^{-1}\left(\mathbf{A}^T\right)^{-1}\mathbf{A}^T=\mathbf{A}^{-1}$.

While it is not important to us here, it is worth noting in passing that the combination $\mathbf{A}^T\left(\mathbf{A}\mathbf{A}^T\right)^{-1}$ arises in some applications and it is known as the *right pseudo inverse* of $\mathbf{A}$.

Exercises

4.14.1 Find the least squares solution to the following system

$$\begin{aligned}x_1+2x_2+x_3&=2\\3x_1+x_2-2x_3&=1\\4x_1-3x_2-x_3&=3\\2x_1+4x_2+2x_3&=4\end{aligned}\tag{4.14.35}$$

The answer you should obtain is $\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}1\\0\\1\end{bmatrix}$.

4.14.2 Find the least squares solution to the following system

$$\begin{aligned}x_2-3x_3&=1\\3x_1+12x_3&=3\\x_1+x_2+x_3&=1\\2x_1+3x_2-x_3&=1\end{aligned}\tag{4.14.36}$$

The answer you should obtain is $\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=x_3\begin{bmatrix}-4\\3\\1\end{bmatrix}+\begin{bmatrix}\dfrac{97}{105}\\\dfrac{2}{15}\\0\end{bmatrix}$.

4.14.3 You are given the matrix equation $A\mathbf{x}=\mathbf{b}$, where

$$A=\begin{bmatrix}1 & 1\\ 2 & -3\\ 0 & 0\end{bmatrix} \quad \text{and} \quad \mathbf{b}=\begin{bmatrix}3\\ 1\\ 2\end{bmatrix} \tag{4.14.37}$$

If it exists, determine the solution to the matrix equation $A\mathbf{x}=\mathbf{b}$. If it does not exist, determine an approximate solution based upon the least squared approximation.

4.14.4 As mentioned above, when $K(\mathbf{A})=\{\mathbf{0}\}$, the the rank of $\mathbf{A}:\mathscr{V}\to\mathscr{U}$ and $\mathbf{A}^T\mathbf{A}:\mathscr{V}\to\mathscr{V}$ is equal to $\dim\mathscr{V}$. In this case, the solution (4.14.34) is sometimes expressed in terms of a so called *QR decomposition* or *factorizatio*n. The purpose of this exercise is to show how this factorization is constructed and to show how it simplifies the solution (4.14.34). The argument begins with selecting a basis $\{\mathbf{e}_1,\mathbf{e}_2,...,\mathbf{e}_N\}$ for $\mathscr{V}$ and a basis $\{\mathbf{b}_1,\mathbf{b}_2,...,\mathbf{b}_M\}$ for $\mathscr{U}$. The fundamental formula that defined the components is equation (4.8.1), repeated,

$$\mathbf{A}\mathbf{e}_k=\sum_{j=1}^{M}A^j{}_k\mathbf{b}_j \qquad k=1,2,...,N \tag{4.14.38}$$

The matrix of this linear transformation was defined by equation (4.8.2), repeated,

$$M\left(\mathbf{A},\mathbf{e}_k,\mathbf{b}_j\right)=\begin{bmatrix}A^1{}_1 & A^1{}_2 & \cdot & \cdot & \cdot & A^1{}_N\\ A^2{}_1 & A^2{}_2 & & & & A^2{}_N\\ A^3{}_1 & & A^3{}_3 & & & \cdot\\ \cdot & & & & & \cdot\\ \cdot & & & & & \cdot\\ A^M{}_1 & A^M{}_2 & \cdot & \cdot & \cdot & A^M{}_N\end{bmatrix}=\left[A^j{}_k\right] \tag{4.14.39}$$

The matrix $M\left(\mathbf{A},\mathbf{e}_k,\mathbf{b}_j\right)$, under the assumptions we have made, has rank $N=\dim\mathscr{V}$. If we denote the N linearly independent columns of $M\left(\mathbf{A},\mathbf{e}_k,\mathbf{b}_j\right)$ by

$$\mathbf{f}_k=\begin{bmatrix}A^1{}_k\\ A^2{}_k\\ A^3{}_k\\ \cdot\\ \cdot\\ A^M{}_k\end{bmatrix} \qquad \text{for} \quad k=1,2,...,N \tag{4.14.40}$$

The next formal step is to apply the Gram-Schmidt process explained in Section 4.3 and derive an orthonormal set of column vectors $\{\mathbf{i}_1,...,\mathbf{i}_N\}$. Because the column vectors $\mathbf{f}_k$, $k=1,2,...,N$ are in

$\mathscr{M}^{M\times 1}$, the orthogonal column vectors $\mathbf{i}_k$, $k=1,2,...,N$ are also in $\mathscr{M}^{M\times 1}$. In component form, we shall write these vectors as

$$\mathbf{i}_k = \begin{bmatrix} Q^1{}_k \\ Q^2{}_k \\ Q^3{}_k \\ \cdot \\ \cdot \\ Q^M{}_k \end{bmatrix} \quad \text{for} \quad k=1,2,...,N \tag{4.14.41}$$

The inner product that is used to construct the orthonormal set $\{\mathbf{i}_1,...,\mathbf{i}_N\}$ is the one defined by equation (4.1.13) except that in this section the vector spaces $\mathscr{V}$ and $\mathscr{U}$ are real. As explained in Section 4.3, the two sets of column vectors, $\{\mathbf{f}_1,...,\mathbf{f}_N\}$ and $\{\mathbf{i}_1,...,\mathbf{i}_N\}$, are connected by the formula (4.3.32), repeated,

$$\mathbf{f}_k = \sum_{q=1}^{N} R_k^q \mathbf{i}_q \quad \text{for} \quad k=1,...,N \tag{4.14.42}$$

where the coefficients R_k^q, for $k,q=1,2,...,N$, are determined by (4.3.33). As (4.3.40) illustrates, the transition matrix $R=\left[R_k^q\right]\in\mathscr{M}^{N\times N}$ is upper triangular. Given (4.14.40), (4.14.41) and (4.14.42), the various components are connected by the formula

$$A^s{}_k = \sum_{q=1}^{N} Q^s{}_q R_k^q \qquad \text{for } s=1,2,...,M \text{ and } k=1,2,...,N \tag{4.14.43}$$

Equation (4.14.43) is a decomposition of the matrix $M\left(\mathbf{A},\mathbf{e}_k,\mathbf{b}_j\right)$ into the product of an orthogonal matrix

$$Q = \begin{bmatrix} Q^1{}_1 & Q^1{}_2 & \cdot & \cdot & \cdot & Q^1{}_N \\ Q^2{}_1 & Q^2{}_2 & & & & Q^2{}_N \\ Q^3{}_1 & & Q^3{}_3 & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ Q^M{}_1 & Q^M{}_2 & \cdot & \cdot & \cdot & Q^M{}_N \end{bmatrix} \tag{4.14.44}$$

and the upper triangular matrix R. For matrices, equation (4.14.43) is the QR factorization mentioned above. Given the usual connection between matrices and linear transformations, equation (4.14.43) establishes a corresponding factorization of the linear transformation $\mathbf{A}:\mathscr{V}\rightarrow\mathscr{U}$. The result, which can naturally be written

$$\mathbf{A}=\mathbf{QR} \tag{4.14.45}$$

Because $\mathbf{Q}:\mathscr{V}\rightarrow\mathscr{U}$, is orthogonal, equation (4.11.14), repeated

$$\mathbf{Q}^{T}\mathbf{Q}=\mathbf{I}_{\mathscr{V}} \tag{4.14.46}$$

holds for $\mathbf{Q}$. The linear transformation $\mathbf{R}:\mathscr{V}\rightarrow\mathscr{V}$, which was constructed from the transition matrix $R=\left[R_{k}^{q}\right]\in\mathscr{M}^{N\times N}$ is nonsingular. Given this long preamble to this exercise, use the QR factorization and show that the solution (4.14.34) can be written

$$\mathbf{x}=\mathbf{R}^{-1}\mathbf{Q}^{T}\mathbf{b} \tag{4.14.47}$$

Equation (4.14.47) can have some computational advantages when one is trying to generate the solution of (4.14.34).

Section 4.15. A Curve Fit Example

As the Hooke's Law example showed, a good illustration of the least square method of finding an approximate solution is when one wishes to fit a curve to the data that captures the data trend. In this section, we shall look at another curve fit example. In this example, we shall fit the data to a polynomial of degree of degree S. Polynomials were first discussed in Secion 2.1. In this section, we shall write the polynomial as

$$y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_S x^S \tag{4.15.1}$$

Viewed as a member of a real vector space, the polynomial (4.15.1) is in the vector space $\mathscr{P}_S$ introduced in Section 2.1. Our objective is to utilize the least squares procedure to determine the $S+1$ unknown real numbers that are coefficients in (4.15.1). The regression we shall formulate will be built upon the assumption that we have a data set of K *distinct* points, where $K > S+1$. The data set is displayed in the table

y_1	y_2	y_3	.	.	.	y_K
x_1	x_2	x_3	.	.	.	x_K

As with the Hooke's Law example, we can evaluate the polynomial (4.15.1) at each data pair and obtain the system of K equations for the $S+1$ unknowns

$$\begin{aligned}
a_0 + a_1 x_1 + a_2 x_1^2 + a_3 x_1^3 + \cdots + a_S x_1^S &= y_1 \\
a_0 + a_1 x_2 + a_2 x_2^2 + a_3 x_2^3 + \cdots + a_S x_2^S &= y_2 \\
a_0 + a_1 x_3 + a_2 x_3^2 + a_3 x_3^3 + \cdots + a_S x_3^S &= y_3 \\
a_0 + a_1 x_4 + a_2 x_4^2 + a_3 x_4^3 + \cdots + a_S x_4^S &= y_4 \\
&\;\;. \\
&\;\;. \\
&\;\;. \\
a_0 + a_1 x_K + a_2 x_K^2 + a_3 x_K^3 + \cdots + a_S x_K^S &= y_K
\end{aligned} \tag{4.15.2}$$

This result can be written as the matrix equation

$$\underbrace{\begin{bmatrix} 1 & x_1 & x_1^2 & \cdot & \cdot & x_1^S \\ 1 & x_2 & x_2^2 & \cdot & \cdot & x_2^S \\ 1 & x_3 & x_3^2 & \cdot & \cdot & x_3^N \\ \cdot & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & \cdot & & & \cdot \\ 1 & x_K & x_K^2 & \cdot & \cdot & x_K^S \end{bmatrix}}_{K\times(S+1)} \underbrace{\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \cdot \\ \cdot \\ a_S \end{bmatrix}}_{(S+1)\times 1} = \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \cdot \\ \cdot \\ \cdot \\ y_K \end{bmatrix}}_{K\times 1} \tag{4.15.3}$$

As an over determined system of K equations in $S+1$ unknowns, (4.15.3) is usually inconsistent and, as such, does not have a solution. Following the formalism of Section 4.14, we can define the residuals as a column vector $\mathbf{r}$ defined by

$$\mathbf{r} = \mathbf{y} - A\mathbf{a} \tag{4.15.4}$$

where

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \cdot \\ \cdot \\ \cdot \\ y_K \end{bmatrix} \tag{4.15.5}$$

$$\mathbf{a} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \cdot \\ \cdot \\ a_S \end{bmatrix} \tag{4.15.6}$$

and

$$A = \begin{bmatrix} 1 & x_1 & x_1^2 & \cdot & \cdot & x_1^S \\ 1 & x_2 & x_2^2 & \cdot & \cdot & x_2^S \\ 1 & x_3 & x_3^2 & \cdot & \cdot & x_3^S \\ \cdot & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & \cdot & & & \cdot \\ 1 & x_K & x_K^2 & \cdot & \cdot & x_K^S \end{bmatrix} \tag{4.15.7}$$

Note that the $(S+1)\times K$ matrix A is similar in form to the transpose of the square Vandermonde matrix introduced in Section (1.10). Because the points $x_1, x_2, \ldots, x_K$ are distinct, it is possible to conclude that the rank of A is $S+1$ and, thus, A is one to one.

The normal equation (4.14.20), rewritten in the above notation is

$$\underbrace{A^T A}_{\substack{(S+1)\times(S+1) \\ \text{Symmetric}}} \underbrace{\mathbf{a}}_{(S+1)\times 1} = \underbrace{A^T}_{(S+1)\times K} \underbrace{\mathbf{y}}_{K\times 1} \tag{4.15.8}$$

Example 4.15.1: Given the data set

x	5	10	15	20	25	30	35	40	45	50
y	49	50	46	43	39	36	33	30	22	19

we will try to fit a *cubic* to the given data. Therefore, you are asked to determine coefficients

$$\mathbf{a} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} \tag{4.15.9}$$

in the cubic equation

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \tag{4.15.10}$$

The matrix A in the normal equation is

$$A=\begin{bmatrix}1 & x_1 & x_1^2 & x_1^3\\ 1 & x_2 & x_2^2 & x_2^3\\ 1 & x_3 & x_3^2 & x_3^3\\ 1 & x_4 & x_4^2 & x_4^3\\ 1 & x_5 & x_5^2 & x_5^3\\ 1 & x_6 & x_6^2 & x_6^3\\ 1 & x_7 & x_7^2 & x_7^3\\ 1 & x_8 & x_8^2 & x_8^3\\ 1 & x_9 & x_9^2 & x_9^3\\ 1 & x_{10} & x_{10}^2 & x_{10}^3\end{bmatrix}=\begin{bmatrix}1 & 5 & 25 & 125\\ 1 & 10 & 100 & 1000\\ 1 & 15 & 225 & 3375\\ 1 & 20 & 400 & 8000\\ 1 & 25 & 625 & 15625\\ 1 & 30 & 900 & 27000\\ 1 & 35 & 1225 & 42875\\ 1 & 40 & 1600 & 64000\\ 1 & 45 & 2025 & 91125\\ 1 & 50 & 2500 & 125000\end{bmatrix} \tag{4.15.11}$$

and, because the points $x_1, x_2, ..., x_{10}$ are distinct has rank 4. The matrix $\mathbf{y}$, which is *not* in $R(A)$, is

$$\mathbf{y}=\begin{bmatrix}49\\ 50\\ 46\\ 43\\ 39\\ 36\\ 33\\ 30\\ 22\\ 19\end{bmatrix} \tag{4.15.12}$$

The symmetric matrix A^TA in the normal equation (4.15.8) turns out to be

$$A^TA=\begin{bmatrix}10 & 275 & 9625 & 378125\\ 275 & 9625 & 378125 & 15833125\\ 9625 & 378125 & 15833125 & 690078125\\ 378125 & 15833125 & 690078125 & 30912578125\end{bmatrix} \tag{4.15.13}$$

The inverse of the matrix (4.15.13) can be shown to be

$$\left(A^T A\right)^{-1} = \begin{bmatrix} \frac{113}{30} & -\frac{19}{36} & \frac{1}{50} & -\frac{1}{4500} \\ -\frac{19}{36} & \frac{13073}{154440} & -\frac{2}{585} & \frac{761}{19305000} \\ \frac{1}{50} & -\frac{2}{585} & \frac{103}{715000} & -\frac{1}{585000} \\ -\frac{1}{4500} & \frac{761}{19305000} & -\frac{1}{585000} & \frac{1}{48262500} \end{bmatrix}$$

$$= \begin{bmatrix} 3.7667 & -0.5278 & 0.0200 & -0.0002 \\ -0.5278 & 0.0846 & -0.0034 & 0.0000 \\ 0.0200 & -0.0034 & 0.0001 & 0.0000 \\ -0.0002 & 0.0000 & 0.0000 & 0.0000 \end{bmatrix} \tag{4.15.14}$$

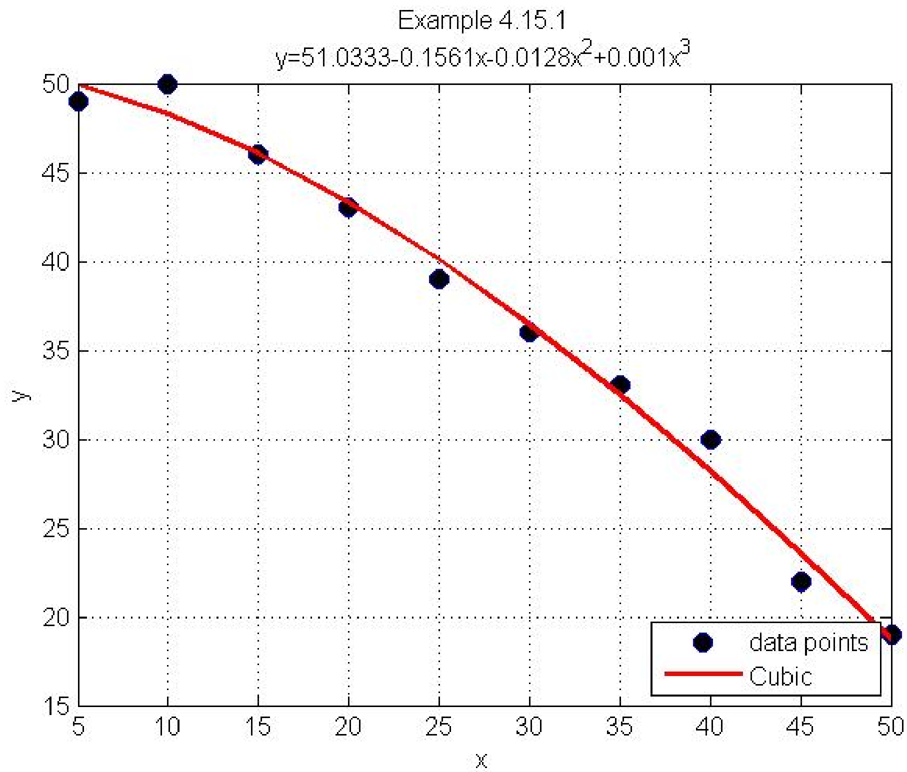
Given (4.15.11), (4.15.12), and (4.15.14), the solution of (4.15.8) can be shown to be

$$\mathbf{a} = \begin{bmatrix} \frac{1531}{30} \\ -\frac{4019}{25740} \\ -\frac{547}{42900} \\ \frac{19}{321750} \end{bmatrix} = \begin{bmatrix} 51.0333 \\ -0.1561 \\ -0.0128 \\ 0.0001 \end{bmatrix} \tag{4.15.15}$$

Therefore, from (4.15.10) and (4.15.15) the least squares approximation to the given data is

$$y = 51.0333 - 0.1561x - 0.0128x^2 + 0.0001x^3 \tag{4.15.16}$$

The plot of the data and this polynomial is as follows:



Example 4.15.1 provides an example that shows how one could use (4.13.11), repeated,

$$\mathbf{u} = \sum_{j=1}^{M} \left\langle \mathbf{v}, \mathbf{e}^{j} \right\rangle \mathbf{e}_{j} \tag{4.15.17}$$

to work least squares problems. If we adapt (4.15.17) to the last example above, then $\mathbf{v}$ represents the vector $\mathbf{y} \in \mathscr{M}^{10\times 1}$ defined by (4.15.12). The vector $\mathbf{u}$ defined by (4.15.17) is the vector in $R(A)$ that is the least squared approximation to $\mathbf{y} = \mathbf{v} \in \mathscr{M}^{10\times 1}$. The basis $\{\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_M\}$ and its reciprocal basis $\{\mathbf{e}^1, \mathbf{e}^2, ..., \mathbf{e}^M\}$ represent bases of the image space, $R(A)$, defined by the matrix (4.15.11). The first step is to determine $M = \dim R(A)$, the rank of A. If the reduced row echelon form of A is calculated by the usual method, the result is

$$U = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{4.15.18}$$

Thus, the rank of A is $M = 4$. Thus, as observed earlier, A is one to one. The basis for $R(A)$ consists of the four columns of A. Therefore,

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{e}_1 = \begin{bmatrix} 5 \\ 10 \\ 15 \\ 20 \\ 25 \\ 30 \\ 35 \\ 40 \\ 45 \\ 50 \end{bmatrix}, \mathbf{e}_3 = \begin{bmatrix} 25 \\ 100 \\ 225 \\ 400 \\ 625 \\ 900 \\ 1225 \\ 1600 \\ 2025 \\ 2500 \end{bmatrix}, \mathbf{e}_4 = \begin{bmatrix} 125 \\ 1000 \\ 3375 \\ 8000 \\ 15625 \\ 27000 \\ 42875 \\ 64000 \\ 91125 \\ 125000 \end{bmatrix} \tag{4.15.19}$$

The inner products $e_{jk} = \langle \mathbf{e}_j, \mathbf{e}_k \rangle$ are given by

$$\left[e_{jk} \right] = \begin{bmatrix} 10 & 275 & 9625 & 378125 \\ 275 & 9625 & 378125 & 15833125 \\ 9625 & 378125 & 15833125 & 690078125 \\ 378125 & 15833125 & 690078125 & 30912578125 \end{bmatrix} \tag{4.15.20}$$

and the resulting reciprocal basis is given by

$$\mathbf{e}^1=\begin{bmatrix}\frac{8}{5}\\ \frac{4}{15}\\ -\frac{2}{5}\\ -\frac{17}{30}\\ -\frac{2}{5}\\ -\frac{1}{15}\\ \frac{4}{15}\\ \frac{13}{30}\\ \frac{4}{15}\\ -\frac{2}{5}\end{bmatrix},\mathbf{e}^2=\begin{bmatrix}-\frac{397}{2145}\\ \frac{19}{1170}\\ \frac{1361}{12870}\\ \frac{2909}{25740}\\ \frac{29}{429}\\ -\frac{2}{2145}\\ -\frac{811}{12870}\\ -\frac{2293}{25740}\\ -\frac{29}{585}\\ \frac{73}{858}\end{bmatrix},\mathbf{e}^3=\begin{bmatrix}\frac{9}{1430}\\ -\frac{16}{10725}\\ -\frac{199}{42900}\\ -\frac{19}{4290}\\ -\frac{23}{10725}\\ \frac{2}{2145}\\ \frac{151}{42900}\\ \frac{31}{7150}\\ \frac{3}{1430}\\ -\frac{16}{3575}\end{bmatrix},\mathbf{e}^4=\begin{bmatrix}-\frac{7}{107250}\\ \frac{7}{321759}\\ \frac{7}{128700}\\ \frac{31}{643500}\\ \frac{1}{53625}\\ -\frac{1}{53625}\\ -\frac{31}{643500}\\ -\frac{7}{128700}\\ -\frac{7}{321759}\\ \frac{7}{107250}\end{bmatrix} \tag{4.15.21}$$

Given (4.15.12), (4.15.19) and (4.15.21), equation (4.15.17) becomes

$$\mathbf{u} = \sum_{j=1}^{4} \langle \mathbf{y}, \mathbf{e}^j \rangle \mathbf{e}_j = \begin{bmatrix} \dfrac{35708}{715} \\ \dfrac{34503}{715} \\ \dfrac{65811}{1430} \\ \dfrac{185683}{4290} \\ \dfrac{85979}{2145} \\ \dfrac{78224}{2145} \\ \dfrac{139343}{4290} \\ \dfrac{120833}{4290} \\ \dfrac{50554}{2145} \\ \dfrac{13393}{715} \end{bmatrix} \tag{4.15.22}$$

Again, $\mathbf{u}$ in (4.15.22) is the vector in $R(A)$ that is the least squared approximation to $\mathbf{y}$ defined by (4.15.12). Given (4.15.22) and the fact that $\mathbf{u} \in R(A)$, insures that there is a vector $\mathbf{a} \in \mathscr{M}^{4\times1}$ that is mapped by A into $\mathbf{u}$, i.e., obeys $A\mathbf{a} = \mathbf{u}$. In order to determina $\mathbf{a}$ we must solve

$$A\mathbf{a} = \begin{bmatrix} 1 & 5 & 25 & 125 \\ 1 & 10 & 100 & 1000 \\ 1 & 15 & 225 & 3375 \\ 1 & 20 & 400 & 8000 \\ 1 & 25 & 625 & 15625 \\ 1 & 30 & 900 & 27000 \\ 1 & 35 & 1225 & 42875 \\ 1 & 40 & 1600 & 64000 \\ 1 & 45 & 2025 & 91125 \\ 1 & 50 & 2500 & 125000 \end{bmatrix} \mathbf{a} = \begin{bmatrix} \frac{35708}{715} \\ \frac{34503}{715} \\ \frac{65811}{1430} \\ \frac{185683}{4290} \\ \frac{85979}{2145} \\ \frac{78224}{2145} \\ \frac{139343}{4290} \\ \frac{120833}{4290} \\ \frac{50554}{2145} \\ \frac{13393}{715} \end{bmatrix} \tag{4.15.23}$$

Repeating our earlier observations, equation (4.15.23) has a solution because we know that its right hand side is in the range of A. Because A is one to one, we know (4.15.23) has a unique solution. This solution is obtained by the method we have used extensively. Namely, construct the augmented matrix and reduce it to row echelon form. The details are tedius at best, but the resulting augmented matrix allows us to replace (4.15.23) with

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \mathbf{a} = \begin{bmatrix} \frac{1531}{30} \\ -\frac{4019}{25740} \\ -\frac{547}{42900} \\ \frac{19}{321750} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{4.15.24}$$

Thus, we again obtain the result (4.15.15). This calculation, perhaps, gives insight into the direct use of the normal equation (4.14.20). It certainly displays the convenience of utilizing (4.14.20) directly rather than the somewhat round about method based upon the use of (4.15.17).

Exercises

4.15.1 You are given a table of data containing K pairs of data:

y_1	y_2	y_3	.	.	.	y_K
x_1	x_2	x_3	.	.	.	x_K

a) Show that if you fit the straight line

$$y = a_0 + a_1 x \tag{4.15.25}$$

to this data, the coefficients a_0 and a_1 are the solution of

$$\begin{bmatrix} 1 & 1 & 1 & \cdot & \cdot & \cdot & 1 \\ x_1 & x_2 & x_3 & \cdot & \cdot & \cdot & x_K \end{bmatrix} \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ 1 & x_3 \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ 1 & x_K \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & \cdot & \cdot & \cdot & 1 \\ x_1 & x_2 & x_3 & \cdot & \cdot & \cdot & x_K \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \cdot \\ \cdot \\ \cdot \\ y_K \end{bmatrix} \tag{4.15.26}$$

b) Show that the solution of (4.15.26) is

$$\begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \frac{1}{K\sum_{i=1}^{K} x_i^2 - \left(\sum_{i=1}^{K} x_i\right)^2} \begin{bmatrix} \sum_{i=1}^{K} x_i^2 & -\sum_{i=1}^{K} x_i \\ -\sum_{i=1}^{K} x_i & K \end{bmatrix} \begin{bmatrix} \sum_{i=1}^{K} y_i \\ \sum_{i=1}^{K} x_i y_i \end{bmatrix} \tag{4.15.27}$$

4.15.2 The numerical example in Section 4.15 provides the opportunity to illustrate the QR decomposition discussed in Exercise 4.14.3. Perform the Gram-Schmidt orthogonalization process on the matrix (4.15.11) and show that

$$Q=\begin{bmatrix} \frac{1}{\sqrt{10}} & -\frac{9}{\sqrt{330}} & \frac{3}{\sqrt{33}} & -\frac{21}{\sqrt{2145}} \\ \frac{1}{\sqrt{10}} & -\frac{7}{\sqrt{330}} & \frac{1}{\sqrt{33}} & \frac{7}{\sqrt{2145}} \\ \frac{1}{\sqrt{10}} & -\frac{5}{\sqrt{330}} & -\frac{1}{2\sqrt{33}} & \frac{35}{2\sqrt{2145}} \\ \frac{1}{\sqrt{10}} & -\frac{3}{\sqrt{330}} & -\frac{3}{2\sqrt{33}} & \frac{31}{2\sqrt{2145}} \\ \frac{1}{\sqrt{10}} & -\frac{1}{\sqrt{330}} & -\frac{2}{\sqrt{33}} & \frac{6}{\sqrt{2145}} \\ \frac{1}{\sqrt{10}} & \frac{1}{\sqrt{330}} & -\frac{2}{\sqrt{33}} & -\frac{6}{\sqrt{2145}} \\ \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{330}} & -\frac{3}{2\sqrt{33}} & -\frac{31}{2\sqrt{2145}} \\ \frac{1}{\sqrt{10}} & \frac{5}{\sqrt{330}} & -\frac{1}{2\sqrt{33}} & -\frac{35}{2\sqrt{2145}} \\ \frac{1}{\sqrt{10}} & \frac{7}{\sqrt{330}} & \frac{1}{\sqrt{33}} & -\frac{7}{\sqrt{2145}} \\ \frac{1}{\sqrt{10}} & \frac{9}{\sqrt{330}} & \frac{3}{\sqrt{33}} & \frac{21}{\sqrt{2145}} \end{bmatrix} \tag{4.15.28}$$

4.15.3 Use the matrix (4.15.11) and the matrix (4.15.28) and show that the matrix R in the QR decomposition is given by

$$R=\begin{bmatrix} \sqrt{10} & \frac{55\sqrt{10}}{2} & \frac{1925\sqrt{10}}{2} & \frac{75625\sqrt{10}}{2} \\ 0 & \frac{5\sqrt{330}}{2} & \frac{275\sqrt{330}}{2} & \frac{13175\sqrt{330}}{2} \\ 0 & 0 & 100\sqrt{33} & 8250\sqrt{33} \\ 0 & 0 & 0 & 150\sqrt{2145} \end{bmatrix} \tag{4.15.29}$$

Chapter 5

EIGENVALUE PROBLEMS

Section 5.1 Eigenvalue Problem Definition and Examples

There is a special problem that arises in a large variety of physical circumstances that we shall now study. It is called the *eigenvalue* or *proper value* problem. In its most basic form, it is a problem that arises when one is given a linear transformation $\mathbf{A}:\mathscr{V}\to\mathscr{V}$, where $\mathscr{V}$ is an inner product space, and you are asked to fine certain *intrinsic directions* for that linear transformation. In particular, directions defined by a vector $\mathbf{v}\in\mathscr{V}$ such that such that when $\mathbf{Av}$ is calculated, you get back a vector proportional to $\mathbf{v}$, i.e., a vector that is parallel to the vector $\mathbf{v}$. This geometric statement is equivalent to the algebraic condition of finding a nonzero vector $\mathbf{v}$

$$\mathbf{Av}=\lambda\mathbf{v} \tag{5.1.1}$$

Eigenvalue problems arise in a wide variety of circumstances. A *partial list* is as follows:

Finite Dimensional: Eigenvalue problems defined on finite dimensional vector spaces.

a) Mechanical Vibrations
   a. These problems typically involve finding the solution to systems of linear constant coefficient ordinary differential equations. We shall look at this kind of application.
b) Rigid Body Dynamics
   a. The application usually involves finding the solution to Euler's rigid body equations when studying, for example, the motion of gyroscopes. As explained in Chapter 3, the angular momentum of a rigid body is given in terms of a linear transformation, known as the moment of inertia, and the angular velocity by equation (3.1.13), repeated,

$$\boldsymbol{\ell}=\mathbf{I}\boldsymbol{\omega} \tag{5.1.2}$$

The eigenvalue problem (5.1.1) seeks to find those directions in the rigid body where the angular momentum vector is parallel to the angular velocity. The directions defined by (5.1.1) are the called the principal directions and the proportional coefficients are called the *principal moments of inertia*.

c) Material Behavior:
   a. Applications that involve modeling material behavior with formulas such as Fourier's Law, equation (3.1.12), and Ohm's Law, equation (3.1.14) often require a calculation of the directions where, for example, the heat flux is parallel to the temperature gradient.

d) Strain Kinematics: In the mechanics of deformable solids, the strain of the body is characterized by a linear transformation. The eigenvalue problem is a way of finding the intrinsic directions where the strains are pure elongations or compressions. In other words, the shear strains are zero. These strains are called the *principal strains*.
e) Stress Tensor: In Example 3.1.7 of Chapter 3, Cauchy's Theorem, equation (3.1.15) was stated as follows:

$$\mathbf{t} = \mathbf{T}\mathbf{n} \tag{5.1.3}$$

The eigenvalue problem arises when one wishes to find the directions where the stress vector $\mathbf{t}$ is parallel to the unit normal $\mathbf{n}$. In such directions, the stresses are pure compressions or tensions. They are called the *principal stresses*.

Infinite Dimensional: Eigenvalue problems defined on infinite dimensional vector spaces.

a) The study of liner partial differential equations is the study of linear transformations defined on infinite dimensional vector spaces. For a certain category of problems, the problem of solving these equations comes down to finding the eigenvalues to these linear transformations.

Our interests here continue to be finite dimensional vector spaces.

A more formal statement of the eigenvalue problem is as follows:

Eigenvalue Problem: Given a linear transformation $\mathbf{A}:\mathscr{V}\rightarrow\mathscr{V}$, where $\mathscr{V}$ is an $N=\dim\mathscr{V}$ dimensional inner product space, find a *nonzero* vector $\mathbf{v}$ such that

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v} \tag{5.1.4}$$

The vector $\mathbf{v}$ which satisfies (5.1.4) is called an *eigenvector* of $\mathbf{A}$. The scalar λ is called the *eigenvalue* of $\mathbf{A}$

Some observations about the eigenvalue problem:

a) The eigenvalue problem may not have a solution. This circumstance arises when one restricts the discussion to real vector spaces. If the vector space is complex, as is the case here, it is a theorem that every linear transformation $\mathbf{A}:\mathscr{V}\rightarrow\mathscr{V}$ has at least one solution to its eigenvalue problem.[1]
b) If there are solutions to the eigenvalue problem, there is no guarantee or requirement that the components of $\mathbf{v}$ be real numbers or that the associated eigenvalues be real numbers. For example, a linear transformation can have the property that its matrix A consists of real numbers and still have eigenvectors with complex components and eigenvalues that are complex numbers.
c) The length of an eigenvector, if it exists, is *not* determined by the defining equation (5.1.4), i.e., by $\mathbf{A}\mathbf{v}=\lambda\mathbf{v}$. It is the *direction* that is determined by $\mathbf{A}\mathbf{v}=\lambda\mathbf{v}$. This fact allows us to

[1] The proof of this theorem can be found in most linear algebra books. For example, the textbook, Vector Spaces of Finite Dimension, by G. C. Shephard has this proof.

normalize the eigenvectors to suit the particular application. Often they are normalized to be unit vectors.

If $\{\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_N\}$ is a basis for the inner product space $\mathscr{V}$, the matrix equivalent of (5.1.4) is

$$A\mathbf{v} = \lambda \mathbf{v} \tag{5.1.5}$$

where

$$A = M\left(\mathbf{A}, \mathbf{e}_j, \mathbf{e}_k\right) \tag{5.1.6}$$

and

$$\mathbf{v} = \begin{bmatrix} \upsilon^1 \\ \upsilon^2 \\ \cdot \\ \cdot \\ \cdot \\ \upsilon^N \end{bmatrix} \tag{5.1.7}$$

As pointed out, there is no assurance that the eigenvalue problem has a solution. However if one is fortunate enough such that a linear transformation (or its matrix) has N such vectors, say $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_N\}$, and one has the additional good fortune that these vectors are linearly independent, then $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_N\}$ is a *basis* for $\mathscr{V}$. It is a basis with the property that

$$\begin{aligned} \mathbf{A}\mathbf{v}_1 &= \lambda_1 \mathbf{v}_1 \\ \mathbf{A}\mathbf{v}_1 &= \lambda_2 \mathbf{v}_1 \\ &\cdot \\ &\cdot \\ &\cdot \\ \mathbf{A}\mathbf{v}_N &= \lambda_N \mathbf{v}_N \end{aligned} \tag{5.1.8}$$

Our definition of the matrix of a linear transformation, equation (3.2.9), tells us that, in this case, the matrix of $\mathbf{A}$ with respect to the basis $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_N\}$ takes the *diagonal* form

$$M\left(\mathbf{A},\mathbf{v}_j,\mathbf{v}_j\right)=\underbrace{\begin{bmatrix}\lambda_1 & 0 & 0 & \cdot & \cdot & 0\\ 0 & \lambda_2 & 0 & \cdot & \cdot & 0\\ 0 & 0 & \lambda_3 & & & 0\\ \cdot & \cdot & \cdot & \cdot & & \cdot\\ \cdot & \cdot & \cdot & & \cdot & \cdot\\ 0 & 0 & 0 & \cdot & \cdot & \lambda_N\end{bmatrix}}_{N\times N\text{ Diagonal Matrix}} \tag{5.1.9}$$

If $\left\{\mathbf{e}_1,\mathbf{e}_2,...,\mathbf{e}_N\right\}$ is an arbitrary basis for $\mathscr{V}$ and we are fortunate enough to have a basis $\left\{\mathbf{v}_1,\mathbf{v}_2,...,\mathbf{v}_N\right\}$ of vectors with the property $\mathbf{A}\mathbf{v}_j=\lambda_j\mathbf{v}_j$, then, by our usual change of basis arguments, i.e., by equation (3.6.18),

$$M\left(\mathbf{A},\mathbf{v}_j,\mathbf{v}_k\right)=T^{-1}M\left(\mathbf{A},\mathbf{e}_j,\mathbf{e}_k\right)T \tag{5.1.10}$$

where, as usual, T is the transition matrix defined by

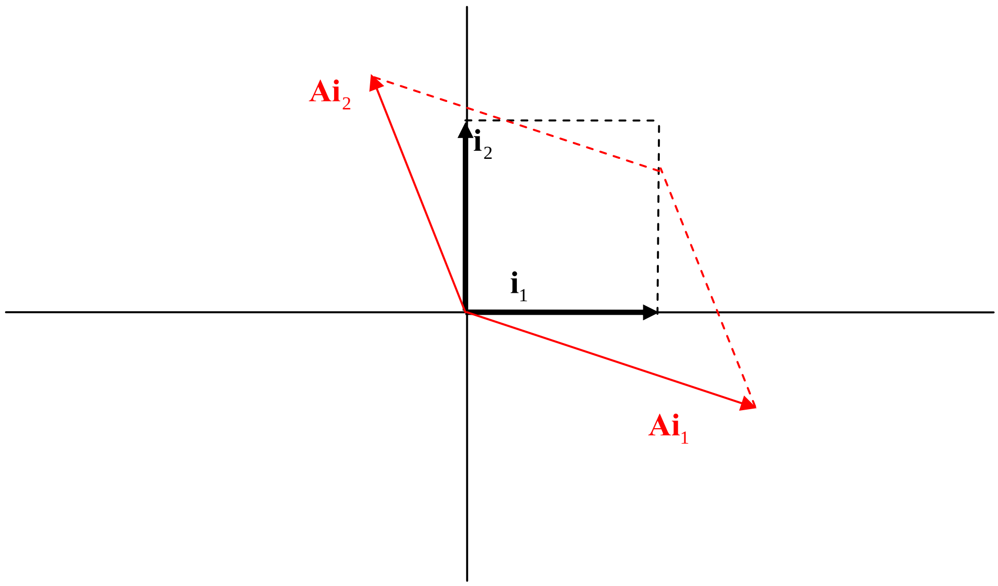
$$\mathbf{v}_j=\sum_{k=1}^{N}T_j^k\mathbf{e}_k \tag{5.1.11}$$

Equation (5.1.10), in the cases where the above construction exists, tells us that the matrix of $\mathbf{A}$ with respect to $\left\{\mathbf{e}_1,\mathbf{e}_2,...,\mathbf{e}_N\right\}$ is *similar to a diagonal matrix*. The construction of this diagonal matrix through a similar transformation is usually referred to as *diagonalizing the matrix*.

Example 5.1.1: A very simple example that illustrates some of the features of the eigenvalue problem is the following. We are given a linear transformation $\mathbf{A}:\mathscr{V}\rightarrow\mathscr{V}$, where $\mathscr{V}$ is a two dimensional inner product space, defined by

$$\begin{aligned}\mathbf{A}\mathbf{i}_1&=\frac{3}{2}\mathbf{i}_1-\frac{1}{2}\mathbf{i}_2\\ \mathbf{A}\mathbf{i}_2&=-\frac{1}{2}\mathbf{i}_1+\frac{3}{2}\mathbf{i}_2\end{aligned} \tag{5.1.12}$$

where $\left\{\mathbf{i}_1,\mathbf{i}_2\right\}$ is an orthonormal basis. The question is whether or not there are any vectors $\mathbf{v}$ for which $\mathbf{A}\mathbf{v}$ is parallel to $\mathbf{v}$. This problem is one where we can display the effect of $\mathbf{A}$ on a vector geometrically. The following figure is a consequence of the above definition (5.1.12) of $\mathbf{A}$:



The linear transformation defined by (5.1.12) deforms the unit square into the parallelogram as the figure shows. The eigenvalue problem is the question whether or not there are vectors $\mathbf{v}$ for which $\mathbf{A}\mathbf{v}$ is parallel to $\mathbf{v}$. For this simple example, the answer is yes. If one takes

$$\mathbf{v}_1 = \frac{1}{\sqrt{2}}\left(\mathbf{i}_1 + \mathbf{i}_2\right) \qquad \text{and} \qquad \mathbf{v}_2 = \frac{1}{\sqrt{2}}\left(-\mathbf{i}_1 + \mathbf{i}_2\right) \tag{5.1.13}$$

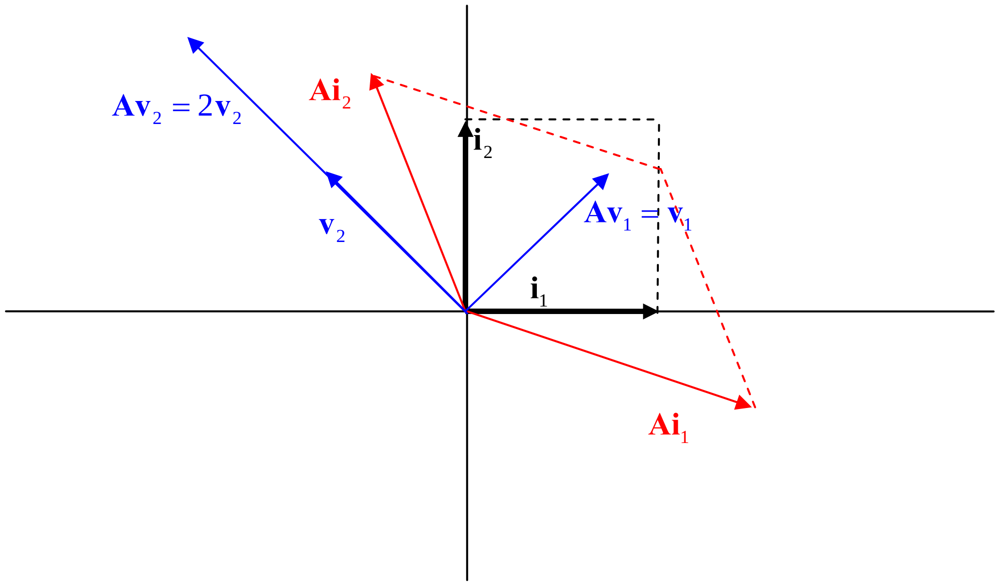
Then,

$$\begin{aligned}\mathbf{A}\mathbf{v}_1 &= \mathbf{A}\left(\left(\frac{1}{\sqrt{2}}\right)\left(\mathbf{i}_1 + \mathbf{i}_1\right)\right) = \left(\frac{1}{\sqrt{2}}\right)\left(\mathbf{A}\left(\mathbf{i}_1\right) + \mathbf{A}\left(\mathbf{i}_2\right)\right) \\ &= \left(\frac{1}{\sqrt{2}}\right)\left(\left(\frac{3}{2}\mathbf{i}_1 - \frac{1}{2}\mathbf{i}_2\right) + \left(-\frac{1}{2}\mathbf{i}_1 + \frac{3}{2}\mathbf{i}_2\right)\right) = \left(\frac{1}{\sqrt{2}}\right)\left(\mathbf{i}_1 + \mathbf{i}\mathbf{e}_2\right) = \mathbf{v}_1\end{aligned} \tag{5.1.14}$$

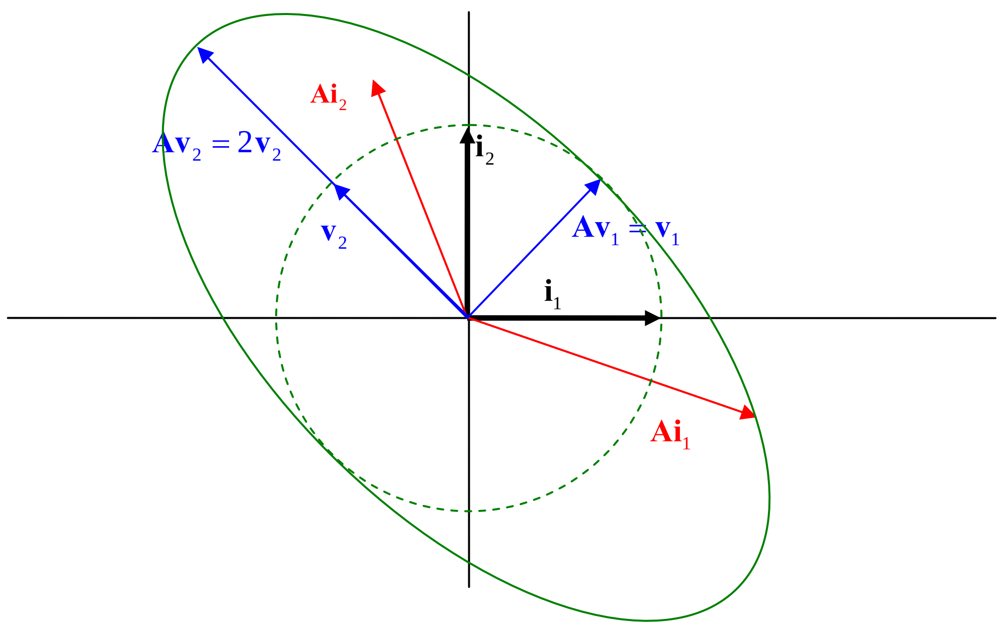
and

$$\begin{aligned}\mathbf{A}\mathbf{v}_2 &= \mathbf{A}\left(\left(\frac{1}{\sqrt{2}}\right)\left(-\mathbf{i}_1 + \mathbf{i}_2\right)\right) = \left(\frac{1}{\sqrt{2}}\right)\left(-\mathbf{A}\left(\mathbf{i}_1\right) + \mathbf{A}\left(\mathbf{i}_2\right)\right) \\ &= \left(\frac{1}{\sqrt{2}}\right)\left(-\left(\frac{3}{2}\mathbf{i}_1 - \frac{1}{2}\mathbf{i}_2\right) + \left(-\frac{1}{2}\mathbf{i}_1 + \frac{3}{2}\mathbf{i}_2\right)\right) = \left(\frac{1}{\sqrt{2}}\right)\left(-2\mathbf{i}_1 + 2\mathbf{i}_2\right) = 2\mathbf{v}_2\end{aligned} \tag{5.1.15}$$

If the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are added to the figure above we get



Another way to geometrically look at what one is doing when the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are plotted is to look at the following figure.



One can think of the unit vectors $\mathbf{i}_1$ and $\mathbf{i}_2$ as defining a unit circle. The linear transformation $\mathbf{A}$ distorts this circle into an ellipse that is rotated relative to the axes defined by $\mathbf{i}_1$ and $\mathbf{i}_2$. In this case the angle of rotation is 45^0 and the major axis of the ellipse is 2 and the minor axis is 1. Notice that in the above example the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are unit vectors. However, any vectors parallel to these will also satisfy the equation $\mathbf{Av} = \lambda \mathbf{v}$.

The matrix of the linear transformation defined by (5.1.12) with respect to the basis $\{\mathbf{i}_1, \mathbf{i}_2\}$ is

$$A = M\left(\mathbf{A}, \mathbf{i}_j, \mathbf{i}_k\right) = \begin{bmatrix} \frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2} \end{bmatrix} \tag{5.1.16}$$

while the matrix of the linear transformation with respect to the basis of eigenvectors $\{\mathbf{v}_1, \mathbf{v}_2\}$ is

$$M\left(\mathbf{A}, \mathbf{v}_j, \mathbf{v}_k\right) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \tag{5.1.17}$$

The transition matrix that connects the two orthonormal bases is given by (5.1.13) and is the orthogonal matrix

$$T = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \tag{5.1.18}$$

Another example, but one without the simple geometric interpretation above is as follows:

Example 5.1.2: We are given a linear transformation $\mathbf{A}: \mathscr{V} \to \mathscr{V}$, where $\mathscr{V}$ is a two dimensional inner product space, defined by

$$\begin{aligned} \mathbf{Ai}_1 &= \frac{1}{\sqrt{2}}\mathbf{i}_1 + \frac{1}{\sqrt{2}}\mathbf{i}_2 \\ \mathbf{Ai}_2 &= -\frac{1}{\sqrt{2}}\mathbf{i}_1 + \frac{1}{\sqrt{2}}\mathbf{i}_2 \end{aligned} \tag{5.1.19}$$

where $\{\mathbf{i}_1, \mathbf{i}_2\}$ is an orthonormal basis. One can show that the linear transformation defined by (5.1.19) is orthogonal. The eigenvectors for this example turn out to be

$$\mathbf{v}_1 = \frac{1}{\sqrt{2}}\left(-i\mathbf{i}_1 + \mathbf{i}_2\right) \quad \text{and} \quad \mathbf{v}_2 = \frac{1}{\sqrt{2}}\left(i\mathbf{i}_1 + \mathbf{i}_2\right) \tag{5.1.20}$$

and the corresponding eigenvalues are

$$\lambda_1 = \frac{1}{\sqrt{2}}(1-i) \quad \text{and} \quad \lambda_2 = \frac{1}{\sqrt{2}}(1+i) \tag{5.1.21}$$

Of course, the matrix of the linear transformation (5.1.19) with respect to the basis (5.1.20) is the diagonal matrix

$$M\left(\mathbf{A}, \mathbf{v}_j, \mathbf{v}_k\right) = \begin{bmatrix} \frac{1}{\sqrt{2}}(1-i) & 0 \\ 0 & \frac{1}{\sqrt{2}}(1+i) \end{bmatrix} \tag{5.1.22}$$

and, from (5.1.20), the transition matrix is

$$T = \begin{bmatrix} -\frac{i}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \tag{5.1.23}$$

This example, because the eigenvalues are complex numbers and the eigenvectors have complex components, does not lend itself to the simple geometric construction used with Example 5.1.2. Never the less, there are important applications for which the linear transformation occurring in the eigenvalue problem is orthogonal.

The fundamental question is now does one find the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$? Also, in Example 5.1.1 the eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ turned out to be orthogonal. This raises the question as to what were the special properties of the linear transformations (5.1.12) that caused the eigenvectors to be linear independent and, in addition, to be orthogonal. In the following Sections we shall examine these questions, among others.

Section 5.2. The Characteristic Polynomial

In this section, we shall discuss a polynomial, called the characteristic polynomial, and its role in the eigenvalue problem. As we shall see, this polynomial plays a fundamental role in the calculation of the eigenvalues and the eigenvectors. Before defining this polynomial, we need a few preliminary definitions and results.

Definition: The *spectrum* of $\mathbf{A}:\mathscr{V}\to\mathscr{V}$ is the set of all eigenvalues of $\mathbf{A}$.

Definition: If λ is an eigenvalue of $\mathbf{A}:\mathscr{V}\to\mathscr{V}$, the *eigenspace* or *characteristic subspace* associated with λ is denoted by $\mathscr{V}(\lambda)$ and is defined by

$$\mathscr{V}(\lambda)=\left\{\mathbf{v}\in\mathscr{V}\,\middle|\,\mathbf{A}\mathbf{v}=\lambda\mathbf{v}\right\} \tag{5.2.1}$$

In words, the characteristic subspace of λ is the subspace of $\mathscr{V}$ consisting of vectors that are eigenvectors associated with the eigenvalue λ. There is nothing in this general discussion that would say that there is only one eigenvector associated with an eigenvalue. In other words, we are not presuming that the dimension of the characteristic subspace of a particular eigenvalue is one.

Definition: The *geometric multiplicity* of an eigenvalue λ is the dimension of its characteristic subspace, i.e. $\dim\mathscr{V}(\lambda)$.

Theorem 5.2.1: The characteristic subspace of an eigenvalue λ is the kernel of the linear transformation $\mathbf{A}-\lambda\mathbf{I}:\mathscr{V}\to\mathscr{V}$, i.e.

$$\mathscr{V}(\lambda)=K(\mathbf{A}-\lambda\mathbf{I}) \tag{5.2.2}$$

To see this result, just write the defining equation (5.1.4) as

$$(\mathbf{A}-\lambda\mathbf{I})\mathbf{v}=\mathbf{0} \tag{5.2.3}$$

It follows from the equation $\mathscr{V}(\lambda)=K(\mathbf{A}-\lambda\mathbf{I})$ that if λ is an eigenvalue of $\mathbf{A}:\mathscr{V}\to\mathscr{V}$, then $\mathbf{A}-\lambda\mathbf{I}$ must be singular and conversely. The condition that $\mathbf{A}-\lambda\mathbf{I}$ is singular is equivalent to the condition

$$\det(\mathbf{A}-\lambda\mathbf{I})=0 \tag{5.2.4}$$

Definition: The N^{th} order polynomial in λ defined by

$$f(\lambda)=\det(\mathbf{A}-\lambda\mathbf{I}) \tag{5.2.5}$$

is the *characteristic polynomial* of $\mathbf{A}:\mathscr{V}\to\mathscr{V}$.

The important result we have obtained is that *the eigenvalues of a linear transformation* $\mathbf{A}:\mathscr{V}\to\mathscr{V}$ *are the roots of the characteristic equation*. There are a couple of ways one can write this polynomial. As an N^{th} order polynomial, it will have N roots. They *need not be distinct* roots and they *need not be real numbers*. If we list these N roots as $\lambda_1,\lambda_2,...,\lambda_N$, then the characteristic polynomial can always be written as

$$f(\lambda)=\det(\mathbf{A}-\lambda\mathbf{I})=\underbrace{(\lambda_1-\lambda)(\lambda_2-\lambda)(\lambda_3-\lambda)\cdots(\lambda_N-\lambda)}_{N\ \text{Factors}} \tag{5.2.6}$$

Example 5.2.1: For Example 5.1.1, the characteristic polynomial is

$$\begin{aligned} f(\lambda)&=\det(\mathbf{A}-\lambda\mathbf{I})=\det\left(M\left(\mathbf{A}-\lambda\mathbf{I},\mathbf{e}_j,\mathbf{e}_k\right)\right)=\det\begin{bmatrix}\frac{3}{2}-\lambda & -\frac{1}{2}\\ -\frac{1}{2} & \frac{3}{2}-\lambda\end{bmatrix}\\ &=\left(\frac{3}{2}-\lambda\right)^2-\frac{1}{4}=\lambda^2-3\lambda+2=(\lambda-1)(\lambda-2)\end{aligned} \tag{5.2.7}$$

Therefore, the roots of the characteristic polynomial are $\lambda_1=1,\lambda_2=2$. Note, in passing, that for this example:

$$\det\mathbf{A}=\begin{vmatrix}\frac{3}{2} & -\frac{1}{2}\\ -\frac{1}{2} & \frac{3}{2}\end{vmatrix}=2=\lambda_1\lambda_2 \quad\text{and}\quad \operatorname{tr}\mathbf{A}=3=\lambda_1+\lambda_2 \tag{5.2.8}$$

Example 5.2.2: For Example 5.1.2, the characteristic polynomial is

$$\begin{aligned} f(\lambda)&=\det(\mathbf{A}-\lambda\mathbf{I})=\det\left(M\left(\mathbf{A}-\lambda\mathbf{I},\mathbf{e}_j,\mathbf{e}_k\right)\right)=\det\begin{bmatrix}\frac{1}{\sqrt{2}}-\lambda & -\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}-\lambda\end{bmatrix}\\ &=\left(\frac{1}{\sqrt{2}}-\lambda\right)^2+\frac{1}{2}=\lambda^2-\sqrt{2}\lambda+1=\left(\lambda-\frac{1}{\sqrt{2}}(1-i)\right)\left(\lambda-\frac{1}{\sqrt{2}}(1+i)\right)\end{aligned} \tag{5.2.9}$$

Therefore, the roots of the characteristic polynomial are $\lambda_1 = \frac{1}{\sqrt{2}}(1-i), \lambda_2 = \frac{1}{\sqrt{2}}(1+i)$. As is a general feature for the roots of polynomials with real coefficients, complex roots occur in complex conjugate pairs. Also note, as with Example 5.2.1, for this example:

$$\det \mathbf{A} = \det \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = 1 = \lambda_1 \lambda_2 \quad \text{and} \quad \operatorname{tr} \mathbf{A} = \sqrt{2} = \lambda_1 + \lambda_2 \tag{5.2.10}$$

The results (5.2.8) and (5.2.10) are special cases of a general feature for polynomials, in this case for the characteristic polynomial. It is a fact that for a linear transformation $\mathbf{A}: \mathscr{V} \to \mathscr{V}$ with eigenvalues $\lambda_1, \lambda_2, ..., \lambda_N$ that

$$\operatorname{tr} \mathbf{A} = \lambda_1 + \lambda_2 + \cdots + \lambda_N \tag{5.2.11}$$

and

$$\det \mathbf{A} = \lambda_1 \lambda_2 \cdots \lambda_N \tag{5.2.12}$$

The details of the derivations of (5.2.11) and (5.2.12) are complicated because of the complications associated with writing down a general expansion of the determinant

$$\begin{aligned} f(\lambda) &= \det(\mathbf{A} - \lambda \mathbf{I}) = \det\left(M\left(\mathbf{A} - \lambda \mathbf{I}, \mathbf{e}_j, \mathbf{e}_k\right)\right) \\ &= \begin{vmatrix} A^1{}_1 - \lambda & A^1{}_2 & A^1{}_3 & \cdot & \cdot & A^1{}_N \\ A^2{}_1 & A^2{}_2 - \lambda & A^2{}_3 & \cdot & \cdot & A^2{}_N \\ A^3{}_1 & A^3{}_2 & A^3{}_3 - \lambda & & & \\ \cdot & & & & & \\ A^N{}_1 & A^N{}_2 & \cdot & \cdot & \cdot & A^N{}_N - \lambda \end{vmatrix} \end{aligned} \tag{5.2.13}$$

and forcing the result, by (5.2.6), to equal $(\lambda_1 - \lambda)(\lambda_2 - \lambda)(\lambda_3 - \lambda)\cdots(\lambda_N - \lambda)$. The expansion of (5.2.13) will always turn out to be on the form

$$f(\lambda) = \det(\mathbf{A} - \lambda \mathbf{I}) = (-\lambda)^N + \mu_1 (-\lambda)^{N-1} + \cdots + \mu_{N-1}(-\lambda) + \mu_N \tag{5.2.14}$$

The algebraic problem is to find formulas for the N coefficients $\mu_1, \mu_2, ..., \mu_N$ in terms of the linear transformation $\mathbf{A}$. While it is perhaps evident from (5.2.14) that

$$\mu_N = \det \mathbf{A} \tag{5.2.15}$$

the complications of the algebra make it not so obvious that

$$\mu_1 = \operatorname{tr} \mathbf{A} \tag{5.2.16}$$

The algebraic details are not excessively complicated if we temporarily restrict the discussion to the case $N = \dim \mathscr{V} = 3$. The algebraic problem above is to extract information by forcing the equality

$$\begin{vmatrix} A^1{}_1 - \lambda & A^1{}_2 & A^1{}_3 \\ A^2{}_1 & A^2{}_2 - \lambda & A^2{}_3 \\ A^3{}_1 & A^3{}_2 & A^3{}_3 - \lambda \end{vmatrix} = (\lambda_1 - \lambda)(\lambda_2 - \lambda)(\lambda_3 - \lambda) \tag{5.2.17}$$

If both sides of this equation are expanded, the result, after some labor, is

$$-\lambda^3 + \mu_1 \lambda^2 - \mu_2 \lambda + \mu_3 = -\lambda^3 + (\lambda_1 + \lambda_2 + \lambda_3)\lambda^2 - (\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3)\lambda + \lambda_1\lambda_2\lambda_3 \tag{5.2.18}$$

where the three coefficients μ_1, μ_2, μ_3 are given by[2]

$$\begin{aligned} \mu_1 &= \operatorname{tr} \mathbf{A} \\ \mu_2 &= \frac{1}{2}\left\{ (\operatorname{tr} \mathbf{A})^2 - \operatorname{tr}(\mathbf{AA}) \right\} \\ \mu_3 &= \det \mathbf{A} \end{aligned} \tag{5.2.19}$$

As an *identity* in the parameter λ, (5.2.18) forces the coefficients of the like powers of λ to be the same. As a result,

$$\begin{aligned} \mu_1 &= \operatorname{tr} \mathbf{A} = \lambda_1 + \lambda_2 + \lambda_3 \\ \mu_2 &= \frac{1}{2}\left\{ (\operatorname{tr} \mathbf{A})^2 - \operatorname{tr}(\mathbf{AA}) \right\} = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3 \\ \mu_3 &= \det \mathbf{A} = \lambda_1\lambda_2\lambda_3 \end{aligned} \tag{5.2.20}$$

---

[2] There is another formula for the second fundamental invariant μ_2 that one sometimes finds. It is

$$\mu_2 = \operatorname{tr}\left(\operatorname{adj} M\left(\mathbf{A}, \mathbf{e}_j, \mathbf{e}_k\right)\right)$$

where $\operatorname{adj} M\left(\mathbf{A}, \mathbf{e}_j, \mathbf{e}_k\right)$ is the adjugate matrix defined in Section 1.10. In terms of the components of the matrix $M\left(\mathbf{A}, \mathbf{e}_j, \mathbf{e}_k\right)$, this formula is

$$\mu_2 = \operatorname{tr}\left(\operatorname{adj} M\left(\mathbf{A}, \mathbf{e}_j, \mathbf{e}_k\right)\right) = \begin{vmatrix} A^2{}_2 & A^2{}_3 \\ A^3{}_2 & A^3{}_3 \end{vmatrix} + \begin{vmatrix} A^1{}_1 & A^1{}_3 \\ A^3{}_1 & A^3{}_3 \end{vmatrix} + \begin{vmatrix} A^1{}_1 & A^1{}_2 \\ A^2{}_1 & A^2{}_2 \end{vmatrix}$$

The three coefficients μ_1, μ_2, μ_3 defined by the above formulas are called the *fundamental invariants* of the linear transformation $\mathbf{A}:\mathscr{V} \to \mathscr{V}$ (remember in the case under discussion $\dim \mathscr{V} = 3$). The first and third are defined in terms of operations that we have used throughout this textbook. In Section 3.6, we showed that these two operations do not depend upon the choice of basis. A similar proof can be given for μ_2, since it is defined in terms of the trace operation. The word "invariant" arises because the three quantities do not depend upon the choice of basis used to represent $\mathbf{A}:\mathscr{V} \to \mathscr{V}$. In the case of a linear transformation $\mathbf{A}:\mathscr{V} \to \mathscr{V}$, where $N = \dim \mathscr{V}$, then one can show that the N coefficients $\mu_1, \mu_2, ..., \mu_N$ do not depend upon the choice of basis. These N quantities are the *fundamental invariants* of $\mathbf{A}:\mathscr{V} \to \mathscr{V}$.

Example 5.2.3: Consider a matrix $A \in \mathscr{M}^{3\times 3}$ defined by

$$A = \begin{bmatrix} 3 & -8 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{5.2.21}$$

The eigenvalues are the roots of

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} 3-\lambda & -8 & 0 \\ 2 & 3-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda)\left((3-\lambda)^2 + 16\right) \\ &= (1-\lambda)\left(\lambda^2 - 6\lambda + 25\right) = (1-\lambda)(\lambda - 3 - 4i)(\lambda - 3 + 4i) \end{aligned} \tag{5.2.22}$$

Therefore, the eigenvalues are given by

$$\begin{aligned} \lambda_1 &= 1 \\ \lambda_2 &= 3 + 4i \\ \lambda_3 &= 3 - 4i \end{aligned} \tag{5.2.23}$$

If we calculate the fundamental invariants using the matrix A and the definitions (5.2.19), we obtain

$$
\begin{aligned}
\mu_1 &= \operatorname{tr} A = 7 \\
\mu_2 &= \frac{1}{2}\left\{\left(\operatorname{tr} A\right)^2 - \operatorname{tr}\left(AA\right)\right\} \\
&= \frac{1}{2}\left\{\left(7\right)^2 - \operatorname{tr}\left(\begin{bmatrix} 3 & -8 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & -8 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\right\} \\
&= \frac{1}{2}\left\{\left(7\right)^2 - \operatorname{tr}\left(\begin{bmatrix} -7 & -48 & 0 \\ 12 & -7 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\right\} \\
&= \frac{1}{2}\left\{49 + 13\right\} = 31 \\
\mu_3 &= \det A = \begin{vmatrix} 3 & -8 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 25
\end{aligned}
\tag{5.2.24}
$$

If we use the formulas for the invariants in terms of the eigenvalues, i.e. equation $(5.2.20)_2$ we obtain

$$
\begin{aligned}
\mu_1 &= \lambda_1 + \lambda_2 + \lambda_3 = 1 + 3 + 4i + 3 - 4i = 7 \\
\mu_2 &= \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3 = \left(1\right)\left(3+4i\right) + \left(3+4i\right)\left(3-4i\right) + \left(1\right)\left(3-4i\right) \\
&= 6 + 9 + 16 = 31 \\
\mu_3 &= \det \mathbf{A} = \lambda_1\lambda_2\lambda_3 = \left(1\right)\left(3+4i\right)\left(3-4i\right) = 25
\end{aligned}
\tag{5.2.25}
$$

This example illustrates a feature observed with Example 5.2.2. Namely, a linear transformation whose matrix has real components can have complex eigenvalues. Because the characteristic polynomial in these examples has real coefficients, the complex roots occur in *complex conjugate pairs*. If we had stated in advance that our inner product space was a real inner product space, we would have to conclude that not all linear transformations on real inner product spaces have eigenvectors. However, if we allow complex scalars, then we are assured that all linear transformations have a least one eigenvector.[3] We shall see examples where the occurrence of complex eigenvalues is an essential feature of the application.

There is nothing in our discussion that requires the eigenvalues, the roots of the characteristic equation (5.2.5) to be unique. Because of this, the factored characteristic equation (5.2.6) will sometimes need to be written to display the multiplicity of its roots. If, for example, the characteristic polynomial has L distinct roots and the j^{th} of these roots is repeated d_j times, then the characteristic polynomial, when factored, looks like

[3] See footnote 1 above.

$$f(\lambda)=\det(\mathbf{A}-\lambda\mathbf{I})=(\lambda_1-\lambda)^{d_1}(\lambda_2-\lambda)^{d_2}(\lambda_3-\lambda)^{d_3}\cdots(\lambda_L-\lambda)^{d_L} \tag{5.2.26}$$

where $d_1, d_2, d_3, ..., d_L$ are positive integers which obey

$$\sum_{j=1}^{L} d_j = N \tag{5.2.27}$$

The positive integer d_j is the *algebraic multiplicity* of the eigenvalue λ_j. It is a theoretical result, which we shall not pursue, that

$$\dim \mathscr{V}(\lambda_j) \le d_j \tag{5.2.28}$$

i.e. the *geometric multiplicity of a particular eigenvalue is less than or equal to its algebraic multiplicity*.[4]

Example 5.2.4: Consider the matrix $A \in \mathscr{M}^{3\times 3}$ defined by

$$A=\begin{bmatrix} 4 & -6 & 2 \\ 2 & -4 & 2 \\ 2 & -6 & 4 \end{bmatrix} \tag{5.2.29}$$

Then, the characteristic equation is

$$\det\begin{bmatrix} 4-\lambda & -6 & 2 \\ 2 & -4-\lambda & 2 \\ 2 & -6 & 4-\lambda \end{bmatrix} = -\lambda(2-\lambda)^2 \tag{5.2.30}$$

Therefore, the factorization shown in (5.2.26) corresponds to

$$\begin{aligned} \lambda_1 &= 0, \ d_1 = 1 \\ \lambda_2 &= 2, \ d_2 = 2 \end{aligned} \tag{5.2.31}$$

where $d_1 + d_2 = 3$.

It is helpful to summarize what we have illustrated for the characteristic polynomial of a linear transformation $\mathbf{A}:\mathscr{V}\to\mathscr{V}$, where $N=\dim\mathscr{V}$,

---

[4] A proof that the geometric multiplicity is always less than or equal to the algebraic multiplicity can be found, for example, on page 157 of Stakgold, I., BOUNDARY VALUE PROBLEMS OF MATHEMATICAL PHYSICS, Volume I, The MacMillan Company, 1-333, 1967

a) There are N roots to the characteristic polynomial.
b) The roots are both real and complex numbers.
c) When the coefficients of the characteristic polynomial are real numbers, the complex roots occur in complex conjugate pairs.
d) The roots need not be distinct.

Exercises

5.2.1 You are given the linear transformation $\mathbf{A}:\mathscr{V}\rightarrow\mathscr{V}$ defined by

$$\begin{aligned}\mathbf{Ai}_1 &= 7\mathbf{i}_1 - 2\mathbf{i}_2 + \mathbf{i}_3 \\ \mathbf{Ai}_2 &= -2\mathbf{i}_1 + 10\mathbf{i}_2 - 2\mathbf{i}_3 \\ \mathbf{Ai}_3 &= \mathbf{i}_1 - 2\mathbf{i}_2 + 7\mathbf{i}_3\end{aligned} \tag{5.2.32}$$

where $\{\mathbf{i}_1,\mathbf{i}_2,\mathbf{i}_3\}$ is an orthonormal basis for $\mathscr{V}$. Calculate the fundamental invariants of the linear transformation $\mathbf{A}:\mathscr{V}\rightarrow\mathscr{V}$.

5.2.2 Given the polynomial (5.2.14) where the invariants are assumed to be real numbers, show that

$$\overline{\det\left(\mathbf{A}-\lambda\mathbf{I}\right)} = \det\left(\mathbf{A}-\bar{\lambda}\mathbf{I}\right) \tag{5.2.33}$$

Use (5.2.33) to prove that the roots, for the situation stated, occur in complex conjugate pairs.

5.2.3 The Cayley-Hamilton Theorem says that a linear transformation satisfies its own characteristic equation. Thus, if the characteristic equation of a linear transformation $\mathbf{A}:\mathscr{V}\rightarrow\mathscr{V}$ is given by equation (5.2.14). The Cayley-Hamilton asserts that

$$\left(-\mathbf{A}\right)^N + \mu_1\left(-\mathbf{A}\right)^{N-1} + \cdots + \mu_{N-1}\left(-\mathbf{A}\right) + \mu_N\mathbf{I} = \mathbf{0} \tag{5.2.34}$$

where the powers of $\mathbf{A}$ are defined by $\mathbf{A}^0 = \mathbf{I}$, $\mathbf{A}^n = \underbrace{\mathbf{AAA}\cdot\cdot\mathbf{A}}_{n\text{ times}}$ for $n = 1,2,\ldots$

Confirm (5.2.34) for the matrix (5.2.29).

Section 5.3. Numerical Examples

In this section, we shall illustrate the procedure for solving eigenvalue problems and use these examples to illustrate some general features of the solution we will examine in detail in later sections.

Example 5.3.1: (A distinct eigenvalue example.) The problem is to solve the eigenvalue problem for the linear transformation $\mathbf{A}:\mathscr{V}\to\mathscr{V}$ defined by

$$\begin{aligned}\mathbf{Ae}_1 &= \mathbf{e}_1+\mathbf{e}_2+4\mathbf{e}_3\\ \mathbf{Ae}_2 &= 2\mathbf{e}_1-4\mathbf{e}_3\\ \mathbf{Ae}_3 &= -\mathbf{e}_1+\mathbf{e}_2+5\mathbf{e}_3\end{aligned} \tag{5.3.1}$$

where $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ is a basis for $\mathscr{V}$. By the usual procedure the matrix of $\mathbf{A}$ with respect to this basis is

$$A = M\left(\mathbf{A},\mathbf{e}_j,\mathbf{e}_k\right)=\begin{bmatrix}1 & 2 & -1\\ 1 & 0 & 1\\ 4 & -4 & 5\end{bmatrix} \tag{5.3.2}$$

Step 1: Form the characteristic equation: The characteristic equation is defined by (5.2.5). Given the matrix (5.3.2), we find

$$\begin{aligned}f(\lambda)=\det\left(\mathbf{A}-\lambda\mathbf{I}\right)=\det\left(M\left(\mathbf{A}-\lambda\mathbf{I}\right)\right)&=\begin{vmatrix}1-\lambda & 2 & -1\\ 1 & -\lambda & 1\\ 4 & -4 & 5-\lambda\end{vmatrix}\\ &=-\lambda^3+6\lambda^2-11\lambda+6=-(\lambda-1)(\lambda-2)(\lambda-3)\end{aligned} \tag{5.3.3}$$

Therefore, the three eigenvalues are

$$\begin{aligned}\lambda_1&=1\\ \lambda_2&=2\\ \lambda_3&=3\end{aligned} \tag{5.3.4}$$

Because the eigenvalues are distinct, the algebraic multiplicity is 1 for each eigenvalue. Also, the ordering of the eigenvalues is arbitrary. As a quick check of the answer (5.3.4) it is good practice, for complicated problems, to check the calculated eigenvalues against the formulas (5.2.20). If this check is made, one will confirm that (5.2.20) is obeyed.

Step 2: Determine the characteristic subspaces. In other words, determine the eigenvectors. This calculation involves solving the defining equation (5.1.5), written

$$\left(\mathbf{A}-\lambda\mathbf{I}\right)\mathbf{v}=\mathbf{0} \tag{5.3.5}$$

for the three different eigenvalues. Of course, we shall solve (5.3.5) as the matrix equation

$$\begin{bmatrix} 1-\lambda & 2 & -1 \\ 1 & -\lambda & 1 \\ 4 & -4 & 5-\lambda \end{bmatrix}\begin{bmatrix} \upsilon^1 \\ \upsilon^2 \\ \upsilon^3 \end{bmatrix}=0 \tag{5.3.6}$$

Case 1: $\lambda=1$. This choice reduces (5.3.6) to

$$\begin{bmatrix} 0 & 2 & -1 \\ 1 & -1 & 1 \\ 4 & -4 & 4 \end{bmatrix}\begin{bmatrix} \upsilon^1{}_{(1)} \\ \upsilon^2{}_{(1)} \\ \upsilon^3{}_{(1)} \end{bmatrix}=0 \tag{5.3.7}$$

The reduced row echelon form of this equation is

$$\begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \upsilon^1{}_{(1)} \\ \upsilon^2{}_{(1)} \\ \upsilon^3{}_{(1)} \end{bmatrix}=0 \tag{5.3.8}$$

Equation (5.3.8) yields

$$\mathbf{v}_1=\upsilon^3{}_{(1)}\left(-\frac{1}{2}\mathbf{e}_1+\frac{1}{2}\mathbf{e}_2+\mathbf{e}_3\right) \tag{5.3.9}$$

Therefore, the characteristic subspace associated with the first eigenvalue is the one dimensional subspace spanned by $-\frac{1}{2}\mathbf{e}_1+\frac{1}{2}\mathbf{e}_2+\mathbf{e}_3$. We can write this result as

$$\mathscr{V}\left(\lambda_1\right)=K\left(\mathbf{A}-\lambda_1\mathbf{I}\right)=\operatorname{span}\left(-\frac{1}{2}\mathbf{e}_1+\frac{1}{2}\mathbf{e}_2+\mathbf{e}_3\right) \tag{5.3.10}$$

For this eigenvalue, the geometric multiplicity is one as is the algebraic multiplicity.

Case 2: $\lambda=2$. This choice reduces (5.3.6) to

$$\begin{bmatrix} -1 & 2 & -1 \\ 1 & -2 & 1 \\ 4 & -4 & 3 \end{bmatrix} \begin{bmatrix} \upsilon^1{}_{(2)} \\ \upsilon^2{}_{(2)} \\ \upsilon^3{}_{(2)} \end{bmatrix} = 0 \tag{5.3.11}$$

The reduced row echelon form of this equation is

$$\begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{4} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \upsilon^1{}_{(2)} \\ \upsilon^2{}_{(2)} \\ \upsilon^3{}_{(2)} \end{bmatrix} = 0 \tag{5.3.12}$$

Equation (5.3.12) yields

$$\mathbf{v}_2 = \upsilon^3{}_{(2)} \left(-\frac{1}{2}\mathbf{e}_1 + \frac{1}{4}\mathbf{e}_2 + \mathbf{e}_3 \right) \tag{5.3.13}$$

Therefore, the characteristic subspace associated with the second eigenvalue is the one dimensional subspace spanned by $-\frac{1}{2}\mathbf{e}_1 + \frac{1}{4}\mathbf{e}_2 + \mathbf{e}_3$. We can write this result as

$$\mathscr{V}\left(\lambda_2\right) = K\left(\mathbf{A} - \lambda_2 \mathbf{I}\right) = \operatorname{span}\left(-\frac{1}{2}\mathbf{e}_1 + \frac{1}{4}\mathbf{e}_2 + \mathbf{e}_3 \right) \tag{5.3.14}$$

For this eigenvalue, the geometric multiplicity is one as is again equal to the algebraic multiplicity.

Case 3: $\lambda = 3$. This choice reduces (5.3.6) to

$$\begin{bmatrix} -2 & 2 & -1 \\ 1 & -3 & 1 \\ 4 & -4 & 2 \end{bmatrix} \begin{bmatrix} \upsilon^1{}_{(3)} \\ \upsilon^2{}_{(3)} \\ \upsilon^3{}_{(3)} \end{bmatrix} = 0 \tag{5.3.15}$$

The reduced row echelon form of this equation is

$$\begin{bmatrix} 1 & 0 & \dfrac{1}{4} \\ 0 & 1 & -\dfrac{1}{4} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \upsilon^1{}_{(3)} \\ \upsilon^2{}_{(3)} \\ \upsilon^3{}_{(3)} \end{bmatrix} = 0 \tag{5.3.16}$$

Equation (5.3.16) yields

$$\mathbf{v}_3 = \upsilon^3{}_{(3)} \left(-\frac{1}{4}\mathbf{e}_1 + \frac{1}{4}\mathbf{e}_2 + \mathbf{e}_3 \right) \tag{5.3.17}$$

Therefore, the characteristic subspace associated with the third eigenvalue is the one dimensional subspace spanned by $-\frac{1}{4}\mathbf{e}_1 + \frac{1}{4}\mathbf{e}_2 + \mathbf{e}_3$. We can write this result as

$$\mathscr{V}(\lambda_3) = K(\mathbf{A} - \lambda_3 \mathbf{I}) = \operatorname{span}\left(-\frac{1}{4}\mathbf{e}_1 + \frac{1}{4}\mathbf{e}_2 + \mathbf{e}_3 \right) \tag{5.3.18}$$

For this eigenvalue, the geometric multiplicity is one as is also the algebraic multiplicity.

This example, which has distinct eigenvalues, yields three linearly independent eigenvectors. We shall prove a theorem later that shows that distinct eigenvalues produce linearly independent eigenvectors. Thus, our example illustrates a general result. Another way to display that the eigenvectors are linearly independent is to use the idea of a direct sum introduced in Section 4.11 and write

$$\mathscr{V} = \mathscr{V}(\lambda_1) \oplus \mathscr{V}(\lambda_2) \oplus \mathscr{V}(\lambda_3) \tag{5.3.19}$$

As has been explained, the eigenvalue problem does not determine the lengths of the eigenvectors. If we make an arbitrary choice of the lengths by defining the three eigenvectors to be

$$\begin{aligned} \mathbf{v}_1 &= -\frac{1}{2}\mathbf{e}_1 + \frac{1}{2}\mathbf{e}_2 + \mathbf{e}_3 \\ \mathbf{v}_2 &= -\frac{1}{2}\mathbf{e}_1 + \frac{1}{4}\mathbf{e}_2 + \mathbf{e}_3 \\ \mathbf{v}_3 &= -\frac{1}{4}\mathbf{e}_1 + \frac{1}{4}\mathbf{e}_2 + \mathbf{e}_3 \end{aligned} \tag{5.3.20}$$

Then the matrix of $\mathbf{A}: \mathscr{V} \to \mathscr{V}$ with respect to this basis is

$$M\left(\mathbf{A},\mathbf{v}_j,\mathbf{v}_k\right)=\begin{bmatrix}1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 3\end{bmatrix} \tag{5.3.21}$$

and the transition matrix is

$$T=\begin{bmatrix}-\frac{1}{2} & -\frac{1}{2} & -\frac{1}{4}\\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4}\\ 1 & 1 & 1\end{bmatrix} \tag{5.3.22}$$

Note that with our normalization leading to (5.3.20), the transition matrix is the square matrix made from the column matrices of the components of the eigenvectors. The formula is

$$T=\begin{bmatrix}\upsilon^1{}_{(1)} & \upsilon^1{}_{(2)} & \upsilon^1{}_{(3)}\\ \upsilon^2{}_{(1)} & \upsilon^2{}_{(2)} & \upsilon^2{}_{(3)}\\ \upsilon^3{}_{(1)} & \upsilon^3{}_{(2)} & \upsilon^3{}_{(3)}\end{bmatrix} \tag{5.3.23}$$

The matrices (5.3.2) and (5.3.21) are connected by (5.1.10), repeated,

$$M\left(\mathbf{A},\mathbf{v}_j,\mathbf{v}_k\right)=T^{-1}M\left(\mathbf{A},\mathbf{e}_j,\mathbf{e}_k\right)T \tag{5.3.24}$$

where the transition matrix is given by (5.3.22).

Example 5.3.2: (A repeated eigenvalues example.) The problem is to solve the eigenvalue problem for the linear transformation $\mathbf{A}:\mathscr{V}\rightarrow\mathscr{V}$ defined by

$$\begin{aligned}\mathbf{A}\mathbf{i}_1 &= \mathbf{i}_1-2\mathbf{i}_2+2\mathbf{i}_3\\ \mathbf{A}\mathbf{i}_2 &= -2\mathbf{i}_1+\mathbf{i}_2+2\mathbf{i}_3\\ \mathbf{A}\mathbf{i}_3 &= 2\mathbf{i}_1+2\mathbf{i}_2+\mathbf{i}_3\end{aligned} \tag{5.3.25}$$

where $\{\mathbf{i}_1,\mathbf{i}_2,\mathbf{i}_3\}$ is an *orthonormal* basis for $\mathscr{V}$. By the usual procedure, the matrix of $\mathbf{A}$ with respect to this basis is

$$A=M\left(\mathbf{A},\mathbf{i}_j,\mathbf{i}_k\right)=\begin{bmatrix}1 & -2 & 2\\ -2 & 1 & 2\\ 2 & 2 & 1\end{bmatrix} \tag{5.3.26}$$

The fact that the matrix (5.3.26) is symmetric will turn out to be an important feature of this example. Based upon our discussion in Section 4.9, in particular equation (4.9.39), because we are using an orthonormal basis, symmetry of the matrix implies symmetry of the linear transformation (5.3.25).

Step 1: Form the characteristic equation: The characteristic equation is defined by (5.2.5). Given the matrix (5.3.26), we find

$$f(\lambda)=\det(\mathbf{A}-\lambda\mathbf{I})=\det\big(M(\mathbf{A}-\lambda\mathbf{I})\big)=\begin{vmatrix}1-\lambda & -2 & 2\\ -2 & 1-\lambda & 2\\ 2 & 2 & 1-\lambda\end{vmatrix}$$
$$=-\lambda^3+3\lambda^2+9\lambda-27=-(\lambda+3)(\lambda-3)^2 \tag{5.3.27}$$

Therefore, the three eigenvalues are

$$\begin{aligned}\lambda_1&=-3\\ \lambda_2&=3\end{aligned} \tag{5.3.28}$$

The algebraic multiplicity for λ_1 is 1 and that for λ_2 is 2.

Step2: Determine the characteristic subspaces. This calculation involves solving the defining equation (5.1.5), written

$$(\mathbf{A}-\lambda\mathbf{I})\mathbf{v}=\mathbf{0} \tag{5.3.29}$$

for the two different eigenvalues. Of course, we shall solve (5.3.5) as the matrix equation

$$\begin{bmatrix}1-\lambda & -2 & 2\\ -2 & 1-\lambda & 2\\ 2 & 2 & 1-\lambda\end{bmatrix}\begin{bmatrix}\upsilon^1\\ \upsilon^2\\ \upsilon^3\end{bmatrix}=0 \tag{5.3.30}$$

Case 1: $\lambda=-3$. This choice reduces (5.3.6) to

$$\begin{bmatrix}4 & -2 & 2\\ -2 & 4 & 2\\ 2 & 2 & 4\end{bmatrix}\begin{bmatrix}\upsilon^1{}_{(1)}\\ \upsilon^2{}_{(1)}\\ \upsilon^3{}_{(1)}\end{bmatrix}=0 \tag{5.3.31}$$

The reduced row echelon form of this equation is

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \upsilon^1{}_{(1)} \\ \upsilon^2{}_{(1)} \\ \upsilon^3{}_{(1)} \end{bmatrix} = 0 \tag{5.3.32}$$

Therefore, (5.3.32) yields

$$\mathbf{v}_1 = \upsilon^3{}_{(1)} \left(-\mathbf{i}_1 - \mathbf{i}_2 + \mathbf{i}_3 \right) \tag{5.3.33}$$

Therefore, the characteristic subspace associated with the first eigenvalue is the one dimensional subspace spanned by $-\mathbf{i}_1 - \mathbf{i}_2 + \mathbf{i}_3$. We can write this result as

$$\mathscr{V}\left(\lambda_1\right) = K\left(\mathbf{A} - \lambda_1 \mathbf{I}\right) = \operatorname{span}\left(-\mathbf{i}_1 - \mathbf{i}_2 + \mathbf{i}_3\right) \tag{5.3.34}$$

For this eigenvalue, the geometric multiplicity is one as is the algebraic multiplicity.

Case 2: $\lambda = 3$. This choice reduces (5.3.6) to

$$\begin{bmatrix} -2 & -2 & 2 \\ -2 & -2 & 1 \\ 2 & 2 & -2 \end{bmatrix} \begin{bmatrix} \upsilon^1{}_{(2)} \\ \upsilon^2{}_{(2)} \\ \upsilon^3{}_{(2)} \end{bmatrix} = 0 \tag{5.3.35}$$

The reduced row echelon form of this equation is

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \upsilon^1{}_{(2)} \\ \upsilon^2{}_{(2)} \\ \upsilon^3{}_{(2)} \end{bmatrix} = 0 \tag{5.3.36}$$

Therefore, (5.3.36) yields

$$\mathbf{v}_2 = \upsilon^2{}_{(2)} \left(-\mathbf{i}_1 + \mathbf{i}_2 \right) + \upsilon^3{}_{(2)} \left(\mathbf{i}_1 + \mathbf{i}_3 \right) \tag{5.3.37}$$

Therefore, the characteristic subspace associated with the second eigenvalue is the *two* dimensional subspace spanned by $-\mathbf{i}_1 + \mathbf{i}_2$ and $\mathbf{i}_1 + \mathbf{i}_3$. We can write this result as

$$\mathscr{V}\left(\lambda_2\right) = K\left(\mathbf{A} - \lambda_2 \mathbf{I}\right) = \operatorname{span}\left(-\mathbf{i}_1 + \mathbf{i}_2, \mathbf{i}_1 + \mathbf{i}_3\right) \tag{5.3.38}$$

For this eigenvalue, the geometric multiplicity is two as is the algebraic multiplicity. Because $\mathscr{V}(\lambda_2)$ is two dimensional, an eigenvector associated with the repeated eigenvalue is necessarily of the form

$$\mathbf{v}_2 = \alpha(-\mathbf{i}_1 + \mathbf{i}_2) + \beta(\mathbf{i}_1 + \mathbf{i}_3) \tag{5.3.39}$$

where α and β are arbitrary. We know that the defining equation for the eigenvalue problem, equation (5.1.4), will never determine the length of the eigenvector. In the repeated root case, equation (5.3.39) has an even greater indeterminacy. *Any* vector in the two dimensional subspace spanned by $\{-\mathbf{i}_1 + \mathbf{i}_2, \mathbf{i}_1 + \mathbf{i}_3\}$ is an eigenvector. Because the subspace is two dimensional, only two of these eigenvectors can be linearly independent. What is done for problems of this type is to select two of these eigenvectors which, when joined with the one for the distinct eigenvalue, form a basis for $\mathscr{V}$. Without loss of generality, one can simply take the eigenvectors associated with the repeated eigenvalue to be

$$\mathbf{v}_2 = -\mathbf{i}_1 + \mathbf{i}_2 \quad \text{and} \quad \mathbf{v}_3 = \mathbf{i}_1 + \mathbf{i}_3 \tag{5.3.40}$$

With the choice (5.3.40), and the choice

$$\mathbf{v}_1 = -\mathbf{i}_1 - \mathbf{i}_2 + \mathbf{i}_3 \tag{5.3.41}$$

the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for $\mathscr{V}$. With respect to this basis, the matrix of the linear transformation defined by (5.3.25) is

$$M(\mathbf{A}, \mathbf{v}_j, \mathbf{v}_k) = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \tag{5.3.42}$$

This example, which has two distinct eigenvalues, yields three linearly independent eigenvectors. However, unlike Example 5.3.1, the *directions* of two of the eigenvectors are *not unique*. A way to display that two of the eigenvectors are not unique is to write $\mathscr{V}$ as the direct sum of two subspaces, one of dimension 1 and one of dimension 2. The result is

$$\mathscr{V} = \mathscr{V}(\lambda_1) \oplus \mathscr{V}(\lambda_2) \tag{5.3.43}$$

Another feature of Example 5.2.2 is that the subspaces $\mathscr{V}(\lambda_1)$ and $\mathscr{V}(\lambda_2)$ are *orthogonal*. In other words $\mathscr{V}(\lambda_2) = \mathscr{V}(\lambda_1)^{\perp}$. This interesting result follows by calculating the inner products $\langle \mathbf{v}_1, \mathbf{v}_2 \rangle$ and $\langle \mathbf{v}_1, \mathbf{v}_3 \rangle$. It is a theorem that, for a self adjoint linear transformation, the characteristic subspaces of distinct eigenvalues are orthogonal. It is also a theorem that the eigenvalues of self adjoint linear transformations are necessarily real. We shall prove these results later. Given that the subspaces are orthogonal and that within $\mathscr{V}(\lambda_2)$ the eigenvectors are not unique, it is often the practice to select a basis for $\mathscr{V}(\lambda_2)$ that is mutually orthogonal. In other words, instead of the choices (5.3.40), select $\mathbf{v}_2$ and $\mathbf{v}_3$ such that $\langle \mathbf{v}_2, \mathbf{v}_3 \rangle = 0$.

Finally, it is a feature of Example 5.3.2 that the algebraic multiplicity of both roots equals their geometric multiplicity. This fact lies at the root of why it was possible to find a basis of eigenvectors. The next example is one where it is not possible to find such a basis.

Example 5.3.3: The problem is to solve the eigenvalue problem for the linear transformation $\mathbf{A}: \mathscr{V} \to \mathscr{V}$ defined by

$$\begin{aligned} \mathbf{A}\mathbf{e}_1 &= \mathbf{e}_1 \\ \mathbf{A}\mathbf{e}_2 &= \mathbf{e}_1 + \mathbf{e}_2 \\ \mathbf{A}\mathbf{e}_3 &= \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 \end{aligned} \tag{5.3.44}$$

where $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is a basis for $\mathscr{V}$. By the usual procedure the matrix of $\mathbf{A}$ with respect to this basis is

$$A = M\left(\mathbf{A}, \mathbf{e}_j, \mathbf{e}_k\right) = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \tag{5.3.45}$$

Step 1: Form the characteristic equation: The characteristic equation is defined by (5.2.5). Given the matrix (5.3.2), we find

$$\begin{aligned} f(\lambda) = \det(\mathbf{A} - \lambda \mathbf{I}) = \det\left(M(\mathbf{A} - \lambda \mathbf{I})\right) &= \begin{vmatrix} 1-\lambda & 1 & 1 \\ 0 & 1-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{vmatrix} \\ &= -(\lambda - 1)^3 \end{aligned} \tag{5.3.46}$$

Therefore, the eigenvalue is

$$\lambda_1 = 1 \tag{5.3.47}$$

and it has an algebraic multiplicity of 3.

Step2: Determine the characteristic subspaces. This calculation involves solving the defining equation (5.1.5), written

$$(\mathbf{A} - \lambda \mathbf{I})\mathbf{v} = \mathbf{0} \tag{5.3.48}$$

for the eigenvalue $\lambda = 1$. The matrix equation we must solve is

$$\begin{bmatrix} 1-\lambda & 1 & 1 \\ 0 & 1-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{bmatrix} \begin{bmatrix} \upsilon^1 \\ \upsilon^2 \\ \upsilon^3 \end{bmatrix} = 0 \tag{5.3.49}$$

Case 1: $\lambda = 1$. This choice reduces (5.3.49) to

$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \upsilon^1{}_{(1)} \\ \upsilon^2{}_{(1)} \\ \upsilon^3{}_{(1)} \end{bmatrix} = 0 \tag{5.3.50}$$

The reduced row echelon form of this equation is

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \upsilon^1{}_{(1)} \\ \upsilon^2{}_{(1)} \\ \upsilon^3{}_{(1)} \end{bmatrix} = 0 \tag{5.3.51}$$

Therefore, the characteristic subspace associated with the eigenvalue $\lambda = 1$ is the one dimensional subspace spanned by $\mathbf{e}_1$. We can write this result as

$$\mathscr{V}(\lambda_1) = K(\mathbf{A} - \lambda_1 \mathbf{I}) = \operatorname{span}(\mathbf{e}_1) \tag{5.3.52}$$

Unlike the two previous examples, Example 5.3.3 is a case where the geometric multiplicity of a characteristic subspace (1 in this case) is not the same as the algebraic multiplicity (3 in this case). Therefore, we are unable to construct a basis of eigenvectors. Thus, there is not a basis for the linear transformation that makes its matrix diagonal. As we shall see when we examine in greater detail the theoretical foundations of the eigenvalue problem, the condition that the geometric multiplicity and the algebraic multiplicity agree is a fundamental property of linear transformations that have a basis of eigenvectors. In circumstances where basis of eigenvectors does not exist, one can formulate a procedure for constructing what is known as the *Jordan Normal Form* of the matrix. This procedure will not be covered here.

Example 5.3.4: (Complex eigenvalues example.) The problem is to solve the eigenvalue problem for the linear transformation $\mathbf{A} : \mathscr{V} \to \mathscr{V}$ defined by

$$\begin{aligned}\mathbf{A}\mathbf{e}_1 &= \frac{\sqrt{2}}{2}\mathbf{e}_1 + \frac{\sqrt{2}}{2}\mathbf{e}_3 \\ \mathbf{A}\mathbf{e}_2 &= \mathbf{e}_2 \\ \mathbf{A}\mathbf{e}_3 &= -\frac{\sqrt{2}}{2}\mathbf{e}_1 + \frac{\sqrt{2}}{2}\mathbf{e}_3\end{aligned} \tag{5.3.53}$$

where $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is a basis for $\mathscr{V}$. By the usual procedure the matrix of $\mathbf{A}$ with respect to this basis is

$$A = M\left(\mathbf{A}, \mathbf{e}_j, \mathbf{e}_k\right) = \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \\ \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{bmatrix} \tag{5.3.54}$$

Step 1: Form the characteristic equation: The characteristic equation is defined by (5.2.5). Given the matrix (5.3.2), we find

$$\begin{aligned} f(\lambda) = \det(\mathbf{A} - \lambda\mathbf{I}) = \det\left(M(\mathbf{A} - \lambda\mathbf{I})\right) &= \begin{vmatrix} \frac{\sqrt{2}}{2} - \lambda & 0 & -\frac{\sqrt{2}}{2} \\ 0 & 1-\lambda & 0 \\ \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} - \lambda \end{vmatrix} \\ = (1-\lambda)\left(\left(\frac{\sqrt{2}}{2} - \lambda\right)^2 + \frac{1}{2}\right) &= (1-\lambda)\left(\frac{\sqrt{2}}{2}(1+i) - \lambda\right)\left(\frac{\sqrt{2}}{2}(1-i) - \lambda\right) \end{aligned} \tag{5.3.55}$$

Therefore, the three eigenvalues are

$$\begin{aligned}\lambda_1 &= \frac{\sqrt{2}}{2}(1+i) \\ \lambda_2 &= 1 \\ \lambda_3 &= \frac{\sqrt{2}}{2}(1-i)\end{aligned} \tag{5.3.56}$$

Because the eigenvalues are distinct, the algebraic multiplicity is 1 for each eigenvalue.

Step2: Determine the characteristic subspaces. This calculation again involves solving the defining equation (5.1.5), written

$$\left(\mathbf{A}-\lambda\mathbf{I}\right)\mathbf{v}=\mathbf{0} \tag{5.3.57}$$

for the three different eigenvalues. Of course, we shall solve (5.3.5) as the matrix equation

$$\begin{bmatrix} \frac{\sqrt{2}}{2}-\lambda & 0 & -\frac{\sqrt{2}}{2} \\ 0 & 1-\lambda & 0 \\ \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2}-\lambda \end{bmatrix}\begin{bmatrix} \upsilon^1 \\ \upsilon^2 \\ \upsilon^3 \end{bmatrix}=0 \tag{5.3.58}$$

Case 1: $\lambda_2=1$. This choice reduces (5.3.58) to

$$\begin{bmatrix} \frac{\sqrt{2}}{2}-1 & 0 & -\frac{\sqrt{2}}{2} \\ 0 & 0 & 0 \\ \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2}-1 \end{bmatrix}\begin{bmatrix} \upsilon^1{}_{(2)} \\ \upsilon^2{}_{(2)} \\ \upsilon^3{}_{(2)} \end{bmatrix}=0 \tag{5.3.59}$$

The reduced row echelon form of this equation is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \upsilon^1{}_{(2)} \\ \upsilon^2{}_{(2)} \\ \upsilon^3{}_{(2)} \end{bmatrix}=0 \tag{5.3.60}$$

Therefore, the characteristic subspace associated with the second eigenvalue is the one dimensional subspace spanned by $\mathbf{e}_2$. We can write this result as

$$\mathscr{V}\left(\lambda_2\right)=K\left(\mathbf{A}-\lambda_2\mathbf{I}\right)=\operatorname{span}\left(\mathbf{e}_2\right) \tag{5.3.61}$$

For this eigenvalue, the geometric multiplicity is one as is the algebraic multiplicity.

Case 2: $\lambda_1=\frac{\sqrt{2}}{2}\left(1+i\right)$. This choice reduces (5.3.58) to

$$\begin{bmatrix} -\frac{\sqrt{2}}{2}i & 0 & -\frac{\sqrt{2}}{2} \\ 0 & 1-\frac{\sqrt{2}}{2}(1+i) & 0 \\ \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2}i \end{bmatrix} \begin{bmatrix} \upsilon^1{}_{(1)} \\ \upsilon^2{}_{(1)} \\ \upsilon^3{}_{(1)} \end{bmatrix} = 0 \tag{5.3.62}$$

The reduced row echelon form of this equation is

$$\begin{bmatrix} 1 & 0 & -i \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \upsilon^1{}_{(1)} \\ \upsilon^2{}_{(1)} \\ \upsilon^3{}_{(1)} \end{bmatrix} = 0 \tag{5.3.63}$$

Therefore, the characteristic subspace associated with the second eigenvalue is the one dimensional subspace spanned by $i\mathbf{e}_1 + \mathbf{e}_3$. We can write this result as

$$\mathscr{V}(\lambda_1) = K(\mathbf{A} - \lambda_1 \mathbf{I}) = \operatorname{span}(i\mathbf{e}_1 + \mathbf{e}_3) \tag{5.3.64}$$

For this eigenvalue, the geometric multiplicity is again one as is the algebraic multiplicity.

Case 3: $\lambda_3 = \frac{\sqrt{2}}{2}(1-i)$. This choice reduces (5.3.58) to

$$\begin{bmatrix} \frac{\sqrt{2}}{2}i & 0 & -\frac{\sqrt{2}}{2} \\ 0 & 1-\frac{\sqrt{2}}{2}(1-i) & 0 \\ \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2}i \end{bmatrix} \begin{bmatrix} \upsilon^1{}_{(3)} \\ \upsilon^2{}_{(3)} \\ \upsilon^3{}_{(3)} \end{bmatrix} = 0 \tag{5.3.65}$$

The reduced row echelon form of this equation is

$$\begin{bmatrix} 1 & 0 & i \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \upsilon^1{}_{(3)} \\ \upsilon^2{}_{(3)} \\ \upsilon^3{}_{(3)} \end{bmatrix} = 0 \tag{5.3.66}$$

Therefore, the characteristic subspace associated with the third eigenvalue is the one dimensional subspace spanned by $-i\mathbf{e}_1+\mathbf{e}_3$. We can write this result as

$$\mathscr{V}\left(\lambda_3\right)=K\left(\mathbf{A}-\lambda_3\mathbf{I}\right)=\operatorname{span}\left(-i\mathbf{e}_1+\mathbf{e}_3\right) \tag{5.3.67}$$

For this eigenvalue, the geometric multiplicity is again one as is the algebraic multiplicity.

As with Example 5.3.1, each characteristic subspace is one dimensional and the three dimensional vector space $\mathscr{V}$ has the direct sum decomposition

$$\mathscr{V}=\mathscr{V}\left(\lambda_1\right)\oplus\mathscr{V}\left(\lambda_2\right)\oplus\mathscr{V}\left(\lambda_3\right) \tag{5.3.68}$$

If we select the three eigenvectors to be the spanning vectors introduced above, i.e.

$$\begin{aligned}\mathbf{v}_1&=i\mathbf{e}_1+\mathbf{e}_3\\ \mathbf{v}_2&=\mathbf{e}_1\\ \mathbf{v}_3&=-i\mathbf{e}_1+\mathbf{e}_3\end{aligned} \tag{5.3.69}$$

The matrix of $\mathbf{A}$ with respect to the matrix of eigenvectors is

$$M\left(\mathbf{A},\mathbf{v}_j,\mathbf{v}_k\right)=\begin{bmatrix}\frac{\sqrt{2}}{2}\left(1+i\right) & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & \frac{\sqrt{2}}{2}\left(1-i\right)\end{bmatrix} \tag{5.3.70}$$

The transition matrix for this problem is given by (5.3.23) which becomes

$$T=\begin{bmatrix}\upsilon^1{}_{(1)} & \upsilon^1{}_{(2)} & \upsilon^1{}_{(3)}\\ \upsilon^2{}_{(1)} & \upsilon^2{}_{(2)} & \upsilon^2{}_{(3)}\\ \upsilon^3{}_{(1)} & \upsilon^3{}_{(2)} & \upsilon^3{}_{(3)}\end{bmatrix}=\begin{bmatrix}i & 0 & -i\\ 0 & 1 & 0\\ 1 & 0 & 1\end{bmatrix} \tag{5.3.71}$$

The inverse of this matrix is

$$T^{-1}=\begin{bmatrix}-\frac{1}{2}i & 0 & \frac{1}{2}\\ 0 & 1 & 0\\ \frac{1}{2}i & 0 & \frac{1}{2}\end{bmatrix} \tag{5.3.72}$$

Given (5.3.54), (5.3.70), (5.3.71) and (5.3.72), equation (5.1.10) takes the explicit form

$$\underbrace{\begin{bmatrix} \frac{\sqrt{2}}{2}(1+i) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{\sqrt{2}}{2}(1-i) \end{bmatrix}}_{M(\mathbf{A},\mathbf{v}_j,\mathbf{v}_k)} = \underbrace{\begin{bmatrix} -\frac{1}{2}i & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2}i & 0 & \frac{1}{2} \end{bmatrix}}_{T^{-1}} \underbrace{\begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \\ \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{bmatrix}}_{M(\mathbf{A},\mathbf{e}_j,\mathbf{v}_k)} \underbrace{\begin{bmatrix} i & 0 & -i \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}}_{T} \tag{5.3.73}$$

Example 5.3.5: (A complicated numerical example.) The problem is to solve the eigenvalue problem for the matrix $A:\mathscr{M}^{6\times1} \to \mathscr{M}^{6\times1}$ defined by

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -2 & 1 & 0 & -.76 & .4 & .06 \\ 1 & -2 & 1 & .4 & -.82 & .4 \\ 0 & .5 & -1.5 & .03 & .2 & -.6 \end{bmatrix} \tag{5.3.74}$$

This particular matrix has its origins in the study of a three degree of freedom vibrations problem. The characteristic polynomial turns out to be

$$\det(A-\lambda I) = \lambda^6 + 1.8400\lambda^5 + 6.2922\lambda^4 + 5.4186\lambda^3 + 9.2928\lambda^2 + 3.0800\lambda + 3.5000 \tag{5.3.75}$$

The roots of this sixth order polynomial turn out to be

$$\begin{aligned} \left.\begin{matrix}\lambda_1\\ \lambda_2\end{matrix}\right\} &= -0.6072 \pm 1.6652i \\ \left.\begin{matrix}\lambda_3\\ \lambda_4\end{matrix}\right\} &= -0.2474 \pm 1.2611i \\ \left.\begin{matrix}\lambda_5\\ \lambda_6\end{matrix}\right\} &= -0.0654 \pm 0.8187i \end{aligned} \tag{5.3.76}$$

Because the roots are distinct, the algebraic multiplicity of each root is one and, because of (5.2.28) , the geometric multiplicity of each eigenvalue is one. The eigenvectors associated with these eigenvalues turn out to be

$$\mathbf{v}_1 = \begin{bmatrix} -0.1130 - 0.2939i \\ 0.5581 - 0.0097i \\ 0.1252 + 0.3433i \\ -0.6478 \\ -0.0599 - 0.0717i \\ 0.1557 - 0.0563i \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -0.1130 + 0.2939i \\ 0.5581 + 0.0097i \\ 0.1252 - 0.3433i \\ -0.6478 \\ -0.0599 + 0.0717i \\ 0.1557 + 0.0563i \end{bmatrix} \tag{5.3.77}$$

$$\mathbf{v}_3 = \begin{bmatrix} 0.0834 + 0.4253i \\ -0.5570 \\ -0.0517 + 0.2001i \\ -0.2396 - 0.1147i \\ -0.1491 - 0.3526i \\ 0.4815 - 0.1009i \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} 0.0834 - 0.4253i \\ -0.5570 \\ -0.0517 - 0.2001i \\ -0.2396 + 0.1147i \\ -0.1491 + 0.3526i \\ 0.4815 + 0.1009i \end{bmatrix} \tag{5.3.78}$$

and

$$\mathbf{v}_5 = \begin{bmatrix} 0.4112 - 0.0111i \\ -0.0178 + 0.3374i \\ 0.5535 \\ -0.0362 + 0.4532i \\ 0.3395 + 0.0793i \\ -0.0872 + 0.2728i \end{bmatrix}, \mathbf{v}_6 = \begin{bmatrix} 0.4112 + 0.0111i \\ -0.0178 - 0.3374i \\ 0.5535 \\ -0.0362 - 0.4532i \\ 0.3395 - 0.0793i \\ -0.0872 - 0.2728i \end{bmatrix} \tag{5.3.79}$$

where the six eigenvectors have been normalized to each have a unit length.

If we return to the characteristic polynomial (5.3.75) and compare that formula with the general result (5.2.14), we can identify the six fundamental invariants for this problem. The results are

$$\begin{aligned} \mu_1 &= -1.8400 \\ \mu_2 &= 6.2922 \\ \mu_3 &= -5.4186 \\ \mu_4 &= 9.2928 \\ \mu_5 &= -3.0800 \\ \mu_6 &= 3.5000 \end{aligned} \tag{5.3.80}$$

Because (5.2.16) holds for linear transformations of arbitrary finite dimension, it can be used to confirm $(5.3.80)_1$. This same number can be calculated from the formula

$$\mu_1 = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 \tag{5.3.81}$$

The last invariant, μ_6 can be checked against the general formula

$$\mu_6 = \det A = \lambda_1 \lambda_2 \lambda_3 \lambda_4 \lambda_5 \lambda_6 \tag{5.3.82}$$

Exercises

5.3.1 Determine the eigenvalues and the characteristic subspaces of the linear transformation $\mathbf{A} : \mathscr{V} \to \mathscr{V}$ defined by

$$\begin{aligned} \mathbf{A}\mathbf{i}_1 &= 6\mathbf{i}_1 - 2\mathbf{i}_2 \\ \mathbf{A}\mathbf{i}_2 &= -2\mathbf{i}_1 + 6\mathbf{i}_2 \end{aligned} \tag{5.3.83}$$

where $\{\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3\}$ is an *orthonormal* basis for $\mathscr{V}$.

5.3.2 Determine the eigenvalues and the characteristic subspaces of the linear transformation $\mathbf{A} : \mathscr{V} \to \mathscr{V}$ defined by

$$\begin{aligned} \mathbf{A}\mathbf{i}_1 &= \frac{\sqrt{3}}{2}\mathbf{i}_1 - \frac{1}{2}\mathbf{i}_2 \\ \mathbf{A}\mathbf{i}_2 &= \frac{1}{2}\mathbf{i}_1 + \frac{\sqrt{3}}{2}\mathbf{i}_2 \end{aligned} \tag{5.3.84}$$

where $\{\mathbf{i}_1, \mathbf{i}_2\}$ is an *orthonormal* basis for $\mathscr{V}$.

5.3.3 Determine the eigenvalues and the characteristic subspaces of the linear transformation $\mathbf{A} : \mathscr{V} \to \mathscr{V}$ defined by

$$\mathbf{A}\mathbf{v} = \upsilon^1\left(\frac{5}{3}\mathbf{i}_1 - \frac{1}{3}\mathbf{i}_2 + \frac{1}{3}\mathbf{i}_3\right) + \upsilon^2\left(-\frac{1}{3}\mathbf{i}_1 + \frac{5}{3}\mathbf{i}_2 + \frac{1}{3}\mathbf{i}_3\right) + \upsilon^3\left(\frac{1}{3}\mathbf{i}_1 + \frac{1}{3}\mathbf{i}_2 + \frac{5}{3}\mathbf{i}_3\right) \tag{5.3.85}$$

for all vectors $\mathbf{v} = \upsilon^1\mathbf{i}_1 + \upsilon^2\mathbf{i}_2 + \upsilon^3\mathbf{i}_3$, where $\mathscr{V}$ is an inner product space and $\{\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3\}$ is an orthonormal basis of $\mathscr{V}$.

5.3.4 Determine the eigenvalues and the characteristic subspaces of the linear transformation $\mathbf{A} : \mathscr{V} \to \mathscr{V}$ defined by

$$\mathbf{A}\mathbf{v} = \upsilon^1\left(2\mathbf{i}_1 - \mathbf{i}_2\right) + \upsilon^2\left(\mathbf{i}_3\right) + \upsilon^3\left(-6\mathbf{i}_1 - 3\mathbf{i}_2 - 2\mathbf{i}_3\right) \tag{5.3.86}$$

for all vectors $\mathbf{v} = \upsilon^1\mathbf{i}_1 + \upsilon^2\mathbf{i}_2 + \upsilon^3\mathbf{i}_3$, where $\mathscr{V}$ is an inner product space and $\{\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3\}$ is an orthonormal basis of $\mathscr{V}$.

5.3.5 Determine the eigenvalues and the characteristic subspaces of the linear transformation $\mathbf{A}:\mathscr{V} \to \mathscr{V}$ defined by

$$\mathbf{A}\mathbf{v} = \upsilon^1\left(4\mathbf{i}_1 + \mathbf{i}_2\right) + \upsilon^2\left(-5\mathbf{i}_1 + \mathbf{i}_3\right) + \upsilon^3\left(\mathbf{i}_1 - \mathbf{i}_2 - \mathbf{i}_3\right) \tag{5.3.87}$$

for all vectors $\mathbf{v} = \upsilon^1\mathbf{i}_1 + \upsilon^2\mathbf{i}_2 + \upsilon^3\mathbf{i}_3$, where $\mathscr{V}$ is an inner product space and $\{\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3\}$ is an orthonormal basis of $\mathscr{V}$.

Section 5.4. Some General Theorems for the Eigenvalue Problem

The five examples in Section 5.3 illustrated certain features of the eigenvalue problem that are more general than the examples might indicate. In this section, we shall collect some of these general results. The first is the important one discussed after Example 5.3.1. Namely, that distinct eigenvalues imply linearly independent eigenvectors. The formal statement is the following theorem.

Theorem 5.4.1: If $\lambda_1,...,\lambda_L$ are distinct eigenvalues of $\mathbf{A}:\mathscr{V}\rightarrow\mathscr{V}$ and if $\mathbf{v}_1,...,\mathbf{v}_L$ are the corresponding eigenvectors, then $\left\{\mathbf{v}_1,...,\mathbf{v}_L\right\}$ is a linearly independent set.

Proof: Let $\mathbf{v}_1$ be an eigenvector associated with the eigenvalue λ_1 and, so forth. As with any question of linear independence, we need to examine the implication of the equation

$$\alpha_1\mathbf{v}_1+\alpha_2\mathbf{v}_2+\alpha_3\mathbf{v}_3+\cdots+\alpha_L\mathbf{v}_L=\mathbf{0} \tag{5.4.1}$$

If we can establish that this equation implies $\alpha_1=\alpha_2=\alpha_3=\cdots=\alpha_L=0$, then the theorem will be established. Given that $\mathbf{A}\mathbf{v}_1=\lambda_1\mathbf{v}_1,\mathbf{A}\mathbf{v}_2=\lambda_2\mathbf{v}_2,\mathbf{A}\mathbf{v}_3=\lambda_3\mathbf{v}_3,...,\mathbf{A}\mathbf{v}_L=\lambda_L\mathbf{v}_L$, we can multiply (5.4.1) by $\mathbf{A}$ and obtain

$$\lambda_1\alpha_1\mathbf{v}_1+\lambda_2\alpha_2\mathbf{v}_2+\lambda_3\alpha_3\mathbf{v}_3+\cdots+\lambda_L\alpha_L\mathbf{v}_L=\mathbf{0} \tag{5.4.2}$$

We can repeat this process $L-2$ more times until we obtain the following L equations (including the two above):

$$\begin{aligned}
\alpha_1\mathbf{v}_1+\alpha_2\mathbf{v}_2+\alpha_3\mathbf{v}_3+\cdots+\alpha_L\mathbf{v}_L&=\mathbf{0}\\
\lambda_1\alpha_1\mathbf{v}_1+\lambda_2\alpha_2\mathbf{v}_2+\lambda_3\alpha_3\mathbf{v}_3+\cdots+\lambda_L\alpha_L\mathbf{v}_L&=\mathbf{0}\\
\lambda_1^2\alpha_1\mathbf{v}_1+\lambda_2^2\alpha_2\mathbf{v}_2+\lambda_3^2\alpha_3\mathbf{v}_3+\cdots+\lambda_L^2\alpha_L\mathbf{v}_L&=\mathbf{0}\\
\lambda_1^3\alpha_1\mathbf{v}_1+\lambda_2^3\alpha_2\mathbf{v}_2+\lambda_3^3\alpha_3\mathbf{v}_3+\cdots+\lambda_L^3\alpha_L\mathbf{v}_L&=\mathbf{0}\\
&\cdot\\
&\cdot\\
&\cdot\\
\lambda_1^{L-1}\alpha_1\mathbf{v}_1+\lambda_2^{L-1}\alpha_2\mathbf{v}_2+\lambda_3^{L-1}\alpha_3\mathbf{v}_3+\cdots+\lambda_L^{L-1}\alpha_L\mathbf{v}_L&=\mathbf{0}
\end{aligned} \tag{5.4.3}$$

If we select a basis for $\mathscr{V}$, equations (5.4.3) can be rewritten as the matrix equation

$$\begin{bmatrix} 1 & 1 & 1 & \cdot & \cdot & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 & \cdot & \cdot & \lambda_L \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & & & \lambda_L^2 \\ \cdot & \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & & \cdot & \\ \lambda_1^{L-1} & \lambda_2^{L-1} & \lambda_3^{L-1} & \cdot & \cdot & \lambda_L^{L-1} \end{bmatrix} \begin{bmatrix} \alpha_1 \upsilon^1{}_{(1)} & \alpha_2 \upsilon^1{}_{(2)} & \alpha_3 \upsilon^1{}_{(3)} & \cdot & \cdot & \alpha_L \upsilon^1{}_{(L)} \\ \alpha_1 \upsilon^2{}_{(1)} & \alpha_2 \upsilon^2{}_{(2)} & \alpha_3 \upsilon^2{}_{(3)} & \cdot & \cdot & \alpha_L \upsilon^2{}_{(L)} \\ \alpha_1 \upsilon^3{}_{(1)} & \alpha_2 \upsilon^3{}_{(2)} & \alpha_3 \upsilon^2{}_{(3)} & & & \alpha_L \upsilon^3{}_{(L)} \\ \cdot & \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & & \cdot & \\ \alpha_1 \upsilon^L{}_{(1)} & \alpha_2 \upsilon^L{}_{(2)} & \alpha_3 \upsilon^L{}_{(3)} & \cdot & \cdot & \alpha_L \upsilon^L{}_{(L)} \end{bmatrix} = 0 \quad (5.4.4)$$

The coefficient matrix in (5.4.4) is the Vandermonde matrix introduced in equation (1.10.33). This matrix is nonsingular because, as shown with equation (1.10.34),

$$\det \begin{bmatrix} 1 & 1 & 1 & \cdot & \cdot & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 & \cdot & \cdot & \lambda_L \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & & & \lambda_L^2 \\ \cdot & \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & & \cdot & \\ \lambda_1^{L-1} & \lambda_2^{L-1} & \lambda_3^{L-1} & \cdot & \cdot & \lambda_L^{L-1} \end{bmatrix} = \prod_{\substack{i,j=1 \\ i>j}}^{L} \left(\lambda_i - \lambda_j \right) \quad (5.4.5)$$

and we have postulated that the eigenvalues are distinct. Given that the coefficient matrix in nonsingular, we can multiply (5.4.4) by its inverse and obtain

$$\begin{bmatrix} \alpha_1 \upsilon^1{}_{(1)} & \alpha_2 \upsilon^1{}_{(2)} & \alpha_3 \upsilon^1{}_{(3)} & \cdot & \cdot & \alpha_L \upsilon^1{}_{(L)} \\ \alpha_1 \upsilon^2{}_{(1)} & \alpha_2 \upsilon^2{}_{(2)} & \alpha_3 \upsilon^2{}_{(3)} & \cdot & \cdot & \alpha_L \upsilon^2{}_{(L)} \\ \alpha_1 \upsilon^3{}_{(1)} & \alpha_2 \upsilon^3{}_{(2)} & \alpha_3 \upsilon^2{}_{(3)} & & & \alpha_L \upsilon^3{}_{(L)} \\ \cdot & \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & & \cdot & \\ \alpha_1 \upsilon^L{}_{(1)} & \alpha_2 \upsilon^L{}_{(2)} & \alpha_3 \upsilon^L{}_{(3)} & \cdot & \cdot & \alpha_L \upsilon^L{}_{(L)} \end{bmatrix} = 0 \quad (5.4.6)$$

In order for this $L \times L$ matrix to be zero, each of its L^2 elements must be zero. The L equations in the first column are

$$\alpha_1 \begin{bmatrix} \upsilon^1{}_{(1)} \\ \upsilon^2{}_{(1)} \\ \upsilon^3{}_{(1)} \\ \cdot \\ \cdot \\ \upsilon^L{}_{(1)} \end{bmatrix} = 0 \quad (5.4.7)$$

Because the eigenvector is nonzero, the column matrix of its components is nonzero. Thus, we conclude that $\alpha_1 = 0$. Identical arguments applied to each column yield $\alpha_1 = \alpha_2 = \cdots = \alpha_L = 0$ and the theorem is proven.

This theorem explains why in Example 5.3.1, the eigenvectors were linearly independent. It also explains why the characteristic spaces were all one dimensional. If one of the characteristic subspaces has a dimension greater than one, we would be able to construct a set of linearly independent vectors with more than N members in an N dimensional vector space. Such a construction would be a contradiction. Recall that it was explained, without proof, in Section 5.2 that *geometric multiplicity is always less than or equal to the algebraic multiplicity*. In Example 5.2.1, the algebraic multiplicity of each eigenvalue is one and, consequently, the geometric multiplicity had to be equal to one.

As a corollary to the preceding theorem, we see that if the geometric multiplicity is equal to the algebraic multiplicity for each eigenvalue of $\mathbf{A}$, then the vector space $\mathscr{V}$ admits the direct sum representation

$$\mathscr{V} = \mathscr{V}(\lambda_1) \oplus \mathscr{V}(\lambda_2) \oplus \cdots \oplus \mathscr{V}(\lambda_L) \tag{5.4.8}$$

where $\lambda_1, ..., \lambda_L$ are the distinct eigenvalues of $\mathbf{A}$. The reason for this representation is obvious, since the right-hand side of the above equation is a subspace having the same dimension as $\mathscr{V}$; thus that subspace is equal to $\mathscr{V}$. Whenever the representation holds, we can always choose a basis of $\mathscr{V}$ formed by bases of the subspaces $\mathscr{V}(\lambda_1), ..., \mathscr{V}(\lambda_L)$. Then this basis consists entirely of eigenvectors of $\mathbf{A}$ becomes a diagonal matrix, namely,

$$M(\mathbf{A}, \mathbf{v}_j, \mathbf{v}_k) = \begin{bmatrix}
\lambda_1 &&&&&&&&& \\
& \ddots &&&&&&&& \\
&& \lambda_1 &&&&&&& \\
&&& \lambda_2 &&&&& 0 & \\
&&&& \ddots &&&&& \\
&&&&& \lambda_2 &&&& \\
&&&&&& \ddots &&& \\
&&&&&&& \lambda_L && \\
& 0 &&&&&&& \ddots & \\
&&&&&&&&& \lambda_L
\end{bmatrix} \tag{5.4.9}$$

where each λ_K is repeated d_k times, d_k being the algebraic as well as the geometric multiplicity of λ_K. Of course, the representation of $\mathscr{V}$ by direct sum of eigenspaces of $\mathbf{A}$ is possible if $\mathbf{A}$ has $N = \dim \mathscr{V}$ distinct eigenvalues. In this case the matrix of $\mathbf{A}$ taken with respect to a basis of eigenvectors has the diagonal form

$$M(\mathbf{A}, \mathbf{v}_j, \mathbf{v}_k) = \begin{bmatrix} \lambda_1 & & & & & & 0 \\ & \lambda_2 & & & & & \\ & & \lambda_3 & & & & \\ & & & \cdot & & & \\ & & & & \cdot & & \\ & 0 & & & & \cdot & \\ & & & & & & \lambda_N \end{bmatrix} \tag{5.4.10}$$

If the eigenvalues of $\mathbf{v}$ are not all distinct, then in general the geometric multiplicity of an eigenvalue may be less than the algebraic multiplicity. Whenever the two multiplicities are different for at least one eigenvalue of $\mathbf{A}$, it is no longer possible to find any basis in which the matrix of $\mathbf{A}$ is diagonal. However, if $\mathscr{V}$ is an inner product space and if $\mathbf{A}$ is Hermitian, then a diagonal matrix of $\mathbf{A}$ can always be found; we shall now investigate this problem.

Recall from Section 4.9 that if $\mathbf{u}$ and $\mathbf{v}$ are arbitrary vectors in $\mathscr{V}$, the adjoint $\mathbf{A}^*$ of $\mathbf{A}: \mathscr{V} \to \mathscr{V}$ is a linear transformation $\mathbf{A}^*: \mathscr{V} \to \mathscr{V}$ defined by

$$\langle \mathbf{u}, \mathbf{A}\mathbf{v} \rangle = \langle \mathbf{A}^*\mathbf{u}, \mathbf{v} \rangle \tag{5.4.11}$$

From Theorem 4.9.1, if $\mathbf{A}$ Hermitian, i.e., if $\mathbf{A} = \mathbf{A}^*$, then (5.4.11) reduces to

$$\langle \mathbf{u}, \mathbf{A}\mathbf{v} \rangle = \langle \mathbf{A}\mathbf{u}, \mathbf{v} \rangle \tag{5.4.12}$$

for all $\mathbf{u}, \mathbf{v} \in \mathscr{V}$.

Theorem 5.4.2. The eigenvalues of a Hermitian linear transformation $\mathbf{A}: \mathscr{V} \to \mathscr{V}$ are real.

Proof. Let $\mathbf{A}: \mathscr{V} \to \mathscr{V}$ be Hermitian. Since $\mathbf{A}\mathbf{v} = \lambda \mathbf{v}$ for every eigenvalue λ, we have

$$\lambda = \frac{\langle \mathbf{A}\mathbf{v}, \mathbf{v} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle} \tag{5.4.13}$$

Therefore we must show that $\langle \mathbf{A}\mathbf{v}, \mathbf{v} \rangle$ is real or, equivalently, we must show $\langle \mathbf{A}\mathbf{v}, \mathbf{v} \rangle = \overline{\langle \mathbf{A}\mathbf{v}, \mathbf{v} \rangle}$. This result follows from the definition of a Hermitian linear transformation (5.4.12) and the rearrangement

$$\langle \mathbf{Av}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{Av} \rangle = \overline{\langle \mathbf{Av}, \mathbf{v} \rangle} \tag{5.4.14}$$

where the rule $\langle \mathbf{u}, \mathbf{v} \rangle = \overline{\langle \mathbf{v}, \mathbf{u} \rangle}$ in the definition of an inner product has been used. Equation (5.4.14) yields the desired result.

Theorem 5.4.3. If $\mathbf{A}$ is Hermitian, the characteristic subspaces corresponding to distinct eigenvalues λ_1 and λ_2 are orthogonal.

Proof. Let $\mathbf{Av}_1 = \lambda_1 \mathbf{v}_1$ and $\mathbf{Av}_2 = \lambda_2 \mathbf{v}_2$. Then

$$\lambda_1 \langle \mathbf{v}_1, \mathbf{v}_2 \rangle = \langle \mathbf{Av}_1, \mathbf{v}_2 \rangle = \langle \mathbf{v}_1, \mathbf{Av}_2 \rangle = \lambda_2 \langle \mathbf{v}_1, \mathbf{v}_2 \rangle \tag{5.4.15}$$

Therefore,

$$(\lambda_1 - \lambda_2) \langle \mathbf{v}_1, \mathbf{v}_2 \rangle = 0 \tag{5.4.16}$$

Since $\lambda_1 \neq \lambda_2$, $\langle \mathbf{v}_1 \cdot \mathbf{v}_2 \rangle = 0$, which proves the theorem.

Another important property of Hermitian linear transformations is that *the algebraic multiplicity of each eigenvalue equa1s the geometric multiplicity*. We shall use this result here, however, we shall not give its proof.[5]

The main theorem regarding Hermitian linear transformations $\mathbf{A} : \mathscr{V} \to \mathscr{V}$ is the following.

Theorem 5.4.4. If $\mathbf{A} : \mathscr{V} \to \mathscr{V}$ is a Hermitian linear transformation with distinct eigenvalues $\lambda_1, \lambda_2, ..., \lambda_L$, then $\mathscr{V}$ has the representation

$$\mathscr{V} = \mathscr{V}_1(\lambda_1) \oplus \mathscr{V}(\lambda_2) \oplus \cdots \oplus \mathscr{V}(\lambda_L) \tag{5.4.17}$$

where the eigenspaces $\mathscr{V}(\lambda_k)$ are mutually orthogonal.

The proof of this theorem follows from (5.4.8) which holds when the geometric multiplicity and the algebraic multiplicity are the same and from Theorem 5.4.3.

Another Corollary of Theorem 5.4.3 is that if $\mathbf{A}$ is Hermitian, there exists an orthogonal basis for $\mathscr{V}$ consisting entirely of eigenvectors of $\mathbf{A}$. This feature was observed in our solution to Example 5.3.2. This corollary is clear because each characteristic subspace is orthogonal to the others and within each characteristic subspace an orthogonal basis can be constructed by the Gram-Schmidt procedure discussed in Section 4.3. With respect to this basis of eigenvectors, the matrix of $\mathbf{A}$ is

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[5] The proof can be found in Introduction to Vectors and Tensors, Vol. 1, by Ray M. Bowen and C.-C. Wang.

diagonal. Thus the problem of finding a basis for $\mathscr{V}$ such that $M(\mathbf{A},\mathbf{v}_j,\mathbf{v}_k)$ is diagonal is solved for Hermitian linear transformations.

The Hermitian property causes the eigenvalues to be real and the characteristic subspaces for distinct eigenvalues to be orthogonal. Often in the applications, Hermitian linear transformations have *additional properties* that are important to define. In particular, a Hermitian linear transformation $\mathbf{A}:\mathscr{V}\rightarrow\mathscr{V}$ is defined to be

$$\left\{\begin{array}{l}\text{positive definite}\\ \text{positive semidefinite}\\ \text{negative semidefinite}\\ \text{negative definite}\end{array}\right\} \text{ if } \quad \langle \mathbf{v},\mathbf{A}\mathbf{v}\rangle \left\{\begin{array}{l}>0\\ \geq 0\\ \leq 0\\ <0\end{array}\right\} \tag{5.4.18}$$

all nonzero $\mathbf{v}$, Equation (5.4.18) places conditions on the eigenvalues of a Hermitian linear transformations $\mathbf{A}:\mathscr{V}\rightarrow\mathscr{V}$ that are important to understand. We shall next derive these conditions. Because (5.4.18) holds for all nonzero vectors in $\mathscr{V}$, we can select $\mathbf{v}$ to be an eigenvector of $\mathbf{A}$ and use (5.4.13) to conclude

$$\left\{\begin{array}{l}\text{positive definite}\\ \text{positive semidefinite}\\ \text{negative semidefinite}\\ \text{negative definite}\end{array}\right\} \text{ implies } \quad \lambda \left\{\begin{array}{l}>0\\ \geq 0\\ \leq 0\\ <0\end{array}\right\} \tag{5.4.19}$$

In addition, itt follows from (5.4.17) and the definition of direct sum that any vector $\mathbf{v}\in\mathscr{V}$ can be uniquely written

$$\mathbf{v}=\sum_{j=1}^{L}\mathbf{v}_j \tag{5.4.20}$$

where each $\mathbf{v}_j\in\mathscr{V}_j$. Given $\mathbf{A}\mathbf{v}_j=\lambda_j\mathbf{v}_j$, it follows that

$$\langle \mathbf{v},\mathbf{A}\mathbf{v}\rangle=\left\langle \sum_{j=1}^{L}\mathbf{v}_j,\sum_{k=1}^{L}\mathbf{A}\mathbf{v}_k\right\rangle=\sum_{j,k=1}^{L}\lambda_j\left\langle \mathbf{v}_j,\mathbf{v}_k\right\rangle=\sum_{j=1}^{L}\lambda_j\left\|\mathbf{v}_j\right\|^2 \tag{5.4.21}$$

Equation (5.4.21) allows the definition (5.4.18) to be replaced by

$$\left\{\begin{array}{l}\text{positive definite}\\ \text{positive semidefinite}\\ \text{negative semidefinite}\\ \text{negative definite}\end{array}\right\} \text{ if } \quad \sum_{j=1}^{L}\lambda_j\left\|\mathbf{v}_j\right\|^2 \left\{\begin{array}{l}>0\\ \geq 0\\ \leq 0\\ <0\end{array}\right\} \tag{5.4.22}$$

Equation (5.4.22) shows that if all of the eigenvalues obey $\lambda \begin{Bmatrix} >0 \\ \geq 0 \\ \leq 0 \\ <0 \end{Bmatrix}$, then $\mathbf{A}:\mathscr{V}\rightarrow\mathscr{V}$ is $\begin{Bmatrix} \text{positive definite} \\ \text{positive semidefinite} \\ \text{negative semidefinite} \\ \text{negative definite} \end{Bmatrix}$. Therefore, we have established the following theorem

Theorem 5.4.5 A Hermitian linear transformation $\mathbf{A}:\mathscr{V}\rightarrow\mathscr{V}$ is $\begin{Bmatrix} \text{positive definite} \\ \text{positive semidefinite} \\ \text{negative semidefinite} \\ \text{negative definite} \end{Bmatrix}$ if and only if every eigenvalue of $\mathbf{A}$ is $\begin{Bmatrix} >0 \\ \geq 0 \\ \leq 0 \\ <0 \end{Bmatrix}$

As corollaries to Theorem 5.4.5 it follows that positive-definite and negative-definite Hermitian linear transformations are one to one and, as a consequence, nonsingular.

Positive definite linear transformations occur frequently in the applications. It is often that case that one need the ability to confirm the definiteness property but without going to the trouble of actually calculating the eigenvalues. For this reason, it is important to have criteria that establish when a Hermitian linear transformation is positive definite without actually solving the associated eigenvalue problem. There are tests that can be applied to the linear transformation that achieve this purpose. One of these is as follows:

Theorem 5.4.6: A Hermitian liner transformation $\mathbf{A}:\mathscr{V}\rightarrow\mathscr{V}$ is positive definite if and only if its fundamental invariants $\mu_1,\mu_2,...,\mu_N$ are greater than zero.

Proof: The proof of this result follows from Theorem 5.4.5, the characteristic polynomial (5.2.14), repeated,

$$\det\left(\mathbf{A}-\lambda\mathbf{I}\right)=\left(-\lambda\right)^N+\mu_1\left(-\lambda\right)^{N-1}+\cdots+\mu_{N-1}\left(-\lambda\right)+\mu_N \tag{5.4.23}$$

and the formulas that connect the eigenvalues to the fundamental invariants like (5.2.20) (for the $N=3$ case). These formulas prove one part of the theorem. Namely, if the eigenvalues are greater than to zero, the fundamental invariants are greater than to zero.

The reverse implication follows from (5.4.23). The eigenvalues are the roots of the characteristic equation, i.e., the roots of

$$(-\lambda)^N + \mu_1(-\lambda)^{N-1} + \cdots + \mu_{N-1}(-\lambda) + \mu_N = 0 \tag{5.4.24}$$

If the fundamental invariants are greater than zero, roots less than or equal to zero are not allowed by (5.4.24) because the left side cannot add to zero. If the roots cannot be negative or zero, they must be positive.

Theorem 5.4.6 is also valid is you replace positive definite with positive semidefinite and replace the condition on the fundamental invariants with the requirement that they be nonnegative.

If we adopt an orthonormal basis $\{\mathbf{i}_1, \mathbf{i}_2, ..., \mathbf{i}_N\}$, we can display the matrix of the Hermitian linear transformation $\mathbf{A}: \mathscr{V} \to \mathscr{V}$ by the formula

$$M\left(\mathbf{A}, \mathbf{i}_j, \mathbf{i}_k\right) = \begin{bmatrix} A_{11} & A_{12} & A_{13} & \cdot & \cdot & A_{1N} \\ \overline{A_{12}} & A_{22} & A_{23} & & & A_{2N} \\ \overline{A_{13}} & \overline{A_{23}} & A_{33} & & & A_{3N} \\ \cdot & & & \cdot & & \cdot \\ \cdot & & & & \cdot & \cdot \\ \overline{A_{1N}} & \overline{A_{2N}} & \overline{A_{3N}} & \cdot & \cdot & A_{NN} \end{bmatrix} \tag{5.4.25}$$

The second characterization of positive definiteness is a criterion known as *Sylvester's criterion*. This criterion says that a Hermitian liner transformation $\mathbf{A}: \mathscr{V} \to \mathscr{V}$ is positive definite if and only if its matrix with respect to an orthonormal basis is such that its N *leading principal minors are positive*. The leading principal minors of the matrix (5.4.25) are the N determinants formed from the upper left 1×1 corner, the upper left 2×2 corner, the upper left 3×3 corner and so forth. More explicitly, the Hermitian linear transformation $\mathbf{A}: \mathscr{V} \to \mathscr{V}$ is positive definite if and only if

$$
\begin{aligned}
&A_{11} > 0 \\
&\det\begin{vmatrix} A_{11} & A_{12} \\ \overline{A_{12}} & A_{22} \end{vmatrix} > 0 \\
&\det\begin{vmatrix} A_{12} & A_{12} & A_{13} \\ \overline{A_{12}} & A_{22} & A_{23} \\ \overline{A_{13}} & \overline{A_{23}} & A_{33} \end{vmatrix} > 0 \\
&\cdot \\
&\cdot \\
&\cdot \\
&\det\begin{vmatrix} A_{11} & A_{12} & A_{13} & \cdot & \cdot & A_{1N} \\ \overline{A_{12}} & A_{22} & A_{23} & & & A_{2N} \\ \overline{A_{13}} & \overline{A_{23}} & A_{33} & & & A_{3N} \\ \cdot & & & \cdot & & \cdot \\ \cdot & & & & \cdot & \cdot \\ \overline{A_{1N}} & \overline{A_{2N}} & \overline{A_{3N}} & \cdot & \cdot & A_{NN} \end{vmatrix} > 0
\end{aligned}
\tag{5.4.26}
$$

The proof of Sylvester's criteria will not be given here.[6] However, its essential features will be outlined in the exercises below…

Exercises

5.4.1 In Exercise 4.11.1 it was pointed out that when we are given a linear transformation $\mathbf{A}:\mathscr{V}\rightarrow\mathscr{U}$, the linear transformations $\mathbf{AA}^*:\mathscr{U}\rightarrow\mathscr{U}$ and the linear transformation $\mathbf{A}^*\mathbf{A}:\mathscr{V}\rightarrow\mathscr{V}$ are Hermitian. Show that they are also positive semidefinite. As positive semidefinite linear transformations, their invariants are greater than or equal to zero. This fact provides an alternate proof that the definition (4.9.34) does obey the fourth property of an inner product.

5.4.2 Show that for a positive definite Hermitian linear transformation $\mathbf{A}:\mathscr{V}\rightarrow\mathscr{V}$ displayed as the matrix

[6] An elementary proof for symmetric matrices can be found on pages 72-74 of Introduction to Matrix Analysis by Richard Bellman, McGraw-Hill, New York, 1960. A more advanced proof plus additional references can be found at http://en.wikipedia.org/wiki/Sylvester's_criterion.

$$M\left(\mathbf{A},\mathbf{i}_j,\mathbf{i}_k\right)=\begin{bmatrix} A_{11} & A_{12} & A_{13} & \cdot & \cdot & A_{1N} \\ \overline{A_{12}} & A_{22} & A_{23} & & & A_{2N} \\ \overline{A_{13}} & \overline{A_{23}} & A_{33} & & & A_{3N} \\ \cdot & & & \cdot & & \cdot \\ \cdot & & & & \cdot & \cdot \\ \overline{A_{1N}} & \overline{A_{2N}} & \overline{A_{3N}} & \cdot & \cdot & A_{NN} \end{bmatrix} \tag{5.4.27}$$

the leading submatrices $\left[A_{11}\right],\begin{bmatrix} A_{11} & A_{12} \\ \overline{A_{12}} & A_{22}\end{bmatrix},\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ \overline{A_{12}} & A_{22} & A_{23} \\ \overline{A_{13}} & \overline{A_{23}} & A_{33}\end{bmatrix},\ldots,\begin{bmatrix} A_{11} & A_{12} & A_{13} & \cdot & \cdot & A_{1N} \\ \overline{A_{12}} & A_{22} & A_{23} & & & A_{2N} \\ \overline{A_{13}} & \overline{A_{23}} & A_{33} & & & A_{3N} \\ \cdot & & & \cdot & & \cdot \\ \cdot & & & & \cdot & \cdot \\ \overline{A_{1N}} & \overline{A_{2N}} & \overline{A_{3N}} & \cdot & \cdot & A_{NN} \end{bmatrix}$ are all positive definite. As positive definite matrices, their fundamental invariants must be positive. Each matrix has the number of fundamental invariants equal to its dimension. Included in the list of invariants for each matrix is its determinant. Thus, we have established that if the matrix (5.4.27) is positive definite, Sylvester's criterion holds. The converse, namely if Sylvester's criterion holds then the matrix (5.4.27) is positive definite is not established by the arguments given in this problem. A proof of the converse can be constructed a number of ways. A convenient one involves what is known as a *Cholesky decomposition*.[7] Some of the details of this decomposition are developed in the next exercise.

5.4.3 A *Cholesky decomposition* is a decomposition of an $N \times N$ matrix $A \in \mathscr{M}^{N\times N}$ into the product[8]

$$A = L_1 L_1^* \tag{5.4.28}$$

where $L_1 \in \mathscr{M}^{N\times N}$ is a lower triangular matrix. It is a theorem that every Hermitian positive definite matrix $A \in \mathscr{M}^{M\times N}$ has a unique Cholesky decomposition. The purpose of this exercise is to establish this result.

a) You are given a positive definite Hermitian matrix $A \in \mathscr{M}^{N\times N}$. If you apply the results in Section 1.7, the matrix A has the LU decomposition

[7] Information about Andre-Louis Cholesky can be found at http://en.wikipedia.org/wiki/Andr%C3%A9-Louis_Cholesky.

[8] Like the LU decomposition, the Cholesky decomposition is useful when finding the solution of $A\mathbf{u}=\mathbf{b}$ in the case where A is positive definite and Hermitian. The system $A\mathbf{u}=\mathbf{b}$ is replaced by $L_1\left(L_1^*\mathbf{u}\right)=\mathbf{b}$. This equation is solved by forward substitution and that result is solved by back substitution for $\mathbf{u}$.

$$A = LU \tag{5.4.29}$$

where U is an upper triangle $N \times N$ matrix and L is a lower triangle nonsingular $N \times N$ matrix with 1*s* down the diagonal. The fact that A is positive definite Hermitian shows that the elementary LU decomposition exists. It also shows that the matrix U is nonsingular. Given the fact that A is Hermitian, show that

$$L^{-1}U^* = UL^{-1*} \tag{5.4.30}$$

b) Show that the matrix $L^{-1}U^*$ is a lower triangular matrix and that UL^{-1*} is an upper triangular matrix. These two facts and (5.4.30) combine to show that there exists a *diagonal* matrix $D \in \mathscr{M}^{N\times N}$ such that

$$U = DL^* \tag{5.4.31}$$

Show that

$$A = LDL^* \tag{5.4.32}$$

and that D is a positive definite Hermitian matrix.

c) Explain why the diagonal matrix D can always be written as the square of a diagonal matrix $D_1 \in \mathscr{M}^{N\times N}$ and that (5.4.32) can then be written[9]

$$A = LDL^* = LD_1^2L^* = (LD_1)(LD_1)^* = L_1L_1^* \tag{5.4.33}$$

which is the Cholesky decomposition. The uniqueness of this decomposition follows from the uniqueness of the LU decomposition in this case.

5.4.4 You are given the positive definite Hermitian matrix $A \in \mathscr{M}^{3\times 3}$

$$A = \begin{bmatrix} 4 & -14i & -10i \\ 14i & 53 & \dfrac{8i}{17} \\ 10i & -\dfrac{8i}{17} & \dfrac{13297}{289} \end{bmatrix} \tag{5.4.34}$$

You are also given that A has the LU decomposition

---

[9] The Cholesky decomposition can be implemented in other ways. For example, because A is Hermitian, there is a unitry matrix T and a diagonal matrix D of eigenvalues of A such that $A = TDT^*$. Next apply the QR decomposition, discussed in Section 4.1, to the matrix $D^{1/2}T^* = QR$. Then it is readily shown that $A = R^*R$ which is a Cholesky decomposition.

$$A=\begin{bmatrix} 1 & 0 & 0 \\ \dfrac{7i}{2} & 1 & 0 \\ \dfrac{5i}{2} & \dfrac{35-2i}{17} & 1 \end{bmatrix}\begin{bmatrix} 4 & -14i & -10i \\ 0 & 4 & \dfrac{140+8i}{17} \\ 0 & 0 & 4 \end{bmatrix} \tag{5.4.35}$$

Show that, from (5.4.30), that

$$D=L^{-1}U^{*}=UL^{-1*}=\begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} \tag{5.4.36}$$

Also, determine the Cholesky decomposition of A and show that

$$A=L_1L_1^{*}=\begin{bmatrix} 2 & 0 & 0 \\ 7i & 2 & 0 \\ 5i & \dfrac{70-4i}{17} & 2 \end{bmatrix}\begin{bmatrix} 2 & -7i & -5i \\ 0 & 2 & \dfrac{70+4i}{17} \\ 0 & 0 & 2 \end{bmatrix} \tag{5.4.37}$$

Note the general feature displayed by (5.4.35) and (5.4.36) that the diagonal matrix D and the upper triangular matrix U have the same diagonal elements. This fact is readily confirmed from (5.4.30) and the properties of the LU decomposition.

5.4.5 Exercise 5.4.4 established that if the Hermitian matrix (5.4.27) is positive definite, its principal minors are positive, i.e. obey (5.4.26). The converse, namely, that if (5.4.26) is true, the Hermitian matrix (5.4.27) is positive definite. The key to this result is actually the result (5.4.32), repeated,

$$A=LDL^{*} \tag{5.4.38}$$

Show that the elements of the diagonal matrix $D\in\mathscr{M}^{N\times N}$ are related to the principal minors of $A\in\mathscr{M}^{N\times N}$ by the formulas

$$
\begin{aligned}
D_{11} &= A_{11} \\
D_{22} &= \frac{1}{A_{11}} \det \begin{vmatrix} A_{11} & A_{12} \\ \overline{A_{12}} & A_{22} \end{vmatrix} \\
D_{33} &= \frac{\det \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ \overline{A_{12}} & A_{22} & A_{23} \\ \overline{A_{13}} & \overline{A_{23}} & A_{33} \end{vmatrix}}{\det \begin{vmatrix} A_{11} & A_{12} \\ \overline{A_{12}} & A_{22} \end{vmatrix}} \\
&\cdot \\
&\cdot \\
&\cdot \\
D_{NN} &= \frac{\det \begin{vmatrix} A_{11} & A_{12} & A_{13} & \cdot & \cdot & A_{1N} \\ \overline{A_{12}} & A_{22} & A_{23} & & & A_{2N} \\ \overline{A_{13}} & \overline{A_{23}} & A_{33} & & & A_{3N} \\ \cdot & & & \cdot & & \cdot \\ \cdot & & & & \cdot & \cdot \\ \overline{A_{1N}} & \overline{A_{2N}} & \overline{A_{3N}} & \cdot & \cdot & A_{NN} \end{vmatrix}}{\det \begin{vmatrix} A_{11} & A_{12} & A_{13} & \cdot & \cdot & A_{1N-1} \\ \overline{A_{12}} & A_{22} & A_{23} & & & A_{2N-1} \\ \overline{A_{13}} & \overline{A_{23}} & A_{33} & & & A_{3N-1} \\ \cdot & & & \cdot & & \cdot \\ \cdot & & & & \cdot & \cdot \\ \overline{A_{1N-1}} & \overline{A_{2N-1}} & \overline{A_{3N-1}} & \cdot & \cdot & A_{N-1,N-1} \end{vmatrix}}
\end{aligned}
\tag{5.4.39}
$$

These results, when established, show that if the principal minors are positive the matrix $D \in \mathscr{M}^{N\times N}$ is positive definite and, because of (5.4.38), the Hermitian matrix $A \in \mathscr{M}^{N\times N}$ is positive definite.

Hint: In order to establish (5.4.39), recall that the diagonal elements of D and those of U, in the LU decomposition, are the same. Given this fact, the result (1.7.71) gives the elements of U in terms of those of A.

5.4.6 Show that the eigenvalues of a unitary linear transformation have unit magnitude.

5.4.7 Show that a linear transformation $A:\mathscr{V} \rightarrow \mathscr{V}$ is nonsingular if and only if zero is not an eigenvalue.

5.4.8 You are given a nonsingular linear transformation $A:\mathscr{V} \to \mathscr{V}$ and a vector $\mathbf{b} \in \mathscr{V}$. Because $\mathbf{A}$ is nonsingular, the equation $\mathbf{Au} = \mathbf{b}$ has the solution

$$\mathbf{u} = \mathbf{A}^{-1}\mathbf{b} \tag{5.4.40}$$

Show that when $\mathbf{A}$ has N distinct eigenvalues the solution (5.4.40) can be written

$$\mathbf{u} = \sum_{j=1}^{N} \frac{1}{\lambda_j} b^j \mathbf{v}_j \tag{5.4.41}$$

where $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_N\}$ are the eigenvectors of $\mathbf{A}$ and $b^j = \langle \mathbf{b}, \mathbf{v}^j \rangle$ where $\{\mathbf{v}^1, \mathbf{v}^2, \cdots, \mathbf{v}^N\}$ is the basis reciprocal to $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_N\}$.[10]

[10] The result (5.4.41) is generalized in Section 6.8 for the case where the linear transformation $\mathbf{A}$ is singular.

Section 5.5. Constant Coefficient Linear Ordinary Differential Equations

In Section 5.1, it was mentioned that eigenvalue problems arise in problems involving the solution of systems of ordinary differential equations. In particular, they arise when solving systems of constant coefficient linear ordinary differential equation. In this section, we shall see how this particular eigenvalue problem arises.

It is important when studying ordinary differential equations to classify the kinds of systems of ordinary differential equations being considered. We are interested in problems for which the system can be written

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x} + \mathbf{g}(t) \tag{5.5.1}$$

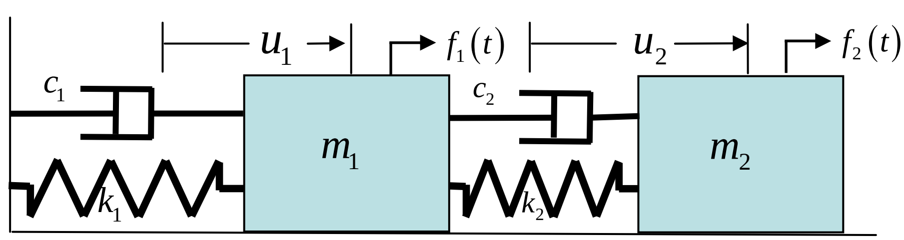
where $\mathbf{x}(t)$ is a $N \times 1$ column matrix, A is an $N \times N$ matrix and $\mathbf{g}(t)$ is a $N \times 1$ column matrix. As the notation suggests, $\mathbf{x}(t)$ and $\mathbf{g}(t)$ are functions of a parameter t, which for simplicity we shall regard as the time. Also note that the matrix A does not depend upon t. The initial value problem associated with (5.5.1) is the problem of finding the function $\mathbf{x} = \mathbf{x}(t)$ such that

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x} + \mathbf{g}(t) \qquad \text{and} \qquad \mathbf{x}(0) = \mathbf{x}_0 \tag{5.5.2}$$

where $\mathbf{x}_0$ is given.

It is typically the case that applications lead to systems of ordinary differential equations of order higher than one. Equation (5.5.1) was selected as the starting place for our discussion because almost all systems of linear constant coefficient ordinary differential equations can be written in the form (5.5.1). Equation (5.5.1) is sometimes referred to as the *normal form* of a system. This designation is probably more common when discussing the more complicated problem of numerical solutions of systems of nonlinear equations. The following example illustrates how a system of two second order linear constant coefficient equations can be put into the form (5.5.1).

Example 5.5.1: Consider the two degree of freedom vibration problem:



where m_1 and m_2 are masses, k_1 and k_2 are spring constants, c_1 and c_2 are damping constants and u_1 and u_2 are displacements. Finally, the functions $f_1(t)$ and $f_2(t)$ are forcing functions. The constants m_1, m_2, k_1, k_2, c_1 and c_2 are positive constants. The ordinary differential equations which govern the motion of this system are[10]

$$
\begin{aligned}
m_1\ddot{u}_1 &= -c_1\dot{u}_1 - k_1u_1 + c_2(\dot{u}_2 - \dot{u}_1) + k_2(u_2 - u_1) + f_1(t) \\
m_2\ddot{u}_2 &= -c_2(\dot{u}_2 - \dot{u}_1) - k_2(u_2 - u_1) + f_2(t)
\end{aligned}
\tag{5.5.3}
$$

where, for example, $\dot{u}_1 = \dfrac{du_1}{dt}$. This second order coupled system of ordinary differential equations can be written in the form (5.5.1) if we define the 4×1 matrix $\mathbf{x}(t)$ by

$$
\mathbf{x}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \dot{u}_1(t) \\ \dot{u}_2(t) \end{bmatrix}
\tag{5.5.4}
$$

The first derivative of (5.5.4) is

$$
\frac{d\mathbf{x}(t)}{dt} = \begin{bmatrix} \dot{u}_1(t) \\ \dot{u}_2(t) \\ \ddot{u}_1(t) \\ \ddot{u}_2(t) \end{bmatrix}
\tag{5.5.5}
$$

We now use (5.5.3) to eliminate $\ddot{u}_1(t)$ and $\ddot{u}_2(t)$. The result of this elimination is

---

[10] Equations (5.5.3) illustrate a general feature of systems of linear equations governing forced vibration problems. These equations always take the general form

$$
M\ddot{\mathbf{u}}(t) + C\dot{\mathbf{u}}(t) + K\mathbf{u}(t) = \mathbf{f}(t)
$$

where M is a symmetric positive definite matrix of masses, C is a symmetric positive semidefinite matrix of damping coefficients and K is a symmetric positive definite matrix of spring constants. The normal form of this equation is

$$
\frac{d}{dt}\begin{bmatrix} \mathbf{u}(t) \\ \dot{\mathbf{u}}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -M^{-1}K & -M^{-1}C \end{bmatrix} \begin{bmatrix} \mathbf{u}(t) \\ \dot{\mathbf{u}}(t) \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ M^{-1}\mathbf{f}(t) \end{bmatrix}
$$

$$\frac{d\mathbf{x}(t)}{dt}=\begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \\ -\frac{c_1}{m_1}\dot{u}_1-\frac{k_1}{m_1}u_1+\frac{c_2}{m_1}(\dot{u}_2-\dot{u}_1)+\frac{k_2}{m_1}(u_2-u_1)+\frac{1}{m_1}f_1(t) \\ -\frac{c_2}{m_2}(\dot{u}_2-\dot{u}_1)-\frac{k_2}{m_2}(u_2-u_1)+\frac{1}{m_2}f_2(t) \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{(k_1+k_2)}{m_1} & \frac{k_2}{m_1} & -\frac{(c_1+c_2)}{m_1} & \frac{c_2}{m_1} \\ \frac{k_2}{m_2} & -\frac{k_2}{m_2} & \frac{c_2}{m_2} & -\frac{c_2}{m_2} \end{bmatrix} \underbrace{\begin{bmatrix} u_1(t) \\ u_2(t) \\ \dot{u}_1(t) \\ \dot{u}_2(t) \end{bmatrix}}_{\mathbf{x}(t)} + \begin{bmatrix} 0 \\ 0 \\ \frac{f_1(t)}{m_1} \\ \frac{f_2(t)}{m_2} \end{bmatrix} \tag{5.5.6}$$

which fits the form of (5.5.1) with the choices

$$A=\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{(k_1+k_2)}{m_1} & \frac{k_2}{m_1} & -\frac{(c_1+c_2)}{m_1} & \frac{c_2}{m_1} \\ \frac{k_2}{m_2} & -\frac{k_2}{m_2} & \frac{c_2}{m_2} & -\frac{c_2}{m_2} \end{bmatrix} \tag{5.5.7}$$

and

$$\mathbf{g}(t)=\begin{bmatrix} 0 \\ 0 \\ \frac{f_1(t)}{m_1} \\ \frac{f_2(t)}{m_2} \end{bmatrix} \tag{5.5.8}$$

Section 5.6. General Solution

Given the fact, as illustrated by Example 5.5.1, one can write systems of ordinary differential equations in the common form (5.5.1), in this section we shall examine how the solution of these systems involve the solution of an eigenvalue problem. Before we display the eigenvalue problem, we need to recall a result from the first course on ordinary differential equations. The result is that linear inhomogeneous ordinary differential equations always have solutions that can be written as the *sum* of a solution of the *homogeneous equation* plus a *particular solution*. In our case, the ordinary differential equation is the system (5.5.1). Therefore, the solution of (5.5.1) will be of the form

$$\mathbf{x}(t)=\mathbf{x}_h(t)+\mathbf{x}_p(t) \tag{5.6.1}$$

where $\mathbf{x}_h$ is the *general solution* , i.e., the solution of the *homogeneous equation*

$$\frac{d\mathbf{x}_h}{dt}=A\mathbf{x}_h \tag{5.6.2}$$

and $\mathbf{x}_p$ is a *particular solution* of

$$\frac{d\mathbf{x}_p}{dt}=A\mathbf{x}_p+\mathbf{g}(t) \tag{5.6.3}$$

This theoretical result is reminiscent of a similar result for systems of linear algebraic equations that we discussed in Section 2.7.

The eigenvalue problem that arises when solving systems of linear ordinary differential equations is when one is *solving for the homogeneous solution*. For simplicity, we shall assume for the moment that our task is to solve a homogeneous equation and we shall delay the problem of finding the particular solution. In other words, we are *temporarily* restricting our discussion to the case

$$\mathbf{g}(t)=\mathbf{0} \tag{5.6.4}$$

We know from our experience with ordinary differential equations that the solution of

$$\frac{d\mathbf{x}}{dt}=A\mathbf{x} \tag{5.6.5}$$

is often an equation of the form

$$\mathbf{x}(t)=\mathbf{v}e^{\lambda t} \tag{5.6.6}$$

where $\mathbf{v}$ is a column matrix to be determined and λ is a scalar, real or complex, which is to be determined. You will recall that the general solution is a linear combination of solutions of the form (5.6.6). For an N^{th} order system one needs N linearly independent solutions of the form (5.6.6) in order to generate the general solution.

If the assumed solution (5.6.6) is substituted into the ordinary differential equation (5.6.5), we see that $\mathbf{v}$ and λ must obey

$$\lambda \mathbf{v} e^{\lambda t} = A \mathbf{v} e^{\lambda t} \tag{5.6.7}$$

Because (5.6.7) must hold for all t, (5.6.7) reduces to the eigenvalue problem

$$A\mathbf{v} = \lambda \mathbf{v} \tag{5.6.8}$$

If (5.6.8) yields N linearly independent eigenvectors, the general solution is

$$\mathbf{x}(t) = c_1 \mathbf{v}_1 e^{\lambda_1 t} + c_2 \mathbf{v}_2 e^{\lambda_2 t} + c_3 \mathbf{v}_3 e^{\lambda_3 t} + \cdots + c_N \mathbf{v}_N e^{\lambda_N t} \tag{5.6.9}$$

where $c_1, c_2, ..., c_N$ are arbitrary constants. The constants are determined by imposing initial conditions. Equation (5.6.9), which does not utilize the initial conditions, is usually called the *general solution* to the homogenous problem. The form of the solution (5.6.9) shows that the vector $\mathbf{x}(t)$ is the linear combination of vectors pointing in the direction of the eigenvectors. The individual terms, $\mathbf{v}_1 e^{\lambda_1 t}, \mathbf{v}_2 e^{\lambda_2 t}, \mathbf{v}_3 e^{\lambda_3 t}, ..., \mathbf{v}_N e^{\lambda_N t}$, each pointing in the direction of an eigenvector, representing the building blocks for the full solution $\mathbf{x}(t)$. If the solution to the eigenvalue problem *does not* yield N linearly independent eigenvectors, then the mathematics is telling us that the solution is *not* of the form (5.6.6).

It helps our later manipulations if we rewrite (5.6.9) in a slightly different form. As with our examples in Section 5.3, we can arrange the eigenvectors as columns of a $N \times N$ matrix T. The notation we shall use is

$$T = [\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_N] = \begin{bmatrix} \upsilon^1{}_{(1)} & \upsilon^1{}_{(2)} & \upsilon^1{}_{(3)} & \cdot & \cdot & \cdot & \upsilon^1{}_{(N)} \\ \upsilon^2{}_{(1)} & \upsilon^2{}_{(2)} & \upsilon^2{}_{(3)} & & & & \upsilon^2{}_{(N)} \\ \upsilon^3{}_{(1)} & \upsilon^3{}_{(2)} & \upsilon^3{}_{(3)} & & & & \upsilon^3{}_{(N)} \\ \cdot & \cdot & & \cdot & & & \cdot \\ \cdot & \cdot & & & \cdot & & \cdot \\ \cdot & \cdot & & & & \cdot & \cdot \\ \underbrace{\upsilon^N{}_{(1)}}_{\mathbf{v}_1} & \underbrace{\upsilon^N{}_{(2)}}_{\mathbf{v}_2} & \underbrace{\upsilon^N{}_{(3)}}_{\mathbf{v}_3} & \cdot & \cdot & \cdot & \underbrace{\upsilon^N{}_{(N)}}_{\mathbf{v}_N} \end{bmatrix} \tag{5.6.10}$$

This definition, allows us to write (5.6.9) as

$$\mathbf{x}(t) = T\begin{bmatrix} e^{\lambda_1 t} & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & e^{\lambda_2 t} & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & e^{\lambda_3 t} & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & e^{\lambda_N t} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \cdot \\ \cdot \\ \cdot \\ c_N \end{bmatrix} \tag{5.6.11}$$

Also, given (5.6.10), the solution to eigenvalue problem (5.6.8) is equivalent to the matrix equation

$$AT = TD \tag{5.6.12}$$

where D is the diagonal matrix of the eigenvalues defined by

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & \lambda_2 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & \lambda_3 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & \lambda_N \end{bmatrix} \tag{5.6.13}$$

Of course, equation (5.6.12) is equation (5.1.10) given earlier. It simplifies our notation later if we adopt the symbol e^{Dt} for the diagonal matrix that appears in (5.6.11). Therefore,

$$e^{Dt} = \begin{bmatrix} e^{\lambda_1 t} & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & e^{\lambda_2 t} & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & e^{\lambda_3 t} & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & e^{\lambda_N t} \end{bmatrix} \tag{5.6.14}$$

and the solution (5.6.11) can be written in the more compact form

$$\mathbf{x}(t) = Te^{Dt}\mathbf{c} \tag{5.6.15}$$

where $\mathbf{c}$ is the column matrix

$$\mathbf{c}=\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \cdot \\ \cdot \\ \cdot \\ c_N \end{bmatrix} \tag{5.6.16}$$

Example 5.6.1: You are given the familiar second order ordinary differential equation governing harmonic motion. Namely,

$$\ddot{u}+\omega_0^2 u=0 \tag{5.6.17}$$

where ω_0 is a positive constant. If we define

$$\mathbf{x}(t)=\begin{bmatrix} u(t) \\ \dot{u}(t) \end{bmatrix} \tag{5.6.18}$$

The equation to solve takes the form (5.6.5) with

$$A=\begin{bmatrix} 0 & 1 \\ -\omega_0^2 & 0 \end{bmatrix} \tag{5.6.19}$$

The eigenvalues of A are the roots of

$$\det(A-\lambda I)=\begin{vmatrix} 0-\lambda & 1 \\ -\omega_0^2 & 0-\lambda \end{vmatrix}=\lambda^2+\omega_0^2=(i\omega_0-\lambda)(-i\omega_0-\lambda) \tag{5.6.20}$$

We shall order the eigenvalues as follows:

$$\begin{aligned} \lambda_1 &= i\omega_0 \\ \lambda_2 &= -i\omega_0 \end{aligned} \tag{5.6.21}$$

The eigenvector associated with the first eigenvalue is the solution of

$$\begin{bmatrix} 0-i\omega_0 & 1 \\ -\omega_0^2 & 0-i\omega_0 \end{bmatrix}\begin{bmatrix} \upsilon^1{}_{(1)} \\ \upsilon^2{}_{(1)} \end{bmatrix}=0 \tag{5.6.22}$$

The reduced row echelon form of (5.6.22) is

$$\begin{bmatrix} -i\omega_0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \upsilon^1{}_{(1)} \\ \upsilon^2{}_{(1)} \end{bmatrix} = 0 \tag{5.6.23}$$

Therefore,

$$\begin{bmatrix} \upsilon^1{}_{(1)} \\ \upsilon^2{}_{(1)} \end{bmatrix} = \upsilon^1{}_{(1)}\begin{bmatrix} 1 \\ i\omega_0 \end{bmatrix} \Rightarrow \mathscr{V}\left(\lambda_1\right) = \operatorname{span}\left(\begin{bmatrix} 1 \\ i\omega_0 \end{bmatrix}\right) \tag{5.6.24}$$

Likewise, the characteristic subspace associated with the second eigenvector is

$$\mathscr{V}\left(\lambda_2\right) = \operatorname{span}\left(\begin{bmatrix} 1 \\ -i\omega_0 \end{bmatrix}\right) \tag{5.6.25}$$

These two eigenvectors give the solution to the system of two ordinary differential equations in the form of a superposition of the two possibilities of solutions of the form $\mathbf{x} = \mathbf{v}e^{\lambda t}$. The result, as follows from (5.6.9), is

$$\mathbf{x}\left(t\right) = c_1\begin{bmatrix} 1 \\ i\omega_0 \end{bmatrix}e^{i\omega_0 t} + c_2\begin{bmatrix} 1 \\ -i\omega_0 \end{bmatrix}e^{-i\omega_0 t} \tag{5.6.26}$$

Or, in the equivalent form (5.6.11)

$$\mathbf{x}\left(t\right) = \begin{bmatrix} 1 & 1 \\ i\omega_0 & -i\omega_0 \end{bmatrix}\begin{bmatrix} e^{i\omega_0 t} & 0 \\ 0 & e^{-i\omega_0 t} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \tag{5.6.27}$$

This solution can be written in a more familiar form if we utilize the Euler identity

$$e^{i\omega_0 t} = \cos\left(\omega_0 t\right) + i\sin\left(\omega_0 t\right) \tag{5.6.28}$$

These equations allow the general solution (5.6.9) to be written

$$\begin{aligned}
\mathbf{x}(t) &= c_1\begin{bmatrix}1\\ i\omega_0\end{bmatrix}e^{i\omega_0 t}+c_2\begin{bmatrix}1\\ -i\omega_0\end{bmatrix}e^{-i\omega_0 t}\\
&= c_1\begin{bmatrix}1\\ i\omega_0\end{bmatrix}\left(\cos(\omega_0 t)+i\sin(\omega_0 t)\right)+c_2\begin{bmatrix}1\\ -i\omega_0\end{bmatrix}\left(\cos(\omega_0 t)-i\sin(\omega_0 t)\right)\\
&= \begin{bmatrix}c_1+c_2\\ i\omega_0(c_1-c_2)\end{bmatrix}\cos(\omega_0 t)+\begin{bmatrix}i(c_1-c_2)\\ \omega_0(-c_1-c_2)\end{bmatrix}\sin(\omega_0 t)\\
&= \begin{bmatrix}(c_1+c_2)\cos(\omega_0 t)+i(c_1-c_2)\sin(\omega_0 t)\\ \omega_0(-c_1-c_2)\sin(\omega_0 t)+i\omega_0(c_1-c_2)\cos(\omega_0 t)\end{bmatrix}\\
&= \begin{bmatrix}d_1\cos(\omega_0 t)+d_2\sin(\omega_0 t)\\ -\omega_0 d_1\sin(\omega_0 t)+\omega_0 d_2\cos(\omega_0 t)\end{bmatrix}=\begin{bmatrix}u(t)\\ \dot{u}(t)\end{bmatrix}
\end{aligned} \tag{5.6.29}$$

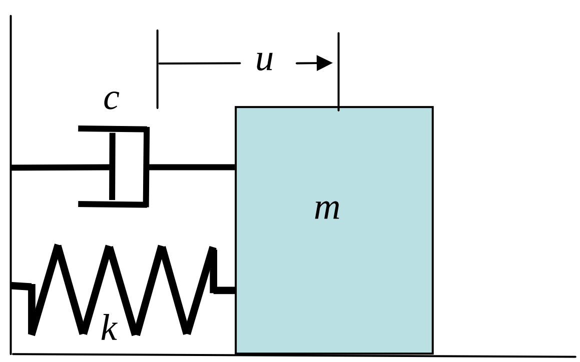
where the complex coefficients c_1 and c_2 are related to the *real coefficients* d_1 and d_2 by the formulas

$$c_1=\frac{1}{2}(d_1-id_2) \qquad \text{and} \qquad c_2=\frac{1}{2}(d_1+id_2) \tag{5.6.30}$$

Example 5.6.2: As a generalization of Example 5.6.1, you are given the problem of *damped harmonic motion* governed by the ordinary differential equation

$$m\ddot{u}+c\dot{u}+ku=0 \tag{5.6.31}$$

where m, c and k are positive constants representing the mass, the damping coefficient and the spring constant, respectively. The figure that is often associated with the ordinary differential equation (5.6.31) is



It is customary to introduce the symbols

$$\omega_0^{\ 2} = \frac{k}{m} \qquad \varsigma = \frac{c}{2m\omega_0} \tag{5.6.32}$$

and write the differential equation (5.6.31) as

$$\ddot{u}(t) + 2\varsigma\omega_0\dot{u}(t) + \omega_0^{\ 2}u(t) = 0 \tag{5.6.33}$$

Equation (5.6.33) modifies the ordinary differential defining harmonic motion, equation (5.6.17), by the inclusion of the first derivative proportional to the dimensionless damping coefficient $(5.6.32)_2$. If we define, as with Example 5.6.1,

$$\mathbf{x}(t) = \begin{bmatrix} u(t) \\ \dot{u}(t) \end{bmatrix} \tag{5.6.34}$$

The equation to solve takes the form (5.6.5) with

$$A = \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & -2\zeta\omega_0 \end{bmatrix} \tag{5.6.35}$$

The eigenvalues of A are the roots of

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} 0-\lambda & 1 \\ -\omega_0^2 & -2\zeta\omega_0 - \lambda \end{vmatrix} = \lambda^2 + 2\varsigma\omega_0\lambda + \omega_0^2 \\ &= \left(-\varsigma\omega_0 + i\omega_0\sqrt{1-\zeta^2} - \lambda\right)\left(-\varsigma\omega_0 - i\omega_0\sqrt{1-\zeta^2} - \lambda\right) \end{aligned} \tag{5.6.36}$$

We shall order the eigenvalues as follows:

$$\begin{aligned} \lambda_1 &= -\zeta\omega_0 + i\omega_0\sqrt{1-\zeta^2} \\ \lambda_2 &= -\zeta\omega_0 - i\omega_0\sqrt{1-\zeta^2} \end{aligned} \tag{5.6.37}$$

Equations (5.6.37) are written to fit what is called the *under damped* case. For this case, the damping is assumed to be such that $\sqrt{1-\zeta^2} > 0$.

The eigenvector associated with the first eigenvalue is the solution of

$$\begin{bmatrix} 0 + \zeta\omega_0 - i\omega_0\sqrt{1-\zeta^2} & 1 \\ -\omega_0^2 & 0 - \zeta\omega_0 - i\omega_0\sqrt{1-\zeta^2} \end{bmatrix} \begin{bmatrix} \upsilon^1_{\ (1)} \\ \upsilon^2_{\ (1)} \end{bmatrix} = 0 \tag{5.6.38}$$

The reduced row echelon form of (5.6.22) is

$$\begin{bmatrix} \zeta\omega_0 - i\omega_0\sqrt{1-\zeta^2} & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \upsilon^1{}_{(1)} \\ \upsilon^2{}_{(1)} \end{bmatrix} = 0 \tag{5.6.39}$$

Therefore,

$$\begin{bmatrix} \upsilon^1{}_{(1)} \\ \upsilon^2{}_{(1)} \end{bmatrix} = \upsilon^1{}_{(1)} \begin{bmatrix} 1 \\ -\zeta\omega_0 + i\omega_0\sqrt{1-\zeta^2} \end{bmatrix} \Rightarrow \mathscr{V}(\lambda_1) = \text{span}\left(\begin{bmatrix} 1 \\ -\zeta\omega_0 + i\omega_0\sqrt{1-\zeta^2} \end{bmatrix} \right) \tag{5.6.40}$$

Likewise, the characteristic subspace associated with the second eigenvector is

$$\mathscr{V}(\lambda_2) = \text{span}\left(\begin{bmatrix} 1 \\ -\zeta\omega_0 - i\omega_0\sqrt{1-\zeta^2} \end{bmatrix} \right) \tag{5.6.41}$$

These two eigenvectors give the solution to the system of two ordinary differential equations in the form of a superposition of the two possibilities of solutions of the form $\mathbf{x} = \mathbf{v}e^{\lambda t}$. The result, as follows from (5.6.9), is

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ -\zeta\omega_0 + i\omega_0\sqrt{1-\zeta^2} \end{bmatrix} e^{\left(-\zeta + i\sqrt{1-\zeta^2}\right)\omega_0 t} + c_2 \begin{bmatrix} 1 \\ -\zeta\omega_0 - i\omega_0\sqrt{1-\zeta^2} \end{bmatrix} e^{\left(-\zeta - i\sqrt{1-\zeta^2}\right)\omega_0 t} \tag{5.6.42}$$

As with Example 5.6.2, this solution can be written in a more familiar form if we utilize the Euler identity

$$e^{i\omega_0\sqrt{1-\zeta^2}\,t} = \cos\left(\omega_0\sqrt{1-\zeta^2}\,t\right) + i\sin\left(\omega_0\sqrt{1-\zeta^2}\,t\right) \tag{5.6.43}$$

These equations allow the general solution (5.6.9) to be written

$$
\begin{aligned}
\mathbf{x}(t) &= c_1\begin{bmatrix} 1 \\ -\zeta\omega_0 + i\omega_0\sqrt{1-\zeta^2}\end{bmatrix} e^{-\zeta\omega_0 t} e^{i\omega_0\sqrt{1-\zeta^2}\,t} + c_2\begin{bmatrix} 1 \\ -\zeta\omega_0 - i\omega_0\sqrt{1-\zeta^2}\end{bmatrix} e^{-\zeta\omega_0 t} e^{-i\omega_0\sqrt{1-\zeta^2}\,t} \\
&= c_1\begin{bmatrix} 1 \\ -\zeta\omega_0 + i\omega_0\sqrt{1-\zeta^2}\end{bmatrix} e^{-\zeta\omega_0 t}\left(\cos\left(\omega_0\sqrt{1-\zeta^2}\,t\right) + i\sin\left(\omega_0\sqrt{1-\zeta^2}\,t\right)\right) \\
&\quad + c_2\begin{bmatrix} 1 \\ -\zeta\omega_0 - i\omega_0\sqrt{1-\zeta^2}\end{bmatrix} e^{-\zeta\omega_0 t}\left(\cos\left(\omega_0\sqrt{1-\zeta^2}\,t\right) - i\sin\left(\omega_0\sqrt{1-\zeta^2}\,t\right)\right) \\
&= \begin{bmatrix} c_1 + c_2 \\ -\zeta\omega_0\left(c_1+c_2\right) + i\omega_0\sqrt{1-\zeta^2}\left(c_1 - c_2\right)\end{bmatrix} e^{-\zeta\omega_0 t}\cos\left(\omega_0\sqrt{1-\zeta^2}\,t\right) \\
&\quad + \begin{bmatrix} i\left(c_1 - c_2\right) \\ -i\zeta\omega_0\left(c_1-c_2\right) + \omega_0\sqrt{1-\zeta^2}\left(-c_1 - c_2\right)\end{bmatrix} e^{-\zeta\omega_0 t}\sin\left(\omega_0\sqrt{1-\zeta^2}\,t\right) \\
&= \begin{bmatrix} e^{-\zeta\omega_0 t}\left(\left(c_1+c_2\right)\cos\left(\omega_0\sqrt{1-\zeta^2}\,t\right) + i\left(c_1-c_2\right)\sin\left(\omega_0\sqrt{1-\zeta^2}\,t\right)\right) \\ \left(\begin{array}{c} e^{-\zeta\omega_0 t}\omega_0\sqrt{1-\zeta^2}\left(\left(-c_1-c_2\right)\sin\left(\omega_0\sqrt{1-\zeta^2}\,t\right) + i\left(c_1-c_2\right)\cos\left(\omega_0\sqrt{1-\zeta^2}\,t\right)\right) \\ -\zeta\omega_0 e^{-\zeta\omega_0 t}\left(\left(c_1+c_2\right)\cos\left(\omega_0\sqrt{1-\zeta^2}\,t\right) + i\left(c_1-c_2\right)\sin\left(\omega_0\sqrt{1-\zeta^2}\,t\right)\right)\end{array}\right)\end{bmatrix} \\
&= \begin{bmatrix} e^{-\zeta\omega_0 t}\left(d_1\cos\left(\omega_0\sqrt{1-\zeta^2}\,t\right) + d_2\sin\left(\omega_0\sqrt{1-\zeta^2}\,t\right)\right) \\ \left(\begin{array}{c} e^{-\zeta\omega_0 t}\omega_0\sqrt{1-\zeta^2}\left(-d_1\sin\left(\omega_0\sqrt{1-\zeta^2}\,t\right) + d_2\cos\left(\omega_0\sqrt{1-\zeta^2}\,t\right)\right) \\ -\zeta\omega_0 e^{-\zeta\omega_0 t}\left(d_1\cos\left(\omega_0\sqrt{1-\zeta^2}\,t\right) + d_2\sin\left(\omega_0\sqrt{1-\zeta^2}\,t\right)\right)\end{array}\right)\end{bmatrix} = \begin{bmatrix} u(t) \\ \dot{u}(t)\end{bmatrix}
\end{aligned}
\tag{5.6.44}
$$

where the complex coefficients c_1 and c_2 are related to the *real coefficients* d_1 and d_2 by the same formulas used with Example 5.6.2, namely,

$$
c_1 = \frac{1}{2}\left(d_1 - id_2\right) \qquad \text{and} \qquad c_2 = \frac{1}{2}\left(d_1 + id_2\right) \tag{5.6.45}
$$

Example 5.6.3:

$$
\begin{aligned}
\frac{dy_1(t)}{dt} &= y_1(t) + 6y_2(t) \\
\frac{dy_2(t)}{dt} &= y_1(t) + 2y_2(t)
\end{aligned}
\tag{5.6.46}
$$

The eigenvalues for the matrix $A=\begin{bmatrix}1 & 6\\ 1 & 2\end{bmatrix}$ are

$$\lambda_1=-1 \qquad \text{and} \qquad \lambda_2=4 \tag{5.6.47}$$

The two characteristic subspaces turn out to be

$$\mathscr{V}\left(\lambda_1\right)=\operatorname{span}\left(\begin{bmatrix}-3\\ 1\end{bmatrix}\right) \quad \text{and} \quad \mathscr{V}\left(\lambda_2\right)=\operatorname{span}\left(\begin{bmatrix}2\\ 1\end{bmatrix}\right) \tag{5.6.48}$$

Therefore, from (5.6.9) the general solution of (5.6.46) is,

$$\mathbf{x}\left(t\right)=c_1\begin{bmatrix}-3\\ 1\end{bmatrix}e^{-t}+c_2\begin{bmatrix}2\\ 1\end{bmatrix}e^{4t} \tag{5.6.49}$$

The three examples above are typical of the solution procedure for finding the homogeneous solution to systems of first order constant ordinary differential equations. The solution procedure works because the *geometric multiplicity of each eigenvalue has equaled its algebraic multiplicity*. In cases where this is not true, the solution is not as simple as the above. The following example illustrates the problems that can arise.

Example 5.6.4

$$\begin{aligned}\frac{dy_1(t)}{dt}&=y_1(t)+y_2(t)+y_3(t)\\ \frac{dy_2(t)}{dt}&=y_2(t)+y_3(t)\\ \frac{dy_3(t)}{dt}&=y_3(t)\end{aligned} \tag{5.6.50}$$

This system can be rewritten as the matrix equation

$$\frac{d\mathbf{x}(t)}{dt}=A\mathbf{x}\left(t\right) \tag{5.6.51}$$

where

$$A=\begin{bmatrix}1 & 1 & 1\\ 0 & 1 & 1\\ 0 & 0 & 1\end{bmatrix} \tag{5.6.52}$$

Equation (5.6.51) is the matrix that we considered in Example 5.3.3. The results of that example are that (5.6.52) has a single eigenvalue $\lambda = 1$ of algebraic multiplicity 3. The characteristic subspace associated with this example has dimension 1. As a result, we cannot represent the solution of (5.6.50) in the form (5.6.9). The general solution in this case turns out to be

$$\mathbf{x}(t) = c_1\begin{bmatrix}1\\0\\0\end{bmatrix}e^t + c_2\begin{bmatrix}t\\1\\0\end{bmatrix}e^t + c_3\begin{bmatrix}t+\frac{1}{2}t^2\\t\\1\end{bmatrix}e^t = \begin{bmatrix}c_1e^t + c_2te^t + c_3\left(t+\frac{1}{2}t^2\right)e^t\\c_2e^t + c_3te^t\\c_3e^t\end{bmatrix} \tag{5.6.53}$$

which is *not* a solution of the form $\mathbf{x} = \mathbf{v}e^{\lambda t}$.

Example 5.6.5: (Complex Eigenvalues) The system or ordinary differential equations is

$$\begin{aligned}\frac{dy_1(t)}{dt} &= y_1(t) + 5y_2(t)\\ \frac{dy_2(t)}{dt} &= -y_1(t) + 5y_2(t)\end{aligned} \tag{5.6.54}$$

In this case, the matrix is given by

$$A = \begin{bmatrix}1 & 5\\-1 & 5\end{bmatrix} \tag{5.6.55}$$

and the eigenvalues are easily shown to be given by

$$\begin{aligned}\lambda_1 &= 3 - i\\ \lambda_2 &= 3 + i\end{aligned} \tag{5.6.56}$$

The characteristic subspaces can be shown to be

$$\mathscr{V}(\lambda_1) = \operatorname{span}\left(\begin{bmatrix}2+i\\1\end{bmatrix}\right) \quad \text{and} \quad \mathscr{V}(\lambda_2) = \operatorname{span}\left(\begin{bmatrix}2-i\\1\end{bmatrix}\right) \tag{5.6.57}$$

Therefore, from (5.6.9) the general solution is

$$\mathbf{x}(t) = c_1\begin{bmatrix}2+i\\1\end{bmatrix}e^{(3-i)t} + c_2\begin{bmatrix}2-i\\1\end{bmatrix}e^{(3+i)t} \tag{5.6.58}$$

The solution is usually written in terms of real coefficients, as with Example 5.5.1, by using

$$e^{(3-i)t} = e^{3t}e^{-it} = e^{3t}\left(\cos t - i\sin t\right) \tag{5.6.59}$$

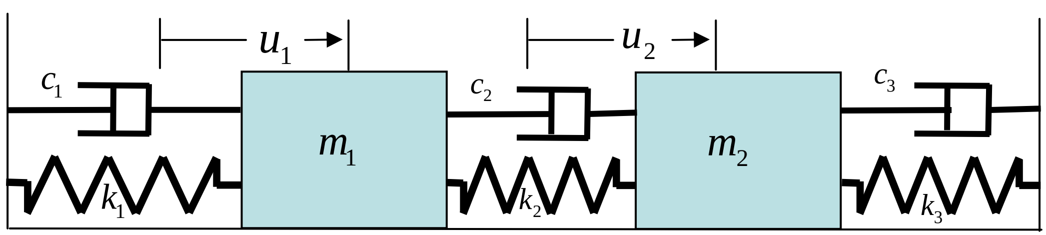
and

$$e^{(3+i)t} = e^{3t}e^{it} = e^{3t}\left(\cos t + i\sin t\right) \tag{5.6.60}$$

Therefore,

$$\begin{aligned}
\mathbf{x}(t) &= c_1\begin{bmatrix} 2+i \\ 1 \end{bmatrix}e^{(3-i)t} + c_2\begin{bmatrix} 2-i \\ 1 \end{bmatrix}e^{(3+i)t} \\
&= c_1\begin{bmatrix} 2+i \\ 1 \end{bmatrix}e^{3t}\left(\cos t - i\sin t\right) + c_2\begin{bmatrix} 2-i \\ 1 \end{bmatrix}e^{3t}\left(\cos t + i\sin t\right) \\
&= \begin{bmatrix} 2\left(c_1+c_2\right)+i\left(c_1-c_2\right) \\ c_1+c_2 \end{bmatrix}e^{3t}\cos t + \begin{bmatrix} c_1+c_2-2i\left(c_1-c_2\right) \\ -i\left(c_1-c_2\right) \end{bmatrix}e^{3t}\sin t \\
&= \begin{bmatrix} 2d_1+d_2 \\ d_1 \end{bmatrix}e^{3t}\cos t + \begin{bmatrix} d_1-2d_2 \\ -d_2 \end{bmatrix}e^{3t}\sin t \\
&= \begin{bmatrix} \left(2d_1+d_2\right)e^{3t}\cos t + \left(d_1-2d_2\right)e^{3t}\sin t \\ d_1e^{3t}\cos t - d_2e^{3t}\sin t \end{bmatrix}
\end{aligned} \tag{5.6.61}$$

Example 5.6.6: (Two Degree of Freedom Free Vibrations) Consider a generalization of the the coupled spring-mass-damper system introduced in Example 5.5.1 shown in the following figure:



The equations of motion for this system are a generalization of (5.5.3)

$$\begin{aligned}
m_1\ddot{u}_1 &= -c_1\dot{u}_1 - k_1u_1 + c_2(\dot{u}_2 - \dot{u}_1) + k_2(u_2 - u_1) \\
m_2\ddot{u}_2 &= -c_2(\dot{u}_2 - \dot{u}_1) - c_3\dot{u}_2 - k_2(u_2 - u_1) - k_3u_2
\end{aligned} \tag{5.6.62}$$

The designation “free vibrations” refers to the fact that this example does not have external applied forces. In other words $f_1\left(t\right) = f_2\left(t\right) = 0$. In order to express the governing equations in normal form define

$$\mathbf{x}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \dot{u}_1(t) \\ \dot{u}_2(t) \end{bmatrix} \tag{5.6.63}$$

and, as a result,

$$\frac{d\mathbf{x}(t)}{dt} = \frac{d}{dt}\begin{bmatrix} u_1(t) \\ u_2(t) \\ \dot{u}_1(t) \\ \dot{u}_2(t) \end{bmatrix} = \begin{bmatrix} \dot{u}_1(t) \\ \dot{u}_2(t) \\ \ddot{u}_1(t) \\ \ddot{u}_2(t) \end{bmatrix} = \begin{bmatrix} \dot{u}_1(t) \\ \dot{u}_2(t) \\ -\frac{c_1}{m_1}\dot{u}_1 - \frac{k_1}{m_1}u_1 + \frac{c_2}{m_1}(\dot{u}_2 - \dot{u}_1) + \frac{k_2}{m_1}(u_2 - u_1) \\ -\frac{c_2}{m_2}(\dot{u}_2 - \dot{u}_1) - \frac{c_3}{m_2}\dot{u}_2 - \frac{k_2}{m_2}(u_2 - u_1) - \frac{k_3}{m_2}u_2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k_1 + k_2}{m_1} & \frac{k_2}{m_1} & -\frac{c_1 + c_2}{m_1} & \frac{c_2}{m_1} \\ \frac{k_2}{m_2} & -\frac{k_2 + k_3}{m_2} & \frac{c_2}{m_2} & -\frac{c_2 + c_3}{m_2} \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \\ \dot{u}_1(t) \\ \dot{u}_2(t) \end{bmatrix}$$

$$= A\mathbf{y}(t)$$

Next, we shall illustrate the solution in the special case where

$$\begin{gathered} \frac{k_1}{m_1} = \frac{k_2}{m_2} = \frac{k_3}{m_3} = 1 \\ c_1 = c_2 = c_3 = 0 \end{gathered} \tag{5.6.64}$$

Therefore,

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k_1+k_2}{m_1} & \frac{k_2}{m_1} & -\frac{c_1+c_2}{m_1} & \frac{c_2}{m_1} \\ \frac{k_2}{m_2} & -\frac{k_2+k_3}{m_2} & \frac{c_2}{m_2} & -\frac{c_2+c_3}{m_2} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 1 & 0 & 0 \\ 1 & -2 & 0 & 0 \end{bmatrix} \tag{5.6.65}$$

The eigenvalues associated with the matrix (5.6.65) are given by

$$\begin{aligned} \begin{vmatrix} 0-\lambda & 0 & 1 & 0 \\ 0 & 0-\lambda & 0 & 1 \\ -2 & 1 & 0-\lambda & 0 \\ 1 & -2 & 0 & 0-\lambda \end{vmatrix} &= -\lambda \begin{vmatrix} -\lambda & 0 & 1 \\ 0 & -\lambda & 1 \\ -2 & 0 & -\lambda \end{vmatrix} + \begin{vmatrix} 0 & -\lambda & 1 \\ -2 & 1 & 0 \\ 1 & -2 & -\lambda \end{vmatrix} \\ &= -\lambda\left(-\lambda \begin{vmatrix} -\lambda & 1 \\ 0 & -\lambda \end{vmatrix} - 2 \begin{vmatrix} 0 & 1 \\ -\lambda & 1 \end{vmatrix} \right) + \left(2 \begin{vmatrix} -\lambda & 1 \\ -2 & -\lambda \end{vmatrix} + \begin{vmatrix} -\lambda & 1 \\ 1 & 0 \end{vmatrix} \right) \\ &= -\lambda\left(-\lambda^3 - 2\lambda\right) + \left(2\left(\lambda^2+2\right) - 1\right) \\ &= \lambda^4 + 4\lambda^2 + 3 = \left(\lambda^2+1\right)\left(\lambda^2+3\right) \\ &= \left(i-\lambda\right)\left(-i-\lambda\right)\left(\sqrt{3}i-\lambda\right)\left(-\sqrt{3}i-\lambda\right) = 0 \end{aligned} \tag{5.6.66}$$

Therefore, we can arbitrarily order the eigenvalues as

$$\begin{aligned} \lambda_1 &= -\sqrt{3}i \\ \lambda_2 &= \sqrt{3}i \\ \lambda_3 &= -i \\ \lambda_4 &= i \end{aligned} \tag{5.6.67}$$

The characteristic subspaces turn out to be

$$\mathscr{V}(\lambda_1)=\operatorname{span}\left(\begin{bmatrix}-\frac{\sqrt{3}}{3}i\\ \frac{\sqrt{3}}{3}i\\ -1\\ 1\end{bmatrix}\right),\mathscr{V}(\lambda_2)=\operatorname{span}\left(\begin{bmatrix}\frac{\sqrt{3}}{3}i\\ -\frac{\sqrt{3}}{3}i\\ -1\\ 1\end{bmatrix}\right),$$
$$\mathscr{V}(\lambda_3)=\operatorname{span}\left(\begin{bmatrix}i\\ i\\ 1\\ 1\end{bmatrix}\right),\mathscr{V}(\lambda_4)=\operatorname{span}\left(\begin{bmatrix}-i\\ -i\\ 1\\ 1\end{bmatrix}\right) \tag{5.6.68}$$

Therefore, from (5.6.9) the general solution is

$$\mathbf{x}(t)=c_1\begin{bmatrix}-\frac{\sqrt{3}}{3}i\\ \frac{\sqrt{3}}{3}i\\ -1\\ 1\end{bmatrix}e^{-\sqrt{3}it}+c_2\begin{bmatrix}\frac{\sqrt{3}}{3}i\\ -\frac{\sqrt{3}}{3}i\\ -1\\ 1\end{bmatrix}e^{\sqrt{3}it}+c_3\begin{bmatrix}i\\ i\\ 1\\ 1\end{bmatrix}e^{-it}+c_4\begin{bmatrix}-i\\ -i\\ 1\\ 1\end{bmatrix}e^{it} \tag{5.6.69}$$

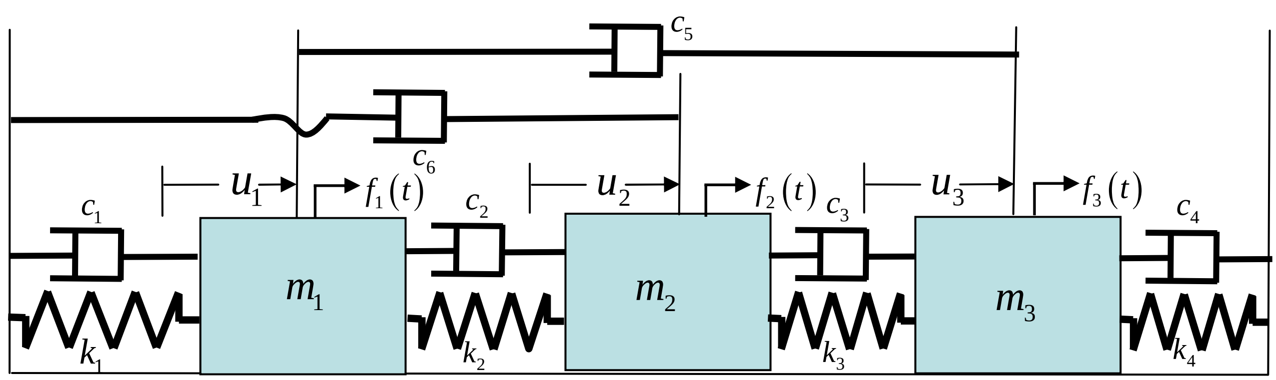
As usual, we can eliminate the complex exponentials by the formulas

$$\begin{aligned}e^{it}&=\cos t+i\sin t\\ e^{-it}&=\cos t-i\sin t\\ e^{\sqrt{3}it}&=\cos\sqrt{3}t+i\sin\sqrt{3}t\\ e^{-\sqrt{3}it}&=\cos\sqrt{3}t-i\sin\sqrt{3}t\end{aligned} \tag{5.6.70}$$

The result of this elimination is

$$\mathbf{x}(t)=\begin{bmatrix}u_1(t)\\ u_2(t)\\ \dot{u}_1(t)\\ \dot{u}_2(t)\end{bmatrix}=\begin{bmatrix}d_1\cos t+d_2\sin t+d_3\cos\sqrt{3}t+d_4\sin\sqrt{3}t\\ d_1\cos t+d_2\sin t-d_3\cos\sqrt{3}t-d_4\sin\sqrt{3}t\\ -d_1\sin t+d_2\cos t-\sqrt{3}d_3\sin\sqrt{3}t+\sqrt{3}d_4\cos\sqrt{3}t\\ -d_1\sin t+d_2\cos t+\sqrt{3}d_3\sin\sqrt{3}t-\sqrt{3}d_4\cos\sqrt{3}t\end{bmatrix} \tag{5.6.71}$$

Example 5.6.7: The following figure shows a possible configuration of a *three degree of freedom* system with linear springs, linear damping and forcing functions.



This example is too complicated to work without the aid of MatLab or something equivalent. In matrix form, the equations of motion for this damped three degree of freedom system are[11]

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}\begin{bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \\ \ddot{u}_3 \end{bmatrix} + \begin{bmatrix} c_1+c_2+c_5 & -c_2 & -c_5 \\ -c_2 & c_2+c_3+c_6 & -c_3 \\ -c_5 & -c_3 & c_3+c_4+c_5 \end{bmatrix}\begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \end{bmatrix}$$
$$+\begin{bmatrix} k_1+k_2 & -k_2 & 0 \\ -k_2 & k_2+k_3 & -k_3 \\ 0 & -k_3 & k_3+k_4 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} f_1(t) \\ f_2(t) \\ f_3(t) \end{bmatrix} \tag{5.6.72}$$

If we define

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \end{bmatrix} \tag{5.6.73}$$

The matrix A in the standard form (5.5.1) is given by

[11] Note that (5.6.72) is in the form mentioned in footnote 6 of Section 5.5.

$$A=\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -\dfrac{k_1+k_2}{m_1} & \dfrac{k_2}{m_1} & 0 & -\dfrac{c_1+c_2+c_5}{m_1} & \dfrac{c_2}{m_1} & \dfrac{c_5}{m_1} \\ \dfrac{k_2}{m_2} & -\dfrac{k_2+k_3}{m_2} & \dfrac{k_3}{m_2} & \dfrac{c_2}{m_2} & -\dfrac{c_2+c_3+c_6}{m_2} & \dfrac{c_3}{m_2} \\ 0 & \dfrac{k_3}{m_3} & -\dfrac{k_3+k_4}{m_3} & \dfrac{c_5}{m_3} & \dfrac{c_3}{m_3} & -\dfrac{c_3+c_4+c_5}{m_3} \end{bmatrix} \tag{5.6.74}$$

and the matrix $\mathbf{g}(t)$ is given by

$$\mathbf{g}(t)=\begin{bmatrix} 0 \\ 0 \\ 0 \\ \dfrac{f_1(t)}{m_1} \\ \dfrac{f_2(t)}{m_2} \\ \dfrac{f_3(t)}{m_3} \end{bmatrix} \tag{5.6.75}$$

If, for the purposes of this example, we adopt the numerical values

$$m_1=m_2=1, m_3=2 \tag{5.6.76}$$

$$k_1=k_2=k_3=1, k_4=2, \tag{5.6.77}$$

$$c_1=.3, c_2=.4, c_3=.4, c_4=.06, c_5=.06, c_6=.02 \tag{5.6.78}$$

The matrix (5.6.74) reduces to

$$A=\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -2 & 1 & 0 & -.76 & .4 & .06 \\ 1 & -2 & 1 & .4 & -.82 & .4 \\ 0 & .5 & -1.5 & .03 & .2 & -.6 \end{bmatrix} \tag{5.6.79}$$

This is the matrix that was adopted when we worked Example 5.3.5. The eigenvalues for this problem were given by equation (5.3.76), repeated,

$$\left.\begin{matrix}\lambda_1\\ \lambda_2\end{matrix}\right\}=-0.6072\pm 1.6652i$$
$$\left.\begin{matrix}\lambda_3\\ \lambda_4\end{matrix}\right\}=-0.2474\pm 1.2611i \tag{5.6.80}$$
$$\left.\begin{matrix}\lambda_5\\ \lambda_6\end{matrix}\right\}=-0.0654\pm 0.8187i$$

and the eigenvectors were given by equations (5.3.77), (5.3.78) and (5.3.79). With this information, one could utilize our solution (5.6.15) and analyze this complicated problem. As indicated above, the problem is too complicated to attempt without the aid of a computer. For our purposes here, we simply want to note that the three degree of freedom system shown in the above figure has six eigenvalues. These six, which occur in complex conjugate pairs, represent three modes of vibration. In each case, the eigenvalues take the general form

$$\lambda_j=-\zeta_j\pm i\omega_j \quad \text{for} \quad j=1,2,3 \tag{5.6.81}$$

The positive numbers

$$\left.\begin{matrix}\zeta_1\\ \zeta_2\\ \zeta_3\end{matrix}\right\}=\left\{\begin{matrix}0.6072\\ 0.2474\\ 0.0654\end{matrix}\right. \tag{5.6.82}$$

are the *damping coefficients*, and the three positive numbers

$$\left.\begin{matrix}\omega_1\\ \omega_2\\ \omega_2\end{matrix}\right\}=\left\{\begin{matrix}1.6652\\ 1.2611\\ .8187\end{matrix}\right. \tag{5.6.83}$$

are the *natural frequencies*.

Exercises

5.6.1 Find the general solution of the system

$$\begin{aligned}\frac{dy_1}{dt} &= 2y_1 - 6y_3\\ \frac{dy_2}{dt} &= y_1 - 3y_3\\ \frac{dy_3}{dt} &= y_2 - 2y_3\end{aligned} \tag{5.6.84}$$

5.6.2 Solve the initial value problem

$$\begin{aligned}\frac{dy_1}{dt} &= \frac{\sqrt{3}}{2}y_1 + \frac{1}{2}y_2\\ \frac{dy_2}{dt} &= -\frac{1}{2}y_1 + \frac{\sqrt{3}}{2}y_2 \qquad y_1(0)=1, y_2(0)=-1\end{aligned} \tag{5.6.85}$$

Section 5.7. Particular Solution

Given the solution to the homogenous equation as generated in Section 5.6, the next question is how to generate the particular solution. This solution, combined with the results of Section 5.6, can then be used along with initial conditions to yield the solution to the initial value problem (5.5.2).

There are various ways to generate the particular solution. The method of *variation of parameters* is probably the best to use at the present point in our understanding of eigenvalue problems. This method builds the particular solution from the solution to the homogeneous equation by a procedure we shall describe. For our purposes, the solution to the homogeneous equation is equation (5.6.15), repeated,

$$\mathbf{x}_h(t) = Te^{Dt}\mathbf{c} \tag{5.7.1}$$

The variation of parameters method seeks a particular solution of the general form of (5.7.1) except that the constants in the matrix $\mathbf{c}$ are replaced by a function that must be determined. The form of the particular solution we shall adopt is

$$\mathbf{x}_p(t) = Te^{Dt}\mathbf{w}(t) \tag{5.7.2}$$

where the $N \times 1$ column matrix of functions $\mathbf{w}(t)$ is the quantity to be determined. If (5.7.2) is substituted into the ordinary differential equation (5.5.1), the result is

$$Te^{Dt}\frac{d\mathbf{w}(t)}{dt} = (AT - TD)e^{Dt}\mathbf{w}(t) + \mathbf{g}(t) \tag{5.7.3}$$

Equation (5.6.12) tells us that the first on the right side of (5.7.3) is zero, and, as a result, the column vector $\mathbf{w}(t)$ is determined by

$$\frac{d\mathbf{w}(t)}{dt} = e^{-Dt}T^{-1}\mathbf{g}(t) \tag{5.7.4}$$

where the notation e^{-Dt} stands for the square matrix

$$e^{-Dt} = \begin{bmatrix} e^{-\lambda_1 t} & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & e^{-\lambda_2 t} & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & e^{-\lambda_3 t} & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & e^{-\lambda_N t} \end{bmatrix} \tag{5.7.5}$$

It is elementary to show that the square matrix e^{-Dt} is the inverse of the matrix e^{Dt} defined by equation (5.6.14). It follows from (5.7.4) that the solution for $\mathbf{w}(t)$ is given by the integral

$$\mathbf{w}(t) = \int_{\tau=0}^{\tau=t} e^{-D\tau} T^{-1} \mathbf{g}(\tau) d\tau \tag{5.7.6}$$

where, without loss of generality, we have taken selected the constant of integration such that $\mathbf{w}(0) = \mathbf{0}$.

Equation (5.7.6) when combined with (5.7.2) gives the particular solution

$$\begin{aligned} \mathbf{x}_p(t) &= Te^{Dt}\mathbf{w}(t) = Te^{Dt}\int_{\tau=0}^{\tau=t} e^{-D\tau} T^{-1}\mathbf{g}(\tau) d\tau \\ &= \int_{\tau=0}^{\tau=t} Te^{-D(\tau-t)} T^{-1}\mathbf{g}(\tau) d\tau \end{aligned} \tag{5.7.7}$$

Given (5.7.1) and (5.7.7) the solution (5.6.1) is

$$\begin{aligned} \mathbf{x}(t) &= \mathbf{x}_h(t) + \mathbf{x}_p(t) \\ &= Te^{Dt}\mathbf{c} + \int_{\tau=0}^{\tau=t} Te^{-D(\tau-t)} T^{-1}\mathbf{g}(\tau) d\tau \end{aligned} \tag{5.7.8}$$

Given (5.7.8), the solution to the *initial value problem* (5.5.2) is obtained by evaluating (5.7.8) at $t = 0$ which yields

$$\mathbf{c} = T^{-1}\mathbf{x}_0 \tag{5.7.9}$$

and allows the solution (5.7.8) to be written in terms of the initial condition as follows:

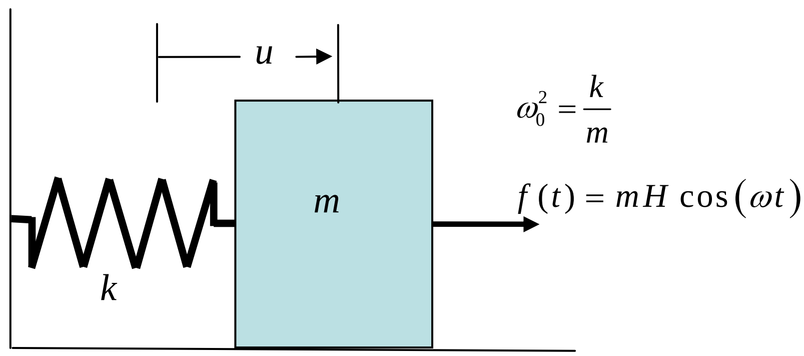
$$\mathbf{x}(t) = Te^{Dt}T^{-1}\mathbf{x}_0 + \int_{\tau=0}^{\tau=t} Te^{-D(\tau-t)} T^{-1}\mathbf{g}(\tau) d\tau \tag{5.7.10}$$

Equation (5.7.10) is extremely general. Other than presuming the matrix A in (5.5.1) can be diagonalized, i.e. has a basis of N linearly independent eigenvectors, it provides the solution to a wide class of systems of inhomogeneous linear constant coefficient ordinary differential equations that occur in the applications. When A cannot be diagonalized, there is a generalization of (5.7.10) that provides the solution. This generalization will be discussed in Chapter 6. In this section, we shall restrict our discussions to cases where (5.7.10) is valid.

Example 5.7.1: As an illustration of how to use (5.7.10), consider the following initial value problem

$$\ddot{u}+\omega_0^2 u = H\cos(\omega t) \qquad u(0)=u_0,\ \dot{u}(0)=\upsilon_0 \tag{5.7.11}$$

We generated the solution for the homogeneous version of this equation in Example 5.6.1. In this example, the forcing function $H\cos(\omega t)$ is included in the ordinary differential equation. The frequency ω is sometimes called the *forcing frequency*. The frequency ω_0 is called the *natural frequency*. The figure that is associated with this ordinary differential equation is



The matrix form of $(5.7.11)_1$ that we are trying to solve is

$$\frac{d\mathbf{x}(t)}{dt} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\omega_0^2 & 0 \end{bmatrix}}_{A}\mathbf{x}(t) + \underbrace{\begin{bmatrix} 0 \\ H\cos(\omega t) \end{bmatrix}}_{\mathbf{g}(\mathbf{t})} \tag{5.7.12}$$

where $\mathbf{x}(t)$ is again defined by (5.6.18). It follows from $(5.7.11)_2$ that the initial condition on (5.7.12) is

$$\mathbf{x}(0) = \mathbf{x}_0 = \begin{bmatrix} u_0 \\ \upsilon_0 \end{bmatrix} \tag{5.7.13}$$

In Example 5.6.1, we looked at the homogeneous version of this problem and derived its general solution. Among the results from Example 5.6.1 that apply to this example is the solution to the homogeneous equation written in the form (5.6.27), repeated,

$$\mathbf{x}_h(t)=\begin{bmatrix}1 & 1\\ i\omega_0 & -i\omega_0\end{bmatrix}\begin{bmatrix}e^{i\omega_0 t} & 0\\ 0 & e^{-i\omega_0 t}\end{bmatrix}\begin{bmatrix}c_1\\ c_2\end{bmatrix} \tag{5.7.14}$$

It follows from (5.7.14) and (5.7.1) that the transition matrix in this case is

$$T=\begin{bmatrix}1 & 1\\ i\omega_0 & -i\omega_0\end{bmatrix} \tag{5.7.15}$$

The inverse of (5.7.15) is the matrix

$$T^{-1}=-\frac{1}{2i\omega_0}\begin{bmatrix}-i\omega_0 & -1\\ -i\omega_0 & 1\end{bmatrix}=\frac{1}{2}\begin{bmatrix}1 & -\dfrac{i}{\omega_0}\\ 1 & \dfrac{i}{\omega_0}\end{bmatrix} \tag{5.7.16}$$

Given (5.7.16), (5.7.15), (5.7.13) and the $\mathbf{g}(t)$ identified in (5.7.12), the solution (5.7.10) takes the form

$$\begin{aligned}\mathbf{x}(t)&=Te^{Dt}T^{-1}\mathbf{x}_0+\int_{\tau=0}^{\tau=t}Te^{-D(\tau-t)}T^{-1}\mathbf{g}(\tau)\,d\tau\\ &=\frac{1}{2}\begin{bmatrix}1 & 1\\ i\omega_0 & -i\omega_0\end{bmatrix}\begin{bmatrix}e^{i\omega_0 t} & 0\\ 0 & e^{-i\omega_0 t}\end{bmatrix}\begin{bmatrix}1 & -\dfrac{i}{\omega_0}\\ 1 & \dfrac{i}{\omega_0}\end{bmatrix}\begin{bmatrix}u_0\\ \upsilon_0\end{bmatrix}\\ &\quad+\frac{1}{2}\int_{\tau=0}^{\tau=t}\begin{bmatrix}1 & 1\\ i\omega_0 & -i\omega_0\end{bmatrix}\begin{bmatrix}e^{-i\omega_0(\tau-t)} & 0\\ 0 & e^{i\omega_0(\tau-t)}\end{bmatrix}\begin{bmatrix}1 & -\dfrac{i}{\omega_0}\\ 1 & \dfrac{i}{\omega_0}\end{bmatrix}\begin{bmatrix}0\\ \cos(\omega\tau)\end{bmatrix}d\tau\end{aligned} \tag{5.7.17}$$

If the various square matrices are multiplied together, (5.7.17) can be written

$$\begin{aligned}\mathbf{x}(t)&=\begin{bmatrix}\dfrac{1}{2}\left(e^{i\omega_0 t}+e^{-i\omega_0 t}\right) & \dfrac{1}{2i\omega_0}\left(e^{i\omega_0 t}-e^{-i\omega_0 t}\right)\\ \dfrac{i\omega_0}{2}\left(e^{i\omega_0 t}-e^{-i\omega_0 t}\right) & \dfrac{1}{2}\left(e^{i\omega_0 t}+e^{-i\omega_0 t}\right)\end{bmatrix}\begin{bmatrix}u_0\\ \upsilon_0\end{bmatrix}\\ &\quad+\int_{\tau=0}^{\tau=t}\begin{bmatrix}\dfrac{1}{2}\left(e^{-i\omega_0(\tau-t)}+e^{i\omega_0(\tau-t)}\right) & \dfrac{1}{2i\omega_0}\left(e^{-i\omega_0(\tau-t)}-e^{i\omega_0(\tau-t)}\right)\\ \dfrac{i\omega_0}{2}\left(e^{-i\omega_0(\tau-t)}-e^{i\omega_0(\tau-t)}\right) & \dfrac{1}{2}\left(e^{-i\omega_0(\tau-t)}+e^{i\omega_0(\tau-t)}\right)\end{bmatrix}\begin{bmatrix}0\\ \cos(\omega\tau)\end{bmatrix}d\tau\end{aligned} \tag{5.7.18}$$

If trigometric functions are introduced in place of the complex exponentials, (5.7.18) simplifies further to

$$\begin{aligned}
\mathbf{x}(t) &= \begin{bmatrix} \cos(\omega_0 t) & \dfrac{1}{\omega_0}\sin(\omega_0 t) \\ -\omega_0 \sin(\omega_0 t) & \cos(\omega_0 t) \end{bmatrix} \begin{bmatrix} u_0 \\ \upsilon_0 \end{bmatrix} \\
&\quad + H\int_{\tau=0}^{\tau=t} \begin{bmatrix} \cos(\omega_0(\tau - t)) & -\dfrac{1}{\omega_0}\sin(\omega_0(\tau - t)) \\ \omega_0 \sin(\omega_0(\tau - t)) & \cos(\omega_0(\tau - t)) \end{bmatrix} \begin{bmatrix} 0 \\ \cos(\omega\tau) \end{bmatrix} d\tau \\
&= \begin{bmatrix} u_0 \cos(\omega_0 t) + \dfrac{\upsilon_0}{\omega_0}\sin(\omega_0 t) \\ -\omega_0 u_0 \sin(\omega_0 t) + \upsilon_0 \cos(\omega_0 t) \end{bmatrix} + H\int_{\tau=0}^{\tau=t} \begin{bmatrix} -\dfrac{1}{\omega_0}\sin(\omega_0(\tau - t))\cos(\omega\tau) \\ \cos(\omega_0(\tau - t))\cos(\omega\tau) \end{bmatrix} d\tau
\end{aligned} \tag{5.7.19}$$

If the integrations in (5.7.19)are carried out, the results are

$$\mathbf{x}(t) = \begin{bmatrix} u(t) \\ \dot{u}(t) \end{bmatrix} = \begin{bmatrix} u_0 \cos(\omega_0 t) + \dfrac{\upsilon_0}{\omega_0}\sin(\omega_0 t) + \dfrac{H}{(\omega^2 - \omega_0^2)}(\cos(\omega_0 t) - \cos(\omega t)) \\ -\omega_0 u_0 \sin(\omega_0 t) + \upsilon_0 \cos(\omega_0 t) + \dfrac{H}{(\omega^2 - \omega_0^2)}(-\omega_0 \sin(\omega_0 t) + \omega \sin(\omega t)) \end{bmatrix} \tag{5.7.20}$$

This result shows that the harmonic motion consists of two components. One that oscillates with the frequency ω_0 and one with the forcing frequency ω. Equation (5.7.20) also displays the phenomena known as *resonance*. As the forcing frequency ω becomes close to the natural frequency ω_0, the amplitude of the displacement grows large.

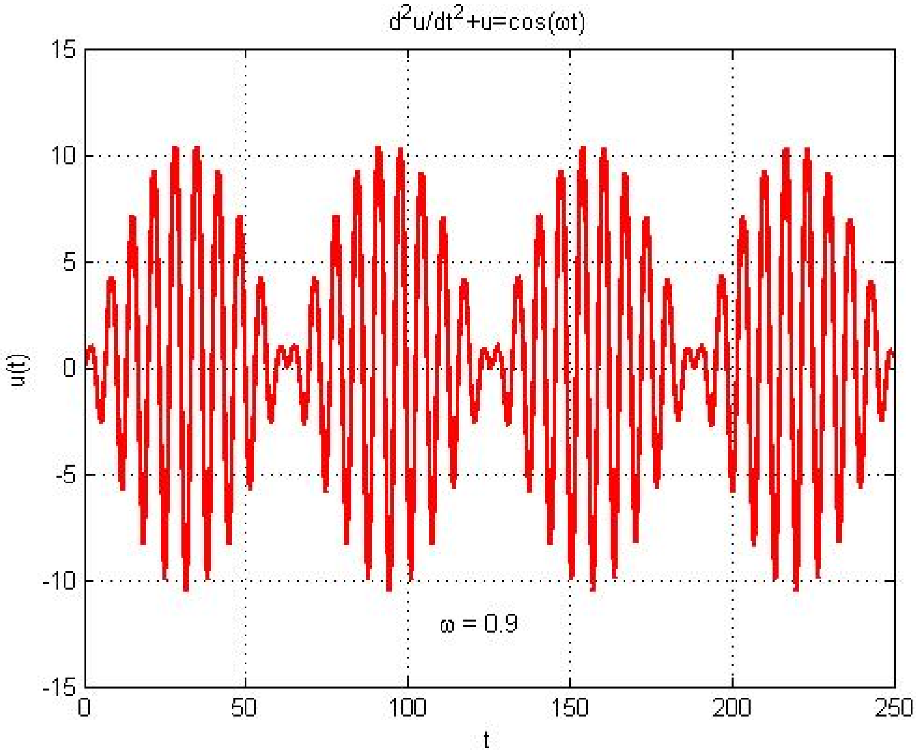
There is an interesting special case of (5.7.20) that we shall examine next. In this special case, we select the initial conditions (5.7.13) to be $u_0 = \upsilon_0 = 0$. These assumptions reduce the first of (5.7.20) to

$$u(t) = \frac{H}{(\omega^2 - \omega_0^2)}(\cos(\omega_0 t) - \cos(\omega t)) \tag{5.7.21}$$

An equivalent form of (5.7.21) is

$$u(t) = \frac{2H}{\omega_0^2 - \omega^2}(\sin(\frac{\omega_0 + \omega}{2}t)\sin(\frac{\omega_0 - \omega}{2}t)) \tag{5.7.22}$$

Thus, the solution is a combination of a high frequency component with frequency $\frac{\omega_0+\omega}{2}$ and a low frequency component with frequency $\frac{\omega_0-\omega}{2}$. In the case where we simplify the numeric's with the choices $H=1$, $\omega_0=1$ and $\omega=0.9$, (5.7.22) produces the figure

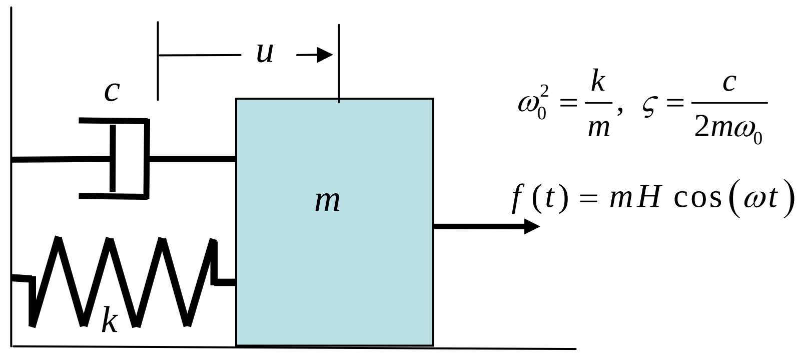


This figure displays the classical "beats" phenomenon which is the result of superimposing two frequencies that are close in value. The figure displays two distinct effects. The effects are a high frequency oscillation contained within a slow oscillation.

Example 5.7.2: If we repeat the above example except adopt as the starting place the ordinary differential equation

$$\ddot{u}(t)+2\varsigma\omega_0\dot{u}(t)+\omega_0{}^2u(t)=H\cos\left(\omega t\right) \tag{5.7.23}$$

we will be studying *damped forced vibrations*. Example 5.6.2 discussed the free vibration or unforced vibration problem for this ordinary differential equation. The following figure applies to this case.



If we define, as with our earlier examples,

$$\mathbf{x}(t) = \begin{bmatrix} u(t) \\ \dot{u}(t) \end{bmatrix} \tag{5.7.24}$$

The matrix form of $(5.7.11)_1$ that we are trying to solve is

$$\frac{d\mathbf{x}(t)}{dt} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\omega_0^2 & -2\zeta\omega_0 \end{bmatrix}}_{A} \mathbf{x}(t) + \underbrace{\begin{bmatrix} 0 \\ H\cos(\omega t) \end{bmatrix}}_{\mathbf{g}(\mathbf{t})} \tag{5.7.25}$$

Recall from Example 5.6.2 that the eigenvalues and the transition matrix for this problem are

$$\begin{aligned} \lambda_1 &= -\zeta\omega_0 + i\omega_0\sqrt{1-\zeta^2} \\ \lambda_2 &= -\zeta\omega_0 - i\omega_0\sqrt{1-\zeta^2} \end{aligned} \tag{5.7.26}$$

and

$$T = \begin{bmatrix} 1 & 1 \\ -\zeta\omega_0 + i\omega_0\sqrt{1-\zeta^2} & -\zeta\omega_0 - i\omega_0\sqrt{1-\zeta^2} \end{bmatrix} \tag{5.7.27}$$

We shall continue to consider the *under damped* case. In other words, we are assuming that $\sqrt{1-\zeta^2} > 0$.

The inverse of (5.7.27) is the matrix

$$T^{-1} = -\frac{1}{2i\omega_0\sqrt{1-\zeta^2}}\begin{bmatrix} -\zeta\omega_0 - i\omega_0\sqrt{1-\zeta^2} & -1 \\ \zeta\omega_0 - i\omega_0\sqrt{1-\zeta^2} & 1 \end{bmatrix} = \frac{1}{2\sqrt{1-\zeta^2}}\begin{bmatrix} i\left(-\zeta - i\sqrt{1-\zeta^2}\right) & -\frac{i}{\omega_0} \\ -i\left(-\zeta + i\sqrt{1-\zeta^2}\right) & \frac{i}{\omega_0} \end{bmatrix} \tag{5.7.28}$$

Given (5.7.28), (5.7.27), (5.7.26) and the $\mathbf{g}(t)$ identified in (5.7.25), the solution (5.7.10) takes the form

$$\begin{aligned}
\mathbf{x}(t) &= Te^{Dt}T^{-1}\mathbf{x}_0 + \int_{\tau=0}^{\tau=t} Te^{-D(\tau-t)}T^{-1}\mathbf{g}(\tau)\,d\tau \\
&= \frac{e^{-\zeta\omega_0 t}}{2\sqrt{1-\zeta^2}}\begin{bmatrix} 1 & 1 \\ -\zeta\omega_0 + i\omega_0\sqrt{1-\zeta^2} & -\zeta\omega_0 - i\omega_0\sqrt{1-\zeta^2} \end{bmatrix}\begin{bmatrix} e^{i\omega_0\sqrt{1-\zeta^2}\,t} & 0 \\ 0 & e^{-i\omega_0\sqrt{1-\zeta^2}\,t} \end{bmatrix}\times \\
&\quad \begin{bmatrix} i\left(-\zeta - i\sqrt{1-\zeta^2}\right) & -\frac{i}{\omega_0} \\ -i\left(-\zeta + i\sqrt{1-\zeta^2}\right) & \frac{i}{\omega_0} \end{bmatrix}\begin{bmatrix} u_0 \\ \upsilon_0 \end{bmatrix} \\
&\quad + \frac{H}{2\sqrt{1-\zeta^2}}\int_{\tau=0}^{\tau=t} e^{\zeta\omega_0(\tau-t)}\left(\begin{matrix} \begin{bmatrix} 1 & 1 \\ -\zeta\omega_0 + i\omega_0\sqrt{1-\zeta^2} & -\zeta\omega_0 - i\omega_0\sqrt{1-\zeta^2} \end{bmatrix}\times \\ \begin{bmatrix} e^{-i\omega_0\sqrt{1-\zeta^2}(\tau-t)} & 0 \\ 0 & e^{i\omega_0\sqrt{1-\zeta^2}(\tau-t)} \end{bmatrix}\times \\ \begin{bmatrix} i\left(-\zeta - i\sqrt{1-\zeta^2}\right) & -\frac{i}{\omega_0} \\ -i\left(-\zeta + i\sqrt{1-\zeta^2}\right) & \frac{i}{\omega_0} \end{bmatrix} \end{matrix}\right)\begin{bmatrix} 0 \\ \cos(\omega\tau) \end{bmatrix}
\end{aligned} \tag{5.7.29}$$

If the various square matrices in (5.7.29)are multiplied together, you obtain

$$\mathbf{x}(t)=\frac{e^{-\zeta\omega_0 t}}{2\sqrt{1-\zeta^2}}\begin{bmatrix}\dfrac{2\sqrt{1-\zeta^2}\cos\left(\omega_0\sqrt{1-\zeta^2}\,t\right)+2\zeta\sin\left(\omega_0\sqrt{1-\zeta^2}\,t\right)}{\sqrt{1-\zeta^2}} & \dfrac{2}{\omega_0}\sin\left(\omega_0\sqrt{1-\zeta^2}\,t\right)\\ -2\omega_0\sin\left(\omega_0\sqrt{1-\zeta^2}\,t\right) & 2\cos\left(\omega_0\sqrt{1-\zeta^2}\,t\right)\left(\sqrt{1-\zeta^2}-2\zeta\sin\left(\omega_0\sqrt{1-\zeta^2}\,t\right)\right)\end{bmatrix}\begin{bmatrix}u_0\\ \upsilon_0\end{bmatrix}$$

$$+\frac{H}{2\sqrt{1-\zeta^2}}\int_{\tau=0}^{\tau=t}e^{\zeta\omega_0(\tau-t)}\begin{bmatrix}\left(\dfrac{2\sqrt{1-\zeta^2}\cos\left(\omega_0\sqrt{1-\zeta^2}\,(\tau-t)\right)-2\zeta\sin\left(\omega_0\sqrt{1-\zeta^2}\,(\tau-t)\right)}{\sqrt{1-\zeta^2}}\right) & -\dfrac{2}{\omega_0}\sin\left(\omega_0\sqrt{1-\zeta^2}\,(\tau-t)\right)\\ 2\omega_0\sin\left(\omega_0\sqrt{1-\zeta^2}\,(\tau-t)\right) & \left(2\cos\left(\omega_0\sqrt{1-\zeta^2}\,(\tau-t)\right)\times\left(\sqrt{1-\zeta^2}+2\zeta\sin\left(\omega_0\sqrt{1-\zeta^2}\,(\tau-t)\right)\right)\right)\end{bmatrix}\begin{bmatrix}0\\ \cos(\omega\tau)\end{bmatrix}\tag{5.7.30}$$

If you keep expanding will eventually find, for the first row of (5.7.30)

$$\begin{aligned}u(t)&=e^{-\zeta\omega_0 t}\left(u_0\cos\left(\omega_0(\sqrt{1-\zeta^2})t\right)+\frac{1}{\sqrt{1-\zeta^2}}\left(\frac{\upsilon_0}{\omega_0}+\zeta u_0\right)\sin\left(\omega_0(\sqrt{1-\zeta^2})t\right)\right)\\&\quad+\frac{H}{\omega_0\sqrt{1-\zeta^2}}\int_{\tau=0}^{\tau=t}e^{-\zeta\omega_0(t-\tau)}\sin\left(\omega_0(\sqrt{1-\zeta^2})(t-\tau)\right)\cos(\omega\tau)\,d\tau\end{aligned}\tag{5.7.31}$$

The integration formula

$$\begin{aligned}&\frac{1}{\omega_0\sqrt{1-\zeta^2}}\int_{\tau=0}^{\tau=t}e^{-\zeta\omega_0(t-\tau)}\sin\left(\omega_0(\sqrt{1-\zeta^2})(t-\tau)\right)\cos(\omega\tau)\,d\tau\\&=\frac{\omega_0^2-\omega^2}{\left(\omega_0^2-\omega^2\right)^2+4\zeta^2\omega_0^2\omega^2}\left(\cos(\omega t)-e^{-\zeta\omega_0 t}\cos\left(\omega_0\sqrt{1-\zeta^2}\,t\right)\right)\\&+\frac{\zeta}{\left(\omega_0^2-\omega^2\right)^2+4\zeta^2\omega_0^2\omega^2}\left(2\omega\omega_0\sin(\omega t)-\frac{\omega_0^2+\omega^2}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_0 t}\sin\left(\omega_0\sqrt{1-\zeta^2}\,t\right)\right)\end{aligned}\tag{5.7.32}$$

can be substituted into (5.7.31) to get the complicated formula [12]

$$
\begin{aligned}
u(t) = {} & e^{-\zeta\omega_0 t}\left(u_0 \cos\left(\omega_0(\sqrt{1-\zeta^2})t\right) + \frac{1}{\sqrt{1-\zeta^2}}\left(\frac{\upsilon_0}{\omega_0} + \zeta u_0\right)\sin\left(\omega_0(\sqrt{1-\zeta^2})t\right)\right) \\
& + H\frac{\omega_0^2 - \omega^2}{\left(\omega_0^2 - \omega^2\right)^2 + 4\zeta^2\omega_0^2\omega^2}\left(\cos(\omega t) - e^{-\zeta\omega_0 t}\cos\left(\omega_0\sqrt{1-\zeta^2}\,t\right)\right) \\
& + H\frac{\zeta}{\left(\omega_0^2 - \omega^2\right)^2 + 4\zeta^2\omega_0^2\omega^2}\left(2\omega\omega_0 \sin(\omega t) - \frac{\omega_0^2 + \omega^2}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_0 t}\sin\left(\omega_0\sqrt{1-\zeta^2}\,t\right)\right)
\end{aligned}
\tag{5.7.33}
$$

This formula reduces to the earlier result, (5.7.20), in the special case $\zeta = 0$. Equation (5.7.33) can be rearranged into the form of a transient solution plus a steady state solution.

$$
\begin{aligned}
u(t) = {} & \underbrace{\begin{aligned} & \left(u_0 - H\frac{\omega_0^2 - \omega^2}{\left(\omega_0^2 - \omega^2\right)^2 + 4\zeta^2\omega_0^2\omega^2}\right) e^{-\zeta\omega_0 t}\cos\left(\omega_0(\sqrt{1-\zeta^2})t\right) \\ & + \left(\frac{1}{\sqrt{1-\zeta^2}}\left(\frac{\upsilon_0}{\omega_0} + \zeta u_0\right) - H\frac{\zeta}{\sqrt{1-\zeta^2}}\frac{\omega_0^2 + \omega^2}{\left(\omega_0^2 - \omega^2\right)^2 + 4\zeta^2\omega_0^2\omega^2}\right) e^{-\zeta\omega_0 t}\sin\left(\omega_0\sqrt{1-\zeta^2}\,t\right) \end{aligned}}_{\text{Goes to zero as } t \text{ grows = Transient Solution}} \\
& + \underbrace{H\left(\frac{\omega_0^2 - \omega^2}{\left(\omega_0^2 - \omega^2\right)^2 + 4\zeta^2\omega_0^2\omega^2}\cos(\omega t) + \frac{2\zeta\omega\omega_0}{\left(\omega_0^2 - \omega^2\right)^2 + 4\zeta^2\omega_0^2\omega^2}\sin(\omega t)\right)}_{\text{Steady State Solution}}
\end{aligned}
\tag{5.7.34}
$$

Within the context of the examples we have been working, equation (5.7.34) is very general. It contains, as special cases, Examples 5.6.1 and 5.6.2 discussed earlier: It is helpful to list these cases more precisely.

Important Special Cases:

a) Undamped Free Vibrations with initial conditions $u(0) = u_0$ and $\dot{u}(0) = \upsilon_0$ (See (5.6.29))

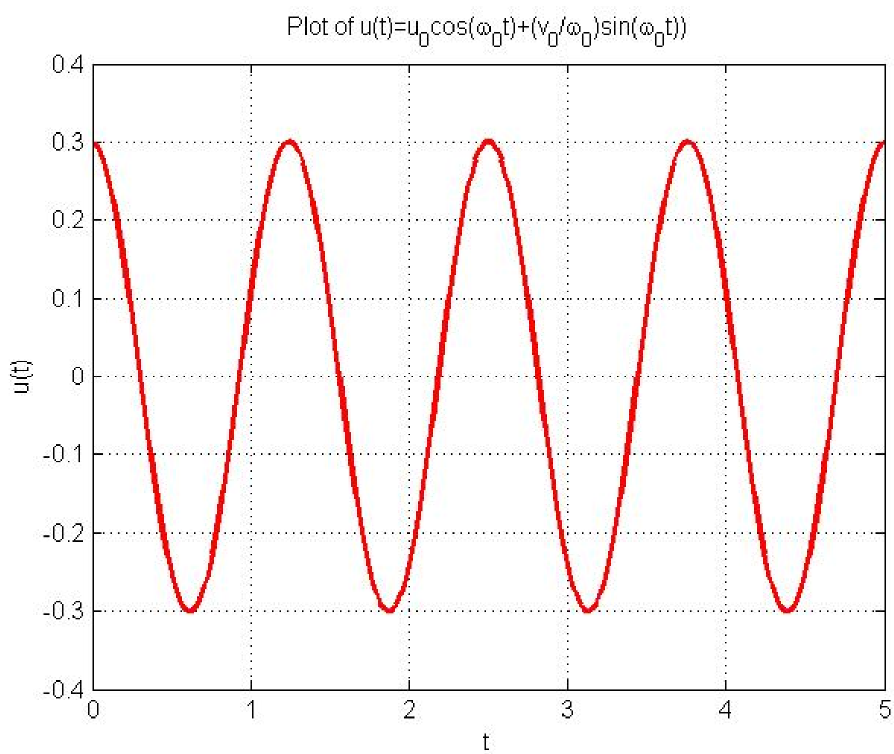
[12] You can find the integral above in a good table of integrals. MatLab will also do the integration.

$$u(t) = u_0 \cos(\omega_0 t) + \frac{\upsilon_0}{\omega_0} \sin(\omega_0 t) \tag{5.7.35}$$

With the numerical values

$$\begin{aligned} &m = 10kg, \quad k = 250kg/\sec^2 \\ &u_0 = \frac{3}{10}m, \quad \upsilon_0 = -\frac{1}{10}m/\sec \end{aligned} \tag{5.7.36}$$

equation (5.7.36) produces the plot



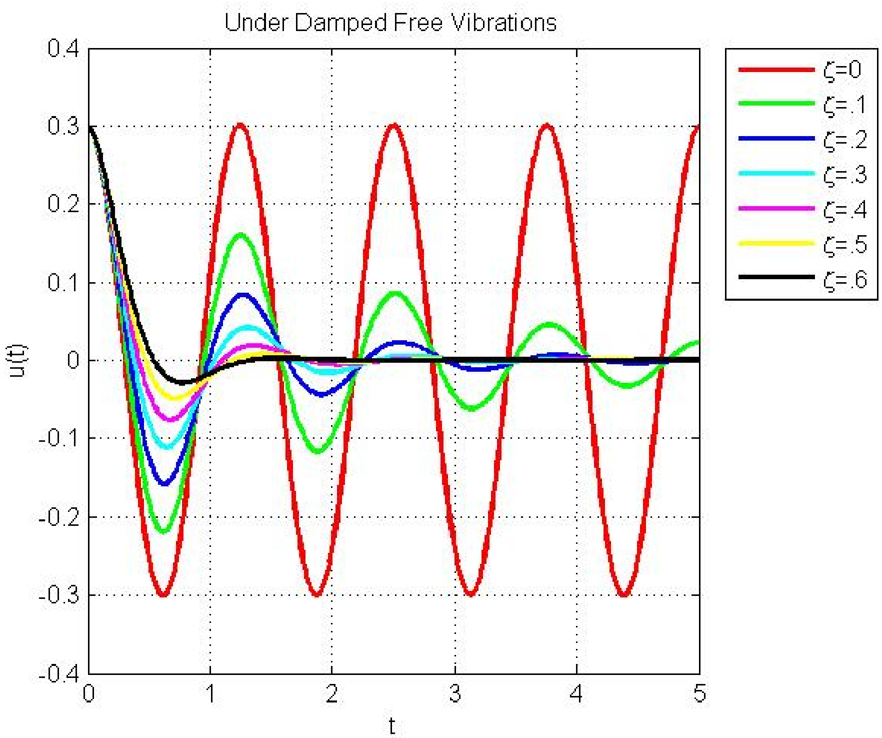
b) Damped Free Vibrations with initial conditions $u(0) = u_0$ and $\dot{u}(0) = \upsilon_0$ (See (5.6.44))

$$u(t) = e^{-\zeta\omega_0 t}\left(u_0 \cos\left(\omega_0(\sqrt{1-\zeta^2})t\right) + \frac{1}{\sqrt{1-\zeta^2}}\left(\frac{\upsilon_0}{\omega_0} + \zeta u_0\right)\sin\left(\omega_0(\sqrt{1-\zeta^2})t\right)\right) \tag{5.7.37}$$

With the numerical values

$$\begin{aligned}&m = 10kg, \quad k = 250kg/\sec^2 \\ &u_0 = \frac{3}{10}m, \quad \upsilon_0 = -\frac{1}{10}m/\sec\end{aligned} \tag{5.7.38}$$

and a family of values of the damping coefficient equation (5.7.37) produces the plots

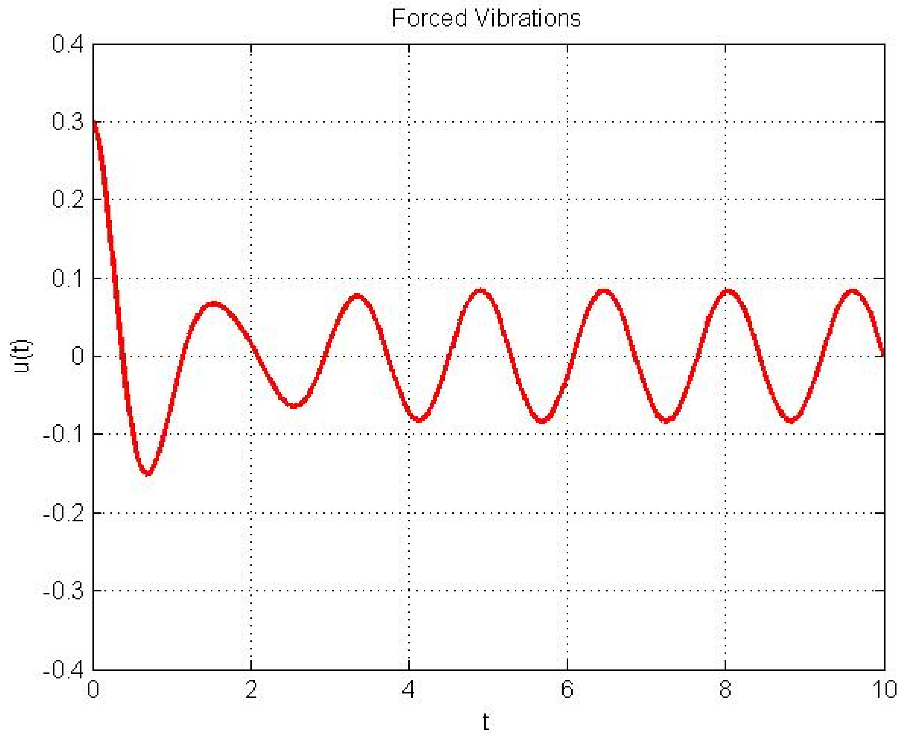


c) Damped Forced Vibrations with initial conditions $u(0) = u_0$ and $\dot{u}(0) = \upsilon_0$ (See (5.7.34)).

With the numerical values

$$\begin{aligned}&m = 10kg, \quad k = 250kg/\sec^2, \quad c = 20, \\ &\omega = 4, \quad H = 1, \\ &u_0 = \frac{3}{10}m, \quad \upsilon_0 = -\frac{1}{10}m/\sec\end{aligned} \tag{5.7.39}$$

the resulting plot of (5.7.34) is



This figure illustrates how the solution quickly evolves to the steady state solution.

Exercises

5.7.1 Find the solution of the initial value problem

$$\begin{aligned}\frac{dy_1}{dt} &= 2y_1 - 6y_3 + 1\\ \frac{dy_2}{dt} &= y_1 - 3y_3 \qquad\qquad y_1(0)=2,\ y_2(0)=y_3(0)=0\\ \frac{dy_3}{dt} &= y_2 - 2y_3\end{aligned} \tag{5.7.40}$$

Answer:

$$\begin{bmatrix} y_1(t)\\ y_2(t)\\ y_3(t)\end{bmatrix} = \begin{bmatrix} e^{-t} + 9e^{t} - 3t - 8\\ \frac{1}{2}e^{-t} + \frac{9}{2}e^{t} - 2t - 5\\ \frac{1}{2}e^{-t} + \frac{3}{2}e^{t} - t - 2\end{bmatrix} \tag{5.7.41}$$

Chapter 6

ADDITIONAL TOPICS RELATING TO EIGENVALUE PROBLEMS

Section 6.1 Characteristic Polynomial and Fundamental Invariants

When the eigenvalue problem was introduced in Chapter 5, we introduced in Section 5.2 the characteristic polynomial and the associated idea of fundamental invariants. In this section, we shall built upon the results in Section 5.2 and introduce a few new results that will be useful in this chapter.

Our formulas are going to involve powers of the linear transformation $\mathbf{A}:\mathscr{V}\rightarrow\mathscr{V}$. As we have utilized earlier with matrices, the n^{th} power of a linear transformation $\mathbf{A}:\mathscr{V}\rightarrow\mathscr{V}$ is defined by

$$\mathbf{A}^0=\mathbf{I},\quad \mathbf{A}^n=\underbrace{\mathbf{AAA}\cdot\cdot\mathbf{A}}_{n\ \text{times}}\quad \text{for}\quad n=1,2,... \tag{6.1.1}$$

It follows from the definition (6.1.1) that the linear transformations $\mathbf{A}^n$ and $\mathbf{A}^m$ commute. In other words,

$$\mathbf{A}^m\mathbf{A}^n=\mathbf{A}^n\mathbf{A}^m=\mathbf{A}^{m+n} \tag{6.1.2}$$

Given (6.1.1), a *polynomial* in $\mathbf{A}:\mathscr{V}\rightarrow\mathscr{V}$ is a linear transformation $g(\mathbf{A})$ of the form

$$g(\mathbf{A})=\alpha_0\mathbf{I}+\alpha_1\mathbf{A}+\alpha_2\mathbf{A}^2+\cdots+\alpha_n\mathbf{A}^n \tag{6.1.3}$$

where n is a positive integer and the coefficients $\alpha_0,\alpha_1,\alpha_2,...,\alpha_n$ are real numbers.

The characteristic polynomial of a linear transformation $\mathbf{A}:\mathscr{V}\rightarrow\mathscr{V}$ is given by equation (5.2.14), repeated,

$$f(\lambda)=\det(\mathbf{A}-\lambda\mathbf{I})=(-\lambda)^N+\mu_1(-\lambda)^{N-1}+\cdots+\mu_{N-1}(-\lambda)+\mu_N \tag{6.1.4}$$

The coefficients $\mu_1,\mu_2,...,\mu_N$ are the fundamental invariants of $\mathbf{A}:\mathscr{V}\rightarrow\mathscr{V}$. In Section 5.2, we observed that

$$\mu_1 = \operatorname{tr}\mathbf{A} \tag{6.1.5}$$

and

$$\mu_N = \det\mathbf{A} \tag{6.1.6}$$

For the case $N=2$, we gave the formula $(5.2.20)_2$ for the invariant μ_2. The explicit formula for the invariants in the general case turns out to be rather complicated. If $\{\mathbf{e}_1,\mathbf{e}_2,\ldots,\mathbf{e}_N\}$ is a basis for $\mathscr{V}$, then it is possible to show that the j^{th} fundamental invariant is given by

$$\mu_j = \frac{1}{j!}\sum_{q_1,q_2,\ldots q_j=1}^{N}\sum_{i_1,i_2,\ldots i_j=1}^{N}\begin{vmatrix} \delta_{i_1}^{q_1} & \delta_{i_2}^{q_1} & \cdot & \cdot & \cdot & \delta_{i_j}^{q_1} \\ \delta_{i_1}^{q_2} & \delta_{i_2}^{q_2} & \cdot & \cdot & \cdot & \delta_{i_j}^{q_2} \\ \cdot & & \cdot & & & \\ \cdot & & & \cdot & & \\ \cdot & & & & \cdot & \\ \delta_{i_1}^{q_j} & \delta_{i_2}^{q_j} & \cdot & \cdot & \cdot & \delta_{i_j}^{q_j} \end{vmatrix} A^{i_1}{}_{q_1}A^{i_2}{}_{q_2}\cdots A^{i_j}{}_{q_j} \tag{6.1.7}$$

As an illustration of (6.1.7), the second invariant, μ_2, is given by

$$\begin{aligned} \mu_2 &= \frac{1}{2}\sum_{q_1,q_2=1}^{N}\sum_{i_1,i_2=1}^{N}\begin{vmatrix} \delta_{i_1}^{q_1} & \delta_{i_2}^{q_1} \\ \delta_{i_1}^{q_2} & \delta_{i_2}^{q_2}\end{vmatrix} A^{i_1}{}_{q_1}A^{i_2}{}_{q_2} = \frac{1}{2}\sum_{q_1,q_2=1}^{N}\sum_{i_1,i_2=1}^{N}\left(\delta_{i_1}^{q_1}\delta_{i_2}^{q_2} - \delta_{i_1}^{q_2}\delta_{i_2}^{q_1}\right)A^{i_1}{}_{q_1}A^{i_2}{}_{q_2} \\ &= \frac{1}{2}\sum_{q_1,q_2=1}^{N}\left(A_{q_1}^{q_1}A_{q_2}^{q_2} - A_{q_1}^{q_2}A_{q_2}^{q_1}\right) = \frac{1}{2}\left(\left(\operatorname{tr}\mathbf{A}\right)^2 - \operatorname{tr}\mathbf{A}^2\right) \end{aligned} \tag{6.1.8}$$

By a much more complicated calculation,

$$\mu_3 = \frac{1}{3}\operatorname{tr}\mathbf{A}^3 - \frac{1}{2}\left(\operatorname{tr}\mathbf{A}\right)\operatorname{tr}\mathbf{A}^2 + \frac{1}{6}\left(\operatorname{tr}\mathbf{A}\right)^3 \tag{6.1.9}$$

It is perhaps evident from (6.1.7) that the fundamental invariants will always be expressible as linear combinations of products of powers of factors like $\operatorname{tr}\mathbf{A},\operatorname{tr}\mathbf{A}^2,\ldots,\operatorname{tr}\mathbf{A}^j$. Equations (6.1.8) and (6.1.9) illustrate this point. Equation (6.1.7) is not always the most convenient formula to use. Fortunately, equations (6.1.8) and (6.1.9) are special cases of formulas that can also be derived from the so called Newton Identities.[1] The iterative formulas that yield these results and others are

$$k\mu_k = \sum_{j=1}^{k}(-1)^{j-1}\mu_{k-j}\operatorname{tr}\mathbf{A}^j \qquad \text{for} \quad k=1,2,\ldots,N \tag{6.1.10}$$

[1] See, for example, http://en.wikipedia.org/wiki/Newton%27s_identities.

where we have adopted the convention $\mu_0 = 1$. It follows from (6.1.10) that

$$
\begin{aligned}
&\mu_1 = \operatorname{tr} \mathbf{A} \\
&2\mu_2 = \mu_1 \operatorname{tr} \mathbf{A} - \operatorname{tr} \mathbf{A}^2 \\
&3\mu_3 = \mu_2 \operatorname{tr} \mathbf{A} - \mu_1 \operatorname{tr} \mathbf{A}^2 + \operatorname{tr} \mathbf{A}^3 \\
&\cdot \\
&\cdot \\
&\cdot \\
&(N-1)\mu_{N-1} = \mu_{N-2} \operatorname{tr} \mathbf{A} - \mu_{N-3} \operatorname{tr} \mathbf{A}^2 + \cdots + (-1)^{N-3} \mu_1 \operatorname{tr} \mathbf{A}^{N-2} + (-1)^{N-2} \operatorname{tr} \mathbf{A}^{N-1} \\
&N\mu_N = \mu_{N-1} \operatorname{tr} \mathbf{A} - \mu_{N-2} \operatorname{tr} \mathbf{A}^2 + \cdots + (-1)^{N-2} \mu_1 \operatorname{tr} \mathbf{A}^{N-1} + (-1)^{N-1} \operatorname{tr} \mathbf{A}^N
\end{aligned}
\tag{6.1.11}
$$

Equation (6.1.9) can be seen to be a result of the substitution of $(6.1.11)_1$ and $(6.1.11)_2$ into $(6.1.11)_3$.

Additional useful information about the fundamental invariants can be obtained if we look again at the factored form of the characteristic equation (5.2.6), repeated,

$$
f(\lambda) = \det(\mathbf{A} - \lambda \mathbf{I}) = \underbrace{(\lambda_1 - \lambda)(\lambda_2 - \lambda)(\lambda_3 - \lambda)\cdots(\lambda_N - \lambda)}_{N \text{ Factors}} \tag{6.1.12}
$$

As illustrated in Section 5 for the special case $N = 3$, we can equate (6.1.4) to (6.1.12), expand the products in (6.1.12) and obtain formulas for the invariants in terms of the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_N$. The results of this multiplication can be written[2]

$$
\mu_j = \sum_{\substack{k_1, k_2, \ldots, k_j = 1 \\ 1 \le k_1 < k_2 < \cdots < k_j \le N}}^{N} \lambda_{k_1} \lambda_{k_2} \cdots \lambda_{k_j} \quad \text{for} \quad j = 1, 2, \ldots, N \tag{6.1.13}
$$

For given $N = \dim \mathscr{V}$ and j, it turns out that there are $\dfrac{N!}{(N-j)!\,j!}$ terms in the sum (6.1.13). It follows from (6.1.13) that

[2] Equation (6.1.13) defines what is known as an elementary symmetric polynomial. A brief but good discussion of these polynomials can be found at http://en.wikipedia.org/wiki/Elementary_symmetric_polynomial.

$$
\begin{aligned}
\mu_1 &= \lambda_1 + \lambda_2 + \cdots + \lambda_N \\
\mu_2 &= \lambda_1\lambda_2 + \lambda_1\lambda_3 + \cdots + \lambda_1\lambda_N + \lambda_2\lambda_3 + \lambda_2\lambda_4 + \cdots + \lambda_2\lambda_N + \cdots + \lambda_{N-1}\lambda_N \\
\mu_3 &= \lambda_1\lambda_2\lambda_3 + \lambda_1\lambda_2\lambda_4 + \cdots + \lambda_1\lambda_2\lambda_N + \lambda_1\lambda_3\lambda_4 + \lambda_1\lambda_3\lambda_5 + \cdots + \lambda_1\lambda_3\lambda_N + \cdots + \lambda_{N-2}\lambda_{N-1}\lambda_N \\
&\cdot \\
&\cdot \\
&\cdot \\
\mu_N &= \lambda_1\lambda_2 \cdots \lambda_N
\end{aligned}
\tag{6.1.14}
$$

It is useful to note in passing that equation (6.1.13) and the expanded version equation (6.1.14) remain valid if the eigenvalues are not distinct.

Section 6.2 The Cayley-Hamilton Theorem

In Section 6.2.1 the idea of a *polynomial* in $\mathbf{A}:\mathscr{V}\to\mathscr{V}$ was mentioned. One polynomial of importance is the one that was mentioned in Exercise 5.2.1. This polynomial is the one formed from the characteristic polynomial by formally replacing the scalar parameter λ by the linear transformation $\mathbf{A}$ and the introduction of the identity linear transformation in the constant term. The resulting theorem of importance involving this polynomial is the Cayley-Hamilton Theorem. This theorem was stated without proof in Exercise 5.2.1. Basically, the Cayley-Hamilton Theorem states that a linear transformation $\mathbf{A}:\mathscr{V}\to\mathscr{V}$ obeys its own characteristic equation. The formal statement of the theorem was given in equation (5.2.33), repeated,[3]

$$\left(-\mathbf{A}\right)^{N}+\mu_{1}\left(-\mathbf{A}\right)^{N-1}+\cdots+\mu_{N-1}\left(-\mathbf{A}\right)+\mu_{N}\mathbf{I}=\mathbf{0} \tag{6.2.1}$$

where the characteristic equation is given by (6.1.4), repeated,

$$f\left(\lambda\right)=\det\left(\mathbf{A}-\lambda\mathbf{I}\right)=\left(-\lambda\right)^{N}+\mu_{1}\left(-\lambda\right)^{N-1}+\cdots+\mu_{N-1}\left(-\lambda\right)+\mu_{N} \tag{6.2.2}$$

In equations (6.2.1) and (6.2.2) $N=\dim\mathscr{V}$. Note that the polynomial (6.2.1) is identically zero *for all* $\mathbf{A}:\mathscr{V}\to\mathscr{V}$. The polynomial (6.2.2) is zero for those λ that are eigenvalues. The formal theorem we shall prove is as follows:

Theorem 6.2.1: (Cayley-Hamilton). If $f(\lambda)$ is the characteristic polynomial (6.2.2) for a linear transformation $\mathbf{A}:\mathscr{V}\to\mathscr{V}$, then

$$f(\mathbf{A})=(-\mathbf{A})^{N}+\mu_{1}(-\mathbf{A})^{N-1}+\cdots+\mu_{N-1}(-\mathbf{A})+\mu_{N}\mathbf{I}=\mathbf{0} \tag{6.2.3}$$

where $N=\dim\mathscr{V}$.

Proof: The proof which we shall now present makes use of equation (1.10.50). If $\operatorname{adj}(\mathbf{A}-\lambda\mathbf{I})$ is the linear transformation whose matrix is $\operatorname{adj}\left[A^{p}{}_{q}-\lambda\delta_{q}^{p}\right]$, where $\left[A^{p}{}_{q}\right]=M(\mathbf{A},\mathbf{e}_{j},\mathbf{e}_{k})$, then by (5.25) and (1.10.50)[4]

$$(\operatorname{adj}(\mathbf{A}-\lambda\mathbf{I}))(\mathbf{A}-\lambda\mathbf{I})=f(\lambda)\mathbf{I} \tag{6.2.4}$$

---

[3] The polynomial $f(\mathbf{A})=(-\mathbf{A})^{N}+\mu_{1}(-\mathbf{A})^{N-1}+\cdots+\mu_{N-1}(-\mathbf{A})+\mu_{N}\mathbf{I}=\mathbf{0}$ is an example of an *annihilating polynomial*. Another such polynomial that is important in linear algebra is the *minimum polynomial* $m\left(\mathbf{A}\right)$. Without attempting a careful definition here, the minimum polynomial of a linear transformation is the lowest order polynomial that annihilates a linear transformation $\mathbf{A}$.

[4] In the case of a three dimensional vector space, equation (6.2.4) is a consequence of equation (4.10.37)

By (1.10.48), it follows that $\operatorname{adj}(\mathbf{A}-\lambda\mathbf{I})$ is a polynomial of degree $N-1$ in λ. Therefore, this polynomial will always take the form

$$\operatorname{adj}(\mathbf{A}-\lambda\mathbf{I}) = \mathbf{B}_0(-\lambda)^{N-1} + \mathbf{B}_1(-\lambda)^{N-2} + \cdots + \mathbf{B}_{N-2}(-\lambda) + \mathbf{B}_{N-1} \tag{6.2.5}$$

where $\mathbf{B}_0,...,\mathbf{B}_{N-1}$ are linear transformations determined by $\mathbf{A}$. If we now substitute (6.2.5) and (6.2.2) into (6.2.4), the result is

$$\begin{aligned}\left(\mathbf{B}_0(-\lambda)^{N-1} + \mathbf{B}_1(-\lambda)^{N-2} + \cdots + \mathbf{B}_{N-2}(-\lambda) + \mathbf{B}_{N-1}\right)(\mathbf{A}-\lambda\mathbf{I}) \\ = \left((-\lambda)^N + \mu_1(-\lambda)^{N-1} + \cdots + \mu_{N-1}(-\lambda) + \mu_N\right)\mathbf{I}\end{aligned} \tag{6.2.6}$$

The next step involves forcing (6.2.6) to hold as an identity for all values of λ. The results of this requirement are

$$\begin{aligned} \mathbf{B}_0 &= \mathbf{I} \\ \mathbf{B}_0\mathbf{A} + \mathbf{B}_1 &= \mu_1\mathbf{I} \\ \mathbf{B}_1\mathbf{A} + \mathbf{B}_2 &= \mu_2\mathbf{I} \\ &\cdot \\ &\cdot \\ &\cdot \\ \mathbf{B}_{N-2}\mathbf{A} + \mathbf{B}_{N-1} &= \mu_{N-1}\mathbf{I} \\ \mathbf{B}_{N-1}\mathbf{A} &= \mu_N\mathbf{I} \end{aligned} \tag{6.2.7}$$

Now we multiply $(6.2.7)_1$ by $(-\mathbf{A})^N$, $(6.2.7)_2$ by $(-\mathbf{A})^{N-1}$, $(6.2.7)_3$ by $(-\mathbf{A})^{N-2}$,…,$(6.2.7)_k$ by $(-\mathbf{A})^{N-k+1}$, etc., and add the resulting N equations, to find

$$(-\mathbf{A})^N + \mu_1(-\mathbf{A})^{N-1} + \cdots + \mu_{N-1}(-\mathbf{A}) + \mu_N\mathbf{I} = \mathbf{0} \tag{6.2.8}$$

which is the desired result. Note that one can form the trace of (6.2.8) and obtain the earlier result $(6.1.11)_N$.

As a polynomial, (6.2.8) can be factored. In the case where we list the N roots as $\lambda_1,\lambda_2,...,\lambda_N$, (6.2.8) factors into the expression

$$\underbrace{(\lambda_1\mathbf{I}-\mathbf{A})(\lambda_2\mathbf{I}-\mathbf{A})(\lambda_3\mathbf{I}-\mathbf{A})\cdots(\lambda_N\mathbf{I}-\mathbf{A})}_{N\ \text{Factors}} = \mathbf{0} \tag{6.2.9}$$

where the order of the factors is unimportant. Equation (6.2.9) is suggested by the characteristic polynomial written as in equation (5.2.6). If we wish to recognize the possibility that the eigenvalues are not distinct, we can express the factorization of (6.2.8) as[5]

$$\left(\lambda_1\mathbf{I}-\mathbf{A}\right)^{d_1}\left(\lambda_2\mathbf{I}-\mathbf{A}\right)^{d_2}\left(\lambda_3\mathbf{I}-\mathbf{A}\right)^{d_3}\cdots\left(\lambda_L\mathbf{I}-\mathbf{A}\right)^{d_L}=\mathbf{0} \tag{6.2.10}$$

where, as in equation (5.2.26), $d_1,d_2,d_3,...,d_L$ are the algebraic multiplicities of each eigenvalue and obey

$$\sum_{j=1}^{L} d_j = N \tag{6.2.11}$$

Again, the order of the factors in (6.2.10) is unimportant.

Equations (6.2.7) contain other results. For example, it follows from (6.2.5) that

$$\mathbf{B}_{N-1} = \operatorname{adj}\mathbf{A} \tag{6.2.12}$$

Given (6.2.12) and $(6.2.7)_4$, it follows that

$$\operatorname{adj}\mathbf{A} = \mu_{N-1}\mathbf{I}-\mathbf{B}_{N-2}\mathbf{A} \tag{6.2.13}$$

Given (6.2.13) and the next equation in the sequence of equations (6.2.7), results in the formula

$$\operatorname{adj}\mathbf{A} = \mu_{N-1}\mathbf{I}-\mathbf{B}_{N-2}\mathbf{A} = \mu_{N-1}\mathbf{I}-\left(\mu_{N-2}\mathbf{I}-\mathbf{B}_{N-3}\mathbf{A}\right)\mathbf{A} \tag{6.2.14}$$

If this sequence of substitutions is continued through the set (6.2.7), the result is

$$\operatorname{adj}\mathbf{A} = (-\mathbf{A})^{N-1}+\mu_1(-\mathbf{A})^{N-2}+\cdots+\mu_{N-2}(-\mathbf{A})+\mu_{N-1}\mathbf{I} \tag{6.2.15}$$

Thus, the linear transformation $\operatorname{adj}\mathbf{A}$ is expressed in terms of powers of $\mathbf{A}$ and the invariants $\mu_1,\mu_2,...,\mu_{N-1}$.

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[5] For certain linear transformations there are lower order polynomials that are obeyed by $\mathbf{A}$. The minimum polynomial, for example, which was mentioned in the above footnote would take the form

$$\left(\lambda_1\mathbf{I}-\mathbf{A}\right)^{r_1}\left(\lambda_2\mathbf{I}-\mathbf{A}\right)^{r_2}\left(\lambda_3\mathbf{I}-\mathbf{A}\right)^{r_3}\cdots\left(\lambda_L\mathbf{I}-\mathbf{A}\right)^{r_L}=\mathbf{0}$$

where $1\le r_j \le d_j$. While we shall not prove it here, when $r_j=1$ for $j=1,2,...,L$ there exists a basis of $\mathscr{V}$ that makes the matrix of $\mathbf{A}$ diagonal.

Given (6.2.7), (6.2.12) and (6.2.15), it follows from (6.2.5) that

$$\begin{aligned}
\operatorname{adj}(\mathbf{A}-\lambda\mathbf{I}) &= \mathbf{B}_0(-\lambda)^{N-1} + \mathbf{B}_1(-\lambda)^{N-2} + \cdots + \mathbf{B}_{N-2}(-\lambda) + \mathbf{B}_{N-1} \\
&= (-\lambda)^{N-1}\mathbf{I} + (-\lambda)^{N-2}\left(\mu_1\mathbf{I} + (-\mathbf{A})\right) \\
&\quad + \cdots + (-\lambda)\left(\mu_{N-2}\mathbf{I} + \mu_{N-3}(-\mathbf{A}) + \mu_{N-4}(-\mathbf{A})^2 + \cdots + \mu_1(-\mathbf{A})^{N-3} + (-\mathbf{A})^{N-2}\right) \\
&\quad + (-\mathbf{A})^{N-1} + \mu_1(-\mathbf{A})^{N-2} + \mu_2(-\mathbf{A})^{N-3} + \cdots + \mu_{N-1}\mathbf{I} \\
&= (-\mathbf{A})^{N-1} + \left((-\lambda)+\mu_1\right)(-\mathbf{A})^{N-2} + \left((-\lambda)^2 + \mu_1(-\lambda) + \mu_2\right)(-\mathbf{A})^{N-3} \\
&\quad + \cdots + \left((-\lambda)^{N-2} + \mu_1(-\lambda)^{N-3} + \cdots + \mu_{N-3}(-\lambda) + \mu_{N-2}\right)(-\mathbf{A}) \\
&\quad + \left((-\lambda)^{N-1} + \mu_1(-\lambda)^{N-2} + \mu_2(-\lambda)^{N-3} + \cdots + \mu_{N-2}(-\lambda) + \mu_{N-1}\right)\mathbf{I}
\end{aligned} \tag{6.2.16}$$

The complicated formula (6.2.16) simplifies considerably if the parameter λ is selected to be an eigenvalue of $\mathbf{A}$. In this case, after (6.1.13) is used to express the fundamental invariants in terms of the eigenvalues of $\mathbf{A}$, the result from (6.2.16) is[6]

$$\begin{aligned}
\operatorname{adj}\left(\mathbf{A}-\lambda_j\mathbf{I}\right) &= \prod_{\substack{k=1\\k\neq j}}^{N}\left(\mathbf{A}-\lambda_k\mathbf{I}\right) \\
&= \left(\mathbf{A}-\lambda_1\mathbf{I}\right)\cdots\left(\mathbf{A}-\lambda_{j-1}\mathbf{I}\right)\left(\mathbf{A}-\lambda_{j+1}\mathbf{I}\right)\cdots\left(\mathbf{A}-\lambda_N\mathbf{I}\right)
\end{aligned} \tag{6.2.17}$$

for the case where $\lambda=\lambda_j$, the j^{th} eigenvalue in the set $\lambda_1,\lambda_2,...,\lambda_N$,

The steps from (6.2.16) to (6.2.17) can best be illustrated if we make the special choice $\lambda=\lambda_1$ and use (6.1.13) or, equivalently, (6.1.14) to evaluate the coefficients in the polynomial of linear transformations $(6.2.16)_2$. The sequence of formulas that result are

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[6] A different derivation of (6.2.17) can be found in Elementary Matrices by Frazier, Duncan and Collar, Cambridge University Press, 1938, Section 3.8.

$$
\begin{aligned}
&-\lambda_1+\mu_1=-\lambda_1+\lambda_1+\lambda_2+\cdots+\lambda_N=\sum_{k=2}^{N}\lambda_k\\
&(-\lambda_1)^2+\mu_1(-\lambda_1)+\mu_2=(-\lambda_1)(-\lambda_1+\mu_1)+\mu_2\\
&=(-\lambda_1)\left(\sum_{k=2}^{N}\lambda_k\right)+\sum_{\substack{k_1,k_2=1\\1\le k_1<k_2\le N}}^{N}\lambda_{k_1}\lambda_{k_2}=\sum_{\substack{k_1,k_2=2\\2\le k_1<k_2\le N}}^{N}\lambda_{k_1}\lambda_{k_2}\\
&.\\
&.\\
&.\\
&(-\lambda_1)^{N-1}+\mu_1(-\lambda_1)^{N-2}+\mu_2(-\lambda_1)^{N-3}+\cdots+\mu_{N-2}(-\lambda_1)+\mu_{N-1}\\
&=(-\lambda_1)\left((-\lambda_1)^{N-2}+\mu_1(-\lambda_1)^{N-3}+\cdots+\mu_{N-2}\right)+\mu_{N-1}\\
&=(-\lambda_1)\left(\sum_{\substack{k_1,k_2,k_3\cdots,k_{N-2}=2\\2\le k_1<k_2<k_3<\cdots<k_{N-2}\le N}}^{N}\lambda_{k_1}\lambda_{k_2}\lambda_{k_3}\cdots\lambda_{k_{N-2}}\right)+\sum_{\substack{k_1,k_2,k_3\cdots,k_{N-2},k_{N-1}=1\\1\le k_1<k_2<k_3<\cdots<k_{N-2}<k_{N-1}\le N}}^{N}\lambda_{k_1}\lambda_{k_2}\lambda_{k_3}\cdots\lambda_{k_{N-2}}\lambda_{k_{N-1}}\\
&=\sum_{\substack{k_1,k_2,k_3\cdots,k_{N-2},k_{N-1}=2\\2\le k_1<k_2<k_3<\cdots<k_{N-2}<k_{N-1}\le N}}^{N}\lambda_{k_1}\lambda_{k_2}\lambda_{k_3}\cdots\lambda_{k_{N-2}}\lambda_{k_{N-1}}=\lambda_2\lambda_3\lambda_4\cdots\lambda_{N-1}\lambda_N
\end{aligned}
\tag{6.2.18}
$$

If these formulas are used in $(6.2.16)_2$, the result can be seen to factor into the form (6.2.17) for $\lambda_j=\lambda_1$.

In Exercise 4.10.5, a formula was asserted for the derivative of a linear transformation that depends upon a parameter. This formula, equation (4.10.39), was stated in the context of three dimensional vector spaces but it was asserted that it holds for vector spaces of arbitrary finite dimension. If we accept this generalization, it follows from (6.2.2) that[7]

$$\mu_{N-1}=-\frac{d}{d\lambda}\det\left(\mathbf{A}-\lambda\mathbf{I}\right)\Big|_{\lambda=0} \tag{6.2.19}$$

If we utilize equation (4.10.39) to evaluate the derivative in (6.2.19), it follows that[8]

$$\mu_{N-1}=\operatorname{tr}(\operatorname{adj}\mathbf{A}) \tag{6.2.20}$$

---

[7] Equation (6.2.19) is a special case of the formula which also follows from (6.2.2) that

$$\mu_j=\frac{(-1)^{N-j}}{(N-j)!}\frac{d^{N-j}}{d\lambda^{N-j}}\det\left(\mathbf{A}-\lambda\mathbf{I}\right)\Big|_{\lambda=0}$$

[8] A special case of equation (6.2.20) was given in footnote 3 of Section 5.2.

If the formula (6.2.14) is used to calculate the right side of (6.2.20), the result is

$$\mu_{N-1} = -\frac{1}{N-1}\left\{\operatorname{tr}(-\mathbf{A})^{N-1} + \mu_1 \operatorname{tr}(-\mathbf{A})^{N-2} + \cdots + \mu_{N-2}\operatorname{tr}(-\mathbf{A})\right\} \tag{6.2.21}$$

which is the result obtained earlier in equation $(6.1.11)_{N-1}$.

Exercises

6.2.1 Consider the case where the linear transformation $\mathbf{A}:\mathscr{V}\to\mathscr{V}$ has L distinct eigenvalues $\lambda_1,\lambda_2,...,\lambda_L$. If λ_j is one of those eigenvalues and its algebraic multiplicity is d_j, use (6.2.4) and (6.2.10) and show that

$$\begin{aligned}\left.\frac{d^{d_j-1}\left(\operatorname{adj}\left(\mathbf{A}-\lambda\mathbf{I}\right)\right)}{d\lambda^{d_j-1}}\right|_{\lambda=\lambda_j} &= (-1)^{N-1}\prod_{\substack{k=1\\k\neq j}}^{L}\left(\lambda_k\mathbf{I}-\mathbf{A}\right)^{d_k}\\ &= (-1)^{N-1}\left(\lambda_1\mathbf{I}-\mathbf{A}\right)^{d_1}\cdots\left(\lambda_{j-1}\mathbf{I}-\mathbf{A}\right)^{d_{j-1}}\left(\lambda_{j+1}\mathbf{I}-\mathbf{A}\right)^{d_{j+1}}\cdots\left(\lambda_L\mathbf{I}-\mathbf{A}\right)^{d_L}\end{aligned} \tag{6.2.22}$$

6.2.2 Consider the linear transformation whose matrix is given by (5.6.65), repeated,

$$A = \begin{bmatrix} 0 & 1 & 0 & 0\\ -2 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 1 & 0 & -2 & 0\end{bmatrix} \tag{6.2.23}$$

In Section 5, we showed that the eigenvalues of this matrix are given by (5.6.68), repeated,

$$\begin{aligned}\lambda_1 &= i\\ \lambda_2 &= -i\\ \lambda_3 &= \sqrt{3}i\\ \lambda_4 &= -\sqrt{3}i\end{aligned} \tag{6.2.24}$$

Show that

$$\operatorname{adj}\left(A-\lambda_1 \mathrm{I}\right)=\begin{bmatrix} i\sqrt{3} & 1 & -i\sqrt{3} & -1 \\ -3 & i\sqrt{3} & 3 & -i\sqrt{3} \\ -i\sqrt{3} & -1 & i\sqrt{3} & 1 \\ 3 & -i\sqrt{3} & -3 & i\sqrt{3} \end{bmatrix} \tag{6.2.25}$$

Confirm (6.2.25) by direct calculation of the left hand side and, also, by use of the identity (6.2.17).

6.2.3 Consider the linear transformation whose matrix is given by (5.3.45), repeated,

$$A=\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \tag{6.2.26}$$

In Section 5, we showed that the eigenvalues of this matrix are given by (5.3.47), repeated,

$$\lambda_1 = 1 \tag{6.2.27}$$

and it has an algebraic multiplicity of 3. Show that

$$\operatorname{adj}\left(A-\lambda_1 \mathrm{I}\right)=\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{6.2.28}$$

Confirm (6.2.28) by direct calculation of the left hand side and, also, by use of the identity (6.2.17).

Section 6.3. The Exponential Linear Transformation

The exponential linear transformation is one example of a function of a linear transformation. In other words, given a linear transformation $\mathbf{A}:\mathscr{V}\rightarrow\mathscr{V}$, the exponential function, which we shall define, is a function g that maps $\mathbf{A}\in\mathscr{L}\left(\mathscr{V};\mathscr{V}\right)$ into a linear transformation $g\left(\mathbf{A}\right)$. The exponential linear transformation is an example of an analytical function of a linear transformation $\mathbf{A}:\mathscr{V}\rightarrow\mathscr{V}$. These are functions that can be expressed as the series

$$g\left(\mathbf{A}\right)=\sum_{n=0}^{\infty}\beta_{n}\mathbf{A}^{n} \tag{6.3.1}$$

Example 6.3.1: A special case of equation (6.3.1) is the polynomial of a linear transformation $\mathbf{A}:\mathscr{V}\rightarrow\mathscr{V}$ defined by

$$g\left(\mathbf{A}\right)=2\mathbf{I}+\mathbf{A}+3\mathbf{A}^{2}+\mathbf{A}^{3} \tag{6.3.2}$$

The *exponential linear transformation* is a special case of (6.3.1) defined by

$$e^{\mathbf{A}}=\mathbf{I}+\mathbf{A}+\frac{1}{2!}\mathbf{A}^{2}+\frac{1}{3!}\mathbf{A}^{3}+\cdots=\sum_{n=0}^{\infty}\frac{1}{n!}\mathbf{A}^{n} \tag{6.3.3}$$

The *motivation* for this definition is the power series representation of the ordinary exponential function e^{λ}. Recall that this series takes the form

$$e^{\lambda}=1+\lambda+\frac{1}{2!}\lambda^{2}+\frac{1}{3!}\lambda^{3}+\cdots=\sum_{n=0}^{\infty}\frac{1}{n!}\lambda^{n} \tag{6.3.4}$$

Equation (6.3.3) is formally obtained from (6.3.4) by replacing λ with $\mathbf{A}$ and the constant term by $\mathbf{I}$. We shall often encounter the matrix exponential in the form

$$e^{\mathbf{A}t}=\mathbf{I}+\mathbf{A}t+\frac{1}{2!}\mathbf{A}^{2}t^{2}+\frac{1}{3!}\mathbf{A}^{3}t^{3}+\cdots=\sum_{n=0}^{\infty}\frac{1}{n!}\mathbf{A}^{n}t^{n} \tag{6.3.5}$$

which you obtain from the definition by the substitution $\mathbf{A}\rightarrow\mathbf{A}t$

The definition (6.3.3) and its equivalent definition (6.3.5) yield the following properties of the exponential linear transformation:

a)

$$e^{\mathbf{0}} = \mathbf{I} \tag{6.3.6}$$

b)

$$e^{\mathbf{I}\lambda} = e^{\lambda}\mathbf{I} \tag{6.3.7}$$

The proof of (6.3.7) follows from (6.3.5) and (6.3.4).

c)

$$e^{\mathbf{A}(t+s)} = e^{\mathbf{A}t}e^{\mathbf{A}s} \tag{6.3.8}$$

The proof of (6.3.8) involves forming $e^{\mathbf{A}(t+s)}$ from the definition (6.3.5) as follow

$$e^{\mathbf{A}(t+s)} = \mathbf{I} + \mathbf{A}(t+s) + \frac{1}{2!}\mathbf{A}^2(t+s)^2 + \frac{1}{3!}\mathbf{A}^3(t+s)^3 + \cdots = \sum_{n=0}^{\infty}\frac{1}{n!}\mathbf{A}^n(t+s)^n \tag{6.3.9}$$

Next, we can again use the definition (6.3.5) and form the product

$$e^{\mathbf{A}t}e^{\mathbf{A}s} = \left(\mathbf{I} + \mathbf{A}t + \frac{1}{2!}\mathbf{A}^2t^2 + \frac{1}{3!}\mathbf{A}^3t^3 + \cdots\right)\left(\mathbf{I} + \mathbf{A}s + \frac{1}{2!}\mathbf{A}^2s^2 + \frac{1}{3!}\mathbf{A}^3s^3 + \cdots\right) \tag{6.3.10}$$

The algebra is not pretty, but if (6.3.10) is expended and compared to (6.3.8) the asserted result is obtained.

d)

$$\left(e^{\mathbf{A}t}\right)^{-1} = e^{-\mathbf{A}t} \tag{6.3.11}$$

This result follows from (6.3.6) and (6.3.8) by the choice $s = -t$ in (6.3.8).

e)

$$e^{(\mathbf{A}+\mathbf{B})t} = e^{\mathbf{A}t}e^{\mathbf{B}t} \quad \text{if} \quad \mathbf{AB} = \mathbf{BA} \tag{6.3.12}$$

The proof of (6.3.12) follows by use of the definition (6.3.5) to form both sides of (6.3.12). The next step is to expand the result and utilize the given condition $\mathbf{AB} = \mathbf{BA}$

f) If $\mathbf{A}$ is given by the product

$$\mathbf{A} = \mathbf{TBT}^{-1} \tag{6.3.13}$$

then

$$\mathbf{A}^n = \mathbf{T}\mathbf{B}^n\mathbf{T}^{-1} \tag{6.3.14}$$

and it follows from (6.3.3) that

$$e^{\mathbf{A}} = \mathbf{T}e^{\mathbf{B}}\mathbf{T}^{-1} \tag{6.3.15}$$

g) As introduced in Section 3.2, if $\{\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_N\}$ is a basis for $\mathscr{V}$, the matrix of $\mathbf{A}: \mathscr{V} \to \mathscr{V}$ with respect to the basis is written $M\left(\mathbf{A}, \mathbf{e}_k, \mathbf{e}_j\right)$. It follows from equation (3.4.19) and the definition (6.3.3) that

$$M\left(e^{\mathbf{A}}, \mathbf{e}_k, \mathbf{e}_j\right) = e^{M\left(\mathbf{A}, \mathbf{e}_k, \mathbf{e}_j\right)} \tag{6.3.16}$$

In words, (6.3.16) says simply that the matrix of the exponential linear transformation is the exponential linear transformation of the matrix.

h) If, for example, the linear transformation $\mathbf{A}: \mathscr{V} \to \mathscr{V}$, has N linearly independent eigenvectors $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_N\}$, then, as explained in Section 5.1, the matrix of $\mathbf{A}$ with respect to this basis takes the *diagonal* form (5.19), repeated,

$$D = M\left(\mathbf{A}, \mathbf{v}_j, \mathbf{v}_j\right) = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdot & \cdot & 0 \\ 0 & \lambda_2 & 0 & \cdot & \cdot & 0 \\ 0 & 0 & \lambda_3 & & & 0 \\ \cdot & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \lambda_N \end{bmatrix} \tag{6.3.17}$$

It is elementary to see that

$$D^n = \begin{bmatrix} \lambda_1^n & 0 & 0 & \cdot & \cdot & 0 \\ 0 & \lambda_2^n & 0 & \cdot & \cdot & 0 \\ 0 & 0 & \lambda_3^n & & & 0 \\ \cdot & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \lambda_N^n \end{bmatrix} \tag{6.3.18}$$

As a result, the definitions (6.3.3) and (6.3.4) combine to yield

$$e^D = \sum_{n=0}^{\infty}\frac{1}{n!}D^n = \begin{bmatrix} \sum_{n=0}^{\infty}\frac{1}{n!}\lambda_1^n & 0 & 0 & \cdot & \cdot & 0 \\ 0 & \sum_{n=0}^{\infty}\frac{1}{n!}\lambda_2^n & 0 & \cdot & \cdot & 0 \\ 0 & 0 & \sum_{n=0}^{\infty}\frac{1}{n!}\lambda_3^n & & & 0 \\ \cdot & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \sum_{n=0}^{\infty}\frac{1}{n!}\lambda_N^n \end{bmatrix}$$

$$= \begin{bmatrix} e^{\lambda_1} & 0 & 0 & \cdot & \cdot & 0 \\ 0 & e^{\lambda_2} & 0 & \cdot & \cdot & 0 \\ 0 & 0 & e^{\lambda_3} & & & 0 \\ \cdot & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & e^{\lambda_N} \end{bmatrix} \tag{6.3.19}$$

i) If we return to the case where $\{\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_N\}$ is an arbitrary basis for $\mathscr{V}$ and continue to assume $\mathbf{A}$ has N linearly independent eigenvectors $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_N\}$, then the matrix of $\mathbf{A}$ with respect to $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_N\}$ is connected to the matrix of $\mathbf{A}$ with respect to $\{\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_N\}$ by (5.1.10), repeated,

$$M\left(\mathbf{A}, \mathbf{v}_j, \mathbf{v}_k\right) = T^{-1}M\left(\mathbf{A}, \mathbf{e}_j, \mathbf{e}_k\right)T \tag{6.3.20}$$

where T is the transition matrix. Given (6.3.13), (6.3.15), (6.3.16) and (6.3.19), it follows that

$$M\left(e^{\mathbf{A}}, \mathbf{e}_k, \mathbf{e}_j\right) = e^{M\left(\mathbf{A}, \mathbf{e}_j, \mathbf{e}_k\right)} = Te^D T^{-1}$$

$$= T\begin{bmatrix} e^{\lambda_1} & 0 & 0 & \cdot & \cdot & 0 \\ 0 & e^{\lambda_2} & 0 & \cdot & \cdot & 0 \\ 0 & 0 & e^{\lambda_3} & & & 0 \\ \cdot & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & e^{\lambda_N} \end{bmatrix}T^{-1} \tag{6.3.21}$$

In the special case where $\mathbf{A}$ has N distinct eigenvalues and, thus, N linearly independent eigenvectors, equation (6.3.21) provides a useful way to actually calculate the exponential linear transformation. The disadvantage of (6.3.21) is that it requires the calculation of the N

eigenvectors in order to construct the transition matrix T. In cases where the eigenvalues of $\mathbf{A}$ are not distinct, the situation is more complicated. For this reason, it is useful to outline an alternate way to determine the exponential linear transformation. This alternate method has the advantage, when applied to the case where the eigenvalues are distinct, that it does not need an explicit calculation of the eigenvectors.

The Cayley-Hamilton theorem, equation (6.2.8), offers a method to simplify the series expansion (6.3.3). Given this simplification, we can then develop a method of calculating the exponential linear transformation. As a first step, it is helpful to illustrate this simplification by an example.

Example 6.3.2: Consider the polynomial defined by (6.3.2) and assume for the moment that we are dealing with a vector space $\mathscr{V}$ such that $N = \dim \mathscr{V} = 2$. In this case, the Cayley-Hamilton Theorem (6.2.3) specializes to

$$\mathbf{A}^2 - \mu_1 \mathbf{A} + \mu_2 \mathbf{I} = \mathbf{0} \tag{6.3.22}$$

This relationship reduces (6.3.2) to

$$\begin{aligned} g(\mathbf{A}) &= 2\mathbf{I} + \mathbf{A} + 3\mathbf{A}^2 + \mathbf{A}^3 \\ &= 2\mathbf{I} + \mathbf{A} + 3(\mu_1 \mathbf{A} - \mu_2 \mathbf{I}) + \mathbf{A}(\mu_1 \mathbf{A} - \mu_2 \mathbf{I}) \\ &= 2\mathbf{I} + \mathbf{A} + 3(\mu_1 \mathbf{A} - \mu_2 \mathbf{I}) + \mu_1 (\mu_1 \mathbf{A} - \mu_2 \mathbf{I}) - \mu_2 \mathbf{A} \\ &= (2 - 3\mu_2 - \mu_1 \mu_2) \mathbf{I} + (1 + 3\mu_1 - \mu_2 + \mu_1^2) \mathbf{A} \end{aligned} \tag{6.3.23}$$

Because the degree of the polynomial was greater than $N = \dim \mathscr{V} = 2$, the terms explicit in $\mathbf{A}^2$ and $\mathbf{A}^3$ can be eliminated by use of the Cayley-Hamilton result (6.3.22).

The point illustrated in Example 6.2.2 can be applied to (6.3.3). Equation (6.2.8), written in the form

$$(-\mathbf{A})^N = -\mu_1 (-\mathbf{A})^{N-1} - \cdots - \mu_{N-1}(-\mathbf{A}) - \mu_N \mathbf{I} \tag{6.3.24}$$

can be used to eliminate the term in (6.3.3) corresponding to $n = N$ in terms of lower powers of $\mathbf{A}$. The next term in the series, the one with the factor $\mathbf{A}^{N+1}$, can be eliminated in terms of lower powers of $\mathbf{A}$ because we can write $\mathbf{A}^{N+1} = \mathbf{A}\mathbf{A}^N$ and again use (6.3.24). If we continue this process, the series (6.3.3) always takes the form

$$e^{\mathbf{A}} = \sum_{n=0}^{N-1} \alpha_n (\mu_1, \mu_2, ..., \mu_N) \mathbf{A}^n \tag{6.3.25}$$

where the coefficients $\alpha_0, \alpha_1, \alpha_2, ..., \alpha_{N-1}$ are complicated functions of the invariants. Given a linear transformation $\mathbf{A} : \mathscr{V} \to \mathscr{V}$, the problem of expressing its exponential as a function in the form

(6.3.25) reduces to finding a method to determine the functions $\alpha_0, \alpha_1, \alpha_2, ..., \alpha_{N-1}$ in the polynomial (6.3.25). It is this calculation that we shall now explain.

Because we do not have anything equivalent to the Cayley-Hamilton theorem for scalars, i.e. for $N = 1$, we have no obvious way to reduce (6.3.4) to a polynomial. We can, however, establish such a result in an important special case. By long division, (6.3.4) can always be written

$$e^{\lambda} = \sum_{n=0}^{\infty} \frac{1}{n!} \lambda^n = Q(\lambda) f(\lambda) + R(\lambda) \tag{6.3.26}$$

The remainder from the division, $R(\lambda)$, is a polynomial of degree $N - 1 = \dim \mathscr{V} - 1$. The quotient $Q(\lambda)$ is an infinite series in λ obtained by dividing the series $\sum_{n=0}^{\infty} \frac{1}{n!} \lambda^n$ by the characteristic polynomial $f(\lambda)$. Given the form of the series (6.3.4) for e^{λ} and the series (6.3.3) for $e^{\mathbf{A}}$, it is also true from (6.3.26) that

$$e^{\mathbf{A}} = \sum_{n=0}^{\infty} \frac{1}{n!} \mathbf{A}^n = Q(\mathbf{A}) f(\mathbf{A}) + R(\mathbf{A}) \tag{6.3.27}$$

However, the Cayley-Hamilton Theorem (6.2.3) tells us that $f(\mathbf{A}) = \mathbf{0}$. Therefore (6.3.27) reduces to

$$e^{\mathbf{A}} = \sum_{n=0}^{\infty} \frac{1}{n!} \mathbf{A}^n = R(\mathbf{A}) = \sum_{n=0}^{N-1} \alpha_n (\mu_1, \mu_2, ..., \mu_N) \mathbf{A}^n \tag{6.3.28}$$

where (6.3.25) has been used. The explicit form of the remainder as a function of $\mathbf{A}$ given in (6.3.28), the formal similarity of (6.3.3) and (6.3.4) and the result (6.3.26) combine to give an explicit form of the remainder as a function of the scalar λ. This explicit form is

$$R(\lambda) = \sum_{n=0}^{N-1} \alpha_n (\mu_1, \mu_2, ..., \mu_N) \lambda^n \tag{6.3.29}$$

This result allow us to write (6.3.26) as

$$e^{\lambda} = \sum_{n=0}^{\infty} \frac{1}{n!} \lambda^n = Q(\lambda) f(\lambda) + \sum_{n=0}^{N-1} \alpha_n (\mu_1, \mu_2, ..., \mu_N) \lambda^n \tag{6.3.30}$$

The utility of (6.3.30) arises when the parameter λ is an eigenvalue of the linear transformation $\mathbf{A} : \mathscr{V} \to \mathscr{V}$. In this special case, λ is a root of the characteristic equation and (6.3.30) reduces to

$$e^{\lambda}=\sum_{n=0}^{\infty}\frac{1}{n!}\lambda^{n}=\sum_{n=0}^{N-1}\alpha_n\left(\mu_1,\mu_2,...,\mu_N\right)\lambda^{n} \quad \text{for } \lambda = \text{ an eigenvalue of } \mathbf{A} \tag{6.3.31}$$

If the linear transformation $\mathbf{A}:\mathscr{V}\to\mathscr{V}$ has $N=\dim\mathscr{V}$ distinct eigenvalues $\lambda_1,\lambda_2,...,\lambda_N$, then we can apply (6.3.31) N times and obtain the system of equations

$$\begin{aligned}
e^{\lambda_1} &= \sum_{n=0}^{N-1}\alpha_n\left(\mu_1,\mu_2,...,\mu_N\right)\lambda_1^{n}\\
e^{\lambda_2} &= \sum_{n=0}^{N-1}\alpha_n\left(\mu_1,\mu_2,...,\mu_N\right)\lambda_2^{n}\\
&\cdot\\
&\cdot\\
&\cdot\\
e^{\lambda_N} &= \sum_{n=0}^{N-1}\alpha_n\left(\mu_1,\mu_2,...,\mu_N\right)\lambda_N^{n}
\end{aligned} \tag{6.3.32}$$

This set of equations, when expressed as a matrix equation, takes the form

$$\begin{bmatrix} 1 & \lambda_1 & \lambda_1^2 & \cdot & \cdot & \lambda_1^{N-1}\\ 1 & \lambda_2 & \lambda_2^2 & & & \lambda_2^{N-1}\\ 1 & \lambda_3 & \lambda_3^2 & & & \lambda_3^{N-1}\\ \cdot & \cdot & & \cdot & & \\ \cdot & \cdot & & & \cdot & \\ 1 & \lambda_N & \lambda_N^2 & & & \lambda_N^{N-1}\end{bmatrix}\begin{bmatrix}\alpha_0\\ \alpha_1\\ \alpha_2\\ \\ \\ \alpha_{N-1}\end{bmatrix}=\begin{bmatrix}e^{\lambda_1}\\ e^{\lambda_2}\\ e^{\lambda_3}\\ \cdot\\ \cdot\\ e^{\lambda_N}\end{bmatrix} \tag{6.3.33}$$

The matrix of coefficients is the transpose of the Vandermondian matrix that was introduced in Section 1.10 and again in Section 5.4. Because the matrix and the transpose have the same determinant, the determinant of the matrix of coefficients, as explained in Section 5.4, is

$$\det\begin{bmatrix} 1 & \lambda_1 & \lambda_1^2 & \cdot & \cdot & \lambda_1^{N-1}\\ 1 & \lambda_2 & \lambda_2^2 & & & \lambda_2^{N-1}\\ 1 & \lambda_3 & \lambda_3^2 & & & \lambda_3^{N-1}\\ \cdot & \cdot & & \cdot & & \\ \cdot & \cdot & & & \cdot & \\ 1 & \lambda_N & \lambda_N^2 & & & \lambda_N^{N-1}\end{bmatrix}=\prod_{\substack{i,j=1\\ i>j}}^{N}\left(\lambda_i-\lambda_j\right) \tag{6.3.34}$$

In the case under discussion, we have assumed the eigenvalues are distinct, thus the determinant (6.3.34) is nonzero, and the system (6.3.33) has a solution. This solution can then be used along with (6.3.28) to determine the exponential solution.

Example 6.3.3: If we examine the case where $N = 2$ and the two eigenvalues are distinct, the system (6.3.32) reduces to

$$\begin{aligned} e^{\lambda_1} &= \alpha_0(\mu_1,\mu_2) + \alpha_1(\mu_1,\mu_2)\lambda_1 \\ e^{\lambda_2} &= \alpha_0(\mu_1,\mu_2) + \alpha_1(\mu_1,\mu_2)\lambda_2 \end{aligned} \tag{6.3.35}$$

and the solution is

$$\begin{aligned} \alpha_0(\mu_1,\mu_2) &= \frac{\lambda_2 e^{\lambda_1} - \lambda_1 e^{\lambda_2}}{\lambda_2 - \lambda_1} \\ \alpha_1(\mu_1,\mu_2) &= \frac{e^{\lambda_2} - e^{\lambda_1}}{\lambda_2 - \lambda_1} \end{aligned} \tag{6.3.36}$$

Given (6.3.36) and our assumption $N = 2$, (6.3.28) reduces to

$$\begin{aligned} e^{\mathbf{A}} &= \alpha_0(\mu_1,\mu_2)\mathbf{I} + \alpha_1(\mu_1,\mu_2)\mathbf{A} \\ &= \frac{\lambda_2 e^{\lambda_1} - \lambda_1 e^{\lambda_2}}{\lambda_2 - \lambda_1}\mathbf{I} + \frac{e^{\lambda_2} - e^{\lambda_1}}{\lambda_2 - \lambda_1}\mathbf{A} \\ &= e^{\lambda_1}\frac{\mathbf{A} - \lambda_2\mathbf{I}}{\lambda_1 - \lambda_2} + e^{\lambda_2}\frac{\mathbf{A} - \lambda_1\mathbf{I}}{\lambda_2 - \lambda_1} \end{aligned} \tag{6.3.37}$$

Example 6.3.3 presumes the two eigenvalues are distinct. If this is not the case, i.e., if $\lambda_1 = \lambda_2$, the logic that led to (6.3.35) only produces a single equation that connects the two unknowns $\alpha_0(\mu_1,\mu_2)$ and $\alpha_1(\mu_1,\mu_2)$. The key to how the argument above is modified in this case is equation (6.3.26) and the characteristic equation in the case of multiple roots, equation (5.2.26). As (5.2.26) illustrates, the polynomial $f(\lambda)$ vanishes at, for example, $\lambda = \lambda_1$. In addition, for the root $\lambda = \lambda_1$,

$$\left.\frac{d^j f(\lambda)}{d\lambda^j}\right|_{\lambda=\lambda_1} = 0 \qquad \text{for} \quad j = 1,2,...,d_1 - 1 \tag{6.3.38}$$

Equations (6.3.38) and (6.3.30) combine to yield

$$e^{\lambda_1} = \left.\frac{d^j}{d\lambda^j}\left(\sum_{n=0}^{N-1}\alpha_n(\mu_1,\mu_2,...,\mu_N)\lambda^n\right)\right|_{\lambda=\lambda_1} \qquad \text{for} \quad j = 1,2,...,d_1 - 1 \tag{6.3.39}$$

with similar results for each of the other repeated roots.

In order to be more explicit, consider the case where the linear transformation $\mathbf{A}:\mathscr{V}\to\mathscr{V}$ has L distinct eigenvalues $\underbrace{\lambda_1,\lambda_1,...,\lambda_1}_{d_1},\underbrace{\lambda_2,...,\lambda_L}_{N-d_1}$, where d_1 is the algebraic multiplicity of the repeated root. For this case, equations (6.3.39) yield the d_1-1 equations

$$
\begin{aligned}
e^{\lambda_1} &= \alpha_1\left(\mu_1,\mu_2,...,\mu_N\right)+2\alpha_2\left(\mu_1,\mu_2,...,\mu_N\right)\lambda_1+\cdots+\left(N-1\right)\alpha_{N-1}\left(\mu_1,\mu_2,...,\mu_N\right)\lambda_1^{N-2}\\
e^{\lambda_1} &= 2\alpha_2\left(\mu_1,\mu_2,...,\mu_N\right)+\cdots+\left(N-1\right)\left(N-2\right)\alpha_{N-1}\left(\mu_1,\mu_2,...,\mu_N\right)\lambda_1^{N-3}\\
&\;\cdot\\
&\;\cdot\\
&\;\cdot\\
e^{\lambda_1} &= \left(N-1\right)\left(N-2\right)\cdots\left(N-d+1\right)\alpha_{N-1}\left(\mu_1,\mu_2,...,\mu_N\right)\lambda_1^{N-d}
\end{aligned}
\tag{6.3.40}
$$

For the distinct roots $\lambda_1,\underbrace{\lambda_2,...,\lambda_L}_{N-d_1}$, we have the $N-d_1+1$ equations

$$
\begin{aligned}
e^{\lambda_1} &= \sum_{n=0}^{N-1}\alpha_n\left(\mu_1,\mu_2,...,\mu_N\right)\lambda_1^n\\
e^{\lambda_2} &= \sum_{n=0}^{N-1}\alpha_n\left(\mu_1,\mu_2,...,\mu_N\right)\lambda_2^n\\
&\;\cdot\\
&\;\cdot\\
&\;\cdot\\
e^{\lambda_L} &= \sum_{n=0}^{N-1}\alpha_n\left(\mu_1,\mu_2,...,\mu_N\right)\lambda_L^n
\end{aligned}
\tag{6.3.41}
$$

The N equations (6.3.40) and (6.3.41) yield the matrix equation for the N unknowns $\alpha_0,\alpha_1,...,\alpha_{N-1}$

$$\begin{bmatrix} 1 & \lambda_1 & \lambda_1^2 & \cdot & \cdot & \lambda_1^{N-1} \\ 0 & 1 & 2\lambda_1 & & & (N-1)\lambda_1^{N-2} \\ 0 & 0 & 2 & & & (N-1)(N-2)\lambda_1^{N-3} \\ \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & (N-1)(N-2)\cdots(N-d_1+1)\lambda_1^{N-d_1} \\ 1 & \lambda_2 & \lambda_2^2 & & & \lambda_2^{N-1} \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ 1 & \lambda_L & \lambda_L^2 & \cdot & \cdot & \lambda_L^{N-1} \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \alpha_{N-2} \\ \alpha_{N-1} \end{bmatrix} = \begin{bmatrix} e^{\lambda_1} \\ e^{\lambda_1} \\ e^{\lambda_1} \\ \cdot \\ \cdot \\ e^{\lambda_1} \\ e^{\lambda_2} \\ \cdot \\ \cdot \\ e^{\lambda_L} \end{bmatrix} \tag{6.3.42}$$

The matrix of coefficients is the (transpose) of a matrix known as a *confluent Vandermondian matrix.* It can be shown to be nonsingular, thus it determines the unknowns $\alpha_0, \alpha_1, ..., \alpha_{N-1}$. As an illustration of the repeated root case, consider the following example:

Example 6.3.4: We again examine the case where $N = 2$ except in this case, the two eigenvalues are not distinct. The system (6.3.32) yields the single equation

$$e^{\lambda_1} = \alpha_0(\mu_1, \mu_2) + \alpha_1(\mu_1, \mu_2)\lambda_1 \tag{6.3.43}$$

The second equation follows from (6.3.39). In this case it yields

$$e^{\lambda_1} = \alpha_1(\mu_1, \mu_2) \tag{6.3.44}$$

Therefore, from (6.3.43)

$$\alpha_0(\mu_1, \mu_2) = (1 - \lambda_1)e^{\lambda_1} \tag{6.3.45}$$

Given (6.3.44), (6.3.45) and our assumption $N = 2$, (6.3.28) reduces to

$$\begin{aligned} e^{\mathbf{A}} &= \alpha_0(\mu_1, \mu_2)\mathbf{I} + \alpha_1(\mu_1, \mu_2)\mathbf{A} \\ &= (1 - \lambda_1)e^{\lambda_1}\mathbf{I} + e^{\lambda_1}\mathbf{A} \\ &= e^{\lambda_1}\left(\mathbf{I} + (\mathbf{A} - \lambda_1\mathbf{I})\right) \end{aligned} \tag{6.3.46}$$

Example 6.3.5: Consider the linear transformation defined in Example 5.3.1. Namely, the linear transformation $\mathbf{A} : \mathscr{V} \to \mathscr{V}$ defined by

$$\begin{aligned}\mathbf{A}\mathbf{e}_1 &= \mathbf{e}_1 + \mathbf{e}_2 + 4\mathbf{e}_3 \\ \mathbf{A}\mathbf{e}_2 &= 2\mathbf{e}_1 - 4\mathbf{e}_3 \\ \mathbf{A}\mathbf{e}_3 &= -\mathbf{e}_1 + \mathbf{e}_2 + 5\mathbf{e}_3\end{aligned} \tag{6.3.47}$$

where $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is a basis for $\mathscr{V}$. The eigenvalues are given by equation (5.3.4), repeated,

$$\begin{aligned}\lambda_1 &= 1 \\ \lambda_2 &= 2 \\ \lambda_3 &= 3\end{aligned} \tag{6.3.48}$$

Because the three eigenvalues are distinct, the exponential linear transformation is given by specializing (6.3.28). The result is

$$e^{\mathbf{A}} = \alpha_0(\mu_1,\mu_2,\mu_3)\mathbf{I} + \alpha_1(\mu_1,\mu_2,\mu_3)\mathbf{A} + \alpha_2(\mu_1,\mu_2,\mu_3)\mathbf{A}^2 \tag{6.3.49}$$

where the coefficients $\alpha_0, \alpha_1, \alpha_2$ are determined from (6.3.31). When (6.3.31) is specialized in this case, the results are

$$\begin{aligned}e^{\lambda_1} &= \alpha_0(\mu_1,\mu_2,\mu_3) + \alpha_1(\mu_1,\mu_2,\mu_3)\lambda_1 + \alpha_2(\mu_1,\mu_2,\mu_3)\lambda_1^2 \\ e^{\lambda_2} &= \alpha_0(\mu_1,\mu_2,\mu_3) + \alpha_1(\mu_1,\mu_2,\mu_3)\lambda_2 + \alpha_2(\mu_1,\mu_2,\mu_3)\lambda_2^2 \\ e^{\lambda_3} &= \alpha_0(\mu_1,\mu_2,\mu_3) + \alpha_1(\mu_1,\mu_2,\mu_3)\lambda_3 + \alpha_2(\mu_1,\mu_2,\mu_3)\lambda_3^2\end{aligned} \tag{6.3.50}$$

where the eigenvalues are given by (6.3.48). If (6.3.48) is used, the solution of (6.3.50) turns out to be

$$\begin{aligned}\begin{bmatrix}\alpha_0(\mu_1,\mu_2,\mu_3) \\ \alpha_1(\mu_1,\mu_2,\mu_3) \\ \alpha_2(\mu_1,\mu_2,\mu_3)\end{bmatrix} &= \begin{bmatrix}1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9\end{bmatrix}^{-1} \begin{bmatrix}e^1 \\ e^2 \\ e^3\end{bmatrix} = \begin{bmatrix}3 & -3 & 1 \\ -\dfrac{5}{2} & 4 & -\dfrac{3}{2} \\ \dfrac{1}{2} & 3 & 9\end{bmatrix}\begin{bmatrix}e^1 \\ e^2 \\ e^3\end{bmatrix} \\ &= \begin{bmatrix}3e - 3e^2 + e^3 \\ -\dfrac{5}{2}e + 4e^2 - \dfrac{3}{2}e^3 \\ \dfrac{1}{2}e - e^2 + \dfrac{1}{2}e^3\end{bmatrix} = \begin{bmatrix}6.0732 \\ -7.3678 \\ 4.0129\end{bmatrix}\end{aligned} \tag{6.3.51}$$

Therefore, (6.3.49) reduces to

$$e^{\mathbf{A}} = 6.0732\mathbf{I} - 7.3678\mathbf{A} + 4.0129\mathbf{A}^2 \tag{6.3.52}$$

If we utilize (6.3.16) and the matrix representation of the linear transformation defined by (6.3.47), equation (5.3.2), it follows from (6.3.52) that

$$\begin{aligned} e^{\begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix}} &= 6.0732\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 7.3678\begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix} + 4.0129\begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix}^2 \\ &= \begin{bmatrix} -5.3078 & 9.3415 & -8.6836 \\ 12.6965 & -1.9525 & 8.6836 \\ 50.7859 & -18.6831 & 37.4528 \end{bmatrix} \end{aligned} \tag{6.3.53}$$

Example 6.3.6: The linear transformation in Example 5.3.2 has the eigenvalues

$$\begin{aligned} \lambda_1 &= -3 \\ \lambda_2 &= 3 \end{aligned} \tag{6.3.54}$$

where the algebraic multiplicity for λ_1 is 1 and that for λ_2 is 2. The exponential linear transformation is given by

$$e^{\mathbf{A}} = \alpha_0(\mu_1,\mu_2,\mu_3)\mathbf{I} + \alpha_1(\mu_1,\mu_2,\mu_3)\mathbf{A} + \alpha_2(\mu_1,\mu_2,\mu_3)\mathbf{A}^2 \tag{6.3.55}$$

where the coefficients $\alpha_0, \alpha_1, \alpha_2$ are determined from (6.3.31) and, because of the repeated root, (6.3.39). When (6.3.31) is specialized in this case, the results are the two equations

$$\begin{aligned} e^{\lambda_1} &= \alpha_0(\mu_1,\mu_2,\mu_3) + \alpha_1(\mu_1,\mu_2,\mu_3)\lambda_1 + \alpha_2(\mu_1,\mu_2,\mu_3)\lambda_1^2 \\ e^{\lambda_2} &= \alpha_0(\mu_1,\mu_2,\mu_3) + \alpha_1(\mu_1,\mu_2,\mu_3)\lambda_2 + \alpha_2(\mu_1,\mu_2,\mu_3)\lambda_2^2 \end{aligned} \tag{6.3.56}$$

The third equation is our set follows from (6.3.39). The result is

$$e^{\lambda_2} = \alpha_1(\mu_1,\mu_2,\mu_3) + 2\alpha_2(\mu_1,\mu_2,\mu_3)\lambda_2 \tag{6.3.57}$$

Given the numerical values in (6.3.54), the solution of (6.3.56) and (6.3.57) is given by

$$\begin{bmatrix} \alpha_0(\mu_1,\mu_2,\mu_3) \\ \alpha_1(\mu_1,\mu_2,\mu_3) \\ \alpha_2(\mu_1,\mu_2,\mu_3) \end{bmatrix} = \begin{bmatrix} 1 & -3 & 9 \\ 1 & 3 & 9 \\ 0 & 1 & 6 \end{bmatrix}^{-1} \begin{bmatrix} e^{-3} \\ e^3 \\ e^3 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{3}{4} & -\frac{3}{2} \\ -\frac{1}{6} & \frac{1}{6} & 0 \\ \frac{1}{36} & -\frac{1}{36} & \frac{1}{6} \end{bmatrix} \begin{bmatrix} e^{-3} \\ e^3 \\ e^3 \end{bmatrix}$$
$$= \begin{bmatrix} \frac{1}{4}e^{-3} - \frac{3}{4}e^3 \\ \frac{1}{6}e^3 - \frac{1}{6}e^{-3} \\ \frac{5}{36}e^3 + \frac{1}{36}e^{-3} \end{bmatrix} = \begin{bmatrix} -15.052 \\ 3.3393 \\ 2.791 \end{bmatrix} \tag{6.3.58}$$

Therefore, (6.3.55) reduces to

$$e^{\mathbf{A}} = -15.052\mathbf{I} + 3.3393\mathbf{A} + 2.791\mathbf{A}^2 \tag{6.3.59}$$

If we utilize (6.3.16) and the matrix representation of the linear transformation defined by (5.3.25), equation (5.3.26), it follows from (6.3.59) that

$$e^{\begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}} = -15.052\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 3.3393\begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} + 2.791\begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}^2$$
$$= \begin{bmatrix} 13.4070 & -6.6786 & 6.6786 \\ -6.6786 & 13.4070 & 6.6786 \\ 6.6786 & 6.6786 & 13.4070 \end{bmatrix} \tag{6.3.60}$$

Exercises

6.3.1 Show that the determinant of the exponential linear transformation obeys

$$\det e^{\mathbf{A}} = e^{\operatorname{tr}\mathbf{A}} \tag{6.3.61}$$

6.3.2 Determine the exponential linear transformation of the linear transformation defined in Example 5.3.3 of Section 5.3.

Answer:

$$e^{\mathbf{A}} = 1.3591\left(\mathbf{I} + \mathbf{A}^2\right) \tag{6.3.62}$$

Section 6.4 More About the Exponential Linear Transformation

In the case when there the eigenvalues are distinct, embedded in the calculations utilizing (6.3.28) and (6.3.32) is a somewhat more elegant way to calculate the exponential matrix. The essential fact that allows this more elegant formula to be obtained is a result known as the Lagrange Interpolation Formula. Lagrange interpolation was mentioned in Exercise 2.4.2 and, later, in Example 2.6.3. The essentials of the derivation of this formula begins with a polynomial of degree $N-1$ that we shall write simply

$$f(x)=\alpha_0+\alpha_1 x+\alpha_2 x^2+\cdots+\alpha_{N-1}x^{N-1} \tag{6.4.1}$$

The idea is to calculate the coefficients $\alpha_0,\alpha_1,\alpha_2,...,\alpha_{N-1}$ from knowledge of the values $f(x_1), f(x_2),..., f(x_N)$ at N *distinct* points $x_1,x_2,...,x_N$. If (6.4.1) is evaluated at these points, the coefficients $\alpha_0,\alpha_1,\alpha_2,...,\alpha_{N-1}$ are the solution of

$$\begin{bmatrix} 1 & x_1 & x_1^2 & \cdot & \cdot & x_1^{N-1} \\ 1 & x_2 & x_2^2 & \cdot & \cdot & x_2^{N-1} \\ 1 & x_3 & x_3^2 & \cdot & \cdot & x_3^{N-1} \\ \cdot & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ 1 & x_N & x_N^2 & \cdot & \cdot & x_N^{N-1} \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \cdot \\ \cdot \\ \alpha_{N-1} \end{bmatrix} = \begin{bmatrix} f(x_1) \\ f(x_2) \\ f(x_2) \\ \cdot \\ \cdot \\ f(x_N) \end{bmatrix} \tag{6.4.2}$$

If we were actually to calculate the coefficients we are led again to the solution of an equation of the form (6.3.33). In this case, however, the polynomial is formally rearranged such that the polynomial (6.4.1) is written

$$f(x)=\alpha_0+\alpha_1 x+\alpha_2 x^2+\cdots+\alpha_{N-1}x^{N-1}=\sum_{j=1}^{N} l_j(x) f(x_j) \tag{6.4.3}$$

where the quantities $l_j(x)$, for $j=1,2,...,N$, are $N-1$ degree polynomials to be determined. One obvious property that follows from (6.4.3) is

$$l_j(x_k)=\begin{cases} 1 & \text{if } j=k \\ 0 & \text{if } j\neq k \end{cases} \tag{6.4.4}$$

These functions are solutions of

$$\begin{bmatrix} 1 & 1 & 1 & \cdot & \cdot & 1 \\ x_1 & x_2 & x_3 & \cdot & \cdot & x_N \\ x_1^2 & x_2^2 & x_3^2 & \cdot & \cdot & x_N^2 \\ \cdot & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ x_1^{N-1} & x_2^{N-1} & x_3^{N-1} & \cdot & \cdot & x_N^{N-1} \end{bmatrix} \begin{bmatrix} l_1(x) \\ l_2(x) \\ l_3(x) \\ \cdot \\ \cdot \\ l_N(x) \end{bmatrix} = \begin{bmatrix} 1 \\ x \\ x^2 \\ \cdot \\ \cdot \\ x^{N-1} \end{bmatrix} \tag{6.4.5}$$

Equation (6.4.5) follows from the multiplication of (6.4.2) by the matrix $\begin{bmatrix} l_1(x) & l_2(x) & l_3(x) & \cdot & \cdot & l_N(x) \end{bmatrix}$, making use of (6.4.3) to eliminate $\sum_{j=1}^{N} l_j(x) f(x_j)$ from the result and forcing the result to hold as an identity in the coefficients $\alpha_0, \alpha_1, \alpha_2, ..., \alpha_{N-1}$. Given what we know about the determinant of the Vandermondian matrix, equation (6.3.34), and Cremers rule, equation (1.11.6), we can write the solution for $l_1(x)$, for example, as

$$l_1(x) = \frac{\begin{vmatrix} 1 & 1 & 1 & \cdot & \cdot & 1 \\ x & x_2 & x_3 & \cdot & \cdot & x_N \\ x^2 & x_2^2 & x_3^2 & \cdot & \cdot & x_N^2 \\ \cdot & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ x^{N-1} & x_2^{N-1} & x_3^{N-1} & \cdot & \cdot & x_N^{N-1} \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 & \cdot & \cdot & 1 \\ x_1 & x_2 & x_3 & \cdot & \cdot & x_N \\ x_1^2 & x_2^2 & x_3^2 & \cdot & \cdot & x_N^2 \\ \cdot & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ x_1^{N-1} & x_2^{N-1} & x_3^{N-1} & \cdot & \cdot & x_N^{N-1} \end{vmatrix}} \tag{6.4.6}$$

with similar formulas for the other functions $l_j(x)$, for $j = 2,...,N$. The two determinants in (6.4.6) can be evaluated by the formula (6.3.34) to yield

$$l_1(x) = \frac{\prod_{\substack{k=1 \\ k\neq 1}}^{N} (x - x_k)}{\prod_{\substack{k=1 \\ k\neq 1}}^{N} (x_1 - x_k)} = \frac{(x - x_2)(x - x_3)\cdots(x - x_N)}{(x_1 - x_2)(x_1 - x_3)\cdots(x_1 - x_N)} \tag{6.4.7}$$

The general expression for the functions $l_j(x)$, for $j=1,2,...,N$ follows by a similar argument and is

$$l_j(x)=\frac{\prod_{\substack{k=1\\k\neq j}}^{N}(x-x_k)}{\prod_{\substack{k=1\\k\neq j}}^{N}(x_j-x_k)}=\frac{(x-x_1)}{(x_j-x_1)}\cdots\frac{(x-x_{j-1})}{(x_j-x_{j-1})}\frac{(x-x_{j+1})}{(x_j-x_{j+1})}\cdots\frac{(x-x_N)}{(x_j-x_N)} \tag{6.4.8}$$

Equations (6.4.8) and $(6.4.3)_2$ combine to yield the Lagrange Interpolation Formula

$$f(x)=\sum_{j=1}^{N}l_j(x)f(x_j)=\sum_{j=1}^{N}f(x_j)\frac{\prod_{\substack{k=1\\k\neq j}}^{N}(x-x_k)}{\prod_{\substack{k=1\\k\neq j}}^{N}(x_j-x_k)} \tag{6.4.9}$$

Equation (6.4.9) is an identity for the $N-1$ degree polynomial (6.4.1). Given the duality between polynomials and polynomials of linear transformations, we can apply (6.4.9) to the polynomial (6.3.28) and write

$$e^{\mathbf{A}}=\sum_{j=1}^{N}e^{\lambda_j}\frac{\prod_{\substack{k=1\\k\neq j}}^{N}(\mathbf{A}-\lambda_k\mathbf{I})}{\prod_{\substack{k=1\\k\neq j}}^{N}(\lambda_j-\lambda_k)} \tag{6.4.10}$$

Equation (6.4.10) is a special case of a result known as Sylvester's Theorem.[9] Equation $(6.3.37)_3$, a result that was derived without the use of (6.4.10), is of the same form. The point is that the results in Section 6.3 contain results like (6.4.10). They are simply not manipulated into the form (6.4.10)[10].

[9] See, for example, Elementary Matrices by Frazier, Duncan and Collar, Cambridge University Press, 1938, Section 3.9.

[10] Sometimes equation (6.2.17) is used to write (6.4.10) in the equivalent form

$$e^{\mathbf{A}}=\sum_{j=1}^{N}e^{\lambda_j}\frac{\operatorname{adj}(\mathbf{A}-\lambda_j\mathbf{I})}{\prod_{\substack{k=1\\k\neq j}}^{N}(\lambda_k-\lambda_j)}$$

As explained, (6.4.10) assumes that the eigenvalues are distinct. We shall not give the generalization of (6.4.10) to the multiple eigenvalue case.[11] It is useful to note in passing that in the special case where a linear transformation has a single eigenvalue λ, the exponential matrix turns out to be

$$e^{\mathbf{A}} = e^{\lambda}\left(\mathbf{I} + (\mathbf{A}-\lambda\mathbf{I}) + \frac{1}{2!}(\mathbf{A}-\lambda\mathbf{I})^2 + \frac{1}{3!}(\mathbf{A}-\lambda\mathbf{I})^3 + \cdots + \frac{1}{(N-1)!}(\mathbf{A}-\lambda\mathbf{I})^{N-1}\right) \quad (6.4.11)$$

Equation (6.3.46) is a special case of (6.4.11).

Exercises

6.4.1 In this exercise, we shall develop the modification of Sylvester's Theorem, equation (6.4.10), in the case where $N=3$ that is appropriate for the case where the characteristic polynomial takes the form

$$f(\lambda) = \det(\mathbf{A}-\lambda\mathbf{I}) = (\lambda_1-\lambda)^2(\lambda_2-\lambda) \quad (6.4.12)$$

In other words, the first eigenvalue has algebraic multiplicity of two and the other eigenvalue has algebraic multiplicity of one. The first step is to find a generalization of the Lagrange interpolation formula to replace (6.4.9). If $f(x)$ is the quadratic

$$f(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 \quad (6.4.13)$$

and we are given *two* values $f(x_1)$ and $f(x_2)$ for distinct values of x_1 and x_2. In order to complete the interpolation based upon (6.4.13), we are also given the slope, $f'(x_1)$, at x_1. These conditions and (6.4.13) yield the following three equations for the unknown coefficients α_1, α_2 and α_3.

$$\begin{bmatrix} 1 & x_1 & x_1^2 \\ 0 & 1 & 2x_1 \\ 1 & x_3 & x_3^2 \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} f(x_1) \\ f'(x_1) \\ f(x_3) \end{bmatrix} \quad (6.4.14)$$

Equation (6.4.14) is an example of a *confluent Vondermonde matrix* (transposed). Show that

[11] See, Elementary Matrices by Frazier, Duncan and Collar, Cambridge University Press, 1938, Section 3.10.

$$\begin{bmatrix} 1 & x_1 & x_1^2 \\ 0 & 1 & 2x_1 \\ 1 & x_3 & x_3^2 \end{bmatrix}^{-1} =$$

$$\begin{bmatrix} -\dfrac{x_2\left(2x_1 - x_2\right)}{\left(x_1 - x_2\right)^2} & \dfrac{x_1 x_2}{\left(x_1 - x_2\right)} & \dfrac{x_1^2}{\left(x_1 - x_2\right)^2} \\ \dfrac{2x_1}{\left(x_1 - x_2\right)^2} & -\dfrac{x_1 + x_2}{\left(x_1 - x_2\right)} & -\dfrac{2x_1}{\left(x_1 - x_2\right)^2\left(x_2 - x_3\right)} \\ -\dfrac{1}{\left(x_1 - x_2\right)^2} & \dfrac{1}{\left(x_1 - x_2\right)} & -\dfrac{1}{\left(x_1 - x_2\right)^2} \end{bmatrix} \tag{6.4.15}$$

As with the derivation of (6.4.9), we are interested in how the polynomial (6.4.13) can be rearranged into the form

$$f\left(x\right) = N_1\left(x\right) f\left(x_1\right) + M_1\left(x\right) f'\left(x_1\right) + N_2\left(x\right) f\left(x_2\right) \tag{6.4.16}$$

where the quadratics $N_1\left(x\right), M_1\left(x\right)$ and $N_2\left(x\right)$ need to be determined. Show that these three quadratics are given by

$$\begin{bmatrix} N_1\left(x\right) \\ M_1\left(x\right) \\ N_2\left(x\right) \end{bmatrix} = \left(\begin{bmatrix} 1 & x_1 & x_1^2 \\ 0 & 1 & 2x_1 \\ 1 & x_3 & x_3^2 \end{bmatrix}^{-1}\right)^T \begin{bmatrix} x \\ x^2 \\ x^3 \end{bmatrix} = \begin{bmatrix} 1 - \dfrac{\left(x_1 - x\right)^2}{\left(x_1 - x_2\right)^2} \\ \dfrac{\left(x_1 - x\right)\left(x_2 - x\right)}{\left(x_1 - x_2\right)} \\ \dfrac{\left(x_1 - x\right)^2}{\left(x_1 - x_2\right)^2} \end{bmatrix} \tag{6.4.17}$$

Finally, show that for the linear transformation $\mathbf{A} : \mathscr{V} \to \mathscr{V}$ whose characteristic polynomial is given by (6.4.12) the exponential linear transformation is given by

$$e^{\mathbf{A}} = e^{\lambda_1}\left(\mathbf{I} - \frac{\left(\mathbf{A} - \lambda_1\mathbf{I}\right)^2}{\left(\lambda_1 - \lambda_2\right)^2}\right) + e^{\lambda_1}\frac{\left(\mathbf{A} - \lambda_1\mathbf{I}\right)\left(\mathbf{A} - \lambda_2\mathbf{I}\right)}{\left(\lambda_1 - \lambda_2\right)} + e^{\lambda_2}\frac{\left(\mathbf{A} - \lambda_1\mathbf{I}\right)^2}{\left(\lambda_1 - \lambda_2\right)^2} \tag{6.4.18}$$

6.4.2 Adapt the results of Example 5.3.2 to the notation used in (6.4.18) and show that the exponential matrix is given by

$$e^{A} = \frac{1}{3}\begin{bmatrix} e^{-3}+2e^{3} & e^{-3}-e^{3} & e^{3}-e^{-3} \\ e^{-3}-e^{3} & e^{-3}+2e^{3} & e^{3}-e^{-3} \\ e^{3}-e^{-3} & e^{3}-e^{-3} & e^{-3}+2e^{3} \end{bmatrix} = \begin{bmatrix} 13.4070 & -6.6786 & 6.6786 \\ -6.6786 & 13.4070 & 6.6786 \\ 6.6786 & 6.6786 & 13.4070 \end{bmatrix} \tag{6.4.19}$$

This exercise is essentially the same as Example 6.3.6 except that here a basis has been selected to represent the linear transformation that was unspecified in Example 6.3.6. Also, the method of solution was built around the method introduced in Section 6.4.

6.4.3 Repeat Exercise 6.4.1 for the case where all of the eigenvalues are identical. In particular, show that

$$e^{\mathbf{A}} = e^{\lambda_1}\left(\mathbf{I} + \left(\mathbf{A} - \lambda_1\mathbf{I}\right) + \frac{1}{2}\left(\mathbf{A} - \lambda_1\mathbf{I}\right)^2\right) \tag{6.4.20}$$

Equation (6.4.20) confirms (6.4.11) for the special case $N = 3$.

Section 6.5 Application of the Exponential Linear Transformation

One important application of the exponential linear transformation is to the study of certain types of systems of linear constant coefficient ordinary differential equations. Our discussion of ordinary differential equations in Sections 5.5 and 5.7 was limited to the case where the matrix that defined the system or ordinary differential equations has N distinct eigenvalues. As illustrated in Section 6.3, the exponential linear transformation can be calculated in cases where the eigenvalues are not distinct and, thus, the eigenvectors do not form a basis. Therefore, the exponential linear transformation can be used to generate the solution of systems of linear constant coefficient ordinary differential equations in those cases where the matrix that defines the system does not have N distinct eigenvalues. Before we illustrate this generalization, we need to establish a few additional properties of the exponential linear transformation.

We begin with the exponential linear transformation in the form (6.3.5). When studying constant coefficient linear ordinary differential equations, one of the most important properties is

$$\frac{de^{\mathbf{A}t}}{dt} = \mathbf{A}e^{\mathbf{A}t} = e^{\mathbf{A}t}\mathbf{A} \tag{6.5.1}$$

The exponential linear transformation is exceptional in that it commutes under multiplication in the special way the last formula shows. This feature of polynomials of linear transformations was pointed out in Section 6.1. The result (6.5.1) follows by differentiation of the formula (6.3.5). The result of this differentiation is

$$\begin{aligned}\frac{de^{\mathbf{A}t}}{dt} &= \mathbf{A} + \frac{2}{2!}\mathbf{A}^2 t + \frac{3}{3!}\mathbf{A}^3 t^2 + \frac{4}{4!}\mathbf{A}^4 t^3 \cdots \\ &= \mathbf{A}\left(\mathbf{I} + \mathbf{A}t + \frac{1}{2!}\mathbf{A}^2 t^2 + \frac{1}{3!}\mathbf{A}^3 t^3 \cdots\right) \\ &= \mathbf{A}e^{\mathbf{A}t}\end{aligned} \tag{6.5.2}$$

Likewise,

$$\begin{aligned}\frac{de^{\mathbf{A}t}}{dt} &= \mathbf{A} + \frac{2}{2!}\mathbf{A}^2 t + \frac{3}{3!}\mathbf{A}^3 t^2 + \frac{4}{4!}\mathbf{A}^4 t^3 \cdots \\ &= \left(\mathbf{I} + \mathbf{A}t + \frac{1}{2!}\mathbf{A}^2 t^2 + \frac{1}{3!}\mathbf{A}^3 t^3 \cdots\right)\mathbf{A} \\ &= e^{\mathbf{A}t}\mathbf{A}\end{aligned} \tag{6.5.3}$$

The fact that $e^{\mathbf{A}t}$ obeys the ordinary differential equation (6.5.1) turns out to make the exponential linear transformation of great importance when one tries to find the solution to (5.5.1).

In Section 5.5, our study of ordinary differential equations involved finding the solution of

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x} + \mathbf{g}(t) \tag{6.5.4}$$

where $\mathbf{x}(t)$ is a $N \times 1$ column matrix, A is an $N \times N$ matrix and $\mathbf{g}(t)$ is a $N \times 1$ column matrix. The initial value problem associated with (6.5.4) is the problem of finding the function $\mathbf{x} = \mathbf{x}(t)$ such that

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x} + \mathbf{g}(t) \qquad \text{and} \quad \mathbf{x}(0) = \mathbf{x}_0 \tag{6.5.5}$$

where $\mathbf{x}_0$ is given. As discussed in Section 5.6, the solution of (6.5.5) will be of the form

$$\mathbf{x}(t) = \mathbf{x}_h(t) + \mathbf{x}_p(t) \tag{6.5.6}$$

where $\mathbf{x}_h$ is the *general solution* , i.e., the solution of the *homogeneous equation*

$$\frac{d\mathbf{x}_h}{dt} = A\mathbf{x}_h \tag{6.5.7}$$

and $\mathbf{x}_p$ is a *particular solution* of

$$\frac{d\mathbf{x}_p}{dt} = A\mathbf{x}_p + \mathbf{g}(t) \tag{6.5.8}$$

If (6.5.7) is multiplied by e^{-At} and (6.5.3) is utilized, we find that $\mathbf{x}_h(t)$ obeys

$$\frac{d\left(e^{-At}\mathbf{x}_h\right)}{dt} = \mathbf{0} \tag{6.5.9}$$

Therefore, after (6.3.11) is utilized, the general solution is

$$\mathbf{x}_h(t) = e^{At}\mathbf{h} \tag{6.5.10}$$

where $\mathbf{h}$ is an arbitrary $N \times 1$ column matrix. The particular solution is obtained from (6.5.8), for example, by the same kind of procedure used in Section 5.7. The result turns out to be

$$\mathbf{x}_p(t) = \int_{\tau=0}^{\tau=t} e^{-A(\tau-t)}\mathbf{g}(\tau)\,d\tau \tag{6.5.11}$$

and, from (6.5.6), (6.5.10) and (6.5.11),

$$\mathbf{x}(t) = e^{At}\mathbf{h} + \int_{\tau=0}^{\tau=t} e^{-A(\tau-t)}\mathbf{g}(\tau)\,d\tau \tag{6.5.12}$$

Finally, if the initial condition $\mathbf{x}(0) = \mathbf{x}_0$ is applied to (6.5.12), the result is $\mathbf{h} = \mathbf{x}_0$ and (6.5.12) becomes

$$\mathbf{x}(t) = e^{At}\mathbf{x}_0 + \int_{\tau=0}^{\tau=t} e^{-A(\tau-t)}\mathbf{g}(\tau)\,d\tau \tag{6.5.13}$$

In the special case where the matrix A has N linearly independent eigenvectors, equation (6.3.21) reduces (6.5.13) to our earlier result (5.7.10).

The method used in Chapter 5, in the special case where the matrix A has N distinct eigenvalues and, thus, N linearly independent eigenvectors, is the preferred method to evaluate the exponential matrix e^{At}. However, when the matrix A does not have N linearly independent eigenvectors, we need a method based upon the discussion in Section 6.2. The fundamental equations are (6.3.25), (6.3.31) and (6.3.39). Even when the matrix has N distinct eigenvalues, equations (6.3.25), (6.3.31) and (6.3.39) or, in this case, (6.4.10) have the advantage of not requiring that the eigenvectors be explicitly calculated. In certain numerical situations, this fact can be beneficial.

Before we utilize (6.3.25), (6.3.31) and (6.3.39) we must adjust these equations to accommodate the substitution $\mathbf{A} \to \mathbf{A}t$. We could rederive these equations for the linear transformation $e^{\mathbf{A}t}$. However, this approach is not necessary. We simply need to agree that the symbols $\mu_1, \mu_2, ..., \mu_N$ are the fundamental invariants of $\mathbf{A}$ and not those of $\mathbf{A}t$. Likewise, the symbols $\lambda_1, \lambda_2, ..., \lambda_N$ are the N, not necessarily distinct, eigenvalues of $\mathbf{A}$ rather than the eigenvalues of $\mathbf{A}t$. With these understandings, the substitution $\mathbf{A} \to \mathbf{A}t$ converts the fundamental invariants by the rule $\mu_j \to t^N \mu_j$ and the eigenvalues by the rule $\lambda_j \to t\lambda_j$. Next, we redefine the unknown coefficients in (6.3.25) to reflect a dependence on t. The result is

$$e^{\mathbf{A}t} = \sum_{n=0}^{N-1} \alpha_n\left(\mu_1, \mu_2, ..., \mu_N, t\right)\mathbf{A}^n \tag{6.5.14}$$

Likewise, (6.3.31) transforms into

$$e^{\lambda t} = \sum_{n=0}^{N-1} \alpha_n\left(\mu_1, \mu_2, ..., \mu_N, t\right)\lambda^n \quad \text{for } \lambda = \text{ an eigenvalue of } \mathbf{A} \tag{6.5.15}$$

and (6.3.39) transforms into

$$\left.\frac{d^j}{d\lambda^j}\left(e^{\lambda t}\right)\right|_{\lambda=\lambda_1} = t^j e^{\lambda_1 t} = \left.\frac{d^j}{d\lambda^j}\left(\sum_{n=0}^{N-1} \alpha_n\left(\mu_1, \mu_2, ..., \mu_N, t\right)\lambda^n\right)\right|_{\lambda=\lambda_1} \quad \text{for} \quad j = 1, 2, ..., d_1 - 1 \tag{6.5.16}$$

The transformation $\mathbf{A} \to \mathbf{A}t$ converts Sylvester's Theorem, equation (6.4.10), to

$$e^{\mathbf{A}t} = \sum_{j=1}^{N} e^{\lambda_j t} \frac{\prod_{\substack{k=1 \\ k\neq j}}^{N} \left(\mathbf{A} - \lambda_k \mathbf{I}\right)}{\prod_{\substack{k=1 \\ k\neq j}}^{N} \left(\lambda_j - \lambda_k\right)} \tag{6.5.17}$$

In the case of the repeated root example discussed in Exercise 6.4.1, the answer (6.4.18) becomes

$$e^{\mathbf{A}t} = e^{\lambda_1 t}\left(\mathbf{I} - \frac{\left(\mathbf{A}-\lambda_1\mathbf{I}\right)^2}{\left(\lambda_1-\lambda_2\right)^2}\right) + te^{\lambda_1 t}\frac{\left(\mathbf{A}-\lambda_1\mathbf{I}\right)\left(\mathbf{A}-\lambda_2\mathbf{I}\right)}{\left(\lambda_1-\lambda_2\right)} + e^{t\lambda_2}\frac{\left(\mathbf{A}-\lambda_1\mathbf{I}\right)^2}{\left(\lambda_1-\lambda_2\right)^2} \tag{6.5.18}$$

Likewise, the result (6.4.11) becomes

$$e^{\mathbf{A}} = e^{\lambda}\left(\mathbf{I} + \left(\mathbf{A}-\lambda\mathbf{I}\right)t + \frac{1}{2!}\left(\mathbf{A}-\lambda\mathbf{I}\right)^2 t^2 + \frac{1}{3!}\left(\mathbf{A}-\lambda\mathbf{I}\right)^3 t^3 + \cdots + \frac{1}{\left(N-1\right)!}\left(\mathbf{A}-\lambda\mathbf{I}\right)^{N-1} t^{N-1}\right) \tag{6.5.19}$$

Example 6.5.1: In Example 5.6.1, we calculated the general solution of the second order ordinary differential equation

$$\ddot{u} + \omega_0^2 u = 0 \tag{6.5.20}$$

where ω_0 is a positive constant. The matrix A in this case is given by equation (5.6.19), repeated,

$$A = \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & 0 \end{bmatrix} \tag{6.5.21}$$

The eigenvalues in this case are given by equation (5.6.21), repeated,

$$\begin{aligned} \lambda_1 &= i\omega_0 \\ \lambda_2 &= -i\omega_0 \end{aligned} \tag{6.5.22}$$

The transition matrix, as follows from (5.6.24) and (5.6.25), is given by

$$T = \begin{bmatrix} 1 & 1 \\ i\omega_0 & -i\omega_0 \end{bmatrix} \tag{6.5.23}$$

Because this example is one where the eigenvalues are distinct, we have three equivalent ways to calculate the exponential matrix e^{At}.

Method 1: The first method is simply to use (6.3.21) which, in this case, yields

$$\begin{aligned}
e^{At} = Te^{Dt}T^{-1} &= \begin{bmatrix} 1 & 1 \\ i\omega_0 & -i\omega_0 \end{bmatrix}\begin{bmatrix} e^{i\omega_0 t} & 0 \\ 0 & e^{-i\omega_0 t} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ i\omega_0 & -i\omega_0 \end{bmatrix}^{-1} \\
&= \frac{1}{2i\omega_0}\begin{bmatrix} 1 & 1 \\ i\omega_0 & -i\omega_0 \end{bmatrix}\begin{bmatrix} e^{i\omega_0 t} & 0 \\ 0 & e^{-i\omega_0 t} \end{bmatrix}\begin{bmatrix} i\omega_0 & 1 \\ i\omega_0 & -1 \end{bmatrix} \\
&= \frac{1}{2i\omega_0}\begin{bmatrix} 1 & 1 \\ i\omega_0 & -i\omega_0 \end{bmatrix}\begin{bmatrix} i\omega_0 e^{i\omega_0 t} & e^{i\omega_0 t} \\ i\omega_0 e^{-i\omega_0 t} & -e^{-i\omega_0 t} \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{2}\left(e^{i\omega_0 t} + e^{-i\omega_0 t}\right) & \frac{1}{2i\omega_0}\left(e^{i\omega_0 t} - e^{-i\omega_0 t}\right) \\ -\frac{\omega_0}{2i}\left(e^{i\omega_0 t} - e^{-i\omega_0 t}\right) & \frac{1}{2}\left(e^{i\omega_0 t} + e^{-i\omega_0 t}\right) \end{bmatrix} \\
&= \begin{bmatrix} \cos(\omega_0 t) & \frac{1}{\omega_0}\sin(\omega_0 t) \\ -\omega_0 \sin(\omega_0 t) & \cos(\omega_0 t) \end{bmatrix}
\end{aligned} \tag{6.5.24}$$

Method 2: The second method utilizes (6.5.14) and (6.5.15). From (6.5.14), the exponential matrix is given by

$$e^{At} = \alpha_0(\mu_1, \mu_2, t) I + \alpha_1(\mu_1, \mu_2, t) A \tag{6.5.25}$$

Because the eigenvalues are distinct, the coefficients $\alpha_0(\mu_1, \mu_2, t)$ and $\alpha_1(\mu_1, \mu_2, t)$ are determined by specializing (6.5.15). The results are

$$\begin{aligned}
e^{i\omega_0 t} &= \alpha_0(\mu_1, \mu_2, t) + \alpha_1(\mu_1, \mu_2, t) i\omega_0 \\
e^{-i\omega_0 t} &= \alpha_0(\mu_1, \mu_2, t) - \alpha_1(\mu_1, \mu_2, t) i\omega_0
\end{aligned} \tag{6.5.26}$$

The solutions of (6.5.26) are

$$\begin{aligned}
\alpha_0(\mu_1, \mu_2, t) &= \frac{1}{2}\left(e^{i\omega_0 t} + e^{-i\omega_0 t}\right) = \cos(\omega_0 t) \\
\alpha_1(\mu_1, \mu_2, t) &= \frac{1}{2i\omega_0}\left(e^{i\omega_0 t} - e^{-i\omega_0 t}\right) = \frac{1}{\omega_0}\sin(\omega_0 t)
\end{aligned} \tag{6.5.27}$$

If (6.5.27) and (6.5.21) are combined with (6.5.25), the result is again

$$
\begin{aligned}
e^{At} &= \alpha_0(\mu_1,\mu_2,t)I + \alpha_1(\mu_1,\mu_2,t)A \\
&= \cos(\omega_0 t)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{1}{\omega_0}\sin(\omega_0 t)\begin{bmatrix} 0 & 1 \\ -\omega_0^2 & 0 \end{bmatrix} \\
&= \begin{bmatrix} \cos(\omega_0 t) & \dfrac{1}{\omega_0}\sin(\omega_0 t) \\ -\omega_0 \sin(\omega_0 t) & \cos(\omega_0 t) \end{bmatrix}
\end{aligned}
\tag{6.5.28}
$$

Method 3: If this problem is worked utilizing Sylvester's Theorem, equation (6.5.17), the result is again

$$
\begin{aligned}
e^{At} &= \sum_{j=1}^{N} e^{\lambda_j t} \frac{\prod_{\substack{k=1 \\ k\neq j}}^{N} (A-\lambda_k I)}{\prod_{\substack{k=1 \\ k\neq j}}^{N} (\lambda_j-\lambda_k)} = e^{\lambda_1 t}\frac{A-\lambda_2 I}{\lambda_1-\lambda_2} + e^{\lambda_2 t}\frac{A-\lambda_1 I}{\lambda_2-\lambda_1} \\
&= e^{i\omega_0 t}\frac{A+i\omega_0 I}{2i\omega_0} - e^{-i\omega_0 t}\frac{A-i\omega_0 I}{2i\omega_0} \\
&= \frac{e^{i\omega_0 t}}{2i\omega_0}\begin{bmatrix} i\omega_0 & 1 \\ -\omega_0^2 & i\omega_0 \end{bmatrix} - \frac{e^{-i\omega_0 t}}{2i\omega_0}\begin{bmatrix} -i\omega_0 & 1 \\ -\omega_0^2 & -i\omega_0 \end{bmatrix} \\
&= \begin{bmatrix} \dfrac{1}{2}\left(e^{i\omega_0 t}+e^{-i\omega_0 t}\right) & \dfrac{1}{2i\omega_0}\left(e^{i\omega_0 t}-e^{-i\omega_0 t}\right) \\ -\dfrac{\omega_0}{2i}\left(e^{i\omega_0 t}-e^{-i\omega_0 t}\right) & \dfrac{1}{2}\left(e^{i\omega_0 t}+e^{-i\omega_0 t}\right) \end{bmatrix} = \begin{bmatrix} \cos(\omega_0 t) & \dfrac{1}{\omega_0}\sin(\omega_0 t) \\ -\omega_0\sin(\omega_0 t) & \cos(\omega_0 t) \end{bmatrix}
\end{aligned}
\tag{6.5.29}
$$

Example 6.5.2: In Example 5.6.2, we calculated the general solution of the second order ordinary differential equation

$$
\ddot{u}(t) + 2\varsigma\omega_0\dot{u}(t) + \omega_0{}^2 u(t) = 0 \tag{6.5.30}
$$

where ω_0 is a positive constant and ζ is a nonnegative constant. The matrix A in this case is given by equation (5.6.35), repeated,

$$
A = \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & -2\zeta\omega_0 \end{bmatrix} \tag{6.5.31}
$$

The eigenvalues in this case are given by equation (5.6.21), repeated,

$$\begin{aligned}\lambda_1 &= -\zeta\omega_0 + i\omega_0\sqrt{1-\zeta^2}\\ \lambda_2 &= -\zeta\omega_0 - i\omega_0\sqrt{1-\zeta^2}\end{aligned} \tag{6.5.32}$$

As with Example 5.6.2, equations (6.5.32) are written to fit what is called the *under damped* case. For this case, the damping is assumed to be such that $\sqrt{1-\zeta^2} > 0$. We shall work this example utilizing equations (6.5.14) and (6.5.15). From (6.5.14), the exponential matrix is given by

$$e^{At} = \alpha_0(\mu_1,\mu_2,t)I + \alpha_1(\mu_1,\mu_2,t)A \tag{6.5.33}$$

Because the eigenvalues are distinct, the coefficients $\alpha_0(\mu_1,\mu_2,t)$ and $\alpha_1(\mu_1,\mu_2,t)$ are determined by specializing (6.5.15). The results are

$$\begin{aligned}e^{-\zeta\omega_0 t}e^{i\omega_0\sqrt{1-\zeta^2}t} &= \alpha_0(\mu_1,\mu_2,t) + \alpha_1(\mu_1,\mu_2,t)\left(-\zeta\omega_0 + i\omega_0\sqrt{1-\zeta^2}\right)\\ e^{-\zeta\omega_0 t}e^{-i\omega_0\sqrt{1-\zeta^2}t} &= \alpha_0(\mu_1,\mu_2,t) + \alpha_1(\mu_1,\mu_2,t)\left(-\zeta\omega_0 - i\omega_0\sqrt{1-\zeta^2}\right)\end{aligned} \tag{6.5.34}$$

The solutions of (6.5.34) are

$$\begin{aligned}\alpha_0(\mu_1,\mu_2,t) &= e^{-\zeta\omega_0 t}\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\sin\left(\omega_0\sqrt{1-\zeta^2}t\right) + \cos\left(\omega_0\sqrt{1-\zeta^2}t\right)\right)\\ \alpha_1(\mu_1,\mu_2,t) &= \frac{e^{-\zeta\omega_0 t}}{\omega_0\sqrt{1-\zeta^2}}\sin\left(\omega_0\sqrt{1-\zeta^2}t\right)\end{aligned} \tag{6.5.35}$$

If (6.5.35) and (6.5.31) are combined with (6.5.33), the result is

$$
\begin{aligned}
e^{At} &= \alpha_0\left(\mu_1,\mu_2,t\right)I + \alpha_1\left(\mu_1,\mu_2,t\right)A \\
&= e^{-\zeta\omega_0 t}\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\sin\left(\omega_0\sqrt{1-\zeta^2}t\right)+\cos\left(\omega_0\sqrt{1-\zeta^2}t\right)\right)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \\
&\quad + \frac{e^{-\zeta\omega_0 t}}{\omega_0\sqrt{1-\zeta^2}}\sin\left(\omega_0\sqrt{1-\zeta^2}t\right)\begin{bmatrix}0 & 1\\ -\omega_0^2 & -2\zeta\omega_0\end{bmatrix} \\
&= \frac{e^{-\zeta\omega_0 t}}{\sqrt{1-\zeta^2}}\begin{bmatrix}\begin{pmatrix}\sqrt{1-\zeta^2}\cos\left(\omega_0\sqrt{1-\zeta^2}t\right)\\ +\zeta\sin\left(\omega_0\sqrt{1-\zeta^2}t\right)\end{pmatrix} & \dfrac{1}{\omega_0}\sin\left(\omega_0\sqrt{1-\zeta^2}t\right) \\ -\omega_0\sin\left(\omega_0\sqrt{1-\zeta^2}t\right) & \begin{pmatrix}\sqrt{1-\zeta^2}\cos\left(\omega_0\sqrt{1-\zeta^2}t\right)\\ -\zeta\sin\left(\omega_0\sqrt{1-\zeta^2}t\right)\end{pmatrix}\end{bmatrix}
\end{aligned}
\tag{6.5.36}
$$

The last two examples have the property that the eigenvalues are distinct and the associated eigenvectors are linearly independent. An example where this is not the case is one based upon Examples 5.3.3 and 5.6.4. This example is as follows:

Example 6.5.3: In Example 5.6.4, we considered the system of first order ordinary differential equations

$$
\begin{aligned}
\frac{dy_1(t)}{dt} &= y_1(t)+y_2(t)+y_3(t) \\
\frac{dy_2(t)}{dt} &= y_2(t)+y_3(t) \\
\frac{dy_3(t)}{dt} &= y_3(t)
\end{aligned}
\tag{6.5.37}
$$

The matrix A in this case is given by equation (5.3.45), repeated,

$$
A=\begin{bmatrix}1 & 1 & 1\\ 0 & 1 & 1\\ 0 & 0 & 1\end{bmatrix}
\tag{6.5.38}
$$

The eigenvalue in this case are given by equation (5.3.47), repeated,

$$
\lambda_1 = 1
\tag{6.5.39}
$$

and it has an algebraic multiplicity of 3. As explained in the discussion of Example 5.3.3, one cannot construct a basis of eigenvectors for this problem. Given this fact, we can never the less proceed with the calculation of e^{At} based upon equations (6.5.14) through (6.5.16). In this case, equation (6.5.14) reduces to

$$e^{At} = \alpha_0\left(\mu_1,\mu_2,\mu_3,t\right)I + \alpha_1\left(\mu_1,\mu_2,\mu_3,t\right)A + \alpha_2\left(\mu_1,\mu_2,\mu_3,t\right)A^2 \tag{6.5.40}$$

For the eigenvalue $\lambda_1 = 1$, equation (6.5.15) reduces to

$$e^{t} = \alpha_0\left(\mu_1,\mu_2,\mu_3,t\right) + \alpha_1\left(\mu_1,\mu_2,\mu_3,t\right) + \alpha_2\left(\mu_1,\mu_2,\mu_3,t\right) \tag{6.5.41}$$

Because the $\lambda_1 = 1$ has a multiplicity of 3, equation (6.5.16) yields

$$\begin{aligned} te^{t} &= \alpha_1\left(\mu_1,\mu_2,\mu_3,t\right) + 2\alpha_2\left(\mu_1,\mu_2,\mu_3,t\right) \\ t^2e^{t} &= 2\alpha_2\left(\mu_1,\mu_2,\mu_3,t\right) \end{aligned} \tag{6.5.42}$$

Equations (6.5.41) and (6.5.42) yield

$$\begin{aligned} \alpha_0\left(\mu_1,\mu_2,\mu_3,t\right) &= \left(1 - t + \frac{1}{2}t^2\right)e^{t} \\ \alpha_1\left(\mu_1,\mu_2,\mu_3,t\right) &= \left(t - t^2\right)e^{t} \\ \alpha_2\left(\mu_1,\mu_2,\mu_3,t\right) &= \frac{1}{2}t^2e^{t} \end{aligned} \tag{6.5.43}$$

These results along with (6.5.38) and (6.5.40) yield

$$\begin{aligned} e^{At} &= \alpha_0\left(\mu_1,\mu_2,\mu_3,t\right)I + \alpha_1\left(\mu_1,\mu_2,\mu_3,t\right)A + \alpha_2\left(\mu_1,\mu_2,\mu_3,t\right)A^2 \\ &= \left(1 - t + \frac{1}{2}t^2\right)e^{t}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \left(t - t^2\right)e^{t}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} + \frac{1}{2}t^2e^{t}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}^2 \\ &= \left(1 - t + \frac{1}{2}t^2\right)e^{t}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \left(t - t^2\right)e^{t}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} + \frac{1}{2}t^2e^{t}\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \\ &= e^{t}\begin{bmatrix} 1 & t & t + \frac{1}{2}t^2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix} \end{aligned} \tag{6.5.44}$$

Equation (6.5.44) also follows from (6.4.11) or from its special case, equation (6.4.20).

Exercises

6.5.1 Calculate e^{At} for the matrix

$$A=\begin{bmatrix}1 & 6\\ 1 & 2\end{bmatrix} \tag{6.5.45}$$

that arose when we worked Exercise 5.6.3. Calculate e^{At} twice, first utilizing the fact that the eigenvalues are distinct and second utilizing the second method used with Examples 6.4.1, 6.4.2 and 6.4.3 above.

Answer: $e^{At}=\begin{bmatrix}\frac{2}{5}e^{4t}+\frac{3}{5}e^{-t} & \frac{6}{5}e^{4t}-\frac{6}{5}e^{-t}\\ \frac{1}{5}e^{4t}-\frac{1}{5}e^{-t} & \frac{3}{5}e^{4t}+\frac{2}{5}e^{-t}\end{bmatrix}$

6.5.2 Calculate e^{At} for the matrix

$$A=\begin{bmatrix}1 & 5\\ -1 & 5\end{bmatrix} \tag{6.5.46}$$

that arose when we worked Exercise 5.6.5. Calculate e^{At} twice, first utilizing the fact that the eigenvalues are distinct and second utilizing the second method used with Examples 6.4.1, 6.4.2 and 6.4.3 above.

Answer: $e^{At}=\begin{bmatrix}e^{3t}\left(\cos t-2\sin t\right) & 5e^{3t}\sin t\\ -3e^{3t}\sin t & e^{3t}\left(\cos t+2\sin t\right)\end{bmatrix}$

6.5.3 Calculate e^{At} for the matrix

$$A=\begin{bmatrix}1 & -2 & 2\\ -2 & 1 & 2\\ 2 & 2 & 1\end{bmatrix} \tag{6.5.47}$$

that arose when we worked Exercise 5.3.2. Calculate e^{At} utilizing the appropriate forms of equations (6.5.14) through (6.5.16). The calculations in this problem are similar to those used in Example 6.4.2.

Answer:[12] $e^{At} = \frac{1}{3}\begin{bmatrix} e^{-3t} + 2e^{3t} & e^{-3t} - e^{3t} & e^{3t} - e^{-3t} \\ e^{-3t} - e^{3t} & e^{-3t} + 2e^{3t} & e^{3t} - e^{-3t} \\ e^{3t} - e^{-3t} & e^{3t} - e^{-3t} & e^{-3t} + 2e^{3t} \end{bmatrix}$

6.5.4. Use the formula (6.4.11), that was given without proof, and derive the result (6.5.44)

6.5.5 Utilize the result (6.5.44) and the general solution (6.5.10) and derive the solution (5.6.53) that was given without proof in Section 5.6.

---

[12] The answer to this problem can also be obtained from (6.5.18). In this case, it can be confirmed that

$$\left(\mathbf{A} - \lambda_1 \mathbf{I}\right)\left(\mathbf{A} - \lambda_2 \mathbf{I}\right) = 0$$

This result is actually the minimum polynomial for this linear transformation.

Section 6.6 Projections and Spectral Decompositions

In this section, we shall look deeper into the eigenvalue problem introduced in Chapter 5. In particular, we shall introduce a special linear transformation called a projection and show how it can be used to produce a decomposition, called a spectral decomposition, of certain linear transformations $\mathbf{A}:\mathscr{V}\to\mathscr{V}$.

Definition: A projection is a linear transformation is a linear transformation $\mathbf{P}:\mathscr{V}\to\mathscr{V}$ that satisfies the condition

$$\mathbf{P}^2=\mathbf{P} \tag{6.6.1}$$

Theorem 6.6.1. If $\mathbf{P}:\mathscr{V}\to\mathscr{V}$ is a projection, then

$$\mathscr{V}=R(\mathbf{P})\oplus K(\mathbf{P}) \tag{6.6.2}$$

Proof. Let $\mathbf{v}$ be an arbitrary vector in $\mathscr{V}$. Let

$$\mathbf{w}=\mathbf{v}-\mathbf{P}\mathbf{v} \tag{6.6.3}$$

Then, by (6.6.1), $\mathbf{Pw}=\mathbf{Pv}-\mathbf{P}(\mathbf{Pv})=\mathbf{Pv}-\mathbf{Pv}=\mathbf{0}$. Thus, $\mathbf{w}\in K(\mathbf{P})$. Since $\mathbf{Pv}\in R(\mathbf{P})$, (6.6.3) implies that

$$\mathscr{V}=R(\mathbf{P})+K(\mathbf{P}) \tag{6.6.4}$$

To show that $R(\mathbf{P})\cap K(\mathbf{P})=\{\mathbf{0}\}$, let $\mathbf{u}\in R(\mathbf{P})\cap K(\mathbf{P})$. Then, since $\mathbf{u}\in R(\mathbf{P})$ for some $\mathbf{v}\in\mathscr{V}$, $\mathbf{u}=\mathbf{Pv}$. But, since $\mathbf{u}$ is also in $K(\mathbf{P})$,

$$\mathbf{0}=\mathbf{Pu}=\mathbf{P}(\mathbf{Pv})=\mathbf{Pv}=\mathbf{u} \tag{6.6.5}$$

which completes the proof.

The name *projection* arises from the geometric interpretation of (6.6.2). Given any $\mathbf{v}\in\mathscr{V}$, then there are unique vectors $\mathbf{u}\in R(\mathbf{P})$ and $\mathbf{w}\in K(\mathbf{P})$ such that

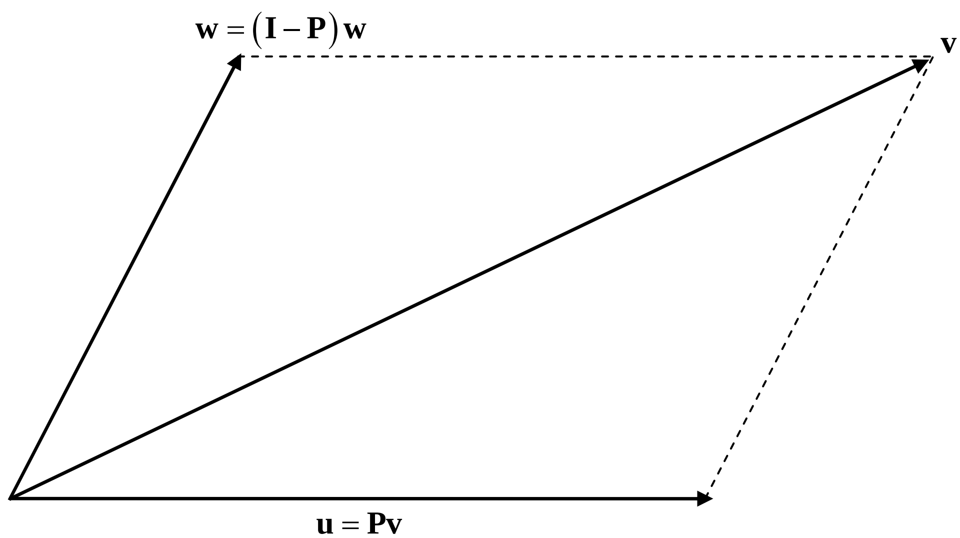
$$\mathbf{v}=\mathbf{u}+\mathbf{w} \tag{6.6.6}$$

where

$$\mathbf{Pu}=\mathbf{u} \qquad \text{and} \qquad \mathbf{Pw}=\mathbf{0} \tag{6.6.7}$$

Viewed as an eigenvalue problem, equation $(6.6.7)_1$ shows that every nonzero vector $\mathbf{u} \in R(\mathbf{P})$ is an eigenvector and the corresponding eigenvalue is 1. Likewise, $(6.6.7)_2$ shows that every nonzero vector $\mathbf{w} \in K(\mathbf{P})$ is an eigenvector and the corresponding eigenvalue is 0.

Geometrically, the linear transformation $\mathbf{P}$ takes $\mathbf{v}$ and projects in onto the subspace $R(\mathbf{P})$ along the subspace $K(\mathbf{P})$. The following figure illustrates this point.



Given a projection $\mathbf{P}$, the linear transformation $\mathbf{I} - \mathbf{P}$ is also a projection. It is easily shown that

$$\mathscr{V} = R(\mathbf{I} - \mathbf{P}) \oplus K(\mathbf{I} - \mathbf{P}) \tag{6.6.8}$$

and

$$R(\mathbf{I} - \mathbf{P}) = K(\mathbf{P}), \quad K(\mathbf{I} - \mathbf{P}) = R(\mathbf{P}) \tag{6.6.9}$$

Matrix of a projection $\mathbf{P} : \mathscr{V} \to \mathscr{V}$ takes a simple form if a judicious choice of made for the basis of $\mathscr{V}$. If $N = \dim \mathscr{V}$ and $R = \dim R(\mathbf{P})$, the basis of $\mathscr{V}$ can always be written $\left\{ \underbrace{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_R}_{\text{Basis for } R(\mathbf{P})}, \underbrace{\mathbf{e}_{R+1}, \mathbf{e}_{R+1}, \ldots, \mathbf{e}_N}_{\text{Basis for } K(\mathbf{P})} \right\}$. It follows from the result (6.6.7) that

$$\mathbf{P}\mathbf{e}_k = \begin{cases} \mathbf{e}_k & \text{for} \quad k = 1, 2, \ldots, R \\ \mathbf{0} & \text{for} \quad k = R+1, \ldots, N \end{cases} \tag{6.6.10}$$

Therefore, the matrix of $\mathbf{P} : \mathscr{V} \to \mathscr{V}$ with respect to the basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_R, \mathbf{e}_{R+1}, \mathbf{e}_{R+1}, \ldots, \mathbf{e}_N\}$ is

$$M\left(\mathbf{P}, \mathbf{e}_j, \mathbf{e}_k\right) = \underbrace{\left[\begin{array}{cccccccc} 1 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & 1 & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & 1 & & & & & \cdot \\ \cdot & \cdot & & \cdot & & & & \cdot \\ \multicolumn{5}{l}{\underbrace{0 \quad 0 \qquad\qquad\quad 1}_{R \times R}} & & & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & 0 & & 0 \\ \cdot & & & & & & 0 & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \end{array}\right]}_{N \times N} \tag{6.6.11}$$

Given the matrix (6.6.11), the characteristic polynomial of the projection is

$$\det\left(\mathbf{P} - \lambda \mathbf{I}\right) = \det\left(M\left(\mathbf{P} - \lambda \mathbf{I}, \mathbf{e}_j, \mathbf{e}_k\right)\right) = \left(-\lambda\right)^{N-R}\left(1-\lambda\right)^{R} \tag{6.6.12}$$

Theorem 6.6.1 is a special case of the following theorem.

Theorem 6.6.2. If $\mathbf{P}_k$, $k = 1, \ldots, R$, are projection operators with the properties that

$$\begin{aligned} \mathbf{P}_k^2 &= \mathbf{P}_k, & \qquad k &= 1, \ldots, R \\ \mathbf{P}_k \mathbf{P}_q &= \mathbf{0}, & \qquad k &\neq q \end{aligned} \tag{6.6.13}$$

and

$$\mathbf{I} = \sum_{k=1}^{R} \mathbf{P}_k \tag{6.6.14}$$

then

$$\mathscr{V} = R\left(\mathbf{P}_1\right) \oplus R\left(\mathbf{P}_2\right) \oplus \cdots \oplus R\left(\mathbf{P}_R\right) \tag{6.6.15}$$

The proof of this theorem is left as an exercise. It is a small generalization of the proof of Theorem 6.6.1. As a converse of Theorem 6.6.2, if $\mathscr{V}$ has the decomposition

$$\mathscr{V} = \mathscr{V}_1 \oplus \cdots \oplus \mathscr{V}_R \tag{6.6.16}$$

then the R linear transformations $\mathbf{P}_k : \mathscr{V} \to \mathscr{V}$ defined by

$$\mathbf{P}_k \mathbf{v} = \mathbf{v}_k, \qquad k = 1,...,R \tag{6.6.17}$$

for all $\mathbf{v} \in \mathscr{V}$, where

$$\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2 + \cdots + \mathbf{v}_R \tag{6.6.18}$$

are projections and satisfy (6.6.14). Moreover, $\mathscr{V}_k = R(\mathbf{P}_k)$, $k = 1,...,R$.

If the vector space $\mathscr{V}$ is an inner product space, the adjoint of a linear transformation $\mathbf{A} : \mathscr{V} \to \mathscr{V}$ is a linear transformation $\mathbf{A}^* : \mathscr{V} \to \mathscr{V}$ defined by the following special case of the definition (4.9.1)

$$\langle \mathbf{v}_2, \mathbf{A}\mathbf{v}_1 \rangle = \langle \mathbf{A}^* \mathbf{v}_2, \mathbf{v}_1 \rangle \quad \text{for all} \quad \mathbf{v}_1, \mathbf{v}_2 \in \mathscr{V} \tag{6.6.19}$$

If we apply the definition (6.6.19) to the projection (6.6.1), the definition becomes

$$\langle \mathbf{v}_2, \mathbf{P}\mathbf{v}_1 \rangle = \langle \mathbf{P}^* \mathbf{v}_2, \mathbf{v}_1 \rangle \quad \text{for all} \quad \mathbf{v}_1, \mathbf{v}_2 \in \mathscr{V} \tag{6.6.20}$$

If we form the inner product $\langle (\mathbf{I} - \mathbf{P})\mathbf{v}_1, \mathbf{P}\mathbf{v}_2 \rangle$, the definition (6.6.20) allows the result to be written

$$\langle (\mathbf{I} - \mathbf{P})\mathbf{v}_1, \mathbf{P}\mathbf{v}_2 \rangle = \langle \mathbf{P}^* (\mathbf{I} - \mathbf{P})\mathbf{v}_1, \mathbf{v}_2 \rangle = \langle (\mathbf{P}^* - \mathbf{P}^*\mathbf{P})\mathbf{v}_1, \mathbf{v}_2 \rangle \tag{6.6.21}$$

The conclusion from (6.6.21) and the definition (6.6.1) is that if the projection $\mathbf{P}$ is self adjoint, i.e., if $\mathbf{P} = \mathbf{P}^*$ then the subspaces $R(\mathbf{P})$ and $K(\mathbf{P})$ are orthogonal. As explained in Section 4.11, this geometric relationship between orthogonal subspaces is written

$$R(\mathbf{P}) = K(\mathbf{P})^{\perp} \tag{6.6.22}$$

In this case, the projection is called an *orthogonal or perpendicular projection.*

In Section 5.4, it was pointed out that for eigenvalue problems for which the geometric multiplicity is equal to the algebraic multiplicity for each eigenvalue of a linear transformation $\mathbf{A} : \mathscr{V} \to \mathscr{V}$, then the vector space $\mathscr{V}$ admits the direct sum representation

$$\mathscr{V} = \mathscr{V}(\lambda_1) \oplus \mathscr{V}(\lambda_2) \oplus \cdots \oplus \mathscr{V}(\lambda_L) \tag{6.6.23}$$

where $\lambda_1,...,\lambda_L$ are the distinct eigenvalues of $\mathbf{A}$. Given the representation (6.6.23), then every $\mathbf{w}\in\mathscr{V}$ has the unique representation

$$\mathbf{w}=\mathbf{w}_1+\mathbf{w}_2+\cdots+\mathbf{w}_L=\sum_{j=1}^{L}\mathbf{w}_j \tag{6.6.24}$$

The representation (6.6.24) allow us to define L projections $\mathbf{P}_k:\mathscr{V}\to\mathscr{V}$ by

$$\mathbf{P}_k\mathbf{w}=\mathbf{w}_k \qquad \text{for} \quad k=1,...,L \tag{6.6.25}$$

The L projections $\mathbf{P}_k:\mathscr{V}\to\mathscr{V}$ defined by (6.6.25) obey (6.6.13) and (6.6.14). Given (6.6.24), it follows that

$$\mathbf{A}\mathbf{w}=\mathbf{A}\left(\sum_{k=1}^{L}\mathbf{w}_j\right)=\sum_{k=1}^{L}\mathbf{A}\mathbf{w}_k=\sum_{k=1}^{L}\lambda_k\mathbf{w}_k \tag{6.6.26}$$

where the defining condition for the k^{th} eigenvector, $\mathbf{A}\mathbf{w}_k=\lambda_k\mathbf{w}_k$, has been used. Given (6.6.25), equation (6.6.26) yields

$$\left(\mathbf{A}-\sum_{k=1}^{L}\lambda_k\mathbf{P}_k\right)\mathbf{w}=\mathbf{0} \tag{6.6.27}$$

Equation (6.6.27) holds for all vectors $\mathbf{w}\in\mathscr{V}$. Therefore, (6.6.27) yields the *spectral decomposition* of $\mathbf{A}:\mathscr{V}\to\mathscr{V}$ in terms of its eigenvalues and projections into its characteristic subspaces. The result is

$$\mathbf{A}=\sum_{k=1}^{L}\lambda_k\mathbf{P}_k \tag{6.6.28}$$

The representation (6.6.28) holds for linear transformations for which the geometric multiplicity is equal to the algebraic multiplicity for each eigenvalue. If this condition does not hold for each eigenvalue, then (6.6.28) is not valid.

Given (6.6.28), it follows from (6.6.13) that

$$\mathbf{A}^n=\sum_{k=1}^{L}\lambda_k^n\mathbf{P}_k \tag{6.6.29}$$

for a positive integer n. Equation (6.6.29) shows that the exponential matrix (6.3.3) takes the form

$$e^{\mathbf{A}} = \sum_{n=0}^{\infty} \frac{1}{n!}\mathbf{A}^n = \sum_{k=1}^{L}\left(\sum_{n=0}^{\infty}\frac{1}{n!}\lambda_k^n\right)\mathbf{P}_k = \sum_{k=1}^{L} e^{\lambda_k}\mathbf{P}_k \tag{6.6.30}$$

where (6.3.4) has been used. It is also true that

$$\mathbf{A}^{\frac{1}{n}} = \sum_{k=1}^{L} \lambda_k^{\frac{1}{n}}\mathbf{P}_k \tag{6.6.31}$$

The quantity $\mathbf{A}^{\frac{1}{n}}$ obeys

$$\left(\mathbf{A}^{\frac{1}{n}}\right)^n = \underbrace{\mathbf{A}^{\frac{1}{n}}\cdots\mathbf{A}^{\frac{1}{n}}}_{n \text{ times}} = \mathbf{A} \tag{6.6.32}$$

and gives an analytical expression for the n^{th} root of a linear transformation that has the spectral decomposition (6.6.28).

Example 6.6.1: In this example, we shall consider again Example 5.3.1. This example solved the eigenvalue problem for the linear transformation $\mathbf{A}:\mathscr{V}\to\mathscr{V}$ defined by (5.3.1), repeated,

$$\begin{aligned} \mathbf{A}\mathbf{e}_1 &= \mathbf{e}_1 + \mathbf{e}_2 + 4\mathbf{e}_3 \\ \mathbf{A}\mathbf{e}_2 &= 2\mathbf{e}_1 - 4\mathbf{e}_3 \\ \mathbf{A}\mathbf{e}_3 &= -\mathbf{e}_1 + \mathbf{e}_2 + 5\mathbf{e}_3 \end{aligned} \tag{6.6.33}$$

where $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ is a basis for $\mathscr{V}$. The problem is to determine the spectral decomposition (6.6.28) for this example. This particular example was one where the three eigenvalues were distinct and the associated characteristic subspaces were one dimensional. In Example 5.3.1, we wrote the three eigenvalues as in (5.3.4), repeated,

$$\begin{aligned} \lambda_1 &= 1 \\ \lambda_2 &= 2 \\ \lambda_3 &= 3 \end{aligned} \tag{6.6.34}$$

and the three eigenvectors as in (5.3.20), repeated,

$$\begin{aligned} \mathbf{v}_1 &= -\frac{1}{2}\mathbf{e}_1 + \frac{1}{2}\mathbf{e}_2 + \mathbf{e}_3 \\ \mathbf{v}_2 &= -\frac{1}{2}\mathbf{e}_1 + \frac{1}{4}\mathbf{e}_2 + \mathbf{e}_3 \\ \mathbf{v}_3 &= -\frac{1}{4}\mathbf{e}_1 + \frac{1}{4}\mathbf{e}_2 + \mathbf{e}_3 \end{aligned} \tag{6.6.35}$$

The three projections $\mathbf{P}_1, \mathbf{P}_2$ and $\mathbf{P}_3$ are defined by (6.6.25). With respect to the basis of eigenvectors, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, their components are given by

$$\mathbf{P}_k \mathbf{v}_j = \begin{cases} \mathbf{v}_k & \text{for} \quad j = k \\ \mathbf{0} & \text{for} \quad j \neq k \end{cases} \tag{6.6.36}$$

Therefore, with respect to the basis of eigenvectors, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, the projections have the following matrices

$$\begin{aligned} M\left(\mathbf{P}_1, \mathbf{v}_j, \mathbf{v}_k\right) &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ M\left(\mathbf{P}_2, \mathbf{v}_j, \mathbf{v}_k\right) &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ M\left(\mathbf{P}_3, \mathbf{v}_j, \mathbf{v}_k\right) &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned} \tag{6.6.37}$$

The matrices of the three projections with respect to the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ are given by the usual basis transformation formula (3.6.18). Therefore,

$$M\left(\mathbf{A}, \mathbf{e}_j, \mathbf{e}_k\right) = TM\left(\mathbf{A}, \mathbf{v}_j, \mathbf{v}_k\right) T^{-1} \tag{6.6.38}$$

where $\mathbf{A}$ is any one of the matrices in (6.6.37) and T is the transition matrix based upon the basis change (6.6.35). This matrix was given earlier in equation (5.3.22), repeated,

$$T = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 1 & 1 & 1 \end{bmatrix} \tag{6.6.39}$$

Given (6.6.37), (6.6.38) and (6.6.39), it follows that

$$M\left(\mathbf{P}_1,\mathbf{e}_j,\mathbf{e}_k\right)=TM\left(\mathbf{P}_1,\mathbf{v}_j,\mathbf{v}_k\right)T^{-1}=\begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 1 & 1 & 1 \end{bmatrix}^{-1}$$

$$=\begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 4 & -1 \\ 4 & -4 & 0 \\ -4 & 0 & 2 \end{bmatrix}=\begin{bmatrix} 0 & -2 & \frac{1}{2} \\ 0 & 2 & -\frac{1}{2} \\ 0 & 4 & -1 \end{bmatrix} \tag{6.6.40}$$

Likewise,

$$M\left(\mathbf{P}_2,\mathbf{e}_j,\mathbf{e}_k\right)=TM\left(\mathbf{P}_2,\mathbf{v}_j,\mathbf{v}_k\right)T^{-1}=\begin{bmatrix} 2 & 2 & 0 \\ -1 & -1 & 0 \\ -4 & -4 & 0 \end{bmatrix} \tag{6.6.41}$$

and

$$M\left(\mathbf{P}_3,\mathbf{e}_j,\mathbf{e}_k\right)=TM\left(\mathbf{P}_3,\mathbf{v}_j,\mathbf{v}_k\right)T^{-1}=\begin{bmatrix} -1 & 0 & -\frac{1}{2} \\ 1 & 0 & \frac{1}{2} \\ 4 & 0 & 2 \end{bmatrix} \tag{6.6.42}$$

It is elementary to show that the relationships (6.6.13) and (6.6.14) are obeyed by the projections whose matrices are given by (6.6.40), (6.6.41) and (6.6.42). In addition, these formulas along with (6.6.34) can be used to confirm the spectral decomposition (6.6.28) in this case. Finally, one can use (6.6.40), (6.6.41) and (6.6.42) along with (6.6.30) and rederive the result (6.3.53) that was obtained earlier by a different method.

Example 6.6.2: In this next example, we shall consider again Example 5.3.2. This example solved the eigenvalue problem for the linear transformation $\mathbf{A}:\mathscr{V}\to\mathscr{V}$ defined by (5.3.25), repeated,

$$\begin{aligned} \mathbf{A}\mathbf{i}_1 &= \mathbf{i}_1 - 2\mathbf{i}_2 + 2\mathbf{i}_3 \\ \mathbf{A}\mathbf{i}_2 &= -2\mathbf{i}_1 + \mathbf{i}_2 + 2\mathbf{i}_3 \\ \mathbf{A}\mathbf{i}_3 &= 2\mathbf{i}_1 + 2\mathbf{i}_2 + \mathbf{i}_3 \end{aligned} \tag{6.6.43}$$

where $\{\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3\}$ is an orthonormal basis for $\mathscr{V}$. The problem is to determine the spectral decomposition for the linear transformation defined by (6.6.43). This particular example was one where there were only two distinct eigenvalues. The two eigenvalues are given by (5.3.28), repeated,

$$\begin{aligned} \lambda_1 &= -3 \\ \lambda_2 &= 3 \end{aligned} \tag{6.6.44}$$

The algebraic multiplicity for λ_1 is 1 and that for λ_2 is 2. The characteristic subspace $\mathscr{V}(\lambda_1)$ is one dimensional and is spanned by the vector, from (5.3.41), repeated,

$$\mathbf{v}_1 = -\mathbf{i}_1 - \mathbf{i}_2 + \mathbf{i}_3 \tag{6.6.45}$$

The characteristic subspace $\mathscr{V}(\lambda_2)$ is two dimensional and is spanned by vectors $\{-\mathbf{i}_1 + \mathbf{i}_2, \mathbf{i}_1 + \mathbf{i}_3\}$. Without loss of generality, we can take the basis of $\mathscr{V}(\lambda_2)$ to be the pair of vectors

$$\mathbf{v}_2 = -\mathbf{i}_1 + \mathbf{i}_2 \quad \text{and} \quad \mathbf{v}_3 = \mathbf{i}_1 + \mathbf{i}_3 \tag{6.6.46}$$

The two projections $\mathbf{P}_1$ and $\mathbf{P}_2$ are defined by (6.6.25). With respect to the basis, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, their components are given by

$$\begin{aligned} \mathbf{P}_1\mathbf{v}_j &= \begin{cases} \mathbf{v}_1 & \text{for } j = 1 \\ \mathbf{0} & \text{for } j = 2,3 \end{cases} \\ \mathbf{P}_2\mathbf{v}_j &= \begin{cases} \mathbf{0} & \text{for } j = 1 \\ \mathbf{v}_j & \text{for } j = 2,3 \end{cases} \end{aligned} \tag{6.6.47}$$

Therefore, with respect to the basis of eigenvectors, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, the projections have the following matrices

$$\begin{aligned} M(\mathbf{P}_1, \mathbf{v}_j, \mathbf{v}_k) &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ M(\mathbf{P}_2, \mathbf{v}_j, \mathbf{v}_k) &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned} \tag{6.6.48}$$

The matrices of the two projections with respect to the basis $\{\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3\}$ are given by the usual basis transformation formula (3.6.18). Therefore,

$$M\left(\mathbf{A},\mathbf{i}_j,\mathbf{i}_k\right)=TM\left(\mathbf{A},\mathbf{v}_j,\mathbf{v}_k\right)T^{-1} \tag{6.6.49}$$

where $\mathbf{A}$ is any one of the matrices in (6.6.48) and T is the transition matrix based upon the basis change (6.6.45) and (6.6.46). This matrix was given by,

$$T=\begin{bmatrix}-1 & -1 & 1\\ -1 & 1 & 0\\ 1 & 0 & 1\end{bmatrix} \tag{6.6.50}$$

Given (6.6.37), (6.6.38) and (6.6.39), it follows that

$$\begin{aligned}M\left(\mathbf{P}_1,\mathbf{i}_j,\mathbf{i}_k\right)=TM\left(\mathbf{P}_1,\mathbf{v}_j,\mathbf{v}_k\right)T^{-1}&=\begin{bmatrix}-1 & -1 & 1\\ -1 & 1 & 0\\ 1 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}-1 & -1 & 1\\ -1 & 1 & 0\\ 1 & 0 & 1\end{bmatrix}^{-1}\\ &=\begin{bmatrix}-1 & -1 & 1\\ -1 & 1 & 0\\ 1 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}-\frac{1}{3} & -\frac{1}{3} & \frac{1}{3}\\ -\frac{1}{3} & \frac{2}{3} & \frac{1}{3}\\ \frac{1}{3} & \frac{1}{3} & \frac{2}{3}\end{bmatrix}=\begin{bmatrix}\frac{1}{3} & \frac{1}{3} & -\frac{1}{3}\\ \frac{1}{3} & \frac{1}{3} & -\frac{1}{3}\\ -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3}\end{bmatrix}\end{aligned} \tag{6.6.51}$$

Likewise,

$$M\left(\mathbf{P}_2,\mathbf{i}_j,\mathbf{i}_k\right)=TM\left(\mathbf{P}_2,\mathbf{v}_j,\mathbf{v}_k\right)T^{-1}=\begin{bmatrix}\frac{2}{3} & -\frac{1}{3} & \frac{1}{3}\\ -\frac{1}{3} & \frac{2}{3} & \frac{1}{3}\\ \frac{1}{3} & \frac{1}{3} & \frac{2}{3}\end{bmatrix} \tag{6.6.52}$$

It is again elementary to show that the relationships (6.6.13) and (6.6.14) are obeyed by the projections whose matrices are given by (6.6.51) and (6.6.52). In addition, these formulas along with (6.6.44) can be used to confirm the spectral decomposition (6.6.28) in this case. Note that the matrices (6.6.51) and (6.6.52) are symmetric matrices. Because the basis $\{\mathbf{i}_1,\mathbf{i}_2,\mathbf{i}_3\}$ is orthonormal, the projection linear transformations are symmetric and, as mentioned above, the image spaces $\mathscr{V}(\lambda_1)$ and $\mathscr{V}(\lambda_2)$ are orthogonal. Finally, one can use (6.6.51) and (6.6.52) along with (6.6.30) to rederive the result (6.3.60) that was derived earlier by a different method.

Example 6.6.3: In Example 5.3.4 the eigenvalue problem was solved for the linear transformation $\mathbf{A}:\mathscr{V}\rightarrow\mathscr{V}$ defined by (5.3.53), repeated,

$$\begin{aligned}\mathbf{A}\mathbf{e}_1 &= \frac{\sqrt{2}}{2}\mathbf{e}_1+\frac{\sqrt{2}}{2}\mathbf{e}_3\\ \mathbf{A}\mathbf{e}_2 &= \mathbf{e}_2\\ \mathbf{A}\mathbf{e}_3 &= -\frac{\sqrt{2}}{2}\mathbf{e}_1+\frac{\sqrt{2}}{2}\mathbf{e}_3\end{aligned} \tag{6.6.53}$$

Of course, $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ is a basis for $\mathscr{V}$. The eigenvalues for this problem are given by equation (5.3.56), repeated,

$$\begin{aligned}\lambda_1 &= \frac{\sqrt{2}}{2}(1+i)\\ \lambda_2 &= 1\\ \lambda_3 &= \frac{\sqrt{2}}{2}(1-i)\end{aligned} \tag{6.6.54}$$

and the three eigenvectors are given by (5.3.69), repeated,

$$\begin{aligned}\mathbf{v}_1 &= i\mathbf{e}_1+\mathbf{e}_3\\ \mathbf{v}_2 &= \mathbf{e}_1\\ \mathbf{v}_3 &= -i\mathbf{e}_1+\mathbf{e}_3\end{aligned} \tag{6.6.55}$$

The three projections $\mathbf{P}_1,\mathbf{P}_2$ and $\mathbf{P}_3$ are defined by (6.6.25). As with Example 6.6.1, with respect to the basis of eigenvectors, $\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$, the projections have the following matrices

$$\begin{aligned}M\left(\mathbf{P}_1,\mathbf{v}_j,\mathbf{v}_k\right) &= \begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}\\ M\left(\mathbf{P}_2,\mathbf{v}_j,\mathbf{v}_k\right) &= \begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}\\ M\left(\mathbf{P}_3,\mathbf{v}_j,\mathbf{v}_k\right) &= \begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}\end{aligned} \tag{6.6.56}$$

The matrices of the three projections with respect to the basis $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ are given by

$$M\left(\mathbf{P}_1,\mathbf{e}_j,\mathbf{e}_k\right)=TM\left(\mathbf{P}_1,\mathbf{v}_j,\mathbf{v}_k\right)T^{-1}=\begin{bmatrix} i & 0 & -i \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} i & 0 & -i \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}^{-1}$$
$$=\begin{bmatrix} i & 0 & -i \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} -\frac{i}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{i}{2} & 0 & \frac{1}{2} \end{bmatrix}=\begin{bmatrix} \frac{1}{2} & 0 & \frac{i}{2} \\ 0 & 0 & 0 \\ -\frac{i}{2} & 0 & \frac{1}{2} \end{bmatrix} \tag{6.6.57}$$

Likewise,

$$M\left(\mathbf{P}_2,\mathbf{e}_j,\mathbf{e}_k\right)=TM\left(\mathbf{P}_2,\mathbf{v}_j,\mathbf{v}_k\right)T^{-1}=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{6.6.58}$$

and

$$M\left(\mathbf{P}_3,\mathbf{e}_j,\mathbf{e}_k\right)=TM\left(\mathbf{P}_3,\mathbf{v}_j,\mathbf{v}_k\right)T^{-1}=\begin{bmatrix} \frac{1}{2} & 0 & -\frac{i}{2} \\ 0 & 0 & 0 \\ \frac{i}{2} & 0 & \frac{1}{2} \end{bmatrix} \tag{6.6.59}$$

Exercises

6.6.1 Consider the special case where the linear transformation $\mathbf{A}:\mathscr{V}\to\mathscr{V}$ has N distinct eigenvalues and show that the projections $\mathbf{P}_j$ for $j=1,2,...,N$ are given by the explicit formulas

$$\mathbf{P}_j=\frac{\prod\limits_{\substack{k=1\\k\neq j}}^{N}\left(\mathbf{A}-\lambda_k\mathbf{I}\right)}{\prod\limits_{\substack{k=1\\k\neq j}}^{N}\left(\lambda_j-\lambda_k\right)} \qquad \text{for} \quad j=1,2,...,N \tag{6.6.60}$$

6.6.2 Example 6.6.2 involved the arbitrary choice (6.6.46) of the basis for the two dimensional characteristic subspace $\mathscr{V}\left(\lambda_2\right)$. Other choices are possible. For example, rather than (6.6.46) one could take

$$\mathbf{v}_2=\mathbf{i}_2+\mathbf{i}_3 \qquad \text{and} \quad \mathbf{v}_3=-2\mathbf{i}_1+\mathbf{i}_2-\mathbf{i}_3 \tag{6.6.61}$$

The two vectors (6.6.61) can be show to span $\mathscr{V}(\lambda_2)$. Show that the two projections $\mathbf{P}_1$ and $\mathbf{P}_2$ are *again* given by (6.6.51) and (6.6.52). The point of this exercise is to illustrate that when the geometric multiplicity is greater than one the corresponding projection into the characteristic subspace does not depend upon the basis selected for the characteristic subspace.

6.6.3 Show that the fundamental invariants of a projection $\mathbf{P}:\mathscr{V}\to\mathscr{V}$ are given by

$$\mu_j = \begin{cases} \dfrac{R!}{(R-j)!\,j!} & \text{for } j=1,2,\dots,R \\ 0 & \text{for } j=R+1,\dots,N \end{cases} \tag{6.6.62}$$

Section 6.7 Tensor Product of Vectors

In this short section, we shall introduce a notation that is convenient in certain applications. The notation is connected to an operation known as the *tensor product* of vectors. In rough terms, if one is given a pair of vectors in an inner product, the tensor product is an operation that creates a linear transformation from the two vectors. The convenience of the notation is that it allows a somewhat more convenient connection to be made between linear transformations and components of linear transformations.

Definition: If $\mathbf{f}$ is a vector in an inner product space $\mathscr{V}$ and $\mathbf{g}$ is in an inner product space $\mathscr{U}$, their *tensor product*, written $\mathbf{g}\otimes\mathbf{f}$, is a linear transformation in $\mathbf{g}\otimes\mathbf{f}:\mathscr{V}\rightarrow\mathscr{U}$ defined by

$$(\mathbf{g}\otimes\mathbf{f})\mathbf{w}=\mathbf{g}\langle\mathbf{w},\mathbf{f}\rangle \tag{6.7.1}$$

for all $\mathbf{w}$ in $\mathscr{V}$.

Because of the properties of an inner product, it should be evident that the quantity $\mathbf{g}\otimes\mathbf{f}$ as defined by (6.7.1) is, in fact, a linear transformation as defined in Section 3.1. It is important to note that $\mathbf{g}\otimes\mathbf{f}\neq\mathbf{f}\otimes\mathbf{g}$.

The definition (6.7.1) and the properties of the inner product combine to show that

$$(\lambda\mathbf{g}_1+\mu\mathbf{g}_2)\otimes\mathbf{f}=\lambda\mathbf{g}_1\otimes\mathbf{f}+\mu\mathbf{g}_2\otimes\mathbf{f} \tag{6.7.2}$$

and

$$\mathbf{g}\otimes(\lambda\mathbf{f}_1+\mu\mathbf{f}_2)=\bar{\lambda}\mathbf{g}\otimes\mathbf{f}_1+\bar{\mu}\mathbf{g}\otimes\mathbf{f}_2 \tag{6.7.3}$$

for scalars $\lambda,\mu\in\mathscr{C}$. It is also true that

$$(\mathbf{g}\otimes\mathbf{f})^*=\mathbf{f}\otimes\mathbf{g} \tag{6.7.4}$$

Equation (6.7.4) follows from the definition of adjoint, equation (4.9.1), and the definition of the tensor product, equation (6.7.1). The following sequence of calculations establishes (6.7.4). From the definition of the adjoint $(\mathbf{g}\otimes\mathbf{f})^*$

$$\langle(\mathbf{g}\otimes\mathbf{f})^*\mathbf{u},\mathbf{v}\rangle=\langle\mathbf{u},(\mathbf{g}\otimes\mathbf{f})\mathbf{v}\rangle \tag{6.7.5}$$

If the definition (6.7.1) is utilized, (6.7.5) can be written

$$\begin{aligned}\langle(\mathbf{g}\otimes\mathbf{f})^*\mathbf{u},\mathbf{v}\rangle &= \langle\mathbf{u},(\mathbf{g}\otimes\mathbf{f})\mathbf{v}\rangle = \langle\mathbf{u},\mathbf{g}\langle\mathbf{v},\mathbf{f}\rangle\rangle \\ &= \langle\mathbf{u},\mathbf{g}\rangle\langle\mathbf{f},\mathbf{v}\rangle = \langle\mathbf{f}\langle\mathbf{u},\mathbf{g}\rangle,\mathbf{v}\rangle = \langle(\mathbf{f}\otimes\mathbf{g})\mathbf{u},\mathbf{v}\rangle\end{aligned} \tag{6.7.6}$$

Because (6.7.6) holds for all vectors $\mathbf{v}\in\mathscr{V}$ and $\mathbf{u}\in\mathscr{U}$, (6.7.4) is obtained. Another relationship that holds is

$$(\mathbf{d}\otimes\mathbf{c})(\mathbf{g}\otimes\mathbf{f}) = \langle\mathbf{g},\mathbf{c}\rangle(\mathbf{d}\otimes\mathbf{f}) \tag{6.7.7}$$

for the product of two linear transformations $\mathbf{g}\otimes\mathbf{f}:\mathscr{V}\rightarrow\mathscr{U}$ and $\mathbf{d}\otimes\mathbf{c}:\mathscr{U}\rightarrow\mathscr{W}$. If $\{\mathbf{e}_1,\mathbf{e}_2,...,\mathbf{e}_N\}$ is a basis for $\mathscr{V}$ and $\{\mathbf{b}_1,\mathbf{b}_2,...,\mathbf{b}_M\}$ is a basis for $\mathscr{U}$, the matrix $M(\mathbf{g}\otimes\mathbf{f},\mathbf{e}_k,\mathbf{b}_j)$ is given by

$$\begin{aligned}M(\mathbf{g}\otimes\mathbf{f},\mathbf{e}_k,\mathbf{b}_j) &= \begin{bmatrix} g^1\bar{f}^1 & g^1\bar{f}^2 & g^1\bar{f}^3 & \cdot & \cdot & \cdot & g^1\bar{f}^N \\ g^2\bar{f}^1 & g^2\bar{f}^2 & g^2\bar{f}^3 & \cdot & \cdot & \cdot & g^2\bar{f}^N \\ g^3\bar{f}^1 & & & & & & \cdot \\ \cdot & & & & & & \cdot \\ \cdot & & & & & & \cdot \\ g^M\bar{f}^1 & g^M\bar{f}^2 & g^M\bar{f}^3 & \cdot & \cdot & \cdot & g^M\bar{f}^N \end{bmatrix} \\ &= \begin{bmatrix} g^1 \\ g^2 \\ g^3 \\ \cdot \\ \cdot \\ g^M \end{bmatrix}\begin{bmatrix}\bar{f}^1 & \bar{f}^2 & \bar{f}^3 & \cdot & \cdot & \bar{f}^N\end{bmatrix}\end{aligned} \tag{6.7.8}$$

The definition (6.7.1) allows the formula (3.2.3), repeated,

$$\mathbf{A}\mathbf{e}_k = \sum_{j=1}^{M} A^j{}_k\mathbf{b}_j \qquad k=1,2,...,N \tag{6.7.9}$$

to be written in an alternate form which is sometimes convenient. If $\{\mathbf{e}^1,\mathbf{e}^2,...,\mathbf{e}^N\}$ is the reciprocal basis to $\{\mathbf{e}_1,\mathbf{e}_2,...,\mathbf{e}_N\}$, then it is true from (4.7.7), repeated,

$$\langle\mathbf{e}^k,\mathbf{e}_q\rangle = \delta^k_q \qquad \text{for} \qquad q,k=1,2,...,N \tag{6.7.10}$$

Therefore, (6.7.9) can be rewritten

$$\begin{aligned}\mathbf{A}\mathbf{e}_k &= \sum_{j=1}^{M} A^j{}_k \mathbf{b}_j = \sum_{j=1}^{M}\sum_{q=1}^{N} A^j{}_q \mathbf{b}_j \delta_k^q = \sum_{j=1}^{M}\sum_{q=1}^{N} A^j{}_q \mathbf{b}_j \left\langle \mathbf{e}_k, \mathbf{e}^q \right\rangle \\ &= \left(\sum_{j=1}^{M}\sum_{q=1}^{N} A^j{}_q \mathbf{b}_j \otimes \mathbf{e}^q \right)\mathbf{e}_k\end{aligned} \tag{6.7.11}$$

An arbitrary vector $\mathbf{v} \in \mathscr{V}$ has the representation $\mathbf{v} = \sum_{k=1}^{N} \upsilon^k \mathbf{e}_k$, it follows from (6.7.11) that

$$\left(\mathbf{A} - \sum_{j=1}^{M}\sum_{q=1}^{N} A^j{}_q \mathbf{b}_j \otimes \mathbf{e}^q \right)\mathbf{v} = \mathbf{0} \tag{6.7.12}$$

for all vectors $\mathbf{v} \in \mathscr{V}$. Therefore,

$$\mathbf{A} = \sum_{j=1}^{M}\sum_{q=1}^{N} A^j{}_q \mathbf{b}_j \otimes \mathbf{e}^q \tag{6.7.13}$$

By the usual changes between basis and reciprocal bases, one can establish the following equivalent forms of (6.7.13)

$$\begin{aligned}\mathbf{A} &= \sum_{j=1}^{M}\sum_{q=1}^{N} A^j{}_q \mathbf{b}_j \otimes \mathbf{e}^q = \sum_{j=1}^{M}\sum_{q=1}^{N} A^{jq} \mathbf{b}_j \otimes \mathbf{e}_q \\ &= \sum_{j=1}^{M}\sum_{q=1}^{N} A_{jq} \mathbf{b}^j \otimes \mathbf{e}^q = \sum_{j=1}^{M}\sum_{q=1}^{N} A_j{}^q \mathbf{b}^j \otimes \mathbf{e}_q\end{aligned} \tag{6.7.14}$$

Example 6.7.1: If $\mathbf{I}: \mathscr{V} \to \mathscr{V}$ is the identity linear transformation and $\{\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_N\}$, then equation (6.7.13) yields

$$\mathbf{I} = \sum_{j=1}^{M}\sum_{q=1}^{N} \delta_q^j \mathbf{e}_j \otimes \mathbf{e}^q = \mathbf{e}_1 \otimes \mathbf{e}^1 + \mathbf{e}_2 \otimes \mathbf{e}^2 + \cdots + \mathbf{e}_N \otimes \mathbf{e}^N \tag{6.7.15}$$

Example 6.7.2: For the eigenvalue problem given in Example 5.3.1, we defined a linear transformation $\mathbf{A}: \mathscr{V} \to \mathscr{V}$ by equation (5.3.1), repeated,

$$\begin{aligned}\mathbf{A}\mathbf{e}_1 &= \mathbf{e}_1 + \mathbf{e}_2 + 4\mathbf{e}_3 \\ \mathbf{A}\mathbf{e}_2 &= 2\mathbf{e}_1 - 4\mathbf{e}_3 \\ \mathbf{A}\mathbf{e}_3 &= -\mathbf{e}_1 + \mathbf{e}_2 + 5\mathbf{e}_3\end{aligned} \tag{6.7.16}$$

If we apply the representation (6.7.13) to the linear transformation (6.7.16), the result is

$$\begin{aligned}\mathbf{A} &= \sum_{j=1}^{3}\sum_{q=1}^{3} A^j{}_q \mathbf{e}_j \otimes \mathbf{e}^q \\ &= \mathbf{e}_1 \otimes \mathbf{e}^1 + 2\mathbf{e}_1 \otimes \mathbf{e}^2 - \mathbf{e}_1 \otimes \mathbf{e}^3 + \mathbf{e}_2 \otimes \mathbf{e}^1 + \mathbf{e}_2 \otimes \mathbf{e}^3 + 4\mathbf{e}_3 \otimes \mathbf{e}^1 - 4\mathbf{e}_3 \otimes \mathbf{e}^2 + 5\mathbf{e}_3 \otimes \mathbf{e}^3\end{aligned} \tag{6.7.17}$$

If we represent $\mathbf{A}:\mathscr{V} \to \mathscr{V}$ with respect to its basis of eigenvectors defined by (5.3.20), repeated,

$$\begin{aligned}\mathbf{v}_1 &= -\frac{1}{2}\mathbf{e}_1 + \frac{1}{2}\mathbf{e}_2 + \mathbf{e}_3 \\ \mathbf{v}_2 &= -\frac{1}{2}\mathbf{e}_1 + \frac{1}{4}\mathbf{e}_2 + \mathbf{e}_3 \\ \mathbf{v}_3 &= -\frac{1}{4}\mathbf{e}_1 + \frac{1}{4}\mathbf{e}_2 + \mathbf{e}_3\end{aligned} \tag{6.7.18}$$

The representation of $\mathbf{A}:\mathscr{V} \to \mathscr{V}$ becomes

$$\mathbf{A} = \sum_{j=1}^{3} \lambda_j \mathbf{v}_j \otimes \mathbf{v}^j = \mathbf{v}_1 \otimes \mathbf{v}^1 + 2\mathbf{v}_2 \otimes \mathbf{v}^2 + 3\mathbf{v}_3 \otimes \mathbf{v}^3 \tag{6.7.19}$$

where the components with respect to the basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ are given by (5.3.21) and where $\{\mathbf{v}^1, \mathbf{v}^2, \mathbf{v}^3\}$ is the reciprocal basis to $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. It is readily established that the three linear transformations $\mathbf{v}_1 \otimes \mathbf{v}^1$, $\mathbf{v}_2 \otimes \mathbf{v}^2$ and $\mathbf{v}_3 \otimes \mathbf{v}^3$ are projections into the characteristic subspaces $\mathscr{V}(\lambda_1)$, $\mathscr{V}(\lambda_2)$ and $\mathscr{V}(\lambda_3)$, respectively.

Example 6.7.3: For the eigenvalue problem given in Example 5.3.2, we defined a linear transformation $\mathbf{A}:\mathscr{V} \to \mathscr{V}$ by equation (5.3.25), repeated,

$$\begin{aligned}\mathbf{A}\mathbf{i}_1 &= \mathbf{i}_1 - 2\mathbf{i}_2 + 2\mathbf{i}_3 \\ \mathbf{A}\mathbf{i}_2 &= -2\mathbf{i}_1 + \mathbf{i}_2 + 2\mathbf{i}_3 \\ \mathbf{A}\mathbf{i}_3 &= 2\mathbf{i}_1 + 2\mathbf{i}_2 + \mathbf{i}_3\end{aligned} \tag{6.7.20}$$

If we represent $\mathbf{A}:\mathscr{V} \to \mathscr{V}$ with respect to its basis of eigenvectors defined by (5.3.40) and 5.3.41), repeated,

$$\begin{aligned}\mathbf{v}_1 &= -\mathbf{i}_1 - \mathbf{i}_2 + \mathbf{i}_3 \\ \mathbf{v}_2 &= -\mathbf{i}_1 + \mathbf{i}_2 \\ \mathbf{v}_3 &= \mathbf{i}_1 + \mathbf{i}_3\end{aligned} \tag{6.7.21}$$

The representation of $\mathbf{A}:\mathscr{V} \to \mathscr{V}$ becomes

$$\mathbf{A} = \sum_{j=1}^{3} \lambda_j \mathbf{v}_j \otimes \mathbf{v}^j = -3\mathbf{v}_1 \otimes \mathbf{v}^1 + 3\left(\mathbf{v}_2 \otimes \mathbf{v}^2 + \mathbf{v}_3 \otimes \mathbf{v}^3\right) \tag{6.7.22}$$

where the components with respect to the basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ are given by (5.3.42) and where $\{\mathbf{v}^1, \mathbf{v}^2, \mathbf{v}^3\}$ is the reciprocal basis to $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. It is readily established that the two linear transformations $\mathbf{v}_1 \otimes \mathbf{v}^1$ and $\mathbf{v}_2 \otimes \mathbf{v}^2 + \mathbf{v}_3 \otimes \mathbf{v}^3$ are projections into the characteristic subspaces $\mathscr{V}(\lambda_1)$ and $\mathscr{V}(\lambda_2)$, respectively. From our earlier discussion of this example in Section 6.7, it should be clear that the matrix of $\mathbf{v}_1 \otimes \mathbf{v}^1$ with respect to the orthonormal basis $\{\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3\}$ is given by (6.6.51) and the matrix of $\mathbf{v}_2 \otimes \mathbf{v}^2 + \mathbf{v}_3 \otimes \mathbf{v}^3$ with respect to the same basis is given by (6.6.52).

Exercises:

6.7.1 If $\mathbf{A} : \mathscr{V} \to \mathscr{U}$ has the representation (6.7.13), show that the adjoint $\mathbf{A}^* : \mathscr{U} \to \mathscr{V}$ has the representation

$$\mathbf{A}^* = \sum_{j=1}^{M} \sum_{q=1}^{N} \overline{A^j{}_q} \mathbf{e}^q \otimes \mathbf{b}_j \tag{6.7.23}$$

6.7.2 If $\mathbf{A} : \mathscr{V} \to \mathscr{U}$ and $\mathbf{B} : \mathscr{U} \to \mathscr{V}$ are linear transformations show that

$$\mathbf{A}(\mathbf{v} \otimes \mathbf{u})\mathbf{B}^* = \mathbf{A}\mathbf{v} \otimes \mathbf{B}\mathbf{u} \tag{6.7.24}$$

where $\mathbf{v} \in \mathscr{V}$ and $\mathbf{u} \notin \mathscr{U}$.

6.7.3 If $\mathbf{f}$ and $\mathbf{g}$ are vectors in an inner product space $\mathscr{V}$, show that

$$\operatorname{tr}(\mathbf{g} \otimes \mathbf{f}) = \langle \mathbf{g}, \mathbf{f} \rangle \tag{6.7.25}$$

6.7.4 If $\mathbf{f}$ is a vector in an inner product space $\mathscr{V}$, $\mathbf{g}$ a vector in an inner product space $\mathscr{U}$, and $\mathbf{B} : \mathscr{U} \to \mathscr{V}$ a linear transformation show that

$$(\mathbf{g} \otimes \mathbf{f})\mathbf{B} = \mathbf{g} \otimes \left(\mathbf{B}^* \mathbf{f}\right) \tag{6.7.26}$$

and

$$\mathbf{B}(\mathbf{g} \otimes \mathbf{f}) = (\mathbf{B}\mathbf{g}) \otimes \mathbf{f} \tag{6.7.27}$$

6.7.5 In Section 4.3, the Gram Schmidt Orthogonalization process was introduced. The fundamental equations that defined that process were equations (4.3.22), repeated,

$$\mathbf{i}_k = \frac{\mathbf{e}_k - \sum_{j=1}^{k-1} \left\langle \mathbf{e}_k, \mathbf{i}_j \right\rangle \mathbf{i}_j}{\left\| \mathbf{e}_k - \sum_{j=1}^{k-1} \left\langle \mathbf{e}_k, \mathbf{i}_j \right\rangle \mathbf{i}_j \right\|} \quad \text{for} \quad k = 1, 2, ..., K \tag{6.7.28}$$

Use the definition of tensor product and show that these equations can be replaced by

$$\mathbf{i}_k = \frac{\mathbf{P}_k \mathbf{e}_k}{\left\| \mathbf{P}_k \mathbf{e}_k \right\|} \quad \text{for} \quad k = 1, 2, ..., K \tag{6.7.29}$$

where $\mathbf{P}_k : \mathscr{V} \rightarrow \mathscr{V}$, for $k = 1, 2, ..., K$, are the orthogonal projections

$$\mathbf{P}_k = \mathbf{I} - \sum_{j=1}^{k-1} \mathbf{i}_j \otimes \mathbf{i}_j \tag{6.7.30}$$

Also, show that the normalizations that appear in (6.7.29) can be evaluated with the formulas

$$\left\| \mathbf{P}_k \mathbf{e}_k \right\| = \sqrt{\left\langle \mathbf{e}_k, \mathbf{P}_k \mathbf{e}_k \right\rangle} \tag{6.7.31}$$

Section 6.8 Singular Value Decomposition

In this section, we shall discuss what is known as the *singular value decomposition* of a matrix. The discussion begins with a linear transformation $\mathbf{A}:\mathscr{V}\to\mathscr{U}$, where $\mathscr{V}$ and $\mathscr{U}$ are finite dimensional inner product spaces. As we have done before, we shall write $N=\dim\mathscr{V}$ and $M=\dim\mathscr{U}$. Also, we shall write $R=\dim R(\mathbf{A})$ for the dimension of the image space $R(\mathbf{A})$. We know from our discussions in Chapters 2 and 3 that

$$R\le \min(M,N) \tag{6.8.1}$$

The singular value decomposition arises from the calculation of the so called *singular values* and the *singular vectors* of $\mathbf{A}:\mathscr{V}\to\mathscr{U}$. These quantities are defined as follows:

Definition: If μ is a nonzero scalar and $\mathbf{v}\in\mathscr{V}$ and $\mathbf{u}\in\mathscr{U}$ are nonzero vectors such that

$$\mathbf{A}\mathbf{v}=\mu\mathbf{u} \tag{6.8.2}$$

and

$$\mathbf{A}^{*}\mathbf{u}=\mu\mathbf{v} \tag{6.8.3}$$

then μ is a *singular value* of $\mathbf{A}$ and $\mathbf{v}$ and $\mathbf{u}$ are a pair of *singular vectors* of $\mathbf{A}$.

It follows from (6.8.2) and (6.8.3) that

$$\mathbf{A}^{*}\mathbf{A}\mathbf{v}=\mu\mathbf{A}^{*}\mathbf{u}=\mu^{2}\mathbf{v} \tag{6.8.4}$$

Likewise, it follows from (6.8.3) and (6.8.2) that

$$\mathbf{A}\mathbf{A}^{*}\mathbf{u}=\mu\mathbf{A}\mathbf{v}=\mu^{2}\mathbf{u} \tag{6.8.5}$$

Equation (6.8.4) shows that μ^2 is an eigenvalue corresponding to the eigenvector $\mathbf{v}$ of the linear transformation $\mathbf{A}^{*}\mathbf{A}:\mathscr{V}\to\mathscr{V}$. Likewise (6.8.5) shows that μ^2 is also an eigenvalue corresponding to the eigenvector $\mathbf{u}$ of the linear transformation $\mathbf{A}\mathbf{A}^{*}:\mathscr{U}\to\mathscr{U}$. If we utilize the identities (4.9.5) and (4.9.9), it is readily established that

$$\left(\mathbf{A}^{*}\mathbf{A}\right)^{*}=\mathbf{A}^{*}\mathbf{A} \tag{6.8.6}$$

and

$$\left(\mathbf{A}\mathbf{A}^*\right)^* = \mathbf{A}\mathbf{A}^* \tag{6.8.7}$$

Therefore, the two linear transformations $\mathbf{A}^*\mathbf{A} : \mathscr{V} \to \mathscr{V}$ and $\mathbf{A}\mathbf{A}^* : \mathscr{U} \to \mathscr{U}$ are Hermitian. They are also positive semidefinite because

$$\left\langle \mathbf{v}, \mathbf{A}^*\mathbf{A}\mathbf{v} \right\rangle = \left\langle \mathbf{A}\mathbf{v}, \mathbf{A}\mathbf{v} \right\rangle = \left\| \mathbf{A}\mathbf{v} \right\|^2 \geq 0 \tag{6.8.8}$$

and

$$\left\langle \mathbf{u}, \mathbf{A}\mathbf{A}^*\mathbf{u} \right\rangle = \left\langle \mathbf{A}^*\mathbf{u}, \mathbf{A}^*\mathbf{u} \right\rangle = \left\| \mathbf{A}^*\mathbf{u} \right\|^2 \geq 0 \tag{6.8.9}$$

These facts and the results summarized in Theorems 5.4.3 through 5.4.6 tell us that the characteristic subspaces of $\mathbf{A}^*\mathbf{A} : \mathscr{V} \to \mathscr{V}$ and $\mathbf{A}\mathbf{A}^* : \mathscr{U} \to \mathscr{U}$, respectively, are mutually orthogonal and that the eigenvalue, μ^2 is greater than or equal to zero. The fact that $\mu^2 \geq 0$ makes it convenient to change notation slightly and write

$$\lambda = \mu^2 \tag{6.8.10}$$

and adopt the convention that we shall always use the positive square root of (6.8.10) and write the singular values as

$$\mu = \sqrt{\lambda} \tag{6.8.11}$$

It is a theoretical result that

$$K\left(\mathbf{A}\right) = K\left(\mathbf{A}^*\mathbf{A}\right) \tag{6.8.12}$$

This result was established in Section 4.4 for the case when $\mathscr{V}$ and $\mathscr{U}$ are real inner product spaces. The particular result is given in equation (4.4.13). That proof is easily modified to fit the case where $\mathscr{V}$ and $\mathscr{U}$ are complex inner product spaces. If we apply the rank nullity theorem, Theorem 3.3.5, to the two linear transformations $\mathbf{A}$ and $\mathbf{A}^*\mathbf{A}$, we can conclude that

$$\dim R(\mathbf{A}) = \dim R(\mathbf{A}^*\mathbf{A}) \tag{6.8.13}$$

The kernel of $\mathbf{A}^*\mathbf{A}$, by definition, consists of those vectors $\mathbf{v} \in \mathscr{V}$ that obey $\mathbf{A}^*\mathbf{A}\mathbf{v} = \mathbf{0}$. The dimension of the kernel, by the rank nullity theorem is $N - R$. Therefore, the linear transformation $\mathbf{A}^*\mathbf{A}$ has zero for an eigenvalue and it has an algebraic multiplicity of $N - R$. From (6.8.12), the vectors in the kernel of $\mathbf{A}^*\mathbf{A}$ are in the kernel of $\mathbf{A}$ and conversely.

In Section 5.2, we introduced the idea of the *spectrum* of a linear transformation as the set of its eigenvalues. Given the fact that $\mathbf{A}^*\mathbf{A}:\mathscr{V}\rightarrow\mathscr{V}$ is Hermitian, positive semidefinite and has $N-R$ zero eigenvalues, the spectrum of $\mathbf{A}^*\mathbf{A}$ is the set of nonnegative real numbers

$$\text{Spectrum}\left(\mathbf{A}^*\mathbf{A}\right)=\left\{\underbrace{\lambda_1,\lambda_2,...,\lambda_R}_{R},\underbrace{0,...,0}_{N-R}\right\} \tag{6.8.14}$$

An entirely similar argument tells us that the spectrum of $\mathbf{AA}^*$ is the set

$$\text{Spectrum}\left(\mathbf{AA}^*\right)=\left\{\underbrace{\lambda_1,\lambda_2,...,\lambda_R}_{R},\underbrace{0,...,0}_{M-R}\right\} \tag{6.8.15}$$

The convention we shall follow is to order the eigenvalues $\{\lambda_1,\lambda_2,...,\lambda_R\}$ such that

$$\lambda_1\geq\lambda_2\geq\lambda_3\geq\cdots\geq\lambda_R>0 \tag{6.8.16}$$

Therefore, the singular values are ordered as follows:

$$\sqrt{\lambda_1}\geq\sqrt{\lambda_2}\geq\sqrt{\lambda_3}\geq\cdots\geq\sqrt{\lambda_R}>0 \tag{6.8.17}$$

We have not assumed that the singular values are distinct. We have, however, ordered the singular values so as to give a list of $R=\dim R\left(\mathbf{A}\right)$ nonzero quantities.

In Section 5.4, it was established that Hermitian linear transformations possess an orthonormal basis consisting entirely of eigenvectors. Given this fact, we can always construct an orthonormal set $\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3,...,\mathbf{v}_R,\mathbf{v}_{R+1},...,\mathbf{v}_N\}$ of N eigenvectors of $\mathbf{A}^*\mathbf{A}$ and an orthonormal set $\{\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3,...,\mathbf{u}_R,\mathbf{u}_{R+1},...,\mathbf{u}_M\}$ of M eigenvectors of $\mathbf{AA}^*$. Given this construction, it is true that

$$\mathbf{A}^*\mathbf{A}\mathbf{v}_j=\begin{cases}\lambda_j\mathbf{v}_j & \text{for } \ j=1,...,R\\ \mathbf{0} & \text{for } \ j=R+1,...,N\end{cases} \tag{6.8.18}$$

and

$$\mathbf{AA}^*\mathbf{u}_j=\begin{cases}\lambda_j\mathbf{u}_j & \text{for } \ j=1,...,R\\ \mathbf{0} & \text{for } \ j=R+1,...,M\end{cases} \tag{6.8.19}$$

It is important when constructing singular values and singular vectors to build the orthonormal sets $\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3,...,\mathbf{v}_R,\mathbf{v}_{R+1},...,\mathbf{v}_N\}$ and $\{\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3,...,\mathbf{u}_R,\mathbf{u}_{R+1},...,\mathbf{u}_M\}$ such that the vectors obey (6.8.2) and (6.8.3). These relationships are summarized by rewriting (6.8.2) and (6.8.3) as

$$\mathbf{A}\mathbf{v}_j = \begin{cases} \sqrt{\lambda_j}\mathbf{u}_j & \text{for } j = 1,...,R \\ \mathbf{0} & \text{for } j = R+1,...,N \end{cases} \tag{6.8.20}$$

and

$$\mathbf{A}^*\mathbf{u}_j = \begin{cases} \sqrt{\lambda_j}\mathbf{v}_j & \text{for } j = 1,...,R \\ \mathbf{0} & \text{for } j = R+1,...,M \end{cases} \tag{6.8.21}$$

The construction of the orthonormal basis $\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3,...,\mathbf{v}_R,\mathbf{v}_{R+1},...,\mathbf{v}_N\}$ of $\mathscr{V}$ and $\{\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3,...,\mathbf{u}_R,\mathbf{u}_{R+1},...,\mathbf{u}_M\}$ of $\mathscr{U}$ yields, among other facts,

$$K(\mathbf{A}) = \operatorname{span}(\mathbf{v}_{R+1},...,\mathbf{v}_N) \tag{6.8.22}$$

$$R(\mathbf{A}) = \operatorname{span}(\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3,...,\mathbf{u}_R) \tag{6.8.23}$$

$$K(\mathbf{A}^*) = R(\mathbf{A})^\perp = \operatorname{span}(\mathbf{u}_{R+1},...,\mathbf{u}_M) \tag{6.8.24}$$

and

$$R(\mathbf{A}^*) = K(\mathbf{A})^\perp = \operatorname{span}(\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3,...,\mathbf{v}_R) \tag{6.8.25}$$

Example 6.8.1 Let $\mathscr{V}$ and $\mathscr{U}$ be real vector spaces such that $\dim\mathscr{V} = 2$ and $\dim\mathscr{U} = 4$. We define $\mathbf{A}:\mathscr{V}\to\mathscr{U}$ by

$$\begin{aligned} \mathbf{A}\mathbf{i}_1 &= \mathbf{j}_1 + 3\mathbf{j}_2 \\ \mathbf{A}\mathbf{i}_2 &= 3\mathbf{j}_1 + \mathbf{j}_2 \end{aligned} \tag{6.8.26}$$

where $\{\mathbf{i}_1,\mathbf{i}_2\}$ and $\{\mathbf{j}_1,\mathbf{j}_2,\mathbf{j}_3,\mathbf{j}_4\}$ are orthonormal bases. The matrix of $\mathbf{A}$ is

$$M(\mathbf{A},\mathbf{i}_q,\mathbf{j}_k) = \begin{bmatrix} 1 & 3 \\ 3 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{6.8.27}$$

The transpose of $\mathbf{A}$, is a linear transformation $\mathbf{A}^T:\mathscr{U}\to\mathscr{V}$ defined by

$$\begin{aligned}\mathbf{A}^T\mathbf{j}_1 &= \mathbf{i}_1 + 3\mathbf{i}_2 \\ \mathbf{A}^T\mathbf{j}_2 &= 3\mathbf{i}_1 + \mathbf{i}_2 \\ \mathbf{A}^T\mathbf{j}_3 &= \mathbf{0} \\ \mathbf{A}^T\mathbf{j}_4 &= \mathbf{0}\end{aligned} \tag{6.8.28}$$

The matrix of $\mathbf{A}^T$ is

$$M\left(\mathbf{A}^T, \mathbf{j}_k, \mathbf{i}_q\right) = \begin{bmatrix} 1 & 3 & 0 & 0 \\ 3 & 1 & 0 & 0 \end{bmatrix} \tag{6.8.29}$$

A straight forward series of calculations yields

$$M\left(\mathbf{A}^T\mathbf{A}, \mathbf{i}_q, \mathbf{i}_p\right) = \begin{bmatrix} 10 & 6 \\ 6 & 10 \end{bmatrix} \tag{6.8.30}$$

and

$$M\left(\mathbf{A}\mathbf{A}^T, \mathbf{j}_k, \mathbf{j}_s\right) = \begin{bmatrix} 10 & 6 & 0 & 0 \\ 6 & 10 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{6.8.31}$$

Note that $N = 2$ and $M = 4$ in this example. The eigenvalue problem for $\mathbf{A}^T\mathbf{A}$ yields

$$\begin{aligned}&\lambda_1 = 16, \lambda_2 = 4 \\ &\mathbf{v}_1 = \frac{1}{\sqrt{2}}\left(\mathbf{i}_1 + \mathbf{i}_2\right), \mathbf{v}_2 = \frac{1}{\sqrt{2}}\left(-\mathbf{i}_1 + \mathbf{i}_2\right)\end{aligned} \tag{6.8.32}$$

As with all eigenvalue problems, the eigenvectors are not uniquely determined. For example, the choices

$$\mathbf{v}_1 = -\frac{1}{\sqrt{2}}\left(\mathbf{i}_1 + \mathbf{i}_2\right), \mathbf{v}_2 = -\frac{1}{\sqrt{2}}\left(-\mathbf{i}_1 + \mathbf{i}_2\right) \tag{6.8.33}$$

are also eigenvectors. We shall continue to use the choices (6.8.32). Given (6.8.32), it follows from (6.8.20) that the vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ are given by

$$\begin{aligned}\mathbf{u}_1 &= \frac{1}{\sqrt{\lambda_1}}\mathbf{A}\mathbf{v}_1 = \frac{1}{4}\mathbf{A}\left(\frac{1}{\sqrt{2}}\left(\mathbf{i}_1+\mathbf{i}_2\right)\right) = \frac{1}{4\sqrt{2}}\left(\mathbf{j}_1+3\mathbf{j}_2+3\mathbf{j}_1+\mathbf{j}_2\right) = \frac{1}{\sqrt{2}}\left(\mathbf{j}_1+\mathbf{j}_2\right)\\ \mathbf{u}_2 &= \frac{1}{\sqrt{\lambda_2}}\mathbf{A}\mathbf{v}_2 = \frac{1}{2}\mathbf{A}\left(\frac{1}{\sqrt{2}}\left(-\mathbf{i}_1+\mathbf{i}_2\right)\right) = \frac{1}{2\sqrt{2}}\left(-\mathbf{j}_1-3\mathbf{j}_2+3\mathbf{j}_1+\mathbf{j}_2\right) = \frac{1}{\sqrt{2}}\left(\mathbf{j}_1-\mathbf{j}_2\right)\end{aligned} \tag{6.8.34}$$

The full set of eigenvectors are obtained by solving the eigenvalue problem for $\mathbf{A}\mathbf{A}^T$. The result of this solution is

$$\begin{aligned}&\lambda_1 = 16, \lambda_2 = 4, \lambda_3 = \lambda_4 = 0\\ &\mathbf{u}_1 = \frac{1}{\sqrt{2}}\left(\mathbf{j}_1+\mathbf{j}_2\right), \mathbf{u}_2 = \frac{1}{\sqrt{2}}\left(\mathbf{j}_1-\mathbf{j}_2\right), \mathbf{u}_3 = \mathbf{j}_3, \mathbf{u}_4 = \mathbf{j}_4\end{aligned} \tag{6.8.35}$$

where the choices (6.8.34) have been made along with the choices $\mathbf{u}_3 = \mathbf{j}_3, \mathbf{u}_4 = \mathbf{j}_4$. Given the fact that there are two nonzero eigenvalues, the rank of the linear transformation $\mathbf{A}:\mathscr{V}\to\mathscr{U}$ is $R = 2$. It follows from the results (6.8.32) and (6.8.35), that the singular values are

$$\sqrt{\lambda_1} = 4, \sqrt{\lambda_2} = 2 \tag{6.8.36}$$

and the singular vectors are

$$\mathbf{v}_1 = \frac{1}{\sqrt{2}}\left(\mathbf{i}_1+\mathbf{i}_2\right), \mathbf{v}_2 = \frac{1}{\sqrt{2}}\left(-\mathbf{i}_1+\mathbf{i}_2\right) \tag{6.8.37}$$

and

$$\mathbf{u}_1 = \frac{1}{\sqrt{2}}\left(\mathbf{j}_1+\mathbf{j}_2\right), \mathbf{u}_2 = \frac{1}{\sqrt{2}}\left(\mathbf{j}_1-\mathbf{j}_2\right) \tag{6.8.38}$$

With respect to the orthonormal bases $\left\{\mathbf{v}_1, \mathbf{v}_2\right\}$ and $\left\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\right\}$, the matrix of $\mathbf{A}:\mathscr{V}\to\mathscr{U}$ is

$$M\left(\mathbf{A}, \mathbf{v}_j, \mathbf{u}_k\right) = \begin{bmatrix} 4 & 0\\ 0 & 2\\ 0 & 0\\ 0 & 0\end{bmatrix} \tag{6.8.39}$$

The matrix (6.8.27) and the matrix (6.8.39) are connected by the usual change of basis expression (3.6.17). In the notation utilized here, (3.6.17) takes the form

$$M\left(\mathbf{A}, \mathbf{i}_q, \mathbf{j}_k\right) = S^{-1} M\left(\mathbf{A}, \mathbf{v}_j, \mathbf{u}_k\right) T \tag{6.8.40}$$

where T is the (orthogonal) transition matrix for the basis change $\{\mathbf{v}_1, \mathbf{v}_2\} \rightarrow \{\mathbf{i}_1, \mathbf{i}_2\}$ and S is the (orthogonal) transition matrix for the basis change $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\} \rightarrow \{\mathbf{j}_1, \mathbf{j}_2, \mathbf{j}_3, \mathbf{j}_4\}$. It follows from (6.8.37) that

$$T^{-1} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \Rightarrow T = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \tag{6.8.41}$$

and, from (6.8.35),

$$S^{-1} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{6.8.42}$$

If we combine (6.8.27), (6.8.39), $(6.8.41)_2$ and (6.8.42) the result (6.8.40) becomes

$$M\left(\mathbf{A}, \mathbf{i}_q, \mathbf{j}_k\right) = \begin{bmatrix} 1 & 3 \\ 3 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \tag{6.8.43}$$

An equivalent version of (6.8.43) is

$$M\left(\mathbf{A}, \mathbf{i}_q, \mathbf{j}_k\right) = \begin{bmatrix} 1 & 3 \\ 3 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \tag{6.8.44}$$

Equation (6.8.43) is an example of a singular value decomposition. It expresses the matrix (6.8.27) as the product of an orthogonal matrix followed by a diagonal matrix followed by an orthogonal matrix.

If we return to the general case characterized by (6.8.20), we can express the linear transformation $\mathbf{A}:\mathscr{V}\to\mathscr{U}$ in terms of its components with respect to the orthonormal bases $\left\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3,...,\mathbf{v}_R,\mathbf{v}_{R+1},...,\mathbf{v}_N\right\}$ and $\left\{\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3,...,\mathbf{u}_R,\mathbf{u}_{R+1},...,\mathbf{u}_M\right\}$. The result is the matrix

$$M\left(\mathbf{A},\mathbf{v}_q,\mathbf{u}_k\right)=\underbrace{\begin{bmatrix} \sqrt{\lambda_1} & 0 & \cdot & \cdot & \cdot & 0 & \cdot & \cdot & 0 \\ 0 & \sqrt{\lambda_2} & \cdot & \cdot & \cdot & 0 & \cdot & \cdot & 0 \\ \cdot & \cdot & \sqrt{\lambda_3} & & & \cdot & & \cdot & 0 \\ \cdot & \cdot & & \cdot & & \cdot & & \cdot & \\ \cdot & \cdot & & & & \cdot & & \cdot & 0 \\ 0 & 0 & \cdot & \cdot & \cdot & \sqrt{\lambda_R} & & \cdot & 0 \\ \cdot & \cdot & & & & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & & & & & & \cdot & \cdot \\ 0 & 0 & & & & \cdot & \cdot & 0 & 0 \end{bmatrix}}_{N} \begin{matrix} \left.\begin{matrix} \\ \\ \\ \\ \\ \\ \end{matrix}\right\} R \\ \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} M-R \end{matrix} \tag{6.8.45}$$

If we adopt the notation of the tensor product introduced in Section 6.7, the linear transformation $\mathbf{A}:\mathscr{V}\to\mathscr{U}$ can, from (6.8.20), be written

$$\mathbf{A}=\sum_{p=1}^{R}\sqrt{\lambda_p}\,\mathbf{u}_p\otimes\mathbf{v}_p \tag{6.8.46}$$

The matrix version of (6.8.46) arises from projecting (6.8.46) into a basis for $\mathscr{V}$ and $\mathscr{U}$. Equation (6.8.45), as explained, represents the matrix with respect to the orthonormal bases $\left\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3,...,\mathbf{v}_R,\mathbf{v}_{R+1},...,\mathbf{v}_N\right\}$ and $\left\{\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3,...,\mathbf{u}_R,\mathbf{u}_{R+1},...,\mathbf{u}_M\right\}$. With respect to an arbitrary basis $\left\{\mathbf{e}_1,\mathbf{e}_2,...,\mathbf{e}_N\right\}$ for $\mathscr{V}$ and $\left\{\mathbf{b}_1,\mathbf{b}_2,...,\mathbf{b}_M\right\}$ for $\mathscr{U}$, the matrix is given by the change of basis formula (6.8.40). Equivalently, the matrix can be derived by projecting the linear transformation in the form (6.8.46) into the given bases. In the applications, the most useful form of (6.8.46) results when orthonormal bases of $\mathscr{V}$ and $\mathscr{U}$ are utilized. If we denote these bases by $\left\{\mathbf{i}_1,\mathbf{i}_2,...,\mathbf{i}_N\right\}$ for $\mathscr{V}$ and $\left\{\mathbf{j}_1,\mathbf{j}_2,...,\mathbf{j}_M\right\}$ for $\mathscr{U}$. If we represent the matrix $M\left(\mathbf{A},\mathbf{i}_q,\mathbf{j}_k\right)$ in components, as in Chapter 3, we can write

$$M\left(\mathbf{A},\mathbf{i}_q,\mathbf{j}_k\right)=\begin{bmatrix} A_{11} & A_{12} & \cdot & \cdot & \cdot & A_{1N} \\ A_{21} & A_{22} & & & & A_{2N} \\ A_{31} & & A_{33} & & & \cdot \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ A_{M1} & A_{M2} & \cdot & \cdot & \cdot & A_{MN} \end{bmatrix} \tag{6.8.47}$$

where, from $(4.8.6)_1$,

$$A_{kq} = \left\langle \mathbf{A}\mathbf{i}_q, \mathbf{j}_k \right\rangle \tag{6.8.48}$$

it follows from (6.8.46) and the definition (6.7.1) that

$$\begin{aligned} A_{kq} &= \left\langle \mathbf{A}\mathbf{i}_q, \mathbf{j}_k \right\rangle = \sum_{p=1}^{R} \left\langle \left(\sqrt{\lambda_p}\, \mathbf{u}_p \otimes \mathbf{v}_p \right) \mathbf{i}_q, \mathbf{j}_k \right\rangle \\ &= \sum_{p=1}^{R} \sqrt{\lambda_p} \left\langle \mathbf{i}_q, \mathbf{v}_p \right\rangle \left\langle \mathbf{u}_p, \mathbf{j}_k \right\rangle = \sum_{p=1}^{R} \sqrt{\lambda_p} \left\langle \mathbf{u}_p, \mathbf{j}_k \right\rangle \overline{\left\langle \mathbf{v}_p, \mathbf{i}_q \right\rangle} \end{aligned} \tag{6.8.49}$$

Equation (6.8.49) is equivalent to the matrix product

$$M\left(\mathbf{A}, \mathbf{i}_q, \mathbf{j}_k\right) = \begin{bmatrix} A_{11} & A_{12} & \cdot & \cdot & \cdot & A_{1N} \\ A_{21} & A_{22} & & & & A_{2N} \\ A_{31} & & A_{33} & & & \cdot \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ A_{M1} & A_{M2} & \cdot & \cdot & \cdot & A_{MN} \end{bmatrix}$$

$$= \begin{bmatrix} \langle \mathbf{u}_1, \mathbf{j}_1 \rangle & \langle \mathbf{u}_2, \mathbf{j}_1 \rangle & \langle \mathbf{u}_3, \mathbf{j}_1 \rangle & \cdot & \cdot & \langle \mathbf{u}_M, \mathbf{j}_1 \rangle \\ \langle \mathbf{u}_1, \mathbf{j}_2 \rangle & \langle \mathbf{u}_2, \mathbf{j}_2 \rangle & \langle \mathbf{u}_3, \mathbf{j}_2 \rangle & & & \langle \mathbf{u}_M, \mathbf{j}_2 \rangle \\ \langle \mathbf{u}_1, \mathbf{j}_3 \rangle & \langle \mathbf{u}_2, \mathbf{j}_3 \rangle & \langle \mathbf{u}_3, \mathbf{j}_3 \rangle & & & \langle \mathbf{u}_M, \mathbf{j}_3 \rangle \\ \cdot & & & \cdot & & \cdot \\ \cdot & & & & \cdot & \cdot \\ \langle \mathbf{u}_1, \mathbf{j}_M \rangle & \langle \mathbf{u}_2, \mathbf{j}_M \rangle & \langle \mathbf{u}_3, \mathbf{j}_M \rangle & \cdot & \cdot & \langle \mathbf{u}_M, \mathbf{j}_M \rangle \end{bmatrix} \begin{bmatrix} \sqrt{\lambda_1} & 0 & \cdot & \cdot & \cdot & 0 & \cdot & 0 \\ 0 & \sqrt{\lambda_2} & & & & 0 & \cdot & 0 \\ 0 & & \cdot & & & \cdot & & \cdot \\ \cdot & & & \cdot & & \cdot & & \cdot \\ \cdot & & & & \sqrt{\lambda_R} & \cdot & & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & 0 & & \\ & & & & & & & \\ 0 & 0 & \cdot & \cdot & \cdot & 0 & \cdot & 0 \end{bmatrix} \times$$

$$\begin{bmatrix} \overline{\langle \mathbf{v}_1, \mathbf{i}_1 \rangle} & \overline{\langle \mathbf{v}_1, \mathbf{i}_2 \rangle} & \overline{\langle \mathbf{v}_1, \mathbf{i}_3 \rangle} & \cdot & \cdot & \overline{\langle \mathbf{v}_1, \mathbf{i}_N \rangle} \\ \overline{\langle \mathbf{v}_2, \mathbf{i}_1 \rangle} & \overline{\langle \mathbf{v}_2, \mathbf{i}_2 \rangle} & \overline{\langle \mathbf{v}_2, \mathbf{i}_3 \rangle} & & & \overline{\langle \mathbf{v}_2, \mathbf{i}_N \rangle} \\ \overline{\langle \mathbf{v}_3, \mathbf{i}_1 \rangle} & \overline{\langle \mathbf{v}_3, \mathbf{i}_2 \rangle} & \overline{\langle \mathbf{v}_3, \mathbf{i}_3 \rangle} & & & \overline{\langle \mathbf{v}_3, \mathbf{i}_N \rangle} \\ \cdot & & & \cdot & & \cdot \\ \cdot & & & & \cdot & \cdot \\ \overline{\langle \mathbf{v}_N, \mathbf{i}_1 \rangle} & \overline{\langle \mathbf{v}_N, \mathbf{i}_2 \rangle} & \overline{\langle \mathbf{v}_N, \mathbf{i}_3 \rangle} & \cdot & \cdot & \overline{\langle \mathbf{v}_N, \mathbf{i}_N \rangle} \end{bmatrix} \tag{6.8.50}$$

Because the bases $\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3,...,\mathbf{v}_R,\mathbf{v}_{R+1},...,\mathbf{v}_N\}$ and $\{\mathbf{i}_1,\mathbf{i}_2,...,\mathbf{i}_N\}$ of $\mathscr{V}$ are orthonormal, the $N\times N$ matrix is $\left[\langle\mathbf{v}_p,\mathbf{i}_q\rangle\right]$ unitary. The proof, which is similar to that used to derive equation (4.4.7), involves connecting the two sets of bases by the usual change of basis formulas we have used many times. In this case, we shall write the change of basis as

$$\mathbf{v}_p=\sum_q^N Q_{qp}\mathbf{i}_q \tag{6.8.51}$$

As a result of (6.8.51), the components of the matrix $\left[\langle\mathbf{v}_p,\mathbf{i}_q\rangle\right]$ are given by

$$Q_{qp}=\langle\mathbf{v}_p,\mathbf{i}_q\rangle \tag{6.8.52}$$

Given the labeling of the inner products as in (6.8.52), we can define the $N\times N$ unitary matrix

$$Q=\left[Q_{qp}\right]=\begin{bmatrix}\langle\mathbf{v}_1,\mathbf{i}_1\rangle & \langle\mathbf{v}_2,\mathbf{i}_1\rangle & \langle\mathbf{v}_3,\mathbf{i}_1\rangle & \cdot & \cdot & \langle\mathbf{v}_N,\mathbf{i}_1\rangle\\ \langle\mathbf{v}_1,\mathbf{i}_2\rangle & \langle\mathbf{v}_2,\mathbf{i}_2\rangle & \langle\mathbf{v}_3,\mathbf{i}_2\rangle & & & \langle\mathbf{v}_N,\mathbf{i}_2\rangle\\ \langle\mathbf{v}_1,\mathbf{i}_3\rangle & \langle\mathbf{v}_2,\mathbf{i}_3\rangle & \langle\mathbf{v}_3,\mathbf{i}_3\rangle & & & \langle\mathbf{v}_N,\mathbf{i}_3\rangle\\ \cdot & & & \cdot & & \cdot\\ \cdot & & & & \cdot & \cdot\\ \langle\mathbf{v}_1,\mathbf{i}_N\rangle & \langle\mathbf{v}_2,\mathbf{i}_N\rangle & \langle\mathbf{v}_3,\mathbf{i}_N\rangle & & & \langle\mathbf{v}_N,\mathbf{i}_N\rangle\end{bmatrix} \tag{6.8.53}$$

In a like fashion, we can characterize the change of basis $\{\mathbf{j}_1,\mathbf{j}_2,...,\mathbf{j}_M\}$ to $\{\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3,...,\mathbf{u}_R,\mathbf{u}_{R+1},...,\mathbf{u}_M\}$ by the formula

$$\mathbf{u}_p=\sum_k^M K_{qp}\mathbf{j}_q \tag{6.8.54}$$

and define the $M\times M$ unitary matrix

$$K=\left[K_{qp}\right]=\begin{bmatrix}\langle\mathbf{u}_1,\mathbf{j}_1\rangle & \langle\mathbf{u}_2,\mathbf{j}_1\rangle & \langle\mathbf{u}_3,\mathbf{j}_1\rangle & \cdot & \cdot & \langle\mathbf{u}_M,\mathbf{j}_1\rangle\\ \langle\mathbf{u}_1,\mathbf{j}_2\rangle & \langle\mathbf{u}_2,\mathbf{j}_2\rangle & \langle\mathbf{u}_3,\mathbf{j}_2\rangle & & & \langle\mathbf{u}_M,\mathbf{j}_2\rangle\\ \langle\mathbf{u}_1,\mathbf{j}_3\rangle & \langle\mathbf{u}_2,\mathbf{j}_3\rangle & \langle\mathbf{u}_3,\mathbf{j}_3\rangle & & & \langle\mathbf{u}_M,\mathbf{j}_3\rangle\\ \cdot & & & \cdot & & \cdot\\ \cdot & & & & \cdot & \cdot\\ \langle\mathbf{u}_1,\mathbf{j}_M\rangle & \langle\mathbf{u}_2,\mathbf{j}_M\rangle & \langle\mathbf{u}_3,\mathbf{j}_M\rangle & & & \langle\mathbf{u}_M,\mathbf{j}_M\rangle\end{bmatrix} \tag{6.8.55}$$

With the notation (6.8.53) and (6.8.55), equation (6.8.50) becomes

$$
\begin{bmatrix}
A_{11} & A_{12} & \cdot & \cdot & \cdot & A_{1N} \\
A_{21} & A_{22} & & & & A_{2N} \\
A_{31} & & A_{33} & & & \cdot \\
\cdot & & & & & \cdot \\
\cdot & & & & & \cdot \\
A_{M1} & A_{M2} & \cdot & \cdot & \cdot & A_{MN}
\end{bmatrix}
= K
\begin{bmatrix}
\sqrt{\lambda_1} & 0 & \cdot & \cdot & \cdot & 0 & \cdot & 0 \\
0 & \sqrt{\lambda_2} & & & & 0 & \cdot & 0 \\
0 & & \cdot & & & \cdot & & \cdot \\
\cdot & & & \cdot & & \cdot & & \cdot \\
\cdot & & & & \sqrt{\lambda_R} & \cdot & & \cdot \\
0 & 0 & \cdot & \cdot & \cdot & 0 & & \\
 & & & & & & & \\
0 & 0 & \cdot & \cdot & \cdot & 0 & \cdot & 0
\end{bmatrix}
Q^* \qquad (6.8.56)
$$

Equation (6.8.56) is the *singular value decomposition* for the matrix $M\left(\mathbf{A},\mathbf{i}_q,\mathbf{j}_k\right)$. It expresses the matrix $M\left(\mathbf{A},\mathbf{i}_q,\mathbf{j}_k\right)$ as the product of a unitary matrix, a diagonal matrix and another unitary matrix. Equation (6.8.56) is illustrated in Example 6.8.1 by equation (6.8.43). As mentioned in Example 6.8.1, it is implicit in eigenvalue problems that the eigenvectors are not unique. This fact results in the matrices K and Q not being unique in the decomposition (6.8.56). The matrix (6.8.45) is unique. The following example further illustrates the construction leading to (6.8.56).

Example 6.8.2: The linear transformation in this case is the matrix $A:\mathscr{M}^{3\times 1}\to\mathscr{M}^{4\times 1}$ defined by

$$
A=\begin{bmatrix}
1 & -i & 7 \\
-8i & 9 & 8i \\
4 & -9 & 15i \\
7i & 4i & 0
\end{bmatrix} \qquad (6.8.57)
$$

The first step is to determine the eigenvalues and eigenvectors of the Hermitian matrix

$$
A^*A=\begin{bmatrix}
130 & -8+71i & -57+60i \\
-8-71i & 179 & -56i \\
-57-60i & 56i & 338
\end{bmatrix} \qquad (6.8.58)
$$

The characteristic polynomial for (6.8.58) is

$$
\begin{aligned}
f(\lambda)=\det\left(A^*A-\lambda I\right)&=-\lambda^3+647\lambda^2-112622\lambda+4106615 \\
&=(377.4466-\lambda)(220.1275-\lambda)(49.4258-\lambda)
\end{aligned} \qquad (6.8.59)
$$

Therefore,

$$\begin{aligned}\lambda_1 &= 377.4466 \Rightarrow \sqrt{\lambda_1} = 19.4280\\ \lambda_2 &= 220.1275 \Rightarrow \sqrt{\lambda_2} = 14.8367\\ \lambda_3 &= 49.4258 \Rightarrow \sqrt{\lambda_3} = 7.0304\end{aligned} \tag{6.8.60}$$

A set of eigenvectors corresponding to these eigenvalues turn out to be

$$\mathbf{v}_1 = \begin{bmatrix} -0.1539+0.2593i \\ 0.0990-0.2159i \\ 0.9234 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -0.2663-0.4758i \\ -0.7696+0.2415i \\ 0.2282 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} -0.6573+0.4239i \\ -0.2728-0.4674i \\ -0.3087 \end{bmatrix} \tag{6.8.61}$$

The eigenvectors in (6.8.61) have been normalized to have unit length. Because the matrix A^*A Hermitian, the eigenvectors are orthogonal. Thus, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthonormal basis. Given (6.8.61), the matrix Q from (6.8.53) is given by

$$Q = \begin{bmatrix} -0.1539+0.2593i & -0.2663-0.4758i & -0.6573+0.4239i \\ 0.0990-0.2159i & -0.7696+0.2415i & -0.2728-0.4674i \\ 0.9234 & 0.2282 & -0.3087 \end{bmatrix} \tag{6.8.62}$$

Choices of the eigenvectors $\mathbf{u}_1, \mathbf{u}_2$ and $\mathbf{u}_3$ are made by utilization of (6.8.20). Therefore,

$$\mathbf{u}_1 = \frac{1}{\sqrt{\lambda_1}} A\mathbf{v}_1 = \frac{1}{\sqrt{377.4466}} \begin{bmatrix} 1 & -i & 7 \\ -8i & 9 & 8i \\ 4 & -9 & 15i \\ 7i & 4i & 0 \end{bmatrix} \begin{bmatrix} -0.1539+0.2593i \\ 0.0990-0.2159i \\ 0.9234 \end{bmatrix} = \begin{bmatrix} 0.3137+0.0083i \\ 0.1562+0.3436i \\ -0.0775+0.8664i \\ -0.0490-0.0351i \end{bmatrix} \tag{6.8.63}$$

$$\mathbf{u}_2 = \frac{1}{\sqrt{\lambda_2}} A\mathbf{v}_2 = \frac{1}{\sqrt{220.1275}} \begin{bmatrix} 1 & -i & 7 \\ -8i & 9 & 8i \\ 4 & -9 & 15i \\ 7i & 4i & 0 \end{bmatrix} \begin{bmatrix} -0.2663-0.4758i \\ -0.7696+0.2415i \\ 0.2282 \end{bmatrix} = \begin{bmatrix} 0.1060+0.0198i \\ -0.7234+0.4131i \\ 0.3951-0.0441i \\ 0.1594-0.331i \end{bmatrix} \tag{6.8.64}$$

and

$$\mathbf{u}_3 = \frac{1}{\sqrt{\lambda_3}} A\mathbf{v}_3 = \frac{1}{\sqrt{49.4258}} \begin{bmatrix} 1 & -i & 7 \\ -8i & 9 & 8i \\ 4 & -9 & 15i \\ 7i & 4i & 0 \end{bmatrix} \begin{bmatrix} -0.6573+0.4239i \\ -0.2728-0.4674i \\ -0.3087 \end{bmatrix} = \begin{bmatrix} -0.4673+0.0991i \\ 0.1331-0.2016i \\ -0.0247+0.1810i \\ -0.1561-0.8097i \end{bmatrix} \tag{6.8.65}$$

The vectors $\mathbf{u}_1, \mathbf{u}_2$ and $\mathbf{u}_3$ are three of the four eigenvectors of the Hermitian matrix

$$AA^* = \begin{bmatrix} 51 & -57i & 4-96i & -4-7i \\ 57i & 209 & 39-32i & -56-36i \\ 4+96i & 39+32i & 322 & 8i \\ -4+7i & -56+35i & -8i & 65 \end{bmatrix} \tag{6.8.66}$$

The solution of this eigenvalue problem yields

$$\begin{aligned} \lambda_1 &= 377.4466 \\ \lambda_2 &= 220.1275 \\ \lambda_3 &= 49.4258 \\ \lambda_4 &= 0 \end{aligned} \tag{6.8.67}$$

for the four eigenvalues and the four eigenvectors given by (6.8.63), (6.8.64) and (6.8.65) and

$$\mathbf{u}_4 = \begin{bmatrix} 0.0298+0.8129i \\ 0.3063+0.1117i \\ 0.2160-0.0735i \\ 0.4242 \end{bmatrix} \tag{6.8.68}$$

for the normalized eigenvectors. Given (6.8.63), (6.8.64), (6.8.65) and (6.8.68) the matrix K in (6.8.55) is given by

$$K = \begin{bmatrix} 0.3137+0.0083i & 0.1060+0.0198i & -0.4673+0.0991i & 0.0298+0.8129i \\ 0.1562+0.3436i & -0.7234+0.4131i & 0.1331-0.2016i & 0.3063+0.1117i \\ -0.0775+0.8664i & 0.3951-0.0441i & -0.0247+0.1810i & 0.2160-0.0735i \\ -0.0490-0.0351i & 0.1594-0.331i & -0.1561-0.8097i & 0.4242 \end{bmatrix} \tag{6.8.69}$$

Given (6.8.62) and (6.8.69), the singular value decomposition of (6.8.57) is

$$A=\begin{bmatrix} 1 & -i & 7 \\ -8i & 9 & 8i \\ 4 & -9 & 15i \\ 7i & 4i & 0 \end{bmatrix}=\begin{bmatrix} 0.3137+0.0083i & 0.1060+0.0198i & -0.4673+0.0991i & 0.0298+0.8129i \\ 0.1562+0.3436i & -0.7234+0.4131i & 0.1331-0.2016i & 0.3063+0.1117i \\ -0.0775+0.8664i & 0.3951-0.0441i & -0.0247+0.1810i & 0.2160-0.0735i \\ -0.0490-0.0351i & 0.1594-0.331i & -0.1561-0.8097i & 0.4242 \end{bmatrix}\times$$
$$\begin{bmatrix} 19.4280 & 0 & 0 \\ 0 & 14.8367 & 0 \\ 0 & 0 & 7.0304 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} -0.1539+0.2593i & 0.0990-0.2159i & 0.9234 \\ -0.2663-0.4758i & -0.7696+0.2415i & 0.2282 \\ -0.6573+0.4239i & -0.2728-0.4674i & -0.3087 \end{bmatrix} \tag{6.8.70}$$

An equivalent version of (6.8.70) is

$$A=\begin{bmatrix} 1 & -i & 7 \\ -8i & 9 & 8i \\ 4 & -9 & 15i \\ 7i & 4i & 0 \end{bmatrix}=\begin{bmatrix} 0.3137+0.0083i & 0.1060+0.0198i & -0.4673+0.0991i \\ 0.1562+0.3436i & -0.7234+0.4131i & 0.1331-0.2016i \\ -0.0775+0.8664i & 0.3951-0.0441i & -0.0247+0.1810i \\ -0.0490-0.0351i & 0.1594-0.331i & -0.1561-0.8097i \end{bmatrix}\times$$
$$\begin{bmatrix} 19.4280 & 0 & 0 \\ 0 & 14.8367 & 0 \\ 0 & 0 & 7.0304 \end{bmatrix}\begin{bmatrix} -0.1539+0.2593i & 0.0990-0.2159i & 0.9234 \\ -0.2663-0.4758i & -0.7696+0.2415i & 0.2282 \\ -0.6573+0.4239i & -0.2728-0.4674i & -0.3087 \end{bmatrix} \tag{6.8.71}$$

The transformation from (6.8.70) to (6.8.71), which is similar to the transformation from (6.8.43) to (6.8.44) simply reflects the result (6.8.46) which does not depend upon the vectors $\{\mathbf{v}_{R+1},...,\mathbf{v}_N\}$ and $\{\mathbf{u}_{R+1},...,\mathbf{u}_M\}$. In our Example 6.8.2, this observation means that the last column of the matrix (6.8.69) does not contribute to the answer.

It was mentioned above that the K and Q are not unique in the decomposition (6.8.56). The matrix (6.8.45) is unique. The source of this lack of uniqueness is the fact we have observed, namely, that an eigenvalue problem does not determine the length of an eigenvector. In the singular value decomposition the vectors $\{\mathbf{u}_1,\mathbf{u}_2,...,\mathbf{u}_R\}$ are determined by the vectors $\{\mathbf{v}_1,\mathbf{v}_2,...,\mathbf{v}_R\}$ by (6.8.20). It follows then that the indeterminacy in the vectors $\{\mathbf{v}_1,\mathbf{v}_2,...,\mathbf{v}_R\}$ is passed to the vectors $\{\mathbf{u}_1,\mathbf{u}_2,...,\mathbf{u}_R\}$. This fact is illustrated by rewriting (6.8.46) as

$$
\begin{aligned}
\mathbf{A} &= \sum_{p=1}^{R} \sqrt{\lambda_p}\,\mathbf{u}_p \otimes \mathbf{v}_p = \sum_{p=1}^{R} \sqrt{\lambda_p}\left(\frac{\sigma_p}{\sigma_p}\right)\mathbf{u}_p \otimes \mathbf{v}_p \\
&= \sum_{p=1}^{R} \sqrt{\lambda_p}\left(\frac{\sigma_p \bar{\sigma}_p}{\left\|\sigma_p\right\|^2}\right)\mathbf{u}_p \otimes \mathbf{v}_p \\
&= \sum_{p=1}^{R} \sqrt{\lambda_p}\left(\frac{\sigma_p \mathbf{u}_p}{\left\|\sigma_p\right\|}\right) \otimes \left(\frac{\sigma_p \mathbf{v}_p}{\left\|\sigma_p\right\|}\right)
\end{aligned}
\tag{6.8.72}
$$

where σ_p, for $p = 1, 2, ..., R$ are arbitrary nonzero complex numbers. The arbitrary complex numbers can be placed in the formula $(6.8.72)_1$ in other ways. The arrangement $(6.8.72)_4$ preserves the orthonormal character of the sets $\left\{\frac{\sigma_1 \mathbf{v}_1}{\left\|\sigma_1\right\|}, \frac{\sigma_2 \mathbf{v}_2}{\left\|\sigma_2\right\|}, ..., \frac{\sigma_R \mathbf{v}_R}{\left\|\sigma_R\right\|}\right\}$ and $\left\{\frac{\sigma_1 \mathbf{u}_1}{\left\|\sigma_1\right\|}, \frac{\sigma_2 \mathbf{u}_2}{\left\|\sigma_2\right\|}, ..., \frac{\sigma_R \mathbf{u}_R}{\left\|\sigma_R\right\|}\right\}$. If , for example, we take

$$
\begin{aligned}
\frac{\sigma_1}{\left\|\sigma_1\right\|} &= 0.5105 + 0.8599i \\
\frac{\sigma_2}{\left\|\sigma_2\right\|} &= -0.4884 + 0.8726i \\
\frac{\sigma_3}{\left\|\sigma_3\right\|} &= 0.8404 + 0.5419i
\end{aligned}
\tag{6.8.73}
$$

then, for Example 6.8.2,

$$\frac{\sigma_1 \mathbf{v}_1}{\|\sigma_1\|} = (0.5105+0.8599i)\begin{bmatrix} -0.1539+0.2593i \\ 0.0990-0.2159i \\ 0.9234 \end{bmatrix} = \begin{bmatrix} -0.3015 \\ 0.2362-0.0251i \\ 0.4714+0.7940i \end{bmatrix}$$

$$\frac{\sigma_2 \mathbf{v}_2}{\|\sigma_2\|} = (-0.4884+0.8726i)\begin{bmatrix} -0.2663-0.4758i \\ -0.7696+0.2415i \\ 0.2282 \end{bmatrix} = \begin{bmatrix} 0.5452 \\ 0.1651-0.7895i \\ -0.1114+0.1991i \end{bmatrix}$$

$$\frac{\sigma_3 \mathbf{v}_3}{\|\sigma_3\|} = (0.8404+0.5419i)\begin{bmatrix} -0.6573+0.4239i \\ -0.2728-0.4674i \\ -0.3087 \end{bmatrix} = \begin{bmatrix} -0.7822 \\ 0.0240-0.5407i \\ -0.2594-0.1673i \end{bmatrix}$$

$$\frac{\sigma_1 \mathbf{u}_1}{\|\sigma_1\|} = (0.5105+0.8599i)\begin{bmatrix} 0.3137+0.0083i \\ 0.1562+0.3436i \\ -0.0775+0.8664i \\ -0.0490-0.0351i \end{bmatrix} = \begin{bmatrix} 0.1530+0.2739i \\ -0.2175+0.3066i \\ -0.7845+0.3756i \\ 0.0052-0.0600i \end{bmatrix}$$

$$\frac{\sigma_2 \mathbf{u}_2}{\|\sigma_2\|} = (-0.4884+0.8726i)\begin{bmatrix} 0.1060+0.0198i \\ -0.7234+0.4131i \\ 0.3951-0.0441i \\ 0.1594-0.331i \end{bmatrix} = \begin{bmatrix} -0.0690+0.0828i \\ -0.0072-0.8330 \\ -0.1544+0.3663i \\ 0.2129+0.3018i \end{bmatrix}$$

$$\frac{\sigma_3 \mathbf{u}_3}{\|\sigma_3\|} = (0.8404+0.5419i)\begin{bmatrix} -0.4673+0.0991i \\ 0.1331-0.2016i \\ -0.0247+0.1810i \\ -0.1561-0.8097i \end{bmatrix} = \begin{bmatrix} -0.4456-0.1700i \\ 0.2211-0.0974i \\ -0.1188+0.1387i \\ 0.3076-0.7651i \end{bmatrix} \tag{6.8.74}$$

Given, the results (6.8.74), the singular decomposition (6.8.71) can be replaced by the equivalent result[13]

$$A = \begin{bmatrix} 1 & -i & 7 \\ -8i & 9 & 8i \\ 4 & -9 & 15i \\ 7i & 4i & 0 \end{bmatrix} = \begin{bmatrix} 0.1530+0.2739i & -0.0690+0.0828i & -0.4456-0.1700i \\ -0.2175+0.3066i & -0.0072-0.8330 & 0.2211-0.0974i \\ -0.7845+0.3756i & -0.1544+0.3663i & -0.1188+0.1387i \\ 0.0052-0.0600i & 0.2129+0.3018i & 0.3076-0.7651i \end{bmatrix} \times \tag{6.8.75}$$
$$\begin{bmatrix} 19.4280 & 0 & 0 \\ 0 & 14.8367 & 0 \\ 0 & 0 & 7.0304 \end{bmatrix}\begin{bmatrix} -0.3015 & 0.2362-0.0251i & 0.4714+0.7940i \\ 0.5452 & 0.1651-0.7895i & -0.1114+0.1991i \\ -0.7822 & 0.0240-0.5407i & -0.2594-0.1673i \end{bmatrix}$$

---

[13] Example 6.8.2 has been worked with the aid of MATLAB to carry out the calculations. The result (6.8.75) is the form of the answer given by MATLAB's built in singular value decomposition command.

An interesting application for the formula (6.8.46) is to derive a representation for the solution of the equation we have discussed throughout this textbook, namely, given a vector $\mathbf{b} \in \mathscr{U}$ and a linear transformation $\mathbf{A}: \mathscr{V} \rightarrow \mathscr{U}$, find a vector $\mathbf{x} \in \mathscr{V}$ such that

$$\mathbf{A}\mathbf{x} = \mathbf{b} \tag{6.8.76}$$

As we discussed in Section 1.8.1 with Theorem 1.8.1, (6.8.76) has a solution if and only if the vector $\mathbf{b} \in R(\mathbf{A})$. This result was again discussed in Section 2.7. If the representation (6.8.46) is used, equation (6.8.76) can be written

$$\left(\sum_{p=1}^{R} \sqrt{\lambda_p} \mathbf{u}_p \otimes \mathbf{v}_p \right) \mathbf{x} = \mathbf{b} \tag{6.8.77}$$

The definition (6.7.1) can be used to rewrite (6.8.77) as

$$\sum_{p=1}^{R} \sqrt{\lambda_p} \left\langle \mathbf{x}, \mathbf{v}_p \right\rangle \mathbf{u}_p = \mathbf{b} \tag{6.8.78}$$

This equation shows that the given $\mathbf{b}$ must be in the span of $\left\{ \mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_R \right\}$, the image space $R(\mathbf{A})$. This result is simply a restatement of the earlier result mentioned above. An equivalent way to assert this fact is to observe that (6.8.78) is equivalent to

$$\left\langle \mathbf{u}_p, \mathbf{b} \right\rangle = \mathbf{0} \qquad \text{for} \quad p = R+1, R+2, ..., M \tag{6.8.79}$$

Equation (6.8.79) asserts that $\mathbf{b} \in K\left(\mathbf{A}^*\right)^{\perp}$. Of course, equation (4.11.8) shows that the result $\mathbf{b} \in K\left(\mathbf{A}^*\right)^{\perp}$ is simply a restatement of $\mathbf{b} \in R(\mathbf{A})$. If the given $\mathbf{b}$ obeys $\mathbf{b} \in R(\mathbf{A})$, it follows from (6.8.78) that

$$\left\langle \mathbf{x}, \mathbf{v}_p \right\rangle = \frac{1}{\sqrt{\lambda_j}} \left\langle \mathbf{b}, \mathbf{u}_p \right\rangle \quad \text{for} \quad p = 1, 2, ..., R \tag{6.8.80}$$

Because $\mathbf{x}$ is a vector in $\mathscr{V}$ and $\left\{ \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, ..., \mathbf{v}_R, \mathbf{v}_{R+1}, ..., \mathbf{v}_N \right\}$ is a basis for $\mathscr{V}$, it has the representation

$$\mathbf{x} = \sum_{p=1}^{N} \left\langle \mathbf{x}, \mathbf{v}_p \right\rangle \mathbf{v}_p = \sum_{p=1}^{R} \left\langle \mathbf{x}, \mathbf{v}_p \right\rangle \mathbf{v}_p + \sum_{p=R+1}^{N} \left\langle \mathbf{x}, \mathbf{v}_p \right\rangle \mathbf{v}_p \tag{6.8.81}$$

If we now use (6.8.80), (6.8.81) can be used to write the solution of $\mathbf{A}\mathbf{x} = \mathbf{b}$ as

$$\mathbf{x} = \underbrace{\sum_{p=1}^{R}\left(\frac{1}{\sqrt{\lambda_p}}\langle \mathbf{b},\mathbf{u}_p\rangle\right)\mathbf{v}_p}_{\text{In } R(\mathbf{A}^*)=K(\mathbf{A})^{\perp}} + \underbrace{\sum_{p=R+1}^{N}\langle \mathbf{x},\mathbf{v}_p\rangle \mathbf{v}_p}_{\text{In } K(\mathbf{A})} \tag{6.8.82}$$

Equation (6.8.82) is a representation of the solution of (6.8.76) in the case where $\mathbf{b} \in R(\mathbf{A})$. It is of the form of the representation given in Theorem 2.7.5.

Example 6.8.3: In Example 2.7.6, we found the solution of equation (2.7.91), repeated,

$$\begin{bmatrix} 1 & 1 & -2 & 1 & 3 \\ 2 & -1 & 2 & 2 & 6 \\ 3 & 2 & -4 & -3 & -9 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \tag{6.8.83}$$

to be

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_3\begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5\begin{bmatrix} 0 \\ 0 \\ 0 \\ -3 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{6.8.84}$$

It is instructive to generate (6.8.84) based upon the singular value decomposition of the matrix of coefficients

$$A = \begin{bmatrix} 1 & 1 & -2 & 1 & 3 \\ 2 & -1 & 2 & 2 & 6 \\ 3 & 2 & -4 & -3 & -9 \end{bmatrix} \tag{6.8.85}$$

The eigenvalues of the matrix

$$A^T A = \begin{bmatrix} 14 & 5 & -10 & -4 & -12 \\ 5 & 6 & -12 & -7 & -21 \\ -10 & -12 & 24 & 14 & 42 \\ -4 & -7 & 14 & 14 & 42 \\ -12 & -21 & 42 & 42 & 126 \end{bmatrix} \tag{6.8.86}$$

are the solution of the fifth order polynomial

$$\begin{aligned} f(\lambda) &= \det\left(A^T A - \lambda I\right) = \lambda^2\left(-\lambda^3 + 184\lambda^2 - 3845\lambda + 16200\right) \\ &= \lambda^2\left(160.70-\lambda\right)\left(17.56-\lambda\right)\left(5.74-\lambda\right) \end{aligned} \tag{6.8.87}$$

and the solution representation (6.8.82). Therefore,

$$\begin{aligned} \lambda_1 &= 160.70 \Rightarrow \sqrt{\lambda_1} = 12.6768 \\ \lambda_2 &= 17.56 \Rightarrow \sqrt{\lambda_2} = 4.1902 \\ \lambda_3 &= 5.74 \Rightarrow \sqrt{\lambda_3} = 2.3962 \\ \lambda_4 &= \lambda_5 = 0 \end{aligned} \tag{6.8.88}$$

Among other things, equation (6.8.88) tells us that the rank of the matrix (6.8.85) is 3. The normalized eigenvalues of (6.8.86) are

$$\mathbf{v}_1 = \begin{bmatrix} -0.1074 \\ -0.1611 \\ 0.3221 \\ 0.2930 \\ 0.8791 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0.7401 \\ 0.2654 \\ -0.5308 \\ 0.1000 \\ 0.3001 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} -0.6638 \\ 0.3219 \\ -0.5438 \\ 0.0641 \\ 0.1942 \end{bmatrix}, \mathbf{v}_4 = \frac{1}{\sqrt{5}}\begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v}_5 = \frac{1}{\sqrt{10}}\begin{bmatrix} 0 \\ 0 \\ 0 \\ -3 \\ 1 \end{bmatrix} \tag{6.8.89}$$

and the corresponding singular vectors are

$$\mathbf{u}_1 = \begin{bmatrix} 0.1592 \\ 0.5089 \\ -0.8460 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 0.7320 \\ 0.5141 \\ 0.4470 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 0.6624 \\ -0.6904 \\ -0.2907 \end{bmatrix} \tag{6.8.90}$$

The next formal step is to utilize (6.8.88), (6.8.89) and (6.8.90) to form the solution (6.8.82). This calculation goes as follows:

$$\mathbf{x}=\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{bmatrix}=\underbrace{\begin{matrix}\dfrac{1}{12.6768}\left(\begin{bmatrix}1&2&3\end{bmatrix}\begin{bmatrix}0.1592\\0.5089\\-0.8460\end{bmatrix}\right)\begin{bmatrix}-0.1074\\-0.1611\\0.3221\\0.2930\\0.8791\end{bmatrix}+\dfrac{1}{4.1902}\left(\begin{bmatrix}1&2&3\end{bmatrix}\begin{bmatrix}0.7320\\0.5141\\0.4470\end{bmatrix}\right)\begin{bmatrix}0.7401\\0.2654\\-0.5308\\0.1000\\0.3001\end{bmatrix}\\+\dfrac{1}{2.3962}\left(\begin{bmatrix}1&2&3\end{bmatrix}\begin{bmatrix}0.6624\\-0.6904\\-0.2907\end{bmatrix}\right)\begin{bmatrix}-0.6638\\0.3219\\-0.5438\\0.0641\\0.1942\end{bmatrix}\end{matrix}}_{\sum_{p=1}^{R}\left(\frac{1}{\sqrt{\lambda_p}}\langle\mathbf{b},\mathbf{u}_p\rangle\right)\mathbf{v}_p}$$

$$+\underbrace{\left(\frac{1}{5}\begin{bmatrix}x_1&x_2&x_3&x_4&x_5\end{bmatrix}\begin{bmatrix}0\\2\\1\\0\\0\end{bmatrix}\right)\begin{bmatrix}0\\2\\1\\0\\0\end{bmatrix}+\left(\frac{1}{10}\begin{bmatrix}x_1&x_2&x_3&x_4&x_5\end{bmatrix}\begin{bmatrix}0\\0\\0\\-3\\1\end{bmatrix}\right)\begin{bmatrix}0\\0\\0\\-3\\1\end{bmatrix}}_{\sum_{p=R+1}^{N}\langle\mathbf{x},\mathbf{v}_p\rangle\mathbf{v}_p} \tag{6.8.91}$$

If this complicated numerical expression is evaluated, one finds

$$\mathbf{x}=\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{bmatrix}=\underbrace{\begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix}}_{\sum_{p=1}^{R}\left(\frac{1}{\sqrt{\lambda_p}}\langle\mathbf{b},\mathbf{u}_p\rangle\right)\mathbf{v}_p}+\underbrace{\frac{1}{5}(2x_2+x_3)\begin{bmatrix}0\\2\\1\\0\\0\end{bmatrix}+\frac{1}{10}(-3x_4+x_5)}_{\sum_{p=R+1}^{N}\langle\mathbf{x},\mathbf{v}_p\rangle\mathbf{v}_p}\begin{bmatrix}0\\0\\0\\-3\\1\end{bmatrix} \tag{6.8.92}$$

This result is entirely equivalent to the earlier answer (6.8.84)

This brief introduction to the singular value decompositions does not do justice to a large and complex topic that has great application. A few minutes searching the internet will reveal many of these applications.[14]

Exercises

6.8.1 Let $\mathscr{V}$ and $\mathscr{U}$ be real vector spaces such that $\dim\mathscr{V}=2$ and $\dim\mathscr{U}=4$. We define $\mathbf{A}:\mathscr{V}\rightarrow\mathscr{U}$ by

$$\begin{aligned}\mathbf{A}\mathbf{i}_1 &= 3\mathbf{j}_1+\mathbf{j}_2\\ \mathbf{A}\mathbf{i}_2 &= 3\mathbf{j}_1+\mathbf{j}_2\end{aligned} \tag{6.8.93}$$

where $\{\mathbf{i}_1,\mathbf{i}_2\}$ and $\{\mathbf{j}_1,\mathbf{j}_2,\mathbf{j}_3,\mathbf{j}_4\}$ are orthonormal bases. Determine the singular values and the singular vectors for the linear transformation $\mathbf{A}$.

6.8.2 In the case where you are given a matrix $A\in\mathscr{M}^{N\times N}$ that has rank N, the singular value decomposition (6.8.56) can be written

$$A=K\begin{bmatrix}\sqrt{\lambda_1} & 0 & \cdot & \cdot & \cdot\\ 0 & \sqrt{\lambda_2} & & & \\ 0 & & \cdot & & \\ \cdot & & & \cdot & \\ \cdot & & & & \sqrt{\lambda_N}\end{bmatrix}Q^* \tag{6.8.94}$$

The matrix $A\in\mathscr{M}^{N\times N}$, by our assumptions is nonsingular. The inverse, from (6.8.94), is given by

[14] A couple of interesting articles that give insights to the singular value decomposition can be found at http://www.mathworks.com/company/newsletters/news_notes/oct06/clevescorner.html and http://www.ams.org/samplings/feature-column/fcarc-svd. The second article contains a link to a New York Times article that discusses a challenge by Netflix that will award a million dollars to anyone that can improve in a specific way its recommendation engine. The article explains how the singular value decomposition is being utilized by individuals working to win the Netflix challenge.

$$A^{-1} = Q\begin{bmatrix} \frac{1}{\sqrt{\lambda_1}} & 0 & \cdot & \cdot & \cdot \\ 0 & \frac{1}{\sqrt{\lambda_2}} & & & \\ 0 & & \cdot & & \\ \cdot & & & \cdot & \\ \cdot & & & & \frac{1}{\sqrt{\lambda_N}} \end{bmatrix}Q^* \tag{6.8.95}$$

Given the matrix from Example 1.6.5, repeated,

$$A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 1 & 1 \\ -2 & 2 & -1 \end{bmatrix} \tag{6.8.96}$$

and utilize (6.8.95) to calculate the inverse of (6.8.96). The answer was given earlier in equation (1.6.35).

6.8.3 Show that the singular value decomposition of the matrix $A:\mathscr{M}^{3\times 1} \to \mathscr{M}^{5\times 1}$ defined by

$$A = \begin{bmatrix} \frac{16}{5} & \frac{1}{5}(1-i) & \frac{1}{5}(1+i) \\ \frac{1}{5}(1+i) & \frac{22}{5} & -\frac{3}{5}i \\ \frac{1}{5}(1-i) & \frac{3}{5}i & \frac{22}{5} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{6.8.97}$$

is

$$A=\begin{bmatrix} 0 & \frac{1}{\sqrt{10}}(1+i) & -\frac{2}{\sqrt{10}}(1+i) & 0 & 0 \\ -\frac{i}{\sqrt{2}} & \sqrt{\frac{2}{5}}i & \frac{1}{\sqrt{10}}i & 0 & 0 \\ \frac{1}{\sqrt{2}} & \sqrt{\frac{2}{5}} & \frac{1}{\sqrt{10}} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 5 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\times \tag{6.8.98}$$

$$\begin{bmatrix} 0 & \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{10}}(1-i) & \sqrt{\frac{2}{5}}i & \sqrt{\frac{2}{5}} \\ -\frac{2}{\sqrt{10}}(1-i) & -\frac{1}{\sqrt{10}}i & \frac{1}{10} \end{bmatrix}$$

Also, show that an equivalent form of (6.8.98) is

$$A=\begin{bmatrix} 0 & \frac{1}{\sqrt{10}}(1+i) & -\frac{2}{\sqrt{10}}(1+i) \\ -\frac{i}{\sqrt{2}} & \sqrt{\frac{2}{5}}i & \frac{1}{\sqrt{10}}i \\ \frac{1}{\sqrt{2}} & \sqrt{\frac{2}{5}} & \frac{1}{\sqrt{10}} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 5 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 0 & \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{10}}(1-i) & \sqrt{\frac{2}{5}}i & \sqrt{\frac{2}{5}} \\ -\frac{2}{\sqrt{10}}(1-i) & -\frac{1}{\sqrt{10}}i & \frac{1}{10} \end{bmatrix} \tag{6.8.99}$$

Section 6.9 The Polar Decomposition Theorem

The polar decomposition is a decomposition of a linear transformation $\mathbf{A}:\mathscr{V}\to\mathscr{U}$ into the product of a Hermitian linear transformation and a unitary linear transformation. The Hermitian linear transformation is positive semidefinite and, depending upon the properties of $\mathbf{A}$, the unitary linear transformation may not be unique. The details of this rough description will be made clear in this section. One of the applications of the polar decomposition theorem is in the case where $\mathbf{A}:\mathscr{V}\to\mathscr{U}$ is one to one and $N=\dim\mathscr{V}=\dim\mathscr{U}$. In other words, when $\mathbf{A}:\mathscr{V}\to\mathscr{U}$ is one to one and onto, thus invertible. This is the case that arises when one studies the kinematics of strain for continuous materials. The formal statement of the polar decomposition theorem in this case is

Theorem 6.9.1: A one to one onto linear transformation $\mathbf{A}:\mathscr{V}\to\mathscr{U}$ has a unique multiplicative decomposition

$$\mathbf{A}=\mathbf{RV} \tag{6.9.1}$$

where $\mathbf{R}:\mathscr{V}\to\mathscr{U}$ is unitary and $\mathbf{V}:\mathscr{V}\to\mathscr{V}$ is Hermitian and positive definite.

Proof: The proof utilizes a construction similar to that used in Section 6.8 for the singular value decomposition. Given a one to one onto linear transformation $\mathbf{A}:\mathscr{V}\to\mathscr{U}$, we can construct Hermitian linear transformation $\mathbf{C}:\mathscr{V}\to\mathscr{V}$ by the definition

$$\mathbf{C}=\mathbf{A}^*\mathbf{A} \tag{6.9.2}$$

By the same argument that produced (6.8.8)

$$\left\langle \mathbf{v},\mathbf{Cv}\right\rangle=\left\langle \mathbf{v},\mathbf{A}^*\mathbf{Av}\right\rangle=\left\langle \mathbf{Av},\mathbf{Av}\right\rangle=\left\|\mathbf{Av}\right\|^2\geq 0 \tag{6.9.3}$$

Because $\mathbf{A}:\mathscr{V}\to\mathscr{U}$ is one to one and onto, $K\left(\mathbf{A}\right)$ only contains the zero vector. As a result $\left\|\mathbf{Av}\right\|^2>0$ for all *non zero* vectors $\mathbf{v}\in\mathscr{V}$. Therefore, the Hermitian linear transformation $\mathbf{C}:\mathscr{V}\to\mathscr{V}$ is positive definite. As a positive definite Hermitian linear transformation, $\mathbf{C}:\mathscr{V}\to\mathscr{V}$ has the spectral representation

$$\mathbf{C}=\sum_{j=1}^{N}\lambda_j\mathbf{v}_j\otimes\mathbf{v}_j \tag{6.9.4}$$

where the positive numbers $\lambda_1,\lambda_2,...,\lambda_N$ are the eigenvalues of $\mathbf{C}$ and $\left\{\mathbf{v}_1,\mathbf{v}_2,...,\mathbf{v}_N\right\}$ is an orthonormal basis for $\mathscr{V}$ consisting of eigenvectors of $\mathbf{C}$. The representation (6.9.4) does not assume the eigenvalues are distinct. If they are, the tensor products in (6.9.4) represent the projections into the characteristic subspaces of $\mathbf{C}$. It is useful to note that we can apply (6.6.31) to the expression (6.9.4) and obtain

$$\mathbf{C}^{-1} = \sum_{j=1}^{N} \frac{1}{\lambda_j} \mathbf{v}_j \otimes \mathbf{v}_j \tag{6.9.5}$$

We can also apply the definition (6.6.31) and define the linear transformation $\mathbf{V}:\mathscr{V} \to \mathscr{V}$ by

$$\mathbf{V} = \mathbf{C}^{1/2} = \sum_{j=1}^{N} \sqrt{\lambda_j}\, \mathbf{v}_j \otimes \mathbf{v}_j \tag{6.9.6}$$

where, by convention, we have used the positive square root of each eigenvalue. It follows from (6.9.6) that

$$\mathbf{V}^{-1} = \sum_{j=1}^{N} \frac{1}{\sqrt{\lambda_j}} \mathbf{v}_j \otimes \mathbf{v}_j \tag{6.9.7}$$

Equation (6.9.6) provides one of the two linear transformations in the decomposition (6.9.1). The next formal step is to define the linear transformation $\mathbf{R}:\mathscr{V} \to \mathscr{U}$ by the formula

$$\mathbf{R} = \mathbf{A}\mathbf{V}^{-1} \tag{6.9.8}$$

Because $\mathbf{A}$ is invertible, $\mathbf{R}$ as defined by (6.9.8) is also invertible. If we can establish that $\mathbf{R}$ is unitary, we will have established (6.9.1). We shall establish that $\mathbf{R}$ is unitary by showing that it obeys (4.10.14), repeated,

$$\mathbf{R}^*\mathbf{R} = \mathbf{I}_{\mathscr{V}} \tag{6.9.9}$$

The definition (6.9.8) yields

$$\begin{aligned} \mathbf{R}^*\mathbf{R} &= \left(\mathbf{A}\mathbf{V}^{-1}\right)^* \mathbf{A}\mathbf{V}^{-1} = \left(\mathbf{V}^{-1}\right)^* \mathbf{A}^*\mathbf{A}\mathbf{V}^{-1} \\ &= \mathbf{V}^{-1}\mathbf{A}^*\mathbf{A}\mathbf{V}^{-1} = \mathbf{V}^{-1}\mathbf{V}^2\mathbf{V}^{-1} \\ &= \left(\mathbf{V}^{-1}\mathbf{V}\right)\left(\mathbf{V}\mathbf{V}^{-1}\right) = \mathbf{I}_{\mathscr{V}} \end{aligned} \tag{6.9.10}$$

where (4.9.10) and (6.9.6) have been used. The uniqueness of the decomposition (6.9.1) is a consequence of (6.9.2), (6.9.6) and (6.9.8).

A corollary to Theorem 6.9.1 is that $\mathbf{A}:\mathscr{V} \to \mathscr{U}$ also has the decomposition

$$\mathbf{A} = \mathbf{U}\mathbf{R} \tag{6.9.11}$$

where $\mathbf{U}:\mathscr{U} \to \mathscr{U}$ is a positive definite Hermitian linear transformation. Equation (6.9.11) results if we simply define $\mathbf{U}$ by the formula

$$\mathbf{U} = \mathbf{R}\mathbf{V}\mathbf{R}^* \tag{6.9.12}$$

It readily follows from this definition that

$$\mathbf{B} = \mathbf{U}^2 \tag{6.9.13}$$

where $\mathbf{B}:\mathscr{U} \to \mathscr{U}$ is the positive definite Hermitian linear transformation defined by

$$\mathbf{B} = \mathbf{A}\mathbf{A}^* \tag{6.9.14}$$

It is possible to show that

$$\begin{aligned}
\mathbf{B} &= \sum_{j=1}^{N} \lambda_j \mathbf{u}_j \otimes \mathbf{u}_j \\
\mathbf{B}^{-1} &= \sum_{j=1}^{N} \frac{1}{\lambda_j} \mathbf{u}_j \otimes \mathbf{u}_j \\
\mathbf{U} = \mathbf{B}^{1/2} &= \sum_{j=1}^{N} \sqrt{\lambda_j}\, \mathbf{u}_j \otimes \mathbf{u}_j \\
\mathbf{U}^{-1} &= \sum_{j=1}^{N} \frac{1}{\sqrt{\lambda_j}} \mathbf{u}_j \otimes \mathbf{u}_j
\end{aligned} \tag{6.9.15}$$

where $\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_N\}$ is an orthonormal basis of $\mathscr{U}$ consisting of eigenvectors of $\mathbf{B}$. The eigenvectors $\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_N\}$ are related to the eigenvectors $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_N\}$ by the formula

$$\begin{aligned}
\mathbf{u}_j &= \frac{1}{\sqrt{\lambda_j}} \mathbf{A}\mathbf{v}_j = \frac{1}{\sqrt{\lambda_j}} \mathbf{R}\mathbf{V}\mathbf{v}_j \\
&= \frac{1}{\sqrt{\lambda_j}} \mathbf{R}\left(\sqrt{\lambda_j}\,\mathbf{v}_j\right) = \mathbf{R}\mathbf{v}_j
\end{aligned} \tag{6.9.16}$$

where (6.9.1) and (6.9.6) have been used. Equation (6.9.16) can also be established from (6.9.12), (6.9.6), (6.9.15) and (6.7.24).

Example 6.9.1: As an illustration of the polar decomposition theorem, consider the linear transformation $\mathbf{A}:\mathscr{V} \to \mathscr{V}$ introduced in Example 5.3.1. The definition of this linear transformation is given in equation (5.3.1), repeated,

$$\begin{aligned}
\mathbf{A}\mathbf{e}_1 &= \mathbf{e}_1 + \mathbf{e}_2 + 4\mathbf{e}_3 \\
\mathbf{A}\mathbf{e}_2 &= 2\mathbf{e}_1 - 4\mathbf{e}_3 \\
\mathbf{A}\mathbf{e}_3 &= -\mathbf{e}_1 + \mathbf{e}_2 + 5\mathbf{e}_3
\end{aligned} \tag{6.9.17}$$

where $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is a basis for $\mathscr{V}$. As explained in Example 4.5.1, the matrix of $\mathbf{A}$ with respect to this basis is

$$A = M\left(\mathbf{A}, \mathbf{e}_j, \mathbf{e}_k\right) = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix} \tag{6.9.18}$$

The linear transformation $\mathbf{C}$ defined by (6.9.2) has the matrix

$$\begin{aligned} C = M\left(\mathbf{C}, \mathbf{e}_j, \mathbf{e}_k\right) &= \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix}^T \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 18 & -14 & 20 \\ -14 & 20 & -22 \\ 20 & -22 & 27 \end{bmatrix} \end{aligned} \tag{6.9.19}$$

The eigenvalues and eigenvectors of the matrix (6.9.19) can be shown to be

$$\lambda_1 = 0.1207, \lambda_2 = 0.4.9776, \lambda_3 = 59.9017 \tag{6.9.20}$$

and

$$T = \begin{bmatrix} \upsilon^1_{(1)} & \upsilon^1_{(2)} & \upsilon^1_{(3)} \\ \upsilon^2_{(1)} & \upsilon^2_{(2)} & \upsilon^2_{(3)} \\ \upsilon^3_{(1)} & \upsilon^3_{(2)} & \upsilon^3_{(3)} \end{bmatrix} = \begin{bmatrix} -0.4164 & 0.7578 & 0.5024 \\ 0.5268 & 0.6515 & -0.5459 \\ 0.7410 & -0.0374 & 0.6705 \end{bmatrix} \tag{6.9.21}$$

where the notation introduced in equation (5.3.23) has been used to label the eigenvectors. The spectral form of (6.9.19) which follows from (6.9.4) is

$$
\begin{aligned}
C = M\left(\mathbf{C},\mathbf{e}_j,\mathbf{e}_k\right) &= \begin{bmatrix} 18 & -14 & 20 \\ -14 & 20 & -22 \\ 20 & -22 & 27 \end{bmatrix} \\
&= \lambda_1 \begin{bmatrix} \upsilon^1{}_{(1)} \\ \upsilon^2{}_{(1)} \\ \upsilon^3{}_{(1)} \end{bmatrix}\left(\begin{bmatrix} \upsilon^1{}_{(1)} \\ \upsilon^2{}_{(1)} \\ \upsilon^3{}_{(1)} \end{bmatrix}\right)^T + \lambda_2 \begin{bmatrix} \upsilon^1{}_{(2)} \\ \upsilon^2{}_{(2)} \\ \upsilon^3{}_{(2)} \end{bmatrix}\left(\begin{bmatrix} \upsilon^1{}_{(2)} \\ \upsilon^2{}_{(2)} \\ \upsilon^3{}_{(2)} \end{bmatrix}\right)^T + \lambda_3 \begin{bmatrix} \upsilon^1{}_{(3)} \\ \upsilon^2{}_{(3)} \\ \upsilon^3{}_{(3)} \end{bmatrix}\left(\begin{bmatrix} \upsilon^1{}_{(3)} \\ \upsilon^2{}_{(3)} \\ \upsilon^3{}_{(3)} \end{bmatrix}\right)^T \\
&= 0.1207\begin{bmatrix} 0.1734 & -0.2194 & -0.3085 \\ -0.2194 & 0.2775 & 0.3904 \\ -0.3085 & 0.3904 & 0.5491 \end{bmatrix} + 4.9776\begin{bmatrix} 0.5742 & 0.4937 & -0.0283 \\ 0.4937 & 0.4244 & -0.0243 \\ -0.0283 & -0.0243 & 0.0014 \end{bmatrix} \\
&\quad +59.9017\begin{bmatrix} 0.2524 & -0.2743 & 0.3369 \\ -0.2743 & 0.2981 & -0.3660 \\ 0.3369 & -0.3660 & 0.4495 \end{bmatrix}
\end{aligned}
\tag{6.9.22}
$$

where (6.7.8) has been used to determine the matrix representation of the tensor products in (6.9.4). From (6.9.6) and (6.9.22), it follows that

$$
\begin{aligned}
V = M\left(\mathbf{V},\mathbf{e}_j,\mathbf{e}_k\right) &= \sqrt{\lambda_1} \begin{bmatrix} \upsilon^1{}_{(1)} \\ \upsilon^2{}_{(1)} \\ \upsilon^3{}_{(1)} \end{bmatrix}\left(\begin{bmatrix} \upsilon^1{}_{(1)} \\ \upsilon^2{}_{(1)} \\ \upsilon^3{}_{(1)} \end{bmatrix}\right)^T + \sqrt{\lambda_2} \begin{bmatrix} \upsilon^1{}_{(2)} \\ \upsilon^2{}_{(2)} \\ \upsilon^3{}_{(2)} \end{bmatrix}\left(\begin{bmatrix} \upsilon^1{}_{(2)} \\ \upsilon^2{}_{(2)} \\ \upsilon^3{}_{(2)} \end{bmatrix}\right)^T + \sqrt{\lambda_3} \begin{bmatrix} \upsilon^1{}_{(3)} \\ \upsilon^2{}_{(3)} \\ \upsilon^3{}_{(3)} \end{bmatrix}\left(\begin{bmatrix} \upsilon^1{}_{(3)} \\ \upsilon^2{}_{(3)} \\ \upsilon^3{}_{(3)} \end{bmatrix}\right)^T \\
&= 0.3475\begin{bmatrix} 0.1734 & -0.2194 & -0.3085 \\ -0.2194 & 0.2775 & 0.3904 \\ -0.3085 & 0.3904 & 0.5491 \end{bmatrix} + 2.311\begin{bmatrix} 0.5742 & 0.4937 & -0.0283 \\ 0.4937 & 0.4244 & -0.0243 \\ -0.0283 & -0.0243 & 0.0014 \end{bmatrix} \\
&\quad +7.7396\begin{bmatrix} 0.2524 & -0.2743 & 0.3369 \\ -0.2743 & 0.2981 & -0.3660 \\ 0.3369 & -0.3660 & 0.4495 \end{bmatrix} \\
&= \begin{bmatrix} 3.2950 & -1.0978 & 2.4368 \\ -1.0978 & 3.3501 & -2.7516 \\ 2.4368 & -2.7516 & 3.6730 \end{bmatrix}
\end{aligned}
\tag{6.9.23}
$$

Finally, the matrix of the orthogonal linear transformation $\mathbf{R}:\mathscr{V}\to\mathscr{V}$ is, from (6.9.8)

$$
\begin{aligned}
R = M\left(\mathbf{R},\mathbf{e}_j,\mathbf{e}_k\right) &= M\left(\mathbf{A},\mathbf{e}_j,\mathbf{e}_k\right)M\left(\mathbf{V},\mathbf{e}_j,\mathbf{e}_k\right)^{-1} \\
&= \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix}\begin{bmatrix} 3.2950 & -1.0978 & 2.4368 \\ -1.0978 & 3.3501 & -2.7516 \\ 2.4368 & -2.7516 & 3.6730 \end{bmatrix}^{-1} \\
&= \begin{bmatrix} 0.7551 & 0.5442 & -0.3656 \\ -0.0682 & 0.6198 & 0.7818 \\ 0.6520 & -0.5654 & 0.5051 \end{bmatrix}
\end{aligned}
\tag{6.9.24}
$$

Therefore, the polar decomposition (6.9.1) is given by (6.9.18), (6.9.24) and (6.9.23). If we utilize (6.9.12) and (6.9.24) it follows that

$$
M\left(\mathbf{U},\mathbf{e}_j,\mathbf{e}_k\right) = M\left(\mathbf{R},\mathbf{e}_j,\mathbf{e}_k\right)M\left(\mathbf{V},\mathbf{e}_j,\mathbf{e}_k\right)M\left(\mathbf{R}^T,\mathbf{e}_k,\mathbf{e}_j\right) \tag{6.9.25}
$$

Equation (6.9.25) creates a small problem because the components of the linear transformation $\mathbf{R}^T$ with respect to the basis $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ are given by (4.9.24) specialized to the case of a real vector space $\mathscr{V}$ and a linear transformation $\mathscr{V} \to \mathscr{V}$. Equation (4.9.24) requires knowledge of the matrix of inner products formed from the basis $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$. Fortunately, we do not need to utilize (4.9.24) in this case because $\mathbf{R}$ is orthogonal and, from (4.10.15),

$$
\mathbf{R}^T = \mathbf{R}^{-1} \tag{6.9.26}
$$

and from (3.5.42)

$$
M\left(\mathbf{R}^T,\mathbf{e}_k,\mathbf{e}_j\right) = M\left(\mathbf{R}^{-1},\mathbf{e}_k,\mathbf{e}_j\right) = \left(M\left(\mathbf{R},\mathbf{e}_j,\mathbf{e}_k\right)\right)^{-1} \tag{6.9.27}
$$

Equation (6.9.27) allows (6.9.25) to be written

$$
M\left(\mathbf{U},\mathbf{e}_j,\mathbf{e}_k\right) = M\left(\mathbf{R},\mathbf{e}_j,\mathbf{e}_k\right)M\left(\mathbf{V},\mathbf{e}_j,\mathbf{e}_k\right)\left(M\left(\mathbf{R},\mathbf{e}_j,\mathbf{e}_k\right)\right)^{-1} \tag{6.9.28}
$$

As a result of (6.9.28) and (6.9.24),

$$
\begin{aligned}
M\left(\mathbf{U},\mathbf{e}_j,\mathbf{e}_k\right) &= \begin{bmatrix} 0.7551 & 0.5442 & -0.3656 \\ -0.0682 & 0.6198 & 0.7818 \\ 0.6520 & -0.5654 & 0.5051 \end{bmatrix} \begin{bmatrix} 3.2950 & -1.0978 & 2.4368 \\ -1.0978 & 3.3501 & -2.7516 \\ 2.4368 & -2.7516 & 3.6730 \end{bmatrix} \begin{bmatrix} 0.7551 & 0.5442 & -0.3656 \\ -0.0682 & 0.6198 & 0.7818 \\ 0.6520 & -0.5654 & 0.5051 \end{bmatrix}^{-1} \\
&= \begin{bmatrix} 0.7551 & 0.5442 & -0.3656 \\ -0.0682 & 0.6198 & 0.7818 \\ 0.6520 & -0.5654 & 0.5051 \end{bmatrix} \begin{bmatrix} 3.2950 & -1.0978 & 2.4368 \\ -1.0978 & 3.3501 & -2.7516 \\ 2.4368 & -2.7516 & 3.6730 \end{bmatrix} \begin{bmatrix} 0.7551 & -0.0682 & 0.6520 \\ 0.5442 & 0.6198 & -0.5654 \\ -0.3656 & 0.7818 & 0.5051 \end{bmatrix} \\
&= \begin{bmatrix} 2.2091 & 0.3896 & -0.9840 \\ 0.3896 & 0.7136 & 1.1571 \\ -0.9840 & 1.1571 & 7.3955 \end{bmatrix}
\end{aligned}
\tag{6.9.29}
$$

Example 6.9.2: Consider the linear transformation $\mathbf{A}:\mathscr{V}\rightarrow\mathscr{V}$ whose matrix with respect to an orthonormal basis $\left\{\mathbf{i}_1,\mathbf{i}_2,\mathbf{i}_3,\mathbf{i}_4\right\}$ is

$$
A = M\left(\mathbf{A},\mathbf{i}_j,\mathbf{i}_k\right) = \begin{bmatrix} 2 & 3i & -2i & 4 \\ 3 & -2 & 1 & 2i \\ 3i & 2 & 3i & 4 \\ -2i & 4 & 0 & 5i \end{bmatrix} \tag{6.9.30}
$$

The linear transformation $\mathbf{C}$ defined by (6.9.2) has the matrix

$$
\begin{aligned}
C = M\left(\mathbf{C},\mathbf{e}_j,\mathbf{e}_k\right) &= \begin{bmatrix} 2 & 3i & -2i & 4 \\ 3 & -2 & 1 & 2i \\ 3i & 2 & 3i & 4 \\ -2i & 4 & 0 & 5i \end{bmatrix}^T \begin{bmatrix} 2 & 3i & -2i & 4 \\ 3 & -2 & 1 & 2i \\ 3i & 2 & 3i & 4 \\ -2i & 4 & 0 & 5i \end{bmatrix} \\
&= \begin{bmatrix} 2 & 3 & -3i & 2i \\ -3i & -2 & 2 & 4 \\ 2i & 1 & -3i & 0 \\ 4 & -2i & 4 & -5i \end{bmatrix} \begin{bmatrix} 2 & 3i & -2i & 4 \\ 3 & -2 & 1 & 2i \\ 3i & 2 & 3i & 4 \\ -2i & 4 & 0 & 5i \end{bmatrix} = \begin{bmatrix} 26 & -6+8i & 12-4i & -2-6i \\ -6+8i & 33 & -8+6i & 8+4i \\ 12+4i & -8-6i & 14 & -2i \\ -2+6i & 8-4i & 2i & 61 \end{bmatrix}
\end{aligned}
\tag{6.9.31}
$$

The eigenvalues and eigenvectors of the matrix (6.9.31) can be shown to be

$$
\lambda_1 = 2.8842, \lambda_2 = 23.1440, \lambda_3 = 42.2155, \lambda_4 = 65.7563 \tag{6.9.32}
$$

and

$$T = \begin{bmatrix} \upsilon^1{}_{(1)} & \upsilon^1{}_{(2)} & \upsilon^1{}_{(3)} & \upsilon^1{}_{(4)} \\ \upsilon^2{}_{(1)} & \upsilon^2{}_{(2)} & \upsilon^2{}_{(3)} & \upsilon^2{}_{(4)} \\ \upsilon^3{}_{(1)} & \upsilon^3{}_{(2)} & \upsilon^3{}_{(3)} & \upsilon^3{}_{(4)} \\ \upsilon^4{}_{(1)} & \upsilon^4{}_{(2)} & \upsilon^4{}_{(3)} & \upsilon^4{}_{(4)} \end{bmatrix}$$

$$= \begin{bmatrix} 0.0880+0.5008i & 0.4924-0.3650i & -0.5494-0.1283i & -0.1498-0.1584i \\ -0.3202-0.0747i & 0.5868-0.2064i & 0.2852+0.5665i & 0.2463+0.2050i \\ -0.1050-0.7851i & 0.1570-0.4337i & -0.1758-0.3270i & -0.0368-0.1437i \\ 0.0769 & -0.1569 & -0.3764 & 0.9098 \end{bmatrix} \tag{6.9.33}$$

where the notation introduced in equation (5.3.23) has been used to label the eigenvectors. The spectral form of (6.9.33) is given by (6.9.4). From (6.9.6) and (6.9.33), it follows that

$$V = M\left(\mathbf{V},\mathbf{i}_j,\mathbf{i}_k\right) = \sqrt{\lambda_1}\begin{bmatrix} \upsilon^1{}_{(1)} \\ \upsilon^2{}_{(1)} \\ \upsilon^3{}_{(1)} \\ \upsilon^4{}_{(1)} \end{bmatrix}\left(\begin{bmatrix} \upsilon^1{}_{(1)} \\ \upsilon^2{}_{(1)} \\ \upsilon^3{}_{(1)} \\ \upsilon^4{}_{(1)} \end{bmatrix}\right)^* + \sqrt{\lambda_2}\begin{bmatrix} \upsilon^1{}_{(2)} \\ \upsilon^2{}_{(2)} \\ \upsilon^3{}_{(2)} \\ \upsilon^4{}_{(2)} \end{bmatrix}\left(\begin{bmatrix} \upsilon^1{}_{(2)} \\ \upsilon^2{}_{(2)} \\ \upsilon^3{}_{(2)} \\ \upsilon^4{}_{(2)} \end{bmatrix}\right)^* + \sqrt{\lambda_3}\begin{bmatrix} \upsilon^1{}_{(3)} \\ \upsilon^2{}_{(3)} \\ \upsilon^3{}_{(3)} \\ \upsilon^4{}_{(3)} \end{bmatrix}\left(\begin{bmatrix} \upsilon^1{}_{(3)} \\ \upsilon^2{}_{(3)} \\ \upsilon^3{}_{(3)} \\ \upsilon^4{}_{(3)} \end{bmatrix}\right)^*$$

$$+\sqrt{\lambda_4}\begin{bmatrix} \upsilon^1{}_{(4)} \\ \upsilon^2{}_{(4)} \\ \upsilon^3{}_{(4)} \\ \upsilon^4{}_{(4)} \end{bmatrix}\left(\begin{bmatrix} \upsilon^1{}_{(4)} \\ \upsilon^2{}_{(4)} \\ \upsilon^3{}_{(4)} \\ \upsilon^4{}_{(4)} \end{bmatrix}\right)^* \tag{6.9.34}$$

$$= \begin{bmatrix} 4.6995 & -0.4113+0.9145i & 1.5795-0.3682i & -0.1218-0.5138i \\ -0.4113-0.9145i & 5.4913 & -0.8112+0.8392i & 0.6347+0.2736i \\ 1.5795+0.3682i & -0.8112-0.8392i & 3.1631 & 0.0264-0.0356i \\ -0.1218+0.5138i & 0.6347-0.2736i & 0.0264+0.0356i & 7.7615 \end{bmatrix}$$

Finally, the matrix of the orthogonal linear transformation $\mathbf{R}:\mathscr{V}\to\mathscr{V}$ is, from (6.9.8)

$$
\begin{aligned}
R = M\left(\mathbf{R},\mathbf{i}_j,\mathbf{i}_k\right) &= M\left(\mathbf{A},\mathbf{i}_j,\mathbf{i}_k\right) M\left(\mathbf{V},\mathbf{i}_j,\mathbf{i}_k\right)^{-1} \\
&= \begin{bmatrix} 2 & 3i & -2i & 4 \\ 3 & -2 & 1 & 2i \\ 3i & 2 & 3i & 4 \\ -2i & 4 & 0 & 5i \end{bmatrix}
\begin{bmatrix} 4.6995 & -0.4113+0.9145i & 1.5795-0.3682i & -0.1218-0.5138i \\ -0.4113-0.9145i & 5.4913 & -0.8112+0.8392i & 0.6347+0.2736i \\ 1.5795+0.3682i & -0.8112-0.8392i & 3.1631 & 0.0264-0.0356i \\ -0.1218+0.5138i & 0.6347-0.2736i & 0.0264+0.0356i & 7.7615 \end{bmatrix}^{-1} \\
&= \begin{bmatrix} 0.33i8+0.2091i & 0.0859+0.4342i & -0.0572-0.6152i & 0.5180-0.0115i \\ 0.7544-0.1556i & -0.4194-0.1684i & -0.1913+0.2303i & 0.0501+0.3321i \\ 0.0466+0.4144i & 0.2809+0.1589i & 0.0386+0.7080i & 0.4679-0.0155i \\ 0.0943-0.2648i & 0.6970-0.0992i & 0.1436-0.0720i & -0.0416+0.6307i \end{bmatrix}
\end{aligned}
\tag{6.9.35}
$$

Therefore, the polar decomposition (6.9.1) is given by (6.9.30), (6.9.34) and (6.9.35). If we utilize (6.9.12) it follows that

$$
M\left(\mathbf{U},\mathbf{i}_j,\mathbf{i}_k\right) = M\left(\mathbf{R},\mathbf{i}_j,\mathbf{i}_k\right) M\left(\mathbf{V},\mathbf{i}_j,\mathbf{i}_k\right) M\left(\mathbf{R}^*,\mathbf{i}_k,\mathbf{i}_j\right) = M\left(\mathbf{R},\mathbf{i}_j,\mathbf{i}_k\right) M\left(\mathbf{V},\mathbf{i}_j,\mathbf{i}_k\right) M\left(\mathbf{R},\mathbf{i}_j,\mathbf{i}_k\right)^{-1}
\tag{6.9.36}
$$

As a result of (6.9.34) and (6.9.35),

$$
M\left(\mathbf{U},\mathbf{i}_j,\mathbf{i}_k\right) = \begin{bmatrix} 5.2688 & 0.7433-1.8926i & 1.0255-0.0011i & -0.1317-0.1896i \\ 0.7433+1.8926i & 3.5748 & -0.4145-0.6976i & 0.2939+0.5847i \\ 1.0255+0.0011i & -0.4145+0.6976i & 5.8007 & 0.2171-1.6108i \\ -0.1317+0.1896i & 0.2939-0.5847i & 0.2171+1.6108i & 6.4712 \end{bmatrix}
\tag{6.9.37}
$$

The proof of the polar decomposition theorem, as shown by the above, involves a construction that is very similar to that used for the singular decomposition theorem of Section 6.8. It is the singular decomposition theorem that generalizes the polar decomposition theorem. Our next discussion will return to the singular decomposition theorem, and it will be used to reprove and generalize the polar decomposition theorem above. The generalization will be that we will not assume that the linear transformation $\mathbf{A}:\mathscr{V}\to\mathscr{U}$ is one to one and onto. The result will be a polar decomposition theorem similar in form to the one above except that the linear transformation $\mathbf{R}:\mathscr{V}\to\mathscr{U}$ is not unique. We begin this discussion by summarizing the results of Section 6.8. If we are given a linear transformation $\mathbf{A}:\mathscr{V}\to\mathscr{U}$, it has the component representation (6.8.46)

$$
\mathbf{A} = \sum_{p=1}^{R} \sqrt{\lambda_p}\,\mathbf{u}_p \otimes \mathbf{v}_p
\tag{6.9.38}
$$

where $R = \dim R(A)$, $\left\{\lambda_1, \lambda_2, ..., \lambda_R, \underbrace{0, 0, ..., 0}_{N-R}\right\}$ are the eigenvalues of $\mathbf{A}^*\mathbf{A} : \mathscr{V} \to \mathscr{V}$, $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_R, \mathbf{v}_{R+1}, ..., \mathbf{v}_N\}$ is an orthonormal basis of $\mathscr{V}$ consisting of eigenvectors of $\mathbf{A}^*\mathbf{A}$ and $\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_R, \mathbf{u}_{R+1}, ..., \mathbf{u}_N\}$ is an orthonormal basis of $\mathscr{U}$ consisting of eigenvectors of $\mathbf{A}\mathbf{A}^*$. The sets of vectors $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_R\}$ and $\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_R\}$ are connected by the relationships $(6.8.18)_1$ and $(6.8.19)_1$.

Given the above construction, we *define* linear transformations $\mathbf{V} : \mathscr{V} \to \mathscr{V}$ and $\mathbf{R} : \mathscr{V} \to \mathscr{U}$ by

$$\mathbf{V} = \sum_{j=1}^{R} \sqrt{\lambda_j}\, \mathbf{v}_j \otimes \mathbf{v}_j \tag{6.9.39}$$

and

$$\mathbf{R} = \sum_{j=1}^{R} \mathbf{u}_j \otimes \mathbf{v}_j \tag{6.9.40}$$

respectively. Because the eigenvalues $\{\lambda_1, \lambda_2, ..., \lambda_R\}$ are positive and by convention we are utilizing their positive square roots, it easily follows that $\mathbf{V}$ is a positive semidefinite Hermitian linear transformation. From the definitions (6.9.39) and (6.9.40) it follows that

$$\begin{aligned}
\mathbf{RV} &= \left(\sum_{j=1}^{R} \mathbf{u}_j \otimes \mathbf{v}_j\right)\left(\sum_{k=1}^{R} \sqrt{\lambda_k}\, \mathbf{v}_k \otimes \mathbf{v}_k\right) = \sum_{j=1}^{R}\sum_{k=1}^{R} \sqrt{\lambda_k}\left(\mathbf{u}_j \otimes \mathbf{v}_j\right)\left(\mathbf{v}_k \otimes \mathbf{v}_k\right) \\
&= \sum_{j=1}^{R}\sum_{k=1}^{R} \sqrt{\lambda_k}\left\langle \mathbf{v}_j, \mathbf{v}_k \right\rangle \mathbf{u}_j \otimes \mathbf{v}_k = \sum_{j=1}^{R}\sum_{k=1}^{R} \sqrt{\lambda_k}\, \delta_{jk} \mathbf{u}_j \otimes \mathbf{v}_k \\
&= \sum_{j=1}^{R} \sqrt{\lambda_j}\, \mathbf{u}_j \otimes \mathbf{v}_j = \mathbf{A}
\end{aligned} \tag{6.9.41}$$

where the fact the basis $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_R, \mathbf{v}_{R+1}, ..., \mathbf{v}_N\}$ is orthonormal has been used. If, as in (6.9.12), we define a positive semidefinite Hermitian linear transformation $\mathbf{U} : \mathscr{U} \to \mathscr{U}$ by

$$\mathbf{U} = \mathbf{RVR}^* \tag{6.9.42}$$

it is possible to show that

$$\mathbf{U} = \sum_{j=1}^{R} \sqrt{\lambda_j}\, \mathbf{u}_j \otimes \mathbf{u}_j \tag{6.9.43}$$

and

$$\mathbf{UR} = \mathbf{A} \tag{6.9.44}$$

Equations (6.9.41) and (6.9.44) represent the same kind of decomposition as we encountered for the first version of the polar decomposition theorem. In this case, as was mentioned, we have not had to assume that $\mathbf{A}:\mathscr{V} \to \mathscr{U}$ is one to one and onto. If it is one to one and onto, then (6.9.41) and (6.9.44) become the results (6.9.1) and (6.9.11) respectively.

If we return to the case where $\mathbf{A}:\mathscr{V} \to \mathscr{U}$ is not one to one and onto, it is interesting to ask the question whether or not the linear transformation $\mathbf{R}:\mathscr{V} \to \mathscr{U}$ is unitary. An answer to this question is obtained if we form the product

$$\begin{aligned}\mathbf{R}^*\mathbf{R} &= \left(\sum_{j=1}^{R}\mathbf{u}_j \otimes \mathbf{v}_j\right)^* \left(\sum_{k=1}^{R}\mathbf{u}_k \otimes \mathbf{v}_k\right) = \sum_{j=1}^{R}\sum_{k1}^{R}\left(\mathbf{v}_j \otimes \mathbf{u}_j\right)\left(\mathbf{u}_k \otimes \mathbf{v}_k\right) \\ &= \sum_{j=1}^{R}\sum_{k1}^{R}\left\langle \mathbf{u}_k, \mathbf{u}_j\right\rangle \mathbf{v}_j \otimes \mathbf{v}_k = \sum_{j=1}^{R}\sum_{k1}^{R}\delta_{kj}\mathbf{v}_j \otimes \mathbf{v}_k = \sum_{j=1}^{R}\mathbf{v}_j \otimes \mathbf{v}_j\end{aligned} \tag{6.9.45}$$

Because $\mathbf{R}:\mathscr{V} \to \mathscr{U}$ does not obey (4.10.14), the conclusion is that it is not unitary. In the special case where $\mathbf{A}:\mathscr{V} \to \mathscr{U}$ is one to one and onto, it is true that $R = N$ and, because of the result (6.7.15), equation (6.9.45) establishes that $\mathbf{R}$ is unitary. Given (6.7.15), we can replace (6.9.45) by

$$\mathbf{R}^*\mathbf{R} = \sum_{j=1}^{R}\mathbf{v}_j \otimes \mathbf{v}_j = \mathbf{I}_{\mathscr{V}} - \sum_{j=R+1}^{N}\mathbf{v}_j \otimes \mathbf{v}_j \tag{6.9.46}$$

Essentially, one can establish that the *restriction* of $\mathbf{R}$ to the subspace spanned by $\left\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_R\right\}$ is a one to one onto linear transformation $\mathbf{R}_{\mathscr{V}}:\operatorname{Span}\left(\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_R\right) \to R\left(\mathbf{A}\right)$ that is unitary.

If one simply defines the linear transformations $\mathbf{V}:\mathscr{V} \to \mathscr{V}$ and $\mathbf{R}:\mathscr{V} \to \mathscr{U}$ by (6.9.39) and (6.9.40), respectively, then the decomposition (6.9.41) is a consequence. If the problem is stated differently, namely, given $\mathbf{A}:\mathscr{V} \to \mathscr{U}$ and $\mathbf{V}:\mathscr{V} \to \mathscr{V}$, is there an $\mathbf{R}:\mathscr{V} \to \mathscr{U}$ such that (6.9.41) holds, then one encounters the problem that $\mathbf{R}$ is not necessarily given by (6.9.40). A similar problem arises when one is given an $\mathbf{A}:\mathscr{V} \to \mathscr{U}$ and $\mathbf{U}:\mathscr{U} \to \mathscr{U}$ and asks whether or not there is an $\mathbf{R}:\mathscr{V} \to \mathscr{U}$ such that (6.9.44) holds. In order to be more precise, assume for the moment that we are given $\mathbf{A}:\mathscr{V} \to \mathscr{U}$, $\mathbf{V}:\mathscr{V} \to \mathscr{V}$ and $\mathbf{U}:\mathscr{U} \to \mathscr{U}$ and two linear transformations $\mathbf{R}:\mathscr{V} \to \mathscr{U}$ and $\hat{\mathbf{R}}:\mathscr{V} \to \mathscr{U}$ such that

$$\mathbf{A} = \mathbf{RV} = \hat{\mathbf{R}}\mathbf{V} \tag{6.9.47}$$

and

$$\mathbf{A} = \mathbf{UR} = \mathbf{U}\hat{\mathbf{R}} \tag{6.9.48}$$

Therefore, the two linear transformations $\mathbf{R}$ and $\hat{\mathbf{R}}$ must obey

$$\left(\hat{\mathbf{R}}-\mathbf{R}\right)\mathbf{V}=\mathbf{0} \tag{6.9.49}$$

and

$$\mathbf{U}\left(\mathbf{R}-\hat{\mathbf{R}}\right)=\mathbf{0} \tag{6.9.50}$$

The question is what linear transformation $\mathbf{Z}=\mathbf{R}-\hat{\mathbf{R}}$ is allowed by the requirements (6.9.49) and (6.9.50) when $\mathbf{V}$ is given by (6.9.39) and $\mathbf{U}$ is given by (6.9.43). With respect to the bases $\left\{\mathbf{v}_1,\mathbf{v}_2,...,\mathbf{v}_R,\mathbf{v}_{R+1},...,\mathbf{v}_N\right\}$ for $\mathscr{V}$ and $\left\{\mathbf{u}_1,\mathbf{u}_2,...,\mathbf{u}_R,\mathbf{u}_{R+1},...,\mathbf{u}_M\right\}$ for $\mathscr{U}$, we can write

$$\mathbf{Z}=\mathbf{R}-\hat{\mathbf{R}}=\sum_{k=1}^{M}\sum_{j=1}^{N}Z_{kj}\mathbf{u}_k\otimes\mathbf{v}_j \tag{6.9.51}$$

If we use (6.9.51) and (6.9.39), equation (6.9.49) takes the form

$$\left(\hat{\mathbf{R}}-\mathbf{R}\right)\mathbf{V}=\sum_{k=1}^{M}\sum_{j=1}^{N}Z_{kj}\mathbf{u}_k\otimes\left(\mathbf{V}\mathbf{v}_j\right)=\sum_{k=1}^{M}\sum_{j=1}^{R}\sqrt{\lambda_j}Z_{kj}\mathbf{u}_k\otimes\mathbf{v}_j=\mathbf{0} \tag{6.9.52}$$

where the identity (6.7.26) has also been used. If we use (6.9.51) and (6.9.43), equation (6.9.50) takes the form

$$\mathbf{U}\left(\mathbf{R}-\hat{\mathbf{R}}\right)=\sum_{k=1}^{M}\sum_{j=1}^{N}Z_{kj}\left(\mathbf{U}\mathbf{u}_k\right)\otimes\mathbf{v}_j=\sum_{k=1}^{R}\sum_{j=1}^{N}\sqrt{\lambda_k}Z_{kj}\mathbf{u}_k\otimes\mathbf{v}_j=\mathbf{0} \tag{6.9.53}$$

where the identity (6.7.27) has also been used. Because the eigenvalues $\lambda_1,\lambda_2,...,\lambda_R$ are nonzero, equations (6.9.52) and (6.9.53) show that

$$Z_{kj}=\begin{cases}0 & \text{for } k=1,2,...,R \text{ and } j=1,2,...,N\\ 0 & \text{for } k=R+1,R+2,...,M \text{ and } j=1,2,...,R\end{cases} \tag{6.9.54}$$

Equivalently, (6.9.51) must be of the form

$$\hat{\mathbf{R}}-\mathbf{R}=\sum_{k=R+1}^{M}\sum_{j=R+1}^{N}Z_{kj}\mathbf{u}_k\otimes\mathbf{v}_j \tag{6.9.55}$$

where the coefficients Z_{kj} for $k=R+1,...,M$ and $j=R+1,...,N$ are arbitrary. Equation (6.9.55) shows to what extent the linear transformation $\mathbf{R}:\mathscr{V}\to\mathscr{U}$ in the decompositions (6.9.41) and (6.9.44) are determined by the linear transformation $\mathbf{A}:\mathscr{V}\to\mathscr{U}$. More specifically, if we agree to construct the decompositions (6.9.41) and (6.9.44) utilizing the definition (6.9.40), then any other decompositions of the form $\mathbf{A} = \hat{\mathbf{R}}\mathbf{V} = \mathbf{U}\hat{\mathbf{R}}$ also hold where

$$\hat{\mathbf{R}} = \sum_{j=1}^{R} \mathbf{u}_j \otimes \mathbf{v}_j + \sum_{k=R+1}^{M} \sum_{j=R+1}^{N} Z_{kj} \mathbf{u}_k \otimes \mathbf{v}_j \tag{6.9.56}$$

and where the coefficients Z_{kj} for $k = 1,2,\ldots,M$ and $j = R+1,\ldots,N$ are arbitrary. The construction (6.9.56) shows the cases where the linear transformation $\mathbf{R} : \mathscr{V} \to \mathscr{U}$ is unique. It is unique in cases where $R = M$ ($\mathbf{A} : \mathscr{V} \to \mathscr{U}$ is onto) and/or $R = N$ ($\mathbf{A} : \mathscr{V} \to \mathscr{U}$ is one to one)

Example 6.9.3: The polar decomposition in the case where the linear transformation $\mathbf{A} : \mathscr{V} \to \mathscr{U}$ is not one to one and onto can be illustrated by the linear transformation $\mathbf{A} : \mathscr{V} \to \mathscr{V}$ whose matrix with respect to an orthonormal basis $\{\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3, \mathbf{i}_4\}$ is

$$A = M\left(\mathbf{A}, \mathbf{i}_j, \mathbf{i}_k\right) = \begin{bmatrix} 36 & -9 & 18 & 9 \\ 42 & -7 & 19 & 8 \\ 48 & -5 & 20 & 7 \\ 58 & -25 & 35 & 22 \end{bmatrix} \tag{6.9.57}$$

Therefore, for this example $M = N = 4$. The matrix of the linear transformation $\mathbf{C} = \mathbf{A}^T \mathbf{A}$ is

$$C = \begin{bmatrix} 36 & -9 & 18 & 9 \\ 42 & -7 & 19 & 8 \\ 48 & -5 & 20 & 7 \\ 58 & -25 & 35 & 22 \end{bmatrix}^T \begin{bmatrix} 36 & -9 & 18 & 9 \\ 42 & -7 & 19 & 8 \\ 48 & -5 & 20 & 7 \\ 58 & -25 & 35 & 22 \end{bmatrix} = \begin{bmatrix} 8728 & -2308 & 4436 & 2272 \\ -2308 & 780 & -1270 & -722 \\ 4436 & -1270 & 2310 & 1224 \\ 2272 & -722 & 1224 & 678 \end{bmatrix} \tag{6.9.58}$$

The eigenvalues and eigenvectors turn out to be

$$\lambda_1 = 12252, \lambda_2 = 243.5946, \lambda_3 = \lambda_4 = 0 \tag{6.9.59}$$

and

$$T = \begin{bmatrix} \upsilon^1_{(1)} & \upsilon^1_{(2)} & \upsilon^1_{(3)} & \upsilon^1_{(4)} \\ \upsilon^2_{(1)} & \upsilon^2_{(2)} & \upsilon^2_{(3)} & \upsilon^2_{(4)} \\ \upsilon^3_{(1)} & \upsilon^3_{(2)} & \upsilon^3_{(3)} & \upsilon^3_{(4)} \\ \upsilon^4_{(1)} & \upsilon^4_{(2)} & \upsilon^4_{(3)} & \upsilon^4_{(4)} \end{bmatrix} = \begin{bmatrix} 0.8416 & -0.4506 & 0.1188 & 0.2730 \\ -0.2314 & -0.7131 & -0.6346 & -0.1876 \\ 0.4328 & 0.2465 & -0.1843 & -0.8473 \\ 0.2254 & 0.4771 & -0.7411 & 0.4152 \end{bmatrix} \tag{6.9.60}$$

The results (6.9.59) show that $R = 2$. The singular vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ corresponding to $\mathbf{v}_1$ and $\mathbf{v}_2$ are given by $(6.8.20)_1$. Therefore,

$$\begin{bmatrix} u^1_{(1)} \\ u^2_{(1)} \\ u^3_{(1)} \\ u^4_{(1)} \end{bmatrix} = \frac{1}{\sqrt{\lambda_1}} A \begin{bmatrix} \upsilon^1_{(1)} \\ \upsilon^2_{(1)} \\ \upsilon^3_{(1)} \\ \upsilon^4_{(1)} \end{bmatrix} = \begin{bmatrix} 0.3812 \\ 0.4246 \\ 0.4679 \\ 0.6749 \end{bmatrix} \tag{6.9.61}$$

and

$$\begin{bmatrix} u^1_{(2)} \\ u^2_{(2)} \\ u^3_{(2)} \\ u^4_{(2)} \end{bmatrix} = \frac{1}{\sqrt{\lambda_2}} A \begin{bmatrix} \upsilon^1_{(2)} \\ \upsilon^2_{(2)} \\ \upsilon^3_{(2)} \\ \upsilon^4_{(2)} \end{bmatrix} = \begin{bmatrix} -0.0688 \\ -0.3482 \\ -0.6276 \\ 0.6929 \end{bmatrix} \tag{6.9.62}$$

The eigenvectors of $\mathbf{B} = \mathbf{AA}^T$ corresponding to the two zero eigenvalues can be shown to be

$$\begin{bmatrix} u^1_{(3)} \\ u^2_{(3)} \\ u^3_{(3)} \\ u^4_{(3)} \end{bmatrix} = \begin{bmatrix} 0.9219 \\ -0.2012 \\ -0.2606 \\ -0.2274 \end{bmatrix} \tag{6.9.63}$$

$$\begin{bmatrix} u^1_{(4)} \\ u^2_{(4)} \\ u^3_{(4)} \\ u^4_{(4)} \end{bmatrix} = \begin{bmatrix} -0.0004 \\ 0.8112 \\ -0.5739 \\ -0.1122 \end{bmatrix} \tag{6.9.64}$$

Equations (6.9.59) through (6.9.64) combine with (6.9.39) to yield

$$
\begin{aligned}
V &= M\left(\mathbf{V},\mathbf{i}_j,\mathbf{i}_k\right)\\
&= \sqrt{\lambda_1}\begin{bmatrix}\upsilon^1{}_{(1)}\\ \upsilon^2{}_{(1)}\\ \upsilon^3{}_{(1)}\\ \upsilon^4{}_{(1)}\end{bmatrix}\left(\begin{bmatrix}\upsilon^1{}_{(1)}\\ \upsilon^2{}_{(1)}\\ \upsilon^3{}_{(1)}\\ \upsilon^4{}_{(1)}\end{bmatrix}^T\right) + \sqrt{\lambda_2}\begin{bmatrix}\upsilon^1{}_{(2)}\\ \upsilon^2{}_{(2)}\\ \upsilon^3{}_{(2)}\\ \upsilon^4{}_{(2)}\end{bmatrix}\left(\begin{bmatrix}\upsilon^1{}_{(2)}\\ \upsilon^2{}_{(2)}\\ \upsilon^3{}_{(2)}\\ \upsilon^4{}_{(2)}\end{bmatrix}^T\right)\\
&= \begin{bmatrix} 87.5731 & -16.5427 & 38.5862 & 17.6429\\ -16.5427 & 13.8637 & -13.8302 & -11.0843\\ 38.5862 & -13.8302 & 21.6838 & 12.6349\\ 17.6429 & -11.0843 & 12.6349 & 9.1775 \end{bmatrix}
\end{aligned} \tag{6.9.65}
$$

The same equations along with the definition (6.9.40) yield

$$
\begin{aligned}
R = M\left(\mathbf{R},\mathbf{i}_j,\mathbf{i}_k\right) &= \begin{bmatrix}u^1{}_{(1)}\\ u^2{}_{(1)}\\ u^3{}_{(1)}\\ u^4{}_{(1)}\end{bmatrix}\left(\begin{bmatrix}\upsilon^1{}_{(1)}\\ \upsilon^2{}_{(1)}\\ \upsilon^3{}_{(1)}\\ \upsilon^4{}_{(1)}\end{bmatrix}^T\right) + \begin{bmatrix}u^1{}_{(2)}\\ u^2{}_{(2)}\\ u^3{}_{(2)}\\ u^4{}_{(2)}\end{bmatrix}\left(\begin{bmatrix}\upsilon^1{}_{(2)}\\ \upsilon^2{}_{(2)}\\ \upsilon^3{}_{(2)}\\ \upsilon^4{}_{(2)}\end{bmatrix}^T\right)\\
&= \begin{bmatrix} 0.3519 & -0.0392 & 0.1480 & 0.0531\\ 0.5142 & 0.1500 & 0.0979 & -0.0704\\ 0.6766 & 0.3392 & 0.0478 & -0.1940\\ 0.2557 & -0.6503 & 0.4679 & 0.4828 \end{bmatrix}
\end{aligned} \tag{6.9.66}
$$

Finally, the definition (6.9.42) along with (6.9.65) and (6.9.66) yield

$$
\begin{aligned}
U &= M\left(\mathbf{U},\mathbf{i}_j,\mathbf{i}_k\right) = RVR^T\\
&= \begin{bmatrix} 16.1624 & 18.2901 & 20.4179 & 27.7373\\ 18.2901 & 21.6439 & 25.3976 & 27.9513\\ 20.4179 & 25.3976 & 30.3774 & 28.1653\\ 27.7373 & 27.9513 & 28.1653 & 57.9145 \end{bmatrix}
\end{aligned} \tag{6.9.67}
$$

If, instead of (6.9.66), one uses a linear transformation $\hat{\mathbf{R}}$ that obeys (6.9.56) it is readily shown that again $\mathbf{A} = \hat{\mathbf{R}}\mathbf{V} = \mathbf{U}\hat{\mathbf{R}}$ where $\mathbf{A}$, $\mathbf{V}$ and $\mathbf{U}$ are given by (6.9.57), (6.9.65) and (6.9.67), respectively.

Example 6.9.4: Another example of the polar decomposition in the case where the linear transformation $\mathbf{A}:\mathscr{V}\to\mathscr{U}$ is not one to one and onto can be illustrated by the linear transformation introduced in Example 6.8.1. In this example $\mathscr{V}$ and $\mathscr{U}$ are real vector spaces with $\dim\mathscr{V}=2$ and $\dim\mathscr{U}=4$. We defined $\mathbf{A}:\mathscr{V}\to\mathscr{U}$ by (6.8.26), repeated,

$$\begin{aligned}\mathbf{A}\mathbf{i}_1 &= \mathbf{j}_1+3\mathbf{j}_2\\ \mathbf{A}\mathbf{i}_2 &= 3\mathbf{j}_1+\mathbf{j}_2\end{aligned} \tag{6.9.68}$$

where $\{\mathbf{i}_1,\mathbf{i}_2\}$ and $\{\mathbf{j}_1,\mathbf{j}_2,\mathbf{j}_3,\mathbf{j}_4\}$ are orthonormal bases. In the solution of Example 6.8.1, we showed that $N=R=2$ and $M=4$ in this example. In addition, we showed that

$$\begin{aligned}&\lambda_1=16,\lambda_2=4\\ &\mathbf{v}_1=\frac{1}{\sqrt{2}}\left(\mathbf{i}_1+\mathbf{i}_2\right),\mathbf{v}_2=\frac{1}{\sqrt{2}}\left(-\mathbf{i}_1+\mathbf{i}_2\right)\end{aligned} \tag{6.9.69}$$

and

$$\mathbf{u}_1=\frac{1}{\sqrt{2}}\left(\mathbf{j}_1+\mathbf{j}_2\right),\mathbf{u}_2=\frac{1}{\sqrt{2}}\left(\mathbf{j}_1-\mathbf{j}_2\right),\mathbf{u}_3=\mathbf{j}_3,\mathbf{u}_4=\mathbf{j}_4 \tag{6.9.70}$$

With respect to the orthonormal bases $\{\mathbf{v}_1,\mathbf{v}_2\}$ and $\{\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3,\mathbf{u}_4\}$, the matrix of $\mathbf{A}:\mathscr{V}\to\mathscr{U}$ is given by (6.9.38) which takes the form

$$\mathbf{A}=\sum_{p=1}^{R}\sqrt{\lambda_p}\,\mathbf{u}_p\otimes\mathbf{v}_p=4\mathbf{u}_1\otimes\mathbf{v}_1+2\mathbf{u}_2\otimes\mathbf{v}_2 \tag{6.9.71}$$

The linear transformation $\mathbf{V}:\mathscr{V}\to\mathscr{V}$, defined by (6.9.39), is given by

$$\mathbf{V}=\sum_{j=1}^{R}\sqrt{\lambda_j}\,\mathbf{v}_j\otimes\mathbf{v}_j=4\mathbf{v}_1\otimes\mathbf{v}_1+2\mathbf{v}_2\otimes\mathbf{v}_2 \tag{6.9.72}$$

The linear transformation $\mathbf{R}:\mathscr{V}\to\mathscr{U}$, defined by (6.9.40), is given by

$$\mathbf{R}=\sum_{j=1}^{R}\mathbf{u}_j\otimes\mathbf{v}_j=\mathbf{u}_1\otimes\mathbf{v}_1+\mathbf{u}_2\otimes\mathbf{v}_2 \tag{6.9.73}$$

Finally, the linear transformation $\mathbf{U}:\mathscr{U}\to\mathscr{U}$, defined by (6.9.42), is given by

$$\begin{aligned}\mathbf{U}=\mathbf{R}\mathbf{V}\mathbf{R}^*&=\left(\mathbf{u}_1\otimes\mathbf{v}_1+\mathbf{u}_2\otimes\mathbf{v}_2\right)\left(4\mathbf{v}_1\otimes\mathbf{v}_1+2\mathbf{v}_2\otimes\mathbf{v}_2\right)\left(\mathbf{v}_1\otimes\mathbf{u}_1+\mathbf{v}_2\otimes\mathbf{u}_2\right)\\ &=4\mathbf{u}_1\otimes\mathbf{u}_1+2\mathbf{u}_2\otimes\mathbf{u}_2\end{aligned} \tag{6.9.74}$$

If we choose to represent the four linear transformations in (6.9.71), (6.9.72), (6.9.73) and (6.9.74) as matrices with respect to the orthonormal bases $\{\mathbf{i}_1,\mathbf{i}_2\}$ and $\{\mathbf{j}_1,\mathbf{j}_2,\mathbf{j}_3,\mathbf{j}_4\}$ the results are

$$A = M\left(\mathbf{A},\mathbf{i}_q,\mathbf{j}_k\right) = \begin{bmatrix} 1 & 3 \\ 3 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$V = M\left(\mathbf{V},\mathbf{i}_q,\mathbf{i}_k\right) = T^{-1}M\left(\mathbf{V},\mathbf{v}_q,\mathbf{v}_k\right)T = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

$$R = M\left(\mathbf{R},\mathbf{i}_q,\mathbf{j}_k\right) = S^{-1}M\left(\mathbf{R},\mathbf{v}_q,\mathbf{u}_k\right)T = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$U = M\left(\mathbf{U},\mathbf{j}_q,\mathbf{j}_k\right) = S^{-1}M\left(\mathbf{U},\mathbf{u}_q,\mathbf{u}_k\right)S = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{6.9.75}$$

$$= \begin{bmatrix} 3 & 1 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Exercises

6.9.1 As an illustration of the polar decomposition theorem for a nonsingular matrix (a matrix whose linear transformation is one to one and onto, consider the matrix of coefficients of the problem introduced in Exercise 1.3.2. The matrix in this case is

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 3 & 0 & 3 & -4 \\ 1 & 1 & 1 & 2 \\ 2 & 3 & 1 & 3 \end{bmatrix} \tag{6.9.76}$$

Show that the polar decomposition theorem (6.9.1) and (6.9.11) are given by

$$R = \begin{bmatrix} -0.5892 & 0.4120 & 0.6884 & 0.0958 \\ 0.5395 & 0.0257 & 0.5367 & -0.6482 \\ 0.3544 & -0.4443 & 0.4758 & 0.6713 \\ 0.4859 & 0.7951 & -0.1081 & 0.3467 \end{bmatrix} \tag{6.9.77}$$

$$V = \begin{bmatrix} 2.9448 & 1.2229 & 1.8696 & -0.5807 \\ 1.2229 & 2.3531 & 0.8398 & 1.8062 \\ 1.8696 & 0.8398 & 2.6661 & -0.8312 \\ -0.5807 & 1.8062 & -0.8312 & 5.0705 \end{bmatrix} \tag{6.9.78}$$

and

$$U = \begin{bmatrix} 1.1962 & -0.0859 & 0.7028 & 1.0333 \\ -0.0859 & 5.8216 & 0.1946 & -0.2520 \\ 0.7028 & 0.1946 & 1.7285 & 1.8656 \\ 1.0333 & -0.2520 & 1.8656 & 4.2881 \end{bmatrix} \tag{6.9.79}$$

6.9.2 Show that a polar decomposition of the matrix introduced in Exercise 6.8.3, i.e.,

$$A = \begin{bmatrix} \frac{16}{5} & \frac{1}{5}(1-i) & \frac{1}{5}(1+i) \\ \frac{1}{5}(1+i) & \frac{22}{5} & -\frac{3}{5}i \\ \frac{1}{5}(1-i) & \frac{3}{5}i & \frac{22}{5} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{6.9.80}$$

is

$$V=\begin{bmatrix} \frac{16}{5} & \frac{1}{5}(1-i) & \frac{1}{5}(1+i) \\ \frac{1}{5}(1+i) & \frac{22}{5} & -\frac{3i}{5} \\ \frac{1}{5}(1-i) & \frac{3i}{5} & \frac{22}{5} \end{bmatrix} \tag{6.9.81}$$

$$R=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{6.9.82}$$

and

$$U=\begin{bmatrix} \frac{16}{5} & \frac{1}{5}(1-i) & \frac{1}{5}(1+i) & 0 & 0 \\ \frac{1}{5}(1+i) & \frac{22}{5} & -\frac{3i}{5} & 0 & 0 \\ \frac{1}{5}(1-i) & \frac{3i}{5} & \frac{22}{5} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{6.9.83}$$

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