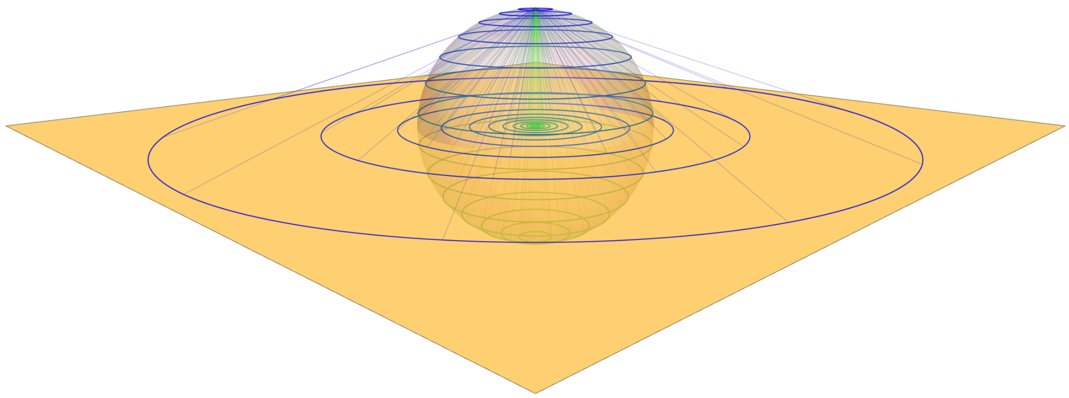
Basic topology

Lecture notes for a 2015 Uppsala University course



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Preface

These are lecture notes written for a course on elementary point set topology given at Uppsala University during the spring of 2015. The notes are, at the time of writing, *not* intended as a full reference for the course. Rather, the course will follow the references [Mun00] and [Fje14], and these notes serve to show which parts of the main references we have covered in the lectures.

The notes will be written as the course moves along and may be discontinued at any moment, depending on time restrictions. As such, they are also likely to contain typos and errors of other kinds; if you come across any such, feel very free to let me know, either by email at s@fuglede.dk, or by letting me know in person. Concrete changes can also be proposed directly at https://github.com/fuglede/basic-topology.

Terms of use

Introduction

This course will mainly be concered with the study of topological spaces. Topological spaces are abstract mathematical concepts whose definition include a sufficient amount of data for them to be called “spaces”. Familiar spaces would be something like $\mathbb{R}^3$ or $\mathbb{R}^n$, but more abstract objects – such as general vector spaces – we also think of as “spaces”. On the other hand, in algebra one encounters objects such as groups and rings that one would generally not think of as spaces, and on the extreme side we talk about sets, which are more general “collections of things” which we may or may not choose to think of as spaces.

From this point of view, one might think of topological spaces with an added bit of *structure* (sv: *struktur*)[1]; a term used throughout mathematics but typically with a rather vague meaning. As such, sets have no interesting structure, but Euclidian space $\mathbb{R}^n$ has plenty: for instance, the usual inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^n$ can be used to talk about angles between vectors. This in turn can be used to define the standard norm $\|\cdot\|$ on $\mathbb{R}^n$ which allows us to talk about lengths of vectors and distances between points. All of this is structure that may not be given to us in a general vector space, but without having this structure on $\mathbb{R}^n$, there would be no such thing as calculus: we wouldn't be able to define things like differentiability and continuity.

For topological spaces we discard all of this fine structure, so that in particular it makes no sense to talk about the distance between two points in a general topological space. The only piece of structure that we will require is that of “open sets”: given a subset of a topological space, we want to be able to tell if it is open or not. This turns out to be the least amount of structure needed to define continuity, so the study of topological spaces is very much the study of continuous functions.

The study of general topological spaces and continuous functions will be contained in Sections 2–5. Simply having open sets turns out also to be sufficient to talk about what it means for a space to be “connected” and “compact” in a way that corresponds to what one would normally associate with those words. More abstractly, we will also look at the notion of separating points, which is less familiar in examples like $\mathbb{R}^n$. These properties of topological spaces will be the basis of Sections 6–8.

The study of general topological spaces and their fundamental properties is often referred to as *point-set topology* (sv: *punktmängdstopologi*) or *general topology* (sv: *allmän topologi*). The less structure a certain space has, the less deep the mathematical results about it tends to be, and out treatment will involve correspondly few deep mathematical theorems; rather, for the first part of the course, one should think of the materials

[1] See https://en.wikipedia.org/wiki/Mathematical_structure for a more precise discussion.

as developing the necessary tools to deal with topological spaces in other contexts. Towards the end of the course, we will remedy this by tying together our theory with other parts of mathematics. Concretely, in Section 10 we touch upon the mathematical area of *algebraic topology* (sv: *algebraisk topologi*) which is concerned with analyzing certain natural algebraic structures that can be associated with topological spaces, and in Section 9, we will study a particular nice family of topological spaces called manifolds that show up in all of geometry.

Before being able to do any of this, though, we need to firmly settle on what sets are, and how one deals with them.

1 Set theory and logic

The theory of sets, typically referred to as *set theory* (sv: *mängdteori*), forms the basic fundament of all of mathematics. As such, it is probably unsurprising that a huge body of work has been devoted to it. This, however, is not obvious if one simply reads these notes as we will not at all deal with many of the important ideas that underly the theory. Moreover, even though set theory plays an essential role in our study, our treatment of set theory will be rather brief, so the inexperienced reader is also strongly encouraged to study [Mun00, §1] in detail.

1.1 Basic notions

Whereas everything that we will during the course will be as mathematically precise as one can get, we will begin by imprecisely considering a *set* (sv: *mängd*) as a "collection of things" (formally, we require a set to satisfy the ZFC axioms but we will not list those here[2]). These "things" will be referred to as the *elements* of the set.

If A is a set, and a is an element of A, we write

$$a \in A.$$

Synonymously, we will sometimes say that a is contained in A. If on the other hand a is not an element of A, then we write

$$a \notin A.$$

If B is another set which contains all the elements of A; that is, if $a \in A$ implies that $a \in B$, then we say that A is a *subset* (sv: *delmängd*) of B and write

$$A \subset B.$$

We will also sometimes say that A is contained in B or that B contains A. With this, we can define two sets A and B to be equal,

$$A = B,$$

if the satisfy that $A \subset B$ and $B \subset A$. If $A \subset B$ but $A \neq B$, we will write

$$A \subsetneq B$$

and say that A is a *proper subset* (sv: *äkta delmängd*) of B.[3] Set-theoretic relationships are often depicted in so-called Euler diagrams; see Figure 1.

Example 1.1. When a set contains only very few elements, one simply lists them. For instance, if A contains only the elements a and b, we write $A = \{a, b\}$. Then if $B = \{a, b, c\}$, we see for instance that $A \subset B$ and since $c \in B$ but $c \notin A$, we have that $A \neq B$, so $A \subsetneq B$.

[2]The interested reader can find the axioms at https://en.wikipedia.org/wiki/ZFC.

[3]Notice that it is also common to use the notation $A \subseteq B$ for the relation "A is a subset of B".

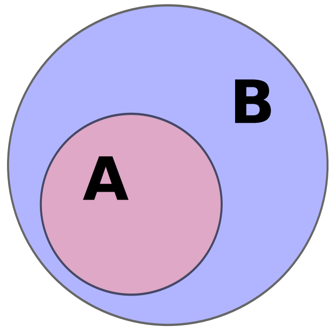

Figure 1: The Euler diagram representing the inclusion $A \subset B$.

Often, sets are given by the properties of their elements, and this will be built into our notation. For instance, the set containing the even numbers will be written

$$\{x \mid x \text{ is an even integer}\},$$

which should be read as "x such that x is an even integer".

We will also often be considering the set which contains no elements at all; this set is called *the empty set* (sv: *den tomma mängden*) and is denoted $\emptyset$. This has the property that $\emptyset \subset X$, no matter what X is.

Definition 1.2. Let X be any set. The *power set* (sv: *potensmängd*) of X, denoted $\mathcal{P}(X)$ is the set of all subsets of X, that is

$$\mathcal{P}(X) = \{U \mid U \subset X\}.$$

Example 1.3. The elements of power sets are themselves sets which in turn may also have elements. Some examples of power sets are the following:

$$\mathcal{P}(\{a\}) = \{\emptyset, \{a\}\},$$
$$\mathcal{P}(\{a, b\}) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}.$$

In general, if X is a finite set containing n elements, then $\mathcal{P}(X)$ contains 2^n elements. The only subset of $\emptyset$ is $\emptyset$ itself, which means that

$$\mathcal{P}(\emptyset) = \{\emptyset\}.$$

Be aware that $\{\emptyset\}$ consists of a single element, namely $\emptyset$, so $\{\emptyset\}$ is not itself empty, even though it is tempting to read the notation like that.

Definition 1.4. Given two sets A and B, we define their *union* (sv: *union*) $A \cup B$ and *intersection* (sv: *snitt*) $A \cap B$ as

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\},$$
$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}.$$

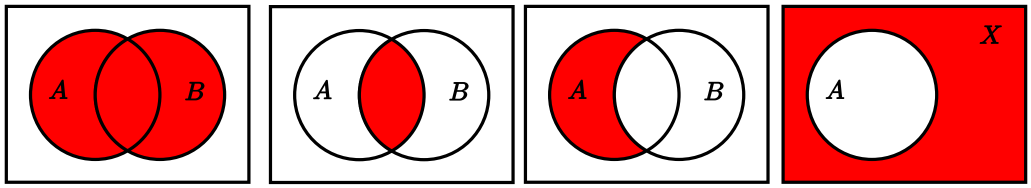

Figure 2: The red sets illustrate $A \cup B$, $A \cap B$, $A \setminus B$, and A^c respectively.

The sets A and B are called *disjoint* (sv: *disjunkt*) if $A \cap B = \emptyset$. We also introduce the *difference* (sv: *differens*) $A \setminus B$, or sometimes $A - B$, given by

$$A \setminus B = \{x \mid x \in A \text{ and } x \notin B\}.$$

If $A \subset X$ for some set X, we introduce the *complement* (sv: *komplement*) of A, written A^c as

$$A^c = X \setminus A.$$

Note that X does not feature in the notation for complements but which set it is will always be clear from the context. See Figure 2 for the corresponding Euler diagrams.

1.2 Set theory and boolean logic

At this point, let us remark that sets and their operations closely mimic logical statements and their boolean logic, and without making this statement too precise, one could summarize their relationship in the following table:

logic	English	set theory
$P \Rightarrow Q$	P implies Q	$A \subset B$
$P \Leftrightarrow Q$	P is equivalent to Q	$A = B$
$P \vee Q$	P or Q	$A \cup B$
$P \wedge Q$	P and Q	$A \cap B$
$\neg P$	not P	A^c

For convenience, and for those who may not have encountered boolean logic before, we recall that these logical operations are defined through truth tables. Here, P and Q are statements that may be either true (T) or false (F), and the values of $P \Rightarrow Q$, $P \Leftrightarrow Q$, etc., are defined accordingly:

P	$\neg P$
T	F
F	T

P	Q	$P \wedge Q$	$P \vee Q$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	F

In these tables, the columns to the left of the vertical bar are the assumptions, and the columns to the right are the definitions. Armed with these definitions, one simply defines $P \Rightarrow Q$ as $(\neg P) \vee Q$, and similarly $P \Leftrightarrow Q$ is defined as $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$.

With these, one sees that the correspondence in the first table in this section is obtained when P is the statement $a \in A$, and Q is the statement $a \in B$.

1.3 Arbitrary unions and intersections

So far, we have considered only operations on pairs of sets, but more often than not we will be dealing with infinite families of sets. First of all, we introduce the notation $\forall$ meaning "for all", and $\exists$ meaning "there exists".

Let I be any set. Then a collection $\{A_i\}_{i \in I}$ of sets A_i is called a family of sets parametrised by I. For such a family, define the infinite union, and the infinite intersection as

$$\bigcup_{i \in I} A_i = \{x \mid \exists i \in I \text{ such that } x \in A_i\},$$
$$\bigcap_{i \in I} A_i = \{x \mid x \in A_i \ \forall i \in I\}.$$

It might be helpful to compare these definitions with the definitions of union and intersection from above, which is the case where I contains two elements.

Proposition 1.5. *For sets A, B, C, and X, for a family $\{A_i\}_{i \in I}$ with $A_i \subset X$ for all $i \in I$, one has the useful identities,*

$$\begin{aligned} A \cap (B \cup C) &= (A \cap B) \cup (A \cap C), \\ A \cup (B \cap C) &= (A \cup B) \cap (A \cup C), \\ X \setminus \bigcup_{i \in I} A_i &= \bigcap_{i \in I} X \setminus A_i, \\ X \setminus \bigcap_{i \in I} A_i &= \bigcup_{i \in I} X \setminus A_i. \end{aligned}$$

Various versions of these equalities are known as De Morgan's laws.

Partial proof. Set theoretical identities like the above are typically shown in the same way: one starts with an element in the left hand side and proves that it is an element of the right hand side, which shows "$\subset$", and one then proceeds to do the same thing the other way around. To convince oneself that the identities indeed hold, it may be helpful to draw the corresponding Euler diagrams.

Let us show for instance that $A \cap (B \cup C) \subset (A \cap B) \cup (A \cap C)$. To do this, let $a \in A \cap (B \cup C)$. This means that $a \in A$ and $a \in B \cup C$. The latter of these means that $a \in B$ or $a \in C$. Together, this says that either $a \in A$ and $a \in B$, or $a \in A$ and $a \in C$. Written in set notation, this says that $a \in (A \cap B) \cup (A \cap C)$.

Let us also show that $X \setminus \bigcup_{i \in I} A_i \subset \bigcap_{i \in I} X \setminus A_i$. That is, let $a \in X \setminus \bigcup_{i \in I} A_i$. This says that $a \notin \bigcup_{i \in I} A_i$. From this we conclude that a is not in any of the A_i (since otherwise a would be in their union), or in symbols, $\forall i \in I$, we have $x \notin A_i$. That is, $\forall i \in I$, we have $x \in X \setminus A_i$, but this exactly means that $x \in \bigcap_{i\in I} X \setminus A_i$.

We leave the remaining six directions for the reader. □

1.4 Cartesian product

Another way of constructing new sets from old sets is through the so-called Cartesian product, often simply called a product. In words, if A and B are two sets, then the *Cartesian product* (sv: *kartesisk produkt*) $A \times B$ is the set of all pairs (a, b), where $a \in A$, or $b \in B$; that is,

$$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}.$$

Be aware that (a, b) is occasionally used as the notation for intervals of real numbers, but this is something completely different.

Example 1.6. One way of defining $\mathbb{R}^2$ is simply as $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$.

Just as for unions and intersections, we would like to be able to talk about infinite products. A little bit of care needs to be taken when defining these. For instance, what is $X \times Y \times Z$? There are two natural definitions: $(X \times Y) \times Z$ and $X \times (Y \times Z)$. The first set consists of elements of the form $((x, y), z)$, while the second one consists of elements of the form $(x, (y, z))$. Clearly, we should be able to think of any of these as simply triples (x, y, z).

To make this precise for an infinite number of sets, consider a family $\{X_i\}_{i\in I}$. The infinite product should then consist of tuples $(x_i)_{i\in I}$ with $x_i \in X_i$ for all i. Such a tuple we can also view as a function $x : I \to \bigcup_{i\in I} X_i$ such that $x(i) \in X_i$ for all i. This brings us to the following.

Definition 1.7. The Cartesian product of a family $\{X_i\}_{i\in I}$ is the set

$$\prod_{i\in I} X_i = \left\{ x : I \to \bigcup_{i\in I} X_i \;\middle|\; x(i) \in X_i \ \forall i \in I \right\}.$$

1.5 Relations

Above, we have talked about various relations between sets; these we will now make mathematically precise. A *binary relation* (sv: *binär relation*) C, often simply called a *relation* (sv: *relation*), on a set A is a subset $C \subset A \times A$. When $(x, y) \in C$, we will often write xCy.

Example 1.8. The subset of $\mathbb{R} \times \mathbb{R}$ given by $C = \{(x, y) \mid x \leq y\}$ is a relation on $\mathbb{R}$, and xCy if and only if $x \leq y$.

Definition 1.9. A relation $C \subseteq A \times A$ is called

- *reflexive* (sv: *reflexiv*) if xCx for all $x \in A$,
- *symmetric* (sv: *symmetrisk*) if xCy implies that yCx for all $x, y \in A$,

- *anti-symmetric* (sv: *antisymmetrisk*) if xCy and yCx implies that $x = y$ for all $x, y \in A$,
- *transitive* (sv: *transitiv*) if xCy and yCz implies that xCz for all $x, y, z \in A$,
- *total* (sv: *total*) if either xCy or yCx when $x, y \in A$.

Example 1.10. The relation $\leq$ from Example 1.8 is reflexive, anti-symmetric, transitive, and total, but it is not symmetric.

Definition 1.11. A relation C on a set A is called a *partial order* (sv: *partiell ordning*) if it is reflexive, anti-symmetric, and transitive. The pair (A, C) is called a *poset* (sv: *pomängd*). If the partial order relation is also total, then it is called a *total order*, and (A, C) is called a *totally ordered set*

We will often denote partial orders by the symbol $\preceq$.

Definition 1.12. An *equivalence relation* (sv: *ekvivalensrelation*) is a relation which is reflexive, symmetric, and transitive.

When C is an equivalence relation, we will use the notation $x \sim y$ for xCy and say that x is equivalent to y.

Example 1.13. Fix a positive integer $p \in \mathbb{N}$, and let $C \subset \mathbb{Z} \times \mathbb{Z}$ be the subset of pairs (m, n) such that $m - n$ is a multiple of p, i.e. $m - n = dp$ for some $d \in \mathbb{Z}$. This is an equivalence relation.

Given any equivalence relation on a set A, it is possible to partition A into smaller sets consisting of elements that are equivalent to each other. More precisely, for $x \in A$, let

$$[x] = \{y \mid y \sim x\}$$

be the so-called *equivalence class* (sv: *ekvivalensklass*) of x. Note that $x \sim x$ by reflexivity so $x \in [x]$ for all $x \in A$.

Lemma 1.14. *Let $\sim$ denote an equivalence relation on a set A. For two elements $x, x' \in A$, the equivalence classes $[x]$ and $[x']$ are either disjoint or equal.*

Proof. Suppose that $[x]$ and $[x']$ are not disjoint and let us show that they must then be equal. That is, let $y \in [x]$ be arbitrary, and let us show that $y \in [x']$. Since $[x]$ and $[x']$ are not disjoint, there is a $z \in A$ such that both $z \in [x]$ and $z \in [x']$. That is, $z \sim x$, and $z \sim x'$. Since $y \in [x]$ we have $y \sim x$, so by symmetry, $x \sim y$, and by transitivity, $z \sim y$. Thus by symmetry, $y \sim z$, and by transitivity $y \sim x'$, but this says that $y \in [x']$. This shows that $[x] \subset [x']$. By the exact same argument one shows that $[x'] \subset [x]$ so that $[x] = [x']$. □

The set of equivalence classes on a set A with respect to an equivalence relation $\sim$ will be denoted $A/\sim$. That is,

$$A/\sim = \{[x] \mid x \in A\}.$$

Example 1.15. Consider the relation $\sim$ from Example 1.13. The equivalence class of an integer $n \in \mathbb{Z}$ is the set of integers

$$[n] = \{\dots, n-2p, n-p, n, n+p, n+2p, \dots\},$$

and we can write $\mathbb{Z}$ as the union of p equivalence classes, as

$$\mathbb{Z} = [0] \cup [1] \cup [2] \cup \cdots \cup [p-1].$$

Similarly,

$$\mathbb{Z}/\!\sim = \{[0], [1], \dots, [p-1]\}.$$

2 Topological spaces

We now turn to the definition of the objects that will be the most interesting to us: topological spaces.

2.1 Definitions and first examples

Definition 2.1. Let X be a set, and let $\mathcal{T} \subset \mathcal{P}(X)$ be a collection of subsets of X. Then $\mathcal{T}$ is called a *topology* (sv: *topologi*) if

(T1) $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$,

(T2) arbitrary unions of elements of $\mathcal{T}$ are once again elements of $\mathcal{T}$; in symbols, if $U_i \in \mathcal{T}$ for $i \in I$, then $\bigcup_{i \in I} U_i \in \mathcal{T}$, and

(T3) *finite* intersections of elements of $\mathcal{T}$ are again elements of $\mathcal{T}$. That is, if $U_1, \dots, U_n \in \mathcal{T}$, then $U_1 \cap U_2 \cap \cdots \cap U_n \in \mathcal{T}$.

If $\mathcal{T}$ is a topology on X, then the pair $(X, \mathcal{T})$ is called a *topological space* (sv: *topologiskt rum*). A set $U \in \mathcal{T}$ is called *open* (sv: *öppen*).

Note that we will often say that X is a topological space when we mean that $(X, \mathcal{T})$ is a topological space. This can be a bit misleading since as the following example shows, a set X might have many different topologies.

Example 2.2. Let $X = \{a, b\}$ be a set containing two elements a and b. Then each of the four following subsets of $\mathcal{P}(X)$ define topologies on X:

$$\begin{aligned}
\mathcal{T}_1 &= \{\emptyset, X\}, \\
\mathcal{T}_2 &= \{\emptyset, \{a\}, X\}, \\
\mathcal{T}_3 &= \{\emptyset, \{b\}, X\}, \\
\mathcal{T}_4 &= \{\emptyset, \{a\}, \{b\}, X\}.
\end{aligned}$$

That is, X is a topological space in at least four different ways. In fact, there are 12 other ways to pick out subsets of $\mathcal{P}(X)$ but it turns out that these four are the only ones that are topologies. (Rather abstractly, topologies are themselves elements in $\mathcal{P}(\mathcal{P}(X))$, which in this case consists of $2^{2^2} = 16$ elements.)

Example 2.3. In fact, any set X can be given a topology in at least two natural ways:

- Let $\mathcal{T} = \{\emptyset, X\} \subset \mathcal{P}(X)$. Then $\mathcal{T}$ is a topology, which is refered to as *the trivial topology* (sv: *den triviala topologin*).
- Let $\mathcal{T} = \mathcal{P}(X)$ itself. Then $\mathcal{T}$ is a topology called *the discrete topology* (sv: *den diskreta topologin*).

Definition 2.4. Let X be a set, and let $\mathcal{T}$ and $\mathcal{T}'$ be topologies on X. If $\mathcal{T} \subset \mathcal{T}'$ then we say that $\mathcal{T}$ is *coarser* (sv: *grövre*) than $\mathcal{T}'$, and that $\mathcal{T}'$ is *finer* (sv: *finare*) than $\mathcal{T}$. If $\mathcal{T} \subsetneq \mathcal{T}'$, we say that $\mathcal{T}$ is *strictly coarser* than $\mathcal{T}'$, and that $\mathcal{T}'$ is *strictly finer* than $\mathcal{T}$. If either $\mathcal{T} \subset \mathcal{T}'$ or $\mathcal{T}' \subset \mathcal{T}$, we say that $\mathcal{T}$ and $\mathcal{T}'$ are *comparable*.

Example 2.5. In Example 2.2, $\mathcal{T}_2$ is strictly coarser than $\mathcal{T}_4$, but $\mathcal{T}_2$ and $\mathcal{T}_3$ are not comparable. The trivial topology on a set is always coarser than the discrete topology, since $\{\emptyset, X\} \subset \mathcal{P}(X)$.

Definition 2.6. A subset $A \subset X$ of a topological space is called *closed* (sv: *sluten*) if A^c is open.

Proposition 2.7. *In a topological space X,*

(T1') $\emptyset$ and X are closed,

(T2') if C_i are closed for $i \in I$, then $\bigcap_{i \in I} C_i$ is closed, and

(T3') if $C_1, \dots, C_n$ are closed, then $C_1 \cup \cdots \cup C_n$ is closed.

Remark 2.8. If one has taken a course on measure theory, the definition of a topological space will look familiar: the σ-algebras appearing for measurable spaces are defined to have particular properties under union, complement, and closure, not unlike topological spaces, but be aware that the two notions are not the same, even though the level of abstraction required to work with them is. However, one could consider the smallest σ-algebra such that all open sets are measurable (obtaining the so-called Borel sets) and thus turn any topological space into a measurable space in a natural manner. This idea of using some measurable sets to generate a full σ-algebra is what we will mimic in the following section.

2.2 Basis for a topology

For a given topological space $(X, \mathcal{T})$ it can often be a bit clumsy to describe *all* open sets. What one does instead is to describe a certain collection of sets that one wants in $\mathcal{T}$, and then includes the sets necessary to obtain a full topology, using the rules (T1)–(T3). This idea is contained in what is called a basis for a topology.

Definition 2.9. Let X be a set, and let $\mathcal{B} \subseteq \mathcal{P}(X)$ be any collection of subsets of X. Then $\mathcal{B}$ is called a *basis* (sv: *bas*) for a topology on X if

(B1) for each $x \in X$, there is a $B \in \mathcal{B}$ such that $x \in B$, and

(B2) if $x \in B_1 \cap B_2$ for $B_1, B_2 \in \mathcal{B}$, then there is a $B_3 \in \mathcal{B}$ such that $x \in B_3 \subset B_1 \cap B_2$.

If $\mathcal{B}$ is a basis, we define $\mathcal{T}_{\mathcal{B}}$, the *topology generated by* (sv: *topologin genererad av*) $\mathcal{B}$ by declaring that $U \in \mathcal{T}_{\mathcal{B}}$ if for every $x \in U$, there is a basis element $B \in \mathcal{B}$ such that $x \in B \subset U$. At first, the condition (B2) might look a little odd but it plays a very explicit role in the proof of the following lemma.

Lemma 2.10. *This collection $\mathcal{T}_{\mathcal{B}} \subset \mathcal{P}(X)$ is a topology.*

Proof. Let us show that $\mathcal{T}_{\mathcal{B}}$ satisfies the properties (T1)–(T3) for a topology.

Notice first that $\emptyset \in \mathcal{T}_{\mathcal{B}}$: a set is in $\mathcal{T}_{\mathcal{B}}$ if all of its elements satisfy a certain condition, but $\emptyset$ contains no elements at all, so the condition is automatically satisfied for all its elements.

That $X \in \mathcal{T}_{\mathcal{B}}$ is exactly (B1). This shows (T1).

To see (T2), let $U_i \in \mathcal{T}_{\mathcal{B}}$ for $i \in I$ and let $x \in \bigcup_{i \in I} U_i$. Then there exists an $i \in I$ so that $x \in U_i$, and since $U_i \in \mathcal{T}_{\mathcal{B}}$ we get a basis element $B \in \mathcal{B}$ so that $x \in B \subset U_i \subset \bigcup_{i \in I} U_i$. But this says exactly that $\bigcup_{i \in I} U_i \in \mathcal{T}_{\mathcal{B}}$, so this shows (T2).

Finally, to see (T3), let us first show that $U_1 \cap U_2 \in \mathcal{T}_{\mathcal{B}}$ whenever $U_1, U_2 \in \mathcal{T}_{\mathcal{B}}$. To do this, let $x \in U_1 \cap U_2$. Then $x \in U_1$ and $x \in U_2$, so we get sets $B_1, B_2 \in \mathcal{B}$ so that $x \in B_1 \subseteq U_1$ and $x \in B_2 \subseteq U_2$. Now, by (B2) we get a set $B_3 \in \mathcal{B}$ so that $x \in B_3 \subset B_1 \cap B_2$. Now clearly, $B_1 \cap B_2 \subset U_1 \cap U_2$ so that we have $x \in B_3 \subset U_1 \cap U_2$, or, in other words, that $U_1 \cap U_2 \in \mathcal{T}_{\mathcal{B}}$.

Finally, let $U_1, \dots, U_n \in \mathcal{T}_{\mathcal{B}}$. Now (T3) follows by induction: if $U_1 \cap \dots \cap U_{n-1} \in \mathcal{T}_{\mathcal{B}}$, then also $U_1 \cap \dots \cap U_n \in \mathcal{T}_{\mathcal{B}}$ since

$$U_1 \cap \dots \cap U_n = (U_1 \cap \dots \cap U_{n-1}) \cap U_n$$

and we now how to handle intersections of only two sets. $\square$

The above proof shows very clearly why we need the condition (B2) in the definition of a basis.

The following result gives what might be an easier way to think about $\mathcal{T}_{\mathcal{B}}$.

Lemma 2.11. *Let $\mathcal{B}$ be the basis for a topology on a set X. Then $U \in \mathcal{T}_{\mathcal{B}}$ if and only if $U = \bigcup_{i \in I} B_i$ for some sets $B_i \in \mathcal{B}$. That is, $\mathcal{T}_{\mathcal{B}}$ consists of all unions of elements from $\mathcal{B}$.*

Proof. First of all, notice that $\emptyset$ is the empty union by convention, so we may assume that U is non-empty.

There are two things to show. First let $U = \bigcup_{i \in I} B_i$ for $B_i \in \mathcal{B}$, and let $x \in U$. Then there is an $i \in I$ so that $x \in B_i \subset U$. This shows that $U \in \mathcal{T}_{\mathcal{B}}$.

On the other hand, let $U \in \mathcal{T}_{\mathcal{B}}$, and let us see that U is a union of basis elements B_i. For every $x \in U$, choose a basis element B_x so that $x \in B_x \subset U$. This is possible since $U \in \mathcal{T}_{\mathcal{B}}$. We now claim that $U = \bigcup_{x \in U} B_x$ which would complete our proof.

To see this, let $y \in U$ be arbitrary. Then $y \in B_y$ and $B_y \subset \bigcup_{x \in U} B_x$, so y is an element of the union. On the other hand, if $y \in \bigcup_{x \in U} B_x$, then there exists a $z \in U$ so that $y \in B_z$, but by our choices of the basis elements, we have that $B_z \subset U$, so $y \in B_z \subset U$. $\square$

While bases are interesting because they allow us to define topologies with less data that we would normally need, we can also go the other way and define a basis that generates a *given* topology; a general way of doing so is the following:

Lemma 2.12. *Let $(X, \mathcal{T})$ be a topological space. Let $\mathcal{C} \subset \mathcal{T}$ be a collection of open sets on X with the following property: for each set $U \in \mathcal{T}$ and each $x \in U$ there is a $C \in \mathcal{C}$ so that $x \in C \subset U$. Then $\mathcal{C}$ is a basis for $\mathcal{T}$.*

Proof. We first show that $\mathcal{C}$ is a basis by showing that it satisfies (B1) and (B2). To see (B1), let $x \in X$. Since $X \in \mathcal{T}$ by (T1) we get a $C \in \mathcal{C}$ so that $x \in C \subset X$ by assumption, so this in particular shows (B1).

Now let $x \in C_1 \cap C_2$ for $C_1, C_2 \in \mathcal{C}$. Since the sets C_1 and C_2 are open by assumption, so is $C_1 \cap C_2$. Therefore we get a $C \in \mathcal{C}$ so that $x \in C \subset C_1 \cap C_2$, which shows (B2).

We now need to show that the topology $\mathcal{T}_{\mathcal{C}}$ that $\mathcal{C}$ generates is actually $\mathcal{T}$. First we show that $\mathcal{T} \subset \mathcal{T}_{\mathcal{C}}$, so let $U \in \mathcal{T}$. Then for any $x \in \mathcal{T}$ we can find a $C \in \mathcal{C}$ so that $x \in C \subset U$ but this is exactly the condition that $U \in \mathcal{T}_{\mathcal{C}}$. On the other hand, if $U \in \mathcal{T}_{\mathcal{C}}$ we know from Lemma 2.11 that U is a union of elements of $\mathcal{C}$. Since $\mathcal{C} \subset \mathcal{T}$ it follows from (T2), applied to $\mathcal{T}$, that $U \in \mathcal{T}$. □

Example 2.13. If $X = \{a, b\}$, then $\mathcal{B} = \{\{a\}, \{b\}\}$ is a basis for a topology on X. The topology $\mathcal{T}_{\mathcal{B}}$ is exactly the discrete topology, $\mathcal{T}_{\mathcal{B}} = \mathcal{P}(X)$. More generally, let X be any set, and let $\mathcal{B}$ consist of those sets that contain only a single element, that is

$$\mathcal{B} = \{\{x\} \mid x \in X\}.$$

Then $\mathcal{B}$ is a basis for a topology, and $\mathcal{T}_{\mathcal{B}}$ is the discrete topology: clearly, every set U in X is a union of sets from this collection since $U = \bigcup_{x \in U} \{x\}$, so it follows that $\mathcal{T}_{\mathcal{B}}$ consists of all subsets of X.

So far, we have been dealing with abstract sets and topological spaces, but at the end of the day, we will be interested in particular topologies on concrete spaces, so at this point, let us use the notion of a basis for a topology to show how we can easily describe a topology on $\mathbb{R}^n$ that agrees with the one we know from analysis.

For $x \in \mathbb{R}^n$ and $r > 0$, let

$$B(x, r) = \{y \in \mathbb{R}^n \mid \|x - y\| < r\}$$

be the open ball centered in x with radius r.

Proposition 2.14. *The collection*

$$\mathcal{B} = \{B(x, r) \mid x \in \mathbb{R}^n, r > 0\}$$

is the basis for a topology on $\mathbb{R}^n$. The resulting topology $\mathcal{T}_{\mathcal{B}}$ is called the standard topology and its open sets are exactly the open sets that one will have encountered in a course on analysis or calculus.

This result will follow from the more general Proposition 2.20 below. While the standard topology is the most interesting one to consider, below we introduce certain other topologies on $\mathbb{R}$.

The following result allows us to compare the topologies generated by bases if we know how to compare the bases. Notice that so far, all the proofs are similar in spirit: the spaces in question are so abstract and have so little structure that one is forced to use the few things that one actually knows about the spaces.

Lemma 2.15. *Let X be a set, and let $\mathcal{B}$ and $\mathcal{B}'$ be bases for topologies $\mathcal{T}$ and $\mathcal{T}'$ respectively; both on X. Then the following are equivalent:*

(1) The topology $\mathcal{T}'$ is finer than $\mathcal{T}$.

(2) For every $x \in X$ and each basis element $B \in \mathcal{B}$ satisfying $x \in B$, there is a basis element $B' \in \mathcal{B}'$ so that $x \in B' \subset B$.

Proof. Assume that $\mathcal{T} \subset \mathcal{T}'$, let x in X, and let $B \in \mathcal{B}$ satisfy $x \in B$. Then since $B \in \mathcal{T}$ (notice that basis elements are always themselves in the topology they generate), it follows that $B \in \mathcal{T}'$. By definition of the topology generated by a basis, this means that there is a basis element $B' \in \mathcal{B}'$ so that $x \in B' \subset B$, which shows one direction.

Assume for the converse that (2) holds, let $U \in \mathcal{T}$, and let $x \in U$ be any element. Then there is a $B \in \mathcal{B}$ with $x \in B \subset U$, and by (2) we get $B' \in \mathcal{B}'$ with $x \in B' \subset B \subset U$, but by definition of $\mathcal{T}'$, this says that $U \in \mathcal{T}'$, so $\mathcal{T} \subset \mathcal{T}'$, which shows (1). □

Example 2.16. We can define a basis for a topology on $\mathbb{R}$ by letting $\mathcal{B}_l$ consist of all sets of the form

$$\{x \in \mathbb{R} \mid a \leq x < b\},$$

where $a, b \in \mathbb{R}$ vary. The topology $\mathcal{T}_l$ generated by $\mathcal{B}_l$ is called the *lower limit topology* (sv: *?*) on $\mathbb{R}$, and we write $\mathbb{R}_l = (\mathbb{R}, \mathcal{T}_l)$.

Example 2.17. Let $K = \{1/n \mid n \in \mathbb{N}\} \subset \mathbb{R}$ and let $\mathcal{B}_K$ consist of all open intervals as well as all sets of the form $(a, b) \setminus K$. Then $\mathcal{B}_K$ is a basis and the topology $\mathcal{T}_K$ that it generates is called the *K-topology* (sv: *K-topologin*) on $\mathbb{R}$. We write $\mathbb{R}_K = (\mathbb{R}, \mathcal{T}_K)$.

So, at this point we have introduced three different topologies on $\mathbb{R}$ and we can now use our results above to compare them.

Lemma 2.18. *The topologies $\mathbb{R}_l$ and $\mathbb{R}_K$ are both strictly finer than the standard topology but are not comparable with each other.*

Proof. We first show that the topology on $\mathbb{R}_l$ is strictly finer than the standard topology. Let $x \in \mathbb{R}$. Let (a, b) be an interval containing x – that is, one of the basis elements for the standard topology. Then $[x, b) \subset (a, b)$ and it follows from Lemma 2.15 that the topology on $\mathbb{R}_l$ is finer than the standard topology. It is strictly finer because $[x, b)$ is open in $\mathbb{R}_l$ but not in the standard topology: There is no open interval B so that $x \in B \subset [x, b)$.

Similarly for $\mathbb{R}_K$: Let $x \in \mathbb{R}$ and let (a, b) contain x. Then this interval itself belongs to $\mathcal{B}_K$ so by Lemma 2.15 we have that the topology on $\mathbb{R}_K$ is finer than the standard topology. To see that it is strictly finer, consider the set $U = (-1, 1) \setminus K \in \mathcal{T}_K$. Then $0 \in U$ but there is no open interval B so that $0 \in B \subset U$.

Finally, one can show that $U \in \mathcal{T}_K$ but $U \notin \mathcal{T}_l$, and that $[1, 2) \in \mathcal{T}_l$ but $[1, 2) \notin \mathcal{T}_K$. $\square$

2.3 Metric spaces

Roughly speaking, metric spaces are spaces where one can always measure distances between two points. This makes them a generalisation of $\mathbb{R}^n$ and in this section we will see that they are special cases of topological spaces. That is, that having a notion of distance is sufficient to obtain a notion of open sets.

Definition 2.19. A *metric space* (sv: *metriskt rum*) (X, d) is a set X together with a non-negative function $d : X \times X \to \mathbb{R}_{\geq 0}$ satisfying for all $x, y, z \in X$ that

(M1) $d(x, y) = 0$ if and only if $x = y$,

(M2) $d(x, y) = d(y, x)$, and

(M3) the triangle inequality $d(x, z) \leq d(x, y) + d(y, z)$.

The function d is called a *metric* (sv: *metrik*), and $d(x, y)$ is called the *distance* (sv: *avstånd*) from x to y.

Having a metric is sufficient to mimic the definition of open balls that we know for $\mathbb{R}^n$. More precisely, for a metric space (X, d) the open *ball* (sv: *boll*) $B_d(x, r)$ centered at x, with radius $r > 0$, with respect to the metric d is defined as

$$B_d(x, r) = \{y \in X \mid d(x, y) < r\}.$$

We will now show how to use the open balls to define a topology, called *the metric topology* (sv: *den metriska topologin*), on any metric space. As promised, this includes Proposition 2.14 as a special case.

Proposition 2.20. *If (X, d) is a metric space, then the collection*

$$\mathcal{B} = \{B_d(x, r) \mid x \in X, r > 0\}$$

is a basis for a topology.

Proof. We need to show that $\mathcal{B}$ satisfies (B1) and (B2). Firstly, (B1) follows since $x \in B_d(x, r)$ for any $r > 0$.

To see (B2), let $x \in B_d(y_1, r_1) \cap B_d(y_2, r_2)$ and let us show that there is a $r > 0$ so that

$$B_d(x, r) \subset B_d(y_1, r_1) \cap B_d(y_2, r_2) \tag{1}$$

Drawing the situation in $\mathbb{R}^2$ one sees that the existence of this r is rather reasonable, and that a good guess would be

$$r = \min(r_1 - d(x, y_1), r_2 - d(x, y_2)),$$

so let us check that this (1) holds with this choice of r. Let $z \in B_d(x,r)$ and let us show that $z \in B_d(y_1, r_1)$ and $z \in B_d(y_2, r_2)$. This follows from (M3) as

$$d(z, y_i) \leq d(z, x) + d(x, y_i) < r + d(x, y_i) < r_i$$

for $i = 1, 2$. $\square$

Remark 2.21. One can show from the definition of the induced topology, that a set U is open in the metric topology if and only if for every point $x \in U$ there is an $r > 0$ so that $B_d(x, r) \subset U$, that is, for the case of $\mathbb{R}^n$, we recover the usual condition for a set to be open.

To see this, suppose that U is open in the metric topology, and let $x \in U$. Since the topology is induced by the basis of open balls, there exists an open ball $B_d(y, \varepsilon)$ so that $x \in B_d(y, \varepsilon) \subset U$. By setting $r = \varepsilon - d(x, y) > 0$ we see that

$$x \in B_d(x, r) \subset B_d(y, \varepsilon) \subset U.$$

Likewise, the other direction follows from the definition of the topology as induced by the basis of open balls.

Example 2.22. As already alluded to above, Euclidean space $\mathbb{R}^n$ is a metric space with metric $d(x, y) = \|x - y\|$.

Example 2.23. Let X be any set. Then we can define a metric on X by

$$d(x, y) = \begin{cases} 0, & \text{if } x = y, \\ 1, & \text{if } x \neq y. \end{cases}$$

The topology induced by this metric is the discrete topology. This follows almost directly from Example 2.13; let us describe the collection of open balls. Let $x \in X$ be arbitrary. If $r \leq 1$, then $B_d(x, r) = \{x\}$ while if $r > 1$ then $B_d(x, r) = X$. Thus the basis of open balls is

$$\mathcal{B} = \{\{x\} \mid x \in X\} \cup \{X\},$$

and as previously, any set $U \subset X$ is the union of sets that are elements of $\mathcal{B}$.

The metric d is often called the *discrete metric* (sv: *diskreta metriken*).

2.4 Continuous functions

As mentioned in the introduction, having the data of open sets turns out to be sufficient to define continuous functions. Recall that if $f : X \to Y$ is a function between two sets, and $A \subset Y$ is a subset, then we define the *preimage* (sv: *urbild*) of A to be the set

$$f^{-1}(A) = \{x \in X \mid f(x) \in A\}.$$

Be aware that the notation f^{-1} is often used for the inverse of an invertible function, but one does *not* need a function to be invertible to talk about preimages.

Proposition 2.24. *The preimage behaves nicely with respect to various operations of sets. In particular, if $f : X \to Y$ and $\{B_i\}_{i \in I}$ is a family of subsets of Y, then*

$$f^{-1}\left(\bigcup_{i \in I} B_i\right) = \bigcup_{i \in I} f^{-1}(B_i), \quad f^{-1}\left(\bigcap_{i \in I} B_i\right) = \bigcap_{i \in I} f^{-1}(B_i).$$

If $B \subset Y$, then $f^{-1}(B^c) = f^{-1}(B)^c$, and if $g : Y \to Z$ is another map and $C \subset Z$, then

$$(g \circ f)^{-1}(C) = f^{-1}(g^{-1}(C)).$$

Definition 2.25. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. A function $f : X \to Y$ is called *continuous* (sv: *kontinuerlig*) if $f^{-1}(U) \in \mathcal{T}_X$ for all $U \in \mathcal{T}_Y$, or in words, if the preimages of open sets are open.

A function $f : X \to Y$ is called *continuous at a point* $x \in X$ if for every $U \in \mathcal{T}_Y$ with $f(x) \in U$ there is a $V \in \mathcal{T}_X$ so that $x \in V$ and $f(V) \subset U$.

Example 2.26. Let X be a topological space. Then the identity map $\mathrm{id} : X \to X$ is continuous since $\mathrm{id}^{-1}(U) = U$ for every subset $U \subset X$.

Example 2.27. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces, and let $y \in Y$. Then the constant map $f : X \to Y$, $f(x) = y$ for all x, is continuous. To see this, let $U \in \mathcal{T}_Y$ and let us consider two cases: if $y \in U$, then $f^{-1}(U) = X$ which is open, and if $y \notin U$, then $f^{-1}(U) = \emptyset$, which is also open.

Example 2.28. Let X have the discrete topology, and let Y be any topological space. Then any map $f : X \to Y$ is continuous, since $f^{-1}(U) \in \mathcal{P}(X)$ no matter what U is.

Example 2.29. Let X be any topological space, and let Y have the trivial topology. Then any map $f : X \to Y$ is continuous since $f^{-1}(\emptyset) = \emptyset$, which is open in X, and $f^{-1}(Y) = X$, which is also open.

We will soon have a huge family of examples of functions which are *not* continuous; thus in particular the last two examples show that the notion of "continuity" depends heavily on the topologies on the spaces under consideration.

Theorem 2.30. *The following properties hold for continuous functions:*

(i) If $f : X \to Y$ and $g : Y \to Z$ are continuous, then so is $g \circ f : X \to Z$.

(ii) A function $f : X \to Y$ is continuous if and only if the preimage of any closed set is closed.

(iii) A function $f : X \to Y$ is continuous if and only if it is continuous at x for all $x \in X$.

Proof. To see the first part, let $U \subset Z$ be open in Z. Then since $U \in \mathcal{T}_Z$ and g is continous, $g^{-1}(U) \in \mathcal{T}_Y$, and since f is continuous, we have $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U)) \in \mathcal{T}_X$.

For the second part, suppose first that f is continuous, and let $C \subset Y$ be closed. Then C^c is open, and $f^{-1}(C)^c = f^{-1}(C^c)$ is open. The other direction is similar.

Finally, suppose that f is continuous, let $x \in X$, and let $U \in \mathcal{T}_Y$ with $f(x) \in U$. Then $V = f^{-1}(U)$ is open in X and $f(V) = U$, so f is continuous at x. Suppose on the other hand that f is continuous at x for all $x \in X$, and let $U \in \mathcal{T}_Y$. Assume without loss of generality that $f^{-1}(U)$ is non-empty, and let $x \in f^{-1}(U)$. Then there exists $V_x \in \mathcal{T}_X$ so that $x \in V_x$ and $f(V_x) \subset U$. Now $f^{-1}(U) = \bigcup_{x \in f^{-1}(U)} V_x$ and thus open by (T2) since each V_x is. □

As the reader will likely have encountered the concept of continuity in other contexts, let us now show that these notions actually coincide. For convenience, let us recall what continuity "normally" means.

Definition 2.31. A function $f : \mathbb{R}^n \to \mathbb{R}$ is called *continuous at a point* $x \in \mathbb{R}^n$ if for every $\varepsilon > 0$ there exists is a $\delta > 0$ such that for every $y \in \mathbb{R}^n$ with $\|x - y\| < \delta$, one has $|f(x) - f(y)| < \varepsilon$. A function is called *continuous* if it is continuous in every point.

This definition can be mirrored for general metric spaces by replacing the distances induced by the norms by the metrics.

Theorem 2.32. *Let (X, d_X) and (Y, d_Y) be metric spaces with their induced metric topologies. Then a function $f : X \to Y$ is continuous, in the sense of Definition 2.25 if and only if*

$$\forall x \in X, \forall \varepsilon > 0, \exists \delta > 0 : d_X(x, y) < \delta \Rightarrow d_Y(f(x), f(y)) < \varepsilon.$$

We will prove this theorem in a second, using the following result.

Lemma 2.33. *Let (X, d_X), (Y, d_Y) be metric spaces with the metric topologies. Then a function $f : X \to Y$ is continuous at a point $x \in X$, in the sense of Definition 2.25, if and only if*

$$\forall \varepsilon > 0, \exists \delta > 0 : f(B_{d_X}(x, \delta)) \subset B_{d_Y}(f(x), \varepsilon). \tag{2}$$

Proof. Suppose first that f is continuous at a given point $x \in X$ and let $\varepsilon > 0$. Since $f(x)$ lies in the open set $B_{d_Y}(f(x), \varepsilon)$, there is an open set V in X such that $x \in V$ and $f(V) \subset B_{d_Y}(f(x), \varepsilon)$. By the discussion in Remark 2.21 the openness of V implies that there is a $\delta > 0$ so that $B_{d_X}(x, \delta) \subset V$, and then in particular $f(B_{d_X}(x, \delta)) \subset f(V) \subset B_{d_Y}(f(x), \varepsilon)$.

Suppose now that (2) holds for f and let U be an open set in Y containing $f(x)$. Once again, by Remark 2.21, this implies that there exists an $\varepsilon > 0$ so that $B_{d_Y}(f(x), \varepsilon) \subset U$. By (2) we then get a $\delta > 0$ with $f(B_{d_X}(x, \delta)) \subset B_{d_Y}(f(x), \varepsilon)$, and since $B_{d_X}(x, \delta)$ is open in X and contains x we are done. □

Proof of Theorem 2.32. This follows by combining Lemma 2.33 and the last part of Theorem 2.30. □

3 Constructing topologies

3.1 The subspace topology

Often, we will be dealing with subsets of topological spaces, and we want to be able to consider these subsets as topological spaces in their own right.

Definition 3.1. Let $(X, \mathcal{T})$ be a topological space, and let $Y \subset X$ be any subset of X. Then the collection

$$\mathcal{T}_Y = \{Y \cap U \mid U \in \mathcal{T}\}$$

is called the *the subspace topology* (sv: *underrumstopologin*).

Lemma 3.2. *The collection $\mathcal{T}_Y$ actually defines a topology on Y.*

Proof. Obviously $\emptyset$ and Y are in $\mathcal{T}_Y$, so (T1) holds. Let $\{U_i\}_{i\in I}$ be a family of subsets $U_i \in \mathcal{T}_Y$. That is, for every $i \in I$ there exists a $V_i \in \mathcal{T}$ so that $U_i = Y \cap V_i$. Then by (an infinite version) of De Morgan's law (Proposition 1.5),

$$\bigcup_{i\in I} U_i = \bigcup_{i\in I} Y \cap V_i = Y \cap \bigcup_{i\in I} V_i,$$

and since $\bigcup_{i\in I} V_i \in \mathcal{T}$ by (T2), applied to $\mathcal{T}$, this shows (T2) for $\mathcal{T}_Y$.

Finally, (T3) follows by the other of De Morgan's laws by the exact same logic. □

Equipping Y with the subspace topology, we will call Y a *subspace* of X. If a set U belongs to $\mathcal{T}_Y$ we will often say that U *is open in* Y.

Example 3.3. As a word of warning, a subspace might have open sets that would not be considered open in the full topological space. For instance, let $X = \mathbb{R}$ and $Y = [0, \infty)$. Then the half-open interval $[0, 1)$ is open in Y since $[0, 1) = Y \cap (-1, 1)$, but $[0, 1)$ is not open in X.

Proposition 3.4. *Let $(X, \mathcal{T})$ be a topological space, and let $(Y, \mathcal{T}_Y)$ be a subspace. Then*

(i) the inclusion map $\iota : Y \to X$ given by $\iota(y) = y$ is continuous,

(ii) if Z is a topological space, and $f : X \to Z$ is a continuous map, then the restriction $f|_Y : Y \to Z$ is also continuous, and

(iii) a set $F \subset Y$ is closed in Y if and only if there is a set $G \subset X$ which is closed in X so that $F = Y \cap G$.

Proof. To see (i), let U be open in X. Then $\iota^{-1}(U) = U \cap Y$, which is open in Y by definition, so ι is continuous.

To see (ii), let U be open in Z. Then $f|_Y^{-1}(U) = f^{-1}(U) \cap Y$, which is open in Y since $f^{-1}(U)$ is open in X.

Finally, let us show (iii). Let $F \subset Y$ be closed in Y. This means that $Y \setminus F$ is open in Y so there is an set U which is open in X and $Y \setminus F = Y \cap U$. Let $G = X \setminus U$. Then G is closed in X and

$$F = Y \setminus (Y \setminus F) = Y \setminus (Y \cap U) = (Y \setminus Y) \cup (Y \setminus U) = Y \setminus U = Y \cap (X \setminus U) = Y \cap G.$$

On the other hand, let $G \subset X$ be closed so that $X \setminus G$ is open, and let $F = Y \cap G$. We have to show that F is closed in Y. We know that $Y \cap (X \setminus G)$ is open in Y and find that

$$Y \cap (X \setminus G) = Y \setminus (Y \cap G) = Y \setminus F,$$

so F is closed in Y. ☐

Example 3.5. The subspace topology on $\mathbb{Z} \subset \mathbb{R}$ is the discrete topology on $\mathbb{Z}$: the set $\{n\}$ is open in $\mathbb{Z}$ for any integer n.

On the other hand, the subspace on $\mathbb{Q} \subset \mathbb{R}$ is *not* the discrete topology, essentially because any non-empty open interval in $\mathbb{R}$ contains infinitely many rational numbers.

The following result is often useful for showing that a given function into a subspace are continuous.

Lemma 3.6. *Let X be a topological space and let Y be a subspace with the inclusion $\iota : Y \to X$. Suppose that Z is a topological space and $f : Z \to Y$ a map. Then f is continuous if and only if $\iota \circ f$ is continuous.*

Proof. Exercise 1.7. ☐

In metric spaces, all subsets are automatically metric spaces as one can restrict metrics to subsets. The following result shows that the subspace topology gives the "right" thing in this case.

Proposition 3.7. *If (X, d) is a metric space and $Y \subset X$, then the metric topology induced by the restricted metric $d|_{Y \times Y}$ is exactly the subspace topology on Y.*

Proof. Left to the reader. ☐

Finally, let us show the following result, which says that to check that a function is continuous, it suffices to check it on a collection of open (or closed) sets that together make up the entire space.

Example 3.8. For the familiar topological spaces, this should not be a big surprise: if a function $f : \mathbb{R} \to \mathbb{R}$ is continuous on $(-\infty, 1)$ and $(-1, \infty)$, then it is continuous on all of $(-\infty, \infty)$.

Lemma 3.9 (The pasting lemma)**.** *Let X and Y be topological spaces, and let $U, V \subset X$ be two open subsets such that $X = U \cup V$. Let $f : U \to Y$ and $g : V \to Y$ be two functions so that $f|_{U \cap V} = g|_{U \cap V}$. Then f and g are continuous with respect to the the subspace topologies on U and V if and only if the function $h : X \to Y$ given by*

$$h(x) = \begin{cases} f(x), & \text{if } x \in U, \\ g(x), & \text{if } x \in V, \end{cases}$$

is continuous.

Proof. Notice first of all that h is well-defined since $f|_{U\cap V} = g|_{U\cap V}$. If h is continuous, then so are f and g by Proposition 3.4.

Assume that f and g are continuous and let $W \subset Y$ be open in Y. Choose open sets U' and V' in X so that $f^{-1}(W) = U \cap U'$ and $g^{-1}(W) = V \cap V'$. Now

$$\begin{aligned} h^{-1}(W) &= \{x \in X \mid h(x) \in W\} = \{x \in U \mid h(x) \in W\} \cup \{x \in V \mid h(x) \in W\} \\ &= \{x \in U \mid f(x) \in W\} \cup \{x \in V \mid g(x) \in W\} \\ &= (f^{-1}(W) \cap U) \cup (g^{-1}(W) \cap W) = (U \cap U') \cup (V \cup V'), \end{aligned}$$

which is open in X since we assumed that both U and V were. $\square$

Remark 3.10. Notice that the exact same result would hold if we replaced "open" with "closed" everywhere in the statement and proof.

Notice also that the result would also be true if we replaced U and V with an infinite collection of open (or closed) sets $\{U_i\}_{i\in I}$ so that $X = \bigcup_{i\in I} U_i$.

3.2 The poset and order topologies

Recall from Definition 1.11 the definition of a poset $(X, \preceq)$.

Proposition 3.11. *Let $(X, \preceq)$ be a poset and define for every $a \in X$ a subset*

$$P_a = \{x \in X \mid a \preceq x\}.$$

Then the collection

$$\mathcal{B} = \{P_a \mid a \in X\}$$

is the basis for a topology $\mathcal{T}_{\mathcal{B}}$ called the poset topology *(sv:* pomängdtopologin*) on X.*

Proof. For every $x \in X$ we have that $x \preceq x$ so $x \in P_x$ which implies (B1).

Suppose that $x \in P_a \cap P_b$ for $a, b \in X$. We claim that $x \in P_x \subset P_a \cap P_b$. This holds since we know that $a \preceq x$ and $b \preceq x$ so for any y with $x \preceq y$, transitivity implies that $a \preceq y$ and $b \preceq y$ so $y \in P_a \cap P_b$. $\square$

For a totally ordered set $(X, \preceq)$, define *intervals*

$$\begin{aligned} [a,b] &= \{x \in X \mid a \preceq x \preceq b\}, \\ [a,b) &= \{x \in X \mid a \preceq x \preceq b, x \neq b\}, \\ (a,b] &= \{x \in X \mid a \preceq x \preceq b, x \neq a\}, \\ (a,b) &= \{x \in X \mid a \preceq x \preceq b, x \neq a, x \neq b\}, \\ (-\infty,b] &= \{x \in X \mid x \preceq b\}, \\ (-\infty,b) &= \{x \in X \mid x \preceq b, x \neq b\}, \\ [a,\infty) &= \{x \in X \mid a \preceq x\}, \\ (a,\infty) &= \{x \in X \mid a \preceq x, x \neq a\}. \end{aligned}$$

We say that an element $a_0 \in X$ is the *smallest element* of X if $a_0 \preceq x$ for all $x \in X$, and similarly, we say that b_0 is the *largest element* of X if $x \preceq b_0$ for all $x \in X$. Notice that the word *the* is justified since there can be at most one smallest and one largest element. There need not be any smallest/largest elements at all though, as is evident from the example of the poset $(\mathbb{R}, \leq)$.

Proposition 3.12. *For a totally ordered $(X, \preceq)$ define a collection $\mathcal{B}$ of subsets to consist of*

- *all open intervals (a, b), $a, b \in X$,*
- *all intervals $[a_0, b)$, $b \in X$, if X has a smallest element a_0, and*
- *all intervals $(a, b_0]$, $a \in X$, if X has a largest element b_0.*

Then $\mathcal{B}$ is the basis for a topological $\mathcal{T}_{\mathcal{B}}$ on X, called the order topology *(sv: ?).*

Proof. One needs to check (B1) and (B2) for $\mathcal{B}$. We leave this to the reader. □

Example 3.13. Since $(\mathbb{R}, \leq)$ has no smallest or largest elements, the basis for its order topology simply consists of all open intervals. That is, the order topology is exactly the standard topology on $\mathbb{R}$.

3.3 The product topology

Recall from Section 1.4 how to define, for any collection of sets, their Cartesian product. In this section we will show how to define a topology on a product of topological spaces.

Definition 3.14. Let X be a set. A *subbasis* (sv: *delbas*) $\mathcal{C}$ for a topology on X is a collection of subsets that cover X, meaning that their union is all of X. The topology $\mathcal{T}_{\mathcal{C}}$ generated by $\mathcal{C}$ consists of all unions of all finite intersections of elements in $\mathcal{C}$. It is the coarsest topology containing $\mathcal{C}$ meaning that is has as few open sets as possible while still including the elements in $\mathcal{C}$ as open sets.

Consider now a cartesian product $X = \prod_{i \in I} X_i$ for a family of sets $\{X_i\}_{i \in I}$. For every $i \in I$, there is a natural map $\pi_i : X \to X_i$, called the projection onto X_i, which maps $\pi_i(x) = x(i)$, where here we think of $x \in X$ as a map $I \to \bigcup_{i \in I} X_i$.

Definition 3.15. Let $\{X_i\}_{i \in I}$ be a family of topological spaces, and let $X = \prod_{i \in I} X_i$. We then define a topology on X, called the *product topology* (sv: *produkttopologin*), to be the coarsest topology such that π_i is continuous for every i.

This definition is rather abstract: rather than describing the actual open sets, we have described a property that we would like the topology to have, namely that all the projections be continuous. Spelled out, the topology on the product X is generated by the subbasis $\mathcal{C}$ which consists of all sets of the form $\pi_i^{-1}(U)$, where U is an open set in X_i.

To make this more concrete, let us consider the case of a product of just two topological spaces $(X_1, \mathcal{T}_{X_1})$ and $(X_2, \mathcal{T}_{X_2})$, and let U and V be open sets in X_1 and X_2

respectively. Then $\pi_1^{-1}(U) = U \times X_2$ and $\pi_2^{-1}(V) = X_1 \times V$ are examples of open sets in $X_1 \times X_2$. Their intersection is the set $\pi_1^{-1}(U) \cap \pi_2^{-1}(V) = U \times V$, and the topology on $X_1 \times X_2$ consists of all unions of this form. In symbols, if we let

$$\mathcal{B} = \{U \times V \mid U \in \mathcal{T}_{X_1}, V \in \mathcal{T}_{X_2}\},$$

then $\mathcal{B}$ is a basis for the product topology on $X_1 \times X_2$. Similarly, open sets in $X = \prod_{i \in I} X_i$ are unions of sets of the form $\prod_{i \in I} U_i$, where U_i is open in X_i for each $i \in I$, and $U_i = X_i$ for all but finitely many i (since we take only finite intersections).

Theorem 3.16. *Let X be a topological space, and let $\{Y_i\}_{i \in I}$ be a family of topological spaces. A function $f : X \to \prod_{i \in I} Y_i$ consists of a family of functions $\{f_i\}_{i \in I}$ where $f_i : X \to Y_i$ for all $i \in I$. Then f is continuous if and only if f_i is continuous for every i.*

Proof. Notice first of all that the maps f_i are exactly the compositions $\pi_i \circ f$.

Suppose that f is continuous. Since each π_i is continuous, so is every f_i by Theorem 2.30.

Suppose now that all the f_i are continuous, and let us show that the preimages of elements of the subbasis are open. That is, let U be an open set in $\prod_{i \in I} Y_i$ of the form $U = \pi_i^{-1}(V)$ where V is open in Y_i. Then $f^{-1}(U) = f^{-1}(\pi_i^{-1}(V)) = f_i^{-1}(V)$, which is open by assumption. A general open set is a union of finite intersections of elements from the subbasis, so the general case follows from Proposition 2.24. $\square$

4 Topological spaces up close

4.1 Interior, closure, boundary, and limit points

Definition 4.1. Let $(X, \mathcal{T})$ be a topological space, and let $Y \subset X$ be a subset. Then

(i) the *interior* (sv: *inre*) of Y, denoted $\mathring{Y}$ or $\operatorname{Int} Y$, is the union of all open subsets in Y,

(ii) the *closure* (sv: *slutna höljet*) of Y, denoted $\overline{Y}$, is the intersection of all closed subsets containing Y.

(iii) The subset Y is called *dense* (sv: *tät*) in X if $\overline{Y} = X$.

Notice that $\operatorname{Int} Y$ is open, since it is a union of open sets. Likewise, $\overline{Y}$ is closed by Proposition 2.7, and we have

$$\operatorname{Int} Y \subset Y \subset \overline{Y}.$$

It also follows directly from the definition that Y is open if and only $Y = \operatorname{Int} Y$ and that Y is closed if and only if $Y = \overline{Y}$. The definition also implies that $\operatorname{Int} Y$ is the largest open subset contained in Y, and that $\overline{Y}$ is the smallest closed subset containing Y.

Furthermore, it is useful to note that the complement of an open set contained in Y is a closed set containing Y^c and on the other hand, the complement of a closed set containing Y is an open set contained in Y^c. This implies that

$$\operatorname{Int} Y = X \setminus \overline{(X \setminus Y)}, \quad \overline{Y} = X \setminus \operatorname{Int}(X \setminus Y), \tag{3}$$

which can also be shown more precisely by using De Morgan's laws.

Proposition 4.2. *Let Y and Z be subsets of a topological space X. Then*

(i) $\overline{Y \cup Z} = \overline{Y} \cup \overline{Z}$,

(ii) $\overline{Y \cap Z} \subset \overline{Y} \cap \overline{Z}$,

(iii) $\operatorname{Int} Y \cup \operatorname{Int} Z \subset \operatorname{Int}(Y \cup Z)$, *and*

(iv) $\operatorname{Int} Y \cap \operatorname{Int} Z = \operatorname{Int}(Y \cap Z)$.

Proof. Let us show (i) and (ii); (iii) and (iv) follow by the same logic. First, note that since $Y \subset \overline{Y}$ and $Z \subset \overline{Z}$ we get that $Y \cup Z \subset \overline{Y} \cup \overline{Z}$. This says that $\overline{Y} \cup \overline{Z}$ is a closed subset containing $Y \cup Z$; since $\overline{Y \cup Z}$ is the smallest closed subset containing $Y \cup Z$, this tells us that $\overline{Y \cup Z} \subset \overline{Y} \cup \overline{Z}$.

On the other hand $Y \subset Y \cup Z \subset \overline{Y \cup Z}$ and the latter set is closed so $\overline{Y} \subset \overline{Y \cup Z}$. For the same reason $\overline{Z} \subset \overline{Y \cup Z}$, and this implies that $\overline{Y} \cup \overline{Z} \subset \overline{Y \cup Z}$.

Since $Y \subset \overline{Y}$ and $Z \subset \overline{Z}$ we have $Y \cap Z \subset \overline{Y} \cap \overline{Z}$. The latter set is closed so $\overline{Y \cap Z} \subset \overline{Y} \cap \overline{Z}$. □

For any topological space X, we say that U is a *neighbourhood* (sv: *omgivning*) of a point x if U is open and $x \in U$.

Definition 4.3. Let Y be a subset of a topological space X. Then

(i) the *boundary* (sv: *rand*) of Y, denoted ∂Y is the set

$$\partial Y = \{x \in X \mid U \cap Y \neq \emptyset \text{ and } U \cap Y^c \neq \emptyset \text{ for all neighbourhoods } U \text{ of } x\}.$$

That is, $x \in \partial Y$ if and only if all neighbourhoods of x intersect both Y and Y^c.

(ii) A *limit point* (sv: *gränspunkt*) of Y is a point $x \in X$ with the property that all its neighbourhoods intersect Y in a point which is not x itself. Let

$$Y' = \{x \in X \mid x \text{ is a limit point of } Y\}.$$

Example 4.4. Let $Y = [0, 1) \cup \{2\} \subset \mathbb{R}$ with the standard topology on $\mathbb{R}$. Then $\operatorname{Int} Y = (0, 1)$, $\overline{Y} = [0, 1] \cup \{2\}$, $\partial Y = \{0, 1, 2\}$, and $Y' = [0, 1]$.

Theorem 4.5. *Let X be a topological space and $Y \subset X$ a subset. Then*

(i) $\partial Y = X \setminus (\operatorname{Int} Y \cup \operatorname{Int}(X \setminus Y)) = \overline{Y} \cap \overline{X \setminus Y}$,

(ii) $\overline{Y} = Y \cup \partial Y$, *and*

(iii) $\overline{Y} = Y \cup Y'$.

Proof. The second equality in (i) is obtained directly from (3), so it suffices to show the first equality. Taking complements, this becomes

$$X \setminus \partial Y = \operatorname{Int} Y \cup \operatorname{Int}(X \setminus Y).$$

Let $x \in X \setminus \partial Y$. This means that there is a neighbourhood U of x so that $x \in U \subset Y$ or $x \in U \subset X \setminus Y$. This means that either $x \in \operatorname{Int} Y$ or $x \in \operatorname{Int}(X \setminus Y)$.

Suppose that either $x \in \operatorname{Int} Y$ or $x \in \operatorname{Int}(X \setminus Y)$. If $x \in \operatorname{Int} Y$ then there is an open set U such that $x \in U \subset Y$, and so $U \cap Y^c = \emptyset$, so $x \notin \partial Y$. Similarly, if $x \in \operatorname{Int}(X \setminus Y)$, one gets an open neighbourhood U of x with $U \cap Y = \emptyset$, so once again $x \notin \partial Y$. This shows (i).

From (i) and (3) it follows that

$$Y \cup \partial Y = Y \cup (\overline{Y} \cap \overline{X \setminus Y}) = (Y \cup \overline{Y}) \cap (Y \cup \overline{X \setminus Y}) = \overline{Y} \cap X = \overline{Y},$$

which is (ii).

Finally, we use (ii) and show that $Y \cup \partial Y = Y \cup Y'$. To see this, it suffices to show that $\partial Y \setminus Y = Y' \setminus Y$ as one can then take the union with Y on both sides. So, let $x \in \partial Y \setminus Y$. Then any neighbourhood U of x intersects Y, and since $x \notin Y$, this necessarily means that U intersects Y in a point which is not x itself, so $x \in Y'$, and since $x \notin Y$, we have $x \in Y' \setminus Y$.

On the other hand, if $x \in Y' \setminus Y$, any neighborhood U of x will intersect Y; it will also intersect $X \setminus Y$, since x belongs to that set. This implies that $x \in \partial Y$, and as before, $x \in \partial Y \setminus Y$. This completes the proof. □

The above theorem provides us with the following useful characterisation of the closure: we see that $x \in \overline{Y}$ if and only if every neighbourhood of x intersects Y.

Example 4.6. We claim that $\mathbb{Q}$ is dense in $\mathbb{R}$, i.e. that $\overline{\mathbb{Q}} = \mathbb{R}$. By the above theorem, it suffices to show that $\partial\mathbb{Q} \cup \mathbb{Q} = \mathbb{R}$. To see this, let $x \in \mathbb{R}$ be arbitrary, and let U be any neighbourhood of x. Now, an open set like U is the union of a number of intervals, any interval contains (an infinite number of) both rational and irrational numbers, that is, $U \cap \mathbb{Q} \neq \emptyset$, and $U \cap (\mathbb{R} \setminus \mathbb{Q}) \neq \emptyset$. This is exactly the condition that $x \in \partial\mathbb{Q}$. Notice, not only did we show that $\partial\mathbb{Q} \cup \mathbb{Q} = \mathbb{R}$; we actually see that $\partial\mathbb{Q} = \mathbb{R}$.

Proposition 4.7. *Let $\{X_i\}_{i\in I}$ be a family of topological spaces, and let $A_i \subset X_i$ be subsets of each of them. Then*

$$\prod_{i\in I} \overline{A_i} = \overline{\prod_{i\in I} A_i}.$$

Proof. Let $x = (x_i)_{i\in I} \in \prod_{i\in I} \overline{A_i}$. Let $U = \prod_{i\in I} U_i$ be any of the basis elements for the product topology $x \in U$, i.e. such that $x_i \in U_i$ and such that U_i is open in X_i for every $i \in I$. Since also $x_i \in \overline{A}_i$ for all i, we can choose a $y_i \in U_i \cap A_i$ for all i, so $y = (y_i)_{i\in I} \in U \cap \prod_{i\in I} A_i$. In other words, any neighbourhood U of x contains points from $\prod_{i\in I} A_i$ so $x \in \overline{\prod_{i\in I} A_i}$.

For the converse, suppose that $x \in \overline{\prod_{i\in I} A_i}$. Let $i \in I$ be given, and let V_i be any open set that contains x_i. Then by definition of the product topology, $\pi_i^{-1}(V_i) \subset X$ is a neighbourhood of x and so contains a point $y \in \prod_{i\in I} A_i$. This says that $y_i \in A_i \cap V_i$ so that $x_i \in \overline{A_i}$ and since i was arbitrary, $x \in \prod_{i\in I} \overline{A_i}$. □

4.2 Separation axioms and Hausdorff spaces

In $\mathbb{R}^n$ we have a very good idea of what it means for two points to be separated: if they're different, they're separated. We now ask ourselves the question if its possible to separate points in general topological spaces, using only the data of open sets. Consider for instance the topological space $X = \{a, b, c\}$ with the topology $\mathcal{T} = \{\emptyset, \{a\}, \{b, c\}, X\}$. From the viewpoint of topology, there is no way to tell the points b and c apart using only open sets; we consider the points inseparable.

Now for general topological spaces, it turns out that there are plenty of meaningful ways – called separation axioms – to then actually separate points, some of which lead to more interesting mathematics than others. In this section, we give the first few of these, including the immensely important Hausdorff axiom.

Definition 4.8. A topological space $(X, \mathcal{T})$ is called

(i) T_0 if for every pair $x, y \in X$, $x \neq y$, there exists a neighbourhood of x that does not contain y, *or* there exists a neighbourhood of y that does not contain x.

(ii) T_1 if for every pair $x, y \in X$, $x \neq y$, x has a neighborhood not containing y, *and* y has a neighbourhood not containing x.

(iii) T_2 or *Hausdorff* if for every pair $x, y \in X$, $x \neq y$, there exists neighbourhoods U_x and U_y of x and y respectively so that $U_x \cap U_y \neq \emptyset$.

Note that clearly, a T_2-space is T_1, and a T_1-space is T_0.

Proposition 4.9. *A topological space X is T_1 if and only if $\{x\}$ is closed for all $x \in X$.*

Proof. Suppose first that $\{x\}$ is closed for all $x \in X$, and let $x, y \in X$, $x \neq y$. Then $X \setminus \{x\}$ is a neighbourhood of y that does not contain x, and $X \setminus \{y\}$ is a neighbourhood of x not containing y, so X is T_1.

For the converse, suppose that X is T_1, and let $x \in X$. Now every $y \in X$ has a neighbourhood U_y that does not contain x, and so

$$\bigcup_{y \neq x} U_y = X \setminus \{x\}$$

is open, and $\{x\}$ is closed. □

Example 4.10. Let X contain at least two points, and endow X with the trivial topology. Then X is not T_0 (or T_1 or T_2), since the only neighborhood of a point x is X.

Example 4.11. If X has the discrete topology, then X is Hausdorff (and T_1 and T_0).

Example 4.12. Any poset $(X, \preceq)$ with the poset topology (Proposition 3.11) is T_0: let $x, y \in X$, $x \neq y$, and suppose that $x \preceq y$. Then $P_y = \{z \in X \mid y \preceq z\}$ is a neighbourhood of y which does not contain x.

Example 4.13. Let $X = \{a, b, c\}$ with the topology

$$\mathcal{T} = \{\emptyset, \{b\}, \{c\}, \{b, c\}, X\}.$$

Then X is T_0 but not T_1.

Example 4.14. All metric spaces (with the metric topology) are Hausdorff (Exercise 1.10).

4.3 Sequences and convergence

When one first encounters sequences in calculus and analysis, their convergence is typically worded in an ε-δ-fashion where one considers a sequence as convergent if it gets arbitrarily close to a given limit. Now, in topological spaces we do not have a concept of "distance" to guide our intuition but just as we were able to recover a natural concept of continuity, so are we still able to discuss convergence.

Definition 4.15. Let X be a topological space. A *sequence* (sv: *följd*) in X is a family $\{x_n\}_{n\in\mathbb{N}}$ of points in X. We say that a sequence $\{x_n\}_{n\in\mathbb{N}}$ *converges* (sv: *konvergera*) to a point $x \in X$, if for any neighbourhood U of x, there is an $N \in \mathbb{N}$ so that $x_n \in U$ for all $n > N$, and in this case, we write $x_n \to x$. A *subsequence* (sv: *delföljd*) $\{y_n\}_{n\in\mathbb{N}}$ is a sequence such that $y_i = x_{n_i}$ for some $n_1 < n_2 < \dots$.

From the definition, one immediately obtains the following result.

Proposition 4.16. *If $\{y_n\}$ is a subsequence of $\{x_n\}$ and $x_n \to x$, then $y_n \to x$.*

The next example shows that the concept of convergence depends on the topology of the underlying topological space.

Example 4.17. In the trivial topology, all sequences converge to any given point. In the discrete topology, for a sequence $\{x_n\}$ to converge to a point x, it has to be constantly equal to x for all large enough n.

Example 4.18. The constant sequence is convergent, regardless of the topology on the space.

Let us show that we recover the possibly well-known definition of convergence for metric spaces and in particular for $\mathbb{R}^n$.

Proposition 4.19. *Let (X, d) be a metric space with the metric topology. Then a sequence $\{x_n\}$ in X converges to $x \in X$ if and only if*

$$\forall \varepsilon > 0, \exists N > 0 : n > N \Rightarrow d(x_n, x) < \varepsilon.$$

Proof. Suppose that $x_n \to x$ and let $\varepsilon > 0$ be given. Let $U = B_d(x, \varepsilon)$. Then by definition of convergence, there exists an $N > 0$ so that $x_n \in U$ for all $n > N$, but this says that $d(x_n, x) < \varepsilon$ for all $n > N$.

For the converse, let U be a neighbourhood of x. As we saw in Remark 2.21, this implies that there exists an $\varepsilon > 0$ so that $B_d(x, \varepsilon) \subset U$. Now by assumption there is an $N > 0$ so that $x_n \in B_d(x, \varepsilon) \subset U$ for all $n > N$. □

In spaces like $\mathbb{R}^n$ we are used to convergent sequences only having a single limit, but Example 4.17 above shows that this need not be the case in general. It is, however, true for Hausdorff spaces.

Proposition 4.20. *Let X be Hausdorff. If $x_n \to x$ and $x_n \to y$ in X, then $x = y$.*

Proof. Suppose that $x \neq y$ and assume that $x_n \to y$ and $x_n \to y$. Choose U and V disjoint neighbourhoods of x and y respectively. By definition of convergence, we get $N_U, N_V > 0$ so that $x_n \in U$ for all $n > N_U$ and $x_n \in V$ for all $n > N_V$. For $n > \max(N_U, N_V)$ we therefore have $x_n \in U \cap V = \emptyset$ which is a contradiction. □

In analysis, one might have encountered the important characterization of continuity in $\mathbb{R}^n$ that a function f is continuous if and only $x_n \to x$ implies that $f(x_n) \to f(x)$. It turns out that this is not quite true for general continuous maps between topological spaces without further assumptions.

Definition 4.21. We say that a topological space X has a *countable basis at $x \in X$* if there is a collection of neighbourhoods $\{B_n\}_{n \in \mathbb{N}}$ of x so that if U is any neighbourhood of x there exists an $n \in \mathbb{N}$ so that $B_n \subset U$. The space X is called *first-countable* (sv: *?*) if it has a countable basis at x for all $x \in X$.

Example 4.22. All metric spaces are first-countable by Exercise 1.14.

Lemma 4.23 (The sequence lemma)**.** *Let X be a topological space and let $A \subset X$. If there is a sequence in A that converges to x then $x \in \overline{A}$. The converse holds if X is first-countable.*

Proof. Suppose that $x_n \to x$ and that $x_n \in A$ for all n. If $x \in A$ we are done, so suppose that $x \in X \setminus A$. Let U be a neighbourhood of x; then there is an $N > 0$ so that $x_n \in U$ for all $n > N$. This implies that $U \cap A \neq \emptyset$ and $U \cap X \setminus A \neq \emptyset$, so $x \in \partial A$, and so $x \in \overline{A}$ by Theorem 4.5.

Suppose that X is first-countable. Let $x \in \overline{A}$ and let us show that there is a sequence $\{x_n\}$ in A with $x_n \to x$. Let $\{B_n\}$ be a countable basis at x and define for every $n \in \mathbb{N}$ an open neighbourhood $U_n = \bigcap_{k=1}^n B_k$ of x. Since $x \in A$ or $x \in \partial A$ by Theorem 4.5 it follows that $U_n \cap A \neq \emptyset$ for every n, and we can choose $x_n \in U_n \cap A$ for every n. We claim that $x_n \to x$. To see this, let U be any neighbourhood of x. Then by definition of first-countability there is an $N \in \mathbb{N}$ so that $B_N \subset U$. Now clearly, $U_n \subset U_N \subset B_N$ for all $n > N$ so $x_n \in B_N \subset U$ for all $n > N$ which means that $x_n \to x$. □

Theorem 4.24. *Let X and Y be topological spaces. If $f : X \to Y$ be continuous, then $x_n \to x$ in X implies that $f(x_n) \to f(x)$ in Y. The converse holds if X is first countable; that is, if if $x_n \to x$ implies that $f(x_n) \to f(x)$ for all convergent sequences $\{x_n\}$, then f is continuous.*

Proof. Suppose that f is continuous, let $\{x_n\}$ be a sequence with $x_n \to x$, and let us show that $f(x_n) \to f(x)$. Let U be a neighborhood of $f(x)$. Then $f^{-1}(U)$ is a neighbourhood of x, and we can choose an $N > 0$ so that $x_n \in f^{-1}(U)$ for all $n > N$. Thus $f(x_n) \in U$ for all $n > N$ so $f(x_n) \to f(x)$.

Suppose that X is first-countable and that $f(x_n) \to x$ whenever $x_n \to x$. Let $B \subset Y$ be a closed set, let $A = f^{-1}(B)$, and let us show that $\overline{A} = A$, so that A is closed, which means that f is continuous. Let $x \in \overline{A}$ be arbitrary. Then by Lemma 4.23, there is a sequence $\{x_n\}$ with $x_n \in A$ so that $x_n \to x$. This means that $f(x_n) \in B$, and since $f(x_n) \to f(x)$, Lemma 4.23 tells us that $f(x) \in \overline{B} = B$, so $x \in A$.

□

5 Homeomorphisms and distinguishability

Often in mathematics, when talking about objects as certains things coming with certain structures, we want to be able to say when two objects are "the same". Consider for instance the two topological spaces $X = \{1, 2, 3\}$ and $Y = \{4, 5, 6\}$ with the topologies

$$\mathcal{T}_X = \{\emptyset, \{1\}, \{2\}, \{1, 2\}, \{1, 2, 3\}\},$$
$$\mathcal{T}_Y = \{\emptyset, \{4\}, \{5\}, \{4, 5\}, \{4, 5, 6\}\}.$$

These spaces are not particularly different for if we identify $1 \leftrightarrow 4$, $2 \leftrightarrow 5$, $3 \leftrightarrow 6$, we have no way to tell them apart. This notion of being the same is made precise in the definition of a "homeomorphism" below.

The well-educated mathematics student will have likely come across this general idea before: we consider two vector spaces the same if there is a linear isomorphism from one to the other, we consider algebraic objects such as groups and rings the same if they are isomorphic, and if all we know about two given sets is that they are in bijection, we may as well treat them as the same. This language of "objects" being "the same" is unified in the branch of mathematics called *category theory* (sv: *kategoriteori*), sometimes referred to as *abstract nonsense*. We will not be discussing category theory in any detail in these notes, but it is useful to be aware of its existence.

5.1 Homeomorphisms

In the example above, we notice that crucial property of two topological spaces that "are the same" is that they are in bijection and have the same open sets. This leads to the following definition.

Definition 5.1. A bijection $f : X \to Y$ between two topological spaces is called a *homeomorphism* (sv: *homeomorfi*) if f and its inverse f^{-1} are continuous. In this case, we say that X and Y are *homeomorphic* (sv: *homeomorfa*) and we write $X \simeq Y$.

Equivalently, since a bijection f always satisfies $f = (f^{-1})^{-1}$, one could define a homeomorphism to be a continuous bijection such that $f(U)$ is open whenever U is. Notice also that $\simeq$ satisfies the property of an equivalence relation.

Example 5.2. In the example in the beginning of this section, the bijection $f : X \to Y$ given by $f(1) = 4$, $f(2) = 5$, $f(3) = 6$ is a homeomorphism.

Example 5.3. Let $f : (-1, 1) \to \mathbb{R}$ be the bijective map

$$f(x) = \tan\left(\frac{\pi x}{2}\right)$$

whose inverse is $f^{-1}(x) = \frac{2}{\pi} \arctan x$. Then both f and f^{-1} are continuous so $(-1, 1)$ and $\mathbb{R}$ are isomorphic.

From the above example we conclude that two spaces that we are otherwise familiar with and think of as different may turn out to be the same from the viewpoint of topology. Roughly, since we don't care about the scale of $(-1, 1)$ but only its open sets, we are able to stretch it as much as we please, and end up with something like $\mathbb{R}$

Bad joke 5.4. *Let A be a typical topologist. Then A is not able to tell the difference between her coffee mug and her donut.*

Proof. The surfaces of the coffee mug and the donut are homeomorphic. See https://upload.wikimedia.org/wikipedia/commons/2/26/Mug_and_Torus_morph.gif. □

Example 5.5. Let $B^n := B(0, 1)$ be the unit ball in $\mathbb{R}^n$. Then $B^n \simeq \mathbb{R}^n$. This can be seen because the map $f : B^n \to \mathbb{R}^n$ given by

$$f(x) = \frac{x}{1 - \|x\|}$$

is a continuous bijection with inverse

$$f^{-1}(x) = \frac{x}{1 + \|x\|}.$$

The graph of f in the case $n = 1$ is shown in Figure 3. The case $n = 2$ is illustrated in Figures 15-16.

We will often be interested in functions that would be homeomorphisms if we were allowed to shrink the codomain appropriately.

Definition 5.6. Let X and Y be topological spaces. A function $f : X \to Y$ is called an *embedding* (sv: *?*) if $f : X \to f(X)$ is a homeomorphism; here $f(X)$ has the subspace topology from Y.

Example 5.7. If X is a topological space and $Y \subset X$ a subspace, then the inclusion $\iota : Y \to X$ given by $\iota(x) = x$ is an embedding.

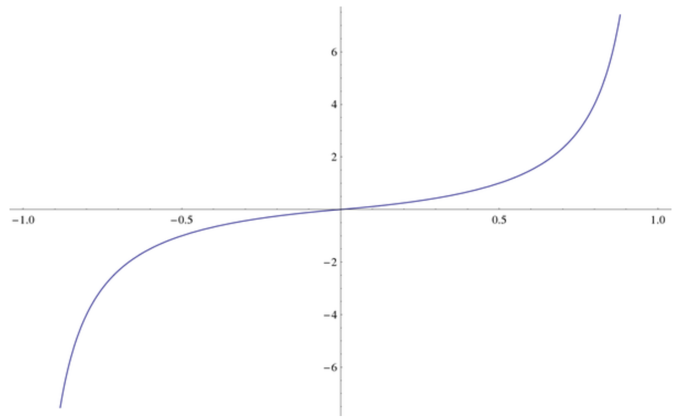

Figure 3: The graph of $f(x) = x/(1 - |x|)$.

5.2 Topological invariants

Above, we have talked about what it means for two topological spaces to be the same. Often, one will be interested in the converse question of telling two topological spaces apart. As such, we consider topological spaces different if they are non-homeomorphic; for instance, if $X = \{a, b\}$ then we obtain two different topological spaces by equipping it with the trivial and the discrete topology.

Definition 5.8. Let **Top** denote the collection of *all* topological spaces. A *topological invariant*, sometimes called a *topological property*, is a function f defined on **Top** so that if $X \simeq Y$, then $f(X) = f(Y)$.

The important thing to note is that if f is a topological invariant and $f(X) \neq f(Y)$, then X and Y are not homeomorphic. Thus we are lucky enough, we can use topological invariants to tell topological spaces apart.

Example 5.9. Let $f : \mathbf{Top} \to \{\text{yes}, \text{no}\}$ be the function given by answering the question "is X Hausdorff?" That is

$$f(X) = \begin{cases} \text{yes}, & \text{if } X \text{ is Hausdorff,} \\ \text{no}, & \text{if } X \text{ is not Hausdorff.} \end{cases}$$

Then f is a topological invariant: if $X \simeq Y$ and X is Hausdorff, then so is Y. For this reason, the property of being Hausdorff is often called a topological property. Again, one can turn this around and say that if X is Hausdorff but Y is not, then X and Y are not homeomorphic. Similarly, being T_0 or T_1 are topological properties. As is being first-countable and so is any other property that is defined using only in terms of open sets.

We will encounter many other topological properties later on, one of the most important ones being the fundamental group, which is to be introduced in Section 10. The reader is encouraged to try to discover these topological properties as we move along.

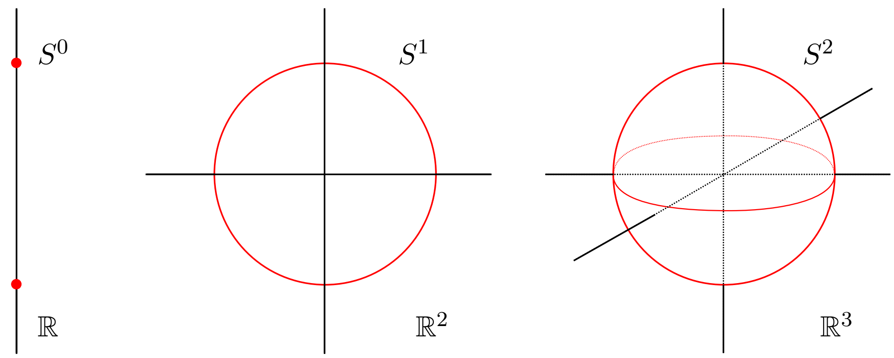

Figure 4: The spheres S^0, S^1, and S^2 in $\mathbb{R}$, $\mathbb{R}^2$, and $\mathbb{R}^3$.

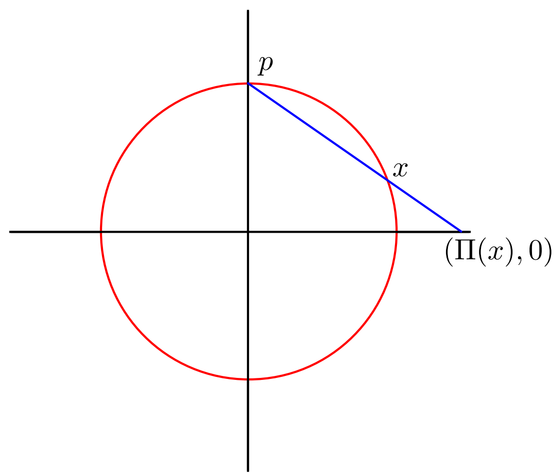

Figure 5: Stereographic projection of S^1.

5.3 The n-dimensional sphere

Definition 5.10. The *n-sphere* is the set

$$S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\} \subset \mathbb{R}^{n+1}$$

with the subspace topology from $\mathbb{R}^{n+1}$ (see Figure 4).

Proposition 5.11. *Let $p = (0, 0, \ldots, 0, 1) \in S^n$ be the "north pole". Then $S^n \setminus \{p\} \simeq \mathbb{R}^n$.*

Proof. We will construct a homeomorphism explicitly, leaving some of the details to the reader. Let $x = (x_1, \ldots, x_{n+1}) \in S^n \setminus \{p\}$ so that $x_{n+1} \neq 1$. We then define the *stereographic projection* (sv: *stereografisk projektion*) of x by

$$\Pi(x) = \Pi(x_1, \ldots, x_n, x_{n+1}) = \frac{1}{1 - x_{n+1}}(x_1, \ldots, x_n) \in \mathbb{R}^n.$$

Geometrically, if one draws a straight line through x and p, then its intersection with $\mathbb{R}^n \times \{0\}$ will be the point $(\Pi(x), 0)$ (see Figure 5 for the case $n = 1$, and the front page for the case $n = 2$). Now Π is continuous because each of its components are (use Proposition 3.4), and one can check that it has an inverse $g : \mathbb{R}^n \to S^n \setminus \{p\}$ given by

$$g(y_1, \ldots, y_n) = (t(y)y_1, \ldots, t(y)y_n, 1 - t(y)),$$

where $t(y) = 2/(1 + \|y\|^2)$. $\square$

Remark 5.12. If $q = (0, \ldots, 0, -1) \in S^n$ is the south pole, then the bijective reflection map $r : (x_1, \ldots, x_n, x_{n+1}) \mapsto (x_1, \ldots, x_n, -x_{n+1})$ is a homeomorphism from $S^n \setminus \{p\}$ to $S^n \setminus \{q\}$, so we also have that $S^n \setminus \{q\}$ is homeomorphic with $\mathbb{R}^n$.

Let us show that $r : S^n \setminus \{p\} \to S^n \setminus \{q\}$ is continuous in detail. First, define $\tilde{r} : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ by

$$\tilde{r}(x_1, \ldots, x_n, x_{n+1}) = (x_1, \ldots, x_n, -x_{n+1}),$$

which is clearly continuous. Let $\iota : S^n \setminus \{q\} \to \mathbb{R}^{n+1}$ denote the inclusion map. Then we have the equality of maps $S^n \setminus \{p\} \to \mathbb{R}^{n+1}$,

$$\tilde{r}|_{S^n \setminus \{p\}} = \iota \circ r.$$

Now $\tilde{r}|_{S^n \setminus \{p\}}$ is continuous by Proposition 3.4 and therefore r is continuous by Lemma 3.6. The same logic applies to show that r^{-1} is continuous so that r is indeed a homeomorphism.

More generally, one can show that $S^n \setminus \{x\} \simeq \mathbb{R}^n$ for all $x \in S^n$.

Example 5.13. Let $f : [0, 1) \to S^1$ be the map $f(x) = (\cos(2\pi x), \sin(2\pi x))$. Then f is a bijection and moreover, just as in the remark above, f is continuous by Proposition 3.4 and Lemma 3.6: consider the map $\tilde{f} : \mathbb{R} \to \mathbb{R}^2$ given by $\tilde{f}(x) = (\cos(2\pi x), \sin(2\pi x))$. This is clearly continuous, so its restriction $\tilde{f}|_{[0,1)} : [0, 1) \to \mathbb{R}^2$ is continuous. If $\iota : S^1 \to \mathbb{R}^2$ denotes the inclusion map, then $\tilde{f}|_{[0,1)} = \iota \circ f$, so f is continuous.

Now $U = [0, \frac{1}{2})$ is open in $[0, 1)$ (recall Example 3.3) but $f(U)$ is not open in S^1 (this is intuitively clear but of course requires a formal proof – try to cook one up!). Thus f is not a homeomorphism, even though it is a continuous bijection.

5.4 The quotient topology

The quotient topology provides us with yet another way of making new topological spaces out of existing ones.

Definition 5.14. Let X and Y be topological spaces, and let $p : X \to Y$ be a surjective map. Then p is called a *quotient map* (sv: *kvotavbildning*) if it has the property that $U \subset Y$ is open if and only if $p^{-1}(U) \subset X$ is open.

Notice that a quotient map is automatically continuous, but it need not be a homeomorphism since it is not necessarily injective; indeed we will be mostly interested in the cases where it's not.

One motivation for studying quotient maps is that they allow us to glue topological spaces together to obtain new ones. In practice, one does this by introducing an equivalence relation whose equivalence classes correspond to the points that we want to glue. An equivalent description is given in terms of general quotient a bit later in this section.

Definition 5.15. Let X be a topological space with an equivalence relation $\sim$. Let $p : X \to X/\sim$ be the map $p(x) = [x]$. The *quotient topology* (sv: *kvottopologin*) on $X/\sim$ is the topology defined by saying that $U \subset X/\sim$ is open if $p^{-1}(U) \subset X$ is open. In other words, it is the unique topology that forces p to be a quotient map.

Example 5.16. We can use the quotient topology to collapse parts of a topological space to a point. Let $U \subset X$ be any subset in a topological space and define an equivalence relation $\sim_U$ on X by $x \sim y$ if $x = y$ or $x, y \in U$. The equivalence class of a point x is

$$[x]_U = \begin{cases} \{x\}, & \text{if } x \notin U, \\ U, & \text{if } x \in U. \end{cases}$$

Intuitively speaking, in $X/\sim_U$ we have collapsed the set U to consist of a single point while we have left the rest of the space unchanged.

We will use the notation $X/U = X/\sim_U$ for the space obtained using the equivalence relation of Example 5.16.

Example 5.17. Let us see what the above construction means for the topology of the space. Let $X = [-1, 1]$ and let $U = \{-1, 1\}$ be the endpoints of the interval. One can then show that $X/U \simeq S^1$: that is, we can tie together the ends of the interval to obtain a circle. We will consider this example again in Example 5.21.

More generally, if $X = D^n \subset \mathbb{R}^n$ is the closed unit ball,

$$D^n = \{x \in \mathbb{R}^n \mid \|x\| \le 1\},$$

then the $(n-1)$-sphere $S^{n-1} \subset D^n$ forms the boundary of D^n in $\mathbb{R}^n$. Now one can show that $D^n/S^{n-1} \simeq S^n$; in fact, we will do so explicitly in Proposition 7.30 below. Picturing the case $n = 2$ is probably helpful.

The following result allows us to determine continuity of functions defined on quotient spaces in terms of the spaces they originate from.

Lemma 5.18. *Let X and Y be a topological spaces, and let $\sim$ be an equivalence relation on X. Suppose that $f : X \to Y$ is a map with the property that $x \sim y$ implies that $f(x) = f(y)$. There then exists a unique map $g : X/\sim \to Y$ so that $f = g \circ p$, where $p : X \to X/\sim$ is the canonical surjection $p(x) = [x]$. Moreover, g is continuous if and only if f is.*

Proof. Let us first define a g that works: let $[x] \in X/\sim$ for $x \in X$. We then define $g([x]) = f(x) \in Y$. The condition on f ensures that g is well-defined, i.e. that if

$[x] = [y]$, then $g([x]) = g([y])$. Now by construction $f = g \circ p$. We will show that g is continuous if and only if f is, and that g is the only map that satisfies $f = g \circ p$.

Let us start with the latter; assume that there is a map $g' : X/{\sim} \to Y$ with $f = g' \circ p$. We then have

$$g([x]) = f(x) = (g' \circ p)(x) = g'(p(x)) = g'([x])$$

for all $x \in X$ so in particular $g([x]) = g'([x])$ for all $[x] \in X/{\sim}$.

Now if g is continuous, so is f since it is a composition of continuous maps.

Assume that f is continuous and let $V \subset Y$ be open. Then $f^{-1}(V) = p^{-1}(g^{-1}(V))$ is open, which implies that $g^{-1}(V)$ is open by definition of the quotient topology, so g is continuous. $\square$

Above we have seen how equivalence relations can be used to define interesting quotient spaces. We will now turn to a result which says that all such quotient spaces may be described in terms of quotient maps.

More precisely, let $f : X \to Y$ be a surjective map. Then we define an equivalence relation $\sim_f$ on X by requiring that $x \sim_f x'$ if and only if $f(x) = f(x')$.

With this relation, the equivalence classes are exactly the sets in X of the form $f^{-1}(\{y\})$ for $y \in Y$, and as before, there is a bijection $g : X/{\sim_f} \to Y$ given by $g(f^{-1}(\{y\})) = y$; that is, $f = g \circ p$ where $p : X \to X/{\sim_f}$ is as before. Notice that we need f to be surjective for this construction to work.

Proposition 5.19. *Let X and Y be topological spaces, let $f : X \to Y$ be a surjective map, and let $g : X/{\sim_f} \to Y$ be the bijection defined by $f^{-1}(\{y\}) \mapsto y$. If f is a quotient map, then g is a homeomorphism.*

Proof. It follows from Lemma 5.18 that g is continuous, since f is. Let $V \subset X/{\sim_f}$ be an open set, and let $U = g(V) \subset Y$. Then we can write $V = g^{-1}(U)$, and

$$p^{-1}(V) = p^{-1}(g^{-1}(U)) = f^{-1}(U)$$

is open since p is continuous. Since f is a quotient map, this implies that $U = g(V)$ is open, so g is a homeomorphism. $\square$

Remark 5.20. In general, if $f : X \to Y$ is a surjection, and $(X, \mathcal{T}_X)$ is a topological space, it is customary to define the quotient topology $\mathcal{T}_Y$ on Y by

$$\mathcal{T}_Y = \{U \subset Y \mid f^{-1}(U) \in \mathcal{T}_X\}.$$

Then Y, with its quotient topology $\mathcal{T}_Y$, is homeomorphic to $X/{\sim_f}$ with its quotient topology (from Definition 5.15).

Example 5.21. Let us show that $[0,1]/\{0,1\}$, with the quotient topology, is homeomorphic to S^1. Let $f : [0,1] \to S^1$ be the map $f(x) = (\cos(2\pi x), \sin(2\pi x))$. Then $f(x) = f(y)$ if and only if $x = y$ or either $x = 1, y = 0$ or $x = 0, y = 1$. This implies that $[0,1]/{\sim_f} = [0,1]/\{0,1\}$. As before, f is continuous, and f is clearly surjective, so the induced map $g : [0,1]/{\sim_f} \to S^1$ is a homeomorphism by Proposition 5.19; we only need to argue that if $f^{-1}(U)$ is open, then so is U; then f is a quotient map. Now one can

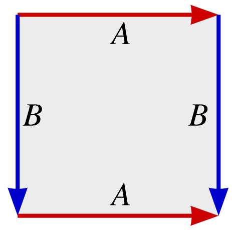

Figure 6: Illustration of $X = [0,1] \times [0,1]/\sim$.

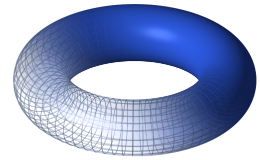

Figure 7: The resulting space $S^1 \times S^1$.

see this by simply examining the open sets in $[0,1]$; these are unions of open intervals, which are allowed to be half-open at 0 and 1. In the sphere these correspond to unions circular intervals which are clearly open. The main point being that the construction ensures that $0 \in f^{-1}(U)$ if and only if $1 \in f^{-1}(U)$ so that we do not run into the same problem as in Example 5.13. While intuitively clear, this is a bit annoying to prove with the tools at hand, and as mentioned, a more precise proof can be found in Proposition 7.30.

Example 5.22. Another way of viewing the same example is as follows: let $\sim$ be the equivalence relation on $\mathbb{R}$ given by $x \sim y$ if $x - y \in \mathbb{Z}$. Define $f : \mathbb{R} \to S^1$ by $f(x) = (\cos(2\pi x), \sin(2\pi x))$ just like above. Then $f(x) = f(y)$ if and only if $x \sim y$, so $\sim = \sim_f$. This implies that $\mathbb{R}/\sim$ is homeomorphic to S^1.

Be aware that the space $\mathbb{R}/\sim$ is often denoted $\mathbb{R}/\mathbb{Z}$, but that this notation does *not* agree with the one from Example 5.16. This somewhat unfortunate coincidence comes from the fact that both $\mathbb{R}$ and $\mathbb{Z}$ are *groups* of which one can form a group quotient. One can combine the study of groups and topological spaces into the study of topological groups. We will not be dealing with those, but the interested reader should check out the end of [Mun00, §22].

Example 5.23. Another important example is the so-called 2-torus T^2, which should be familiar to those that are old enough to know the 1979 arcade shooter Asteroids. It is obtained by gluing together opposing sides of a rectangle $X = [0,1] \times [0,1]$. That is, define an equivalence relation $\sim$ on X by $x \sim y$ of $x = y$, or $x = (p, 0), y = (p, 1)$, or $x = (0, p), y = (1, p)$, see Figure 6. Just like $[0,1]/\{0,1\} \simeq S^1$, one can show that $X/\sim$ is homeomorphic to a circle of circles, $S^1 \times S^1$; we refer to [Mun00, §22] where this is done by examining the open sets in both spaces. See Figure 7 for an illustration of the resulting space.

More generally, we will also be considering the n-torus T^n, which we will simply define to be the product $T^n = S^1 \times \cdots \times S^1$ of n copies of S^1.

Example 5.24. Consider again the $X = [0,1] \times [0,1]$. One could have chosen to identify the opposite sides in various other ways to obtain important spaces. In Figure 8 and Figure 9 two such spaces are shown. The former is the so-called *real projective plane* (sv: *reella projektiva planet*) and the latter is the *Klein bottle* (sv: *Kleinflaska*). The glued-together Klein bottle can be pictured as in Figure 10.

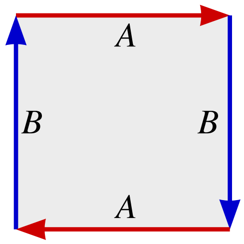

Figure 8: Gluing the real projective plane.

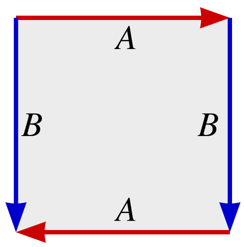

Figure 9: Gluing the Klein bottle.

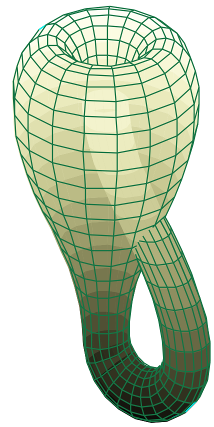

Figure 10: The Klein bottle.

Example 5.25. Finally, an important example is the following generalisation of Example 5.23: notice that we can describe the gluing pattern for the torus by starting at a corner of the square and taking note of the edges that we meet, together with their orientation. For instance, if we begin in the upper left corner of Figure 6 and move clockwise, we encounter the edges $ABA^{-1}B^{-1}$, where we use inverses to denote the orientations of the edges. Similarly, the gluings for the projective plane and the Klein bottle could be described as $ABAB$ and $ABAB^{-1}$ respectively.

Now, consider a $4g$-gon X_g, $g > 0$, and glue together pairs of edges of the boundary according to the rule

$$A_1 B_1 A_1^{-1} B_1^{-1} A_2 B_2 A_2^{-1} B_2^{-1} \cdots A_g B_g A_g^{-1} B_g^{-1},$$

so that for instance, for $g = 1$ we recover the torus example. The resulting space $X_g/\sim$ is called a "surface with g handles", or a "genus g surface". See Figures 11–13 for the examples $g = 1, 2, 3$. How to obtain these pictures is described nicely in [Fje14, Sect. 3.3].

6 Connectedness

In $\mathbb{R}^n$, we have a good intuition about what it means for subsets to be connected or not. For example, the subset $[-2, -1] \cup [1, 2]$ does seem very connected: how would we connect -1 and 1? On the other hand, a set $(-2, 2)$ should probably deserve to be called connected.

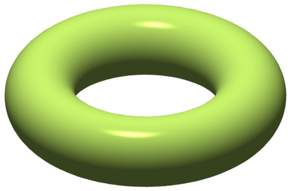

Figure 11: A genus 1 surface.

Figure 12: A genus 2 surface.

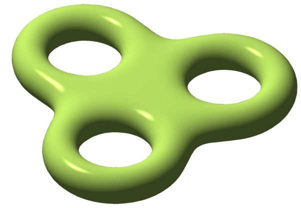

Figure 13: A genus 3 surface.

It turns out that having open sets is sufficient to define a notion of connectedness that agrees with out intuition in the intuitive examples; this is the subject of this sections.

6.1 Connectedness

Definition 6.1. Let X be a topological space. A *separation* (sv: *separation*) of X is a pair U, V of disjoint non-empty open subsets of X so that $X = U \cup V$. We say that X is *connected* (sv: *sammanhängande*) if it has no separation.

In the following we will often be dealing with connectedness of subspaces. Keep in mind that when doing so, the subspace will always be equipped with the subspace topology.

Example 6.2. The subspace $(-2,-1) \cup (1,2) \subset \mathbb{R}$ has a separation.

Notice that if $X = U \cup V$ is a separation, then $U = X \setminus V$ and $V = X \setminus U$. This means that both U and V are both open and closed.

Lemma 6.3. *A topological space X is connected if and only if $\emptyset$ and X are the only subsets of X that are both open and closed.*

Proof. Suppose that $U \subset X$ is both open and closed. Then $V = X \setminus U$ is open, and $X = U \cup V$ is a separation. If X is connected one of U and V must be empty, since otherwise we would have a separation of X. $\square$

Example 6.4. The rational numbers $\mathbb{Q} \subset \mathbb{R}$ are not connected: choose any irrational number $a \in \mathbb{R}$. Then

$$\mathbb{Q} = ((-\infty,a) \cup (a,\infty)) \cap \mathbb{Q} = ((-\infty,a) \cap \mathbb{Q}) \cup (\mathbb{Q} \cap (a,\infty)),$$

which is a separation by definition of the subspace topology on $\mathbb{Q}$.

Example 6.5. If X has the discrete topology and consists of more than two points, then $X = \{x\} \cup (X \setminus \{x\})$ is a separation of X, so X is not connected.

Example 6.6. Let $I \subset \mathbb{R}$ be any interval (bounded, unbounded, open, closed, or half-open). We claim that I is connected. We will use the (hopefully) well-known fact that any subset $A \subset \mathbb{R}$ satisfies $\sup A \in \overline{A}$ (since, for instance, $\sup A$ is a limit of a sequence in A so that we can use Lemma 4.23).

Now to see the claim, assume that $I = U \cup V$ is a separation of I, let $x \in U$, $y \in V$, and assume without loss of generality that $x < y$. Then we have $[x, y] \subset I$. Let $U_0 = [x, y] \cap U$, $V_0 = [x, y] \cap V$ so that $[x, y] = U_0 \cup V_0$ is a separation of $[x, y]$, and define $z = \sup U_0 \in [x, y]$. Now as noticed above, U_0 is both open and closed in $[x, y]$ so in particular U_0 is also closed in $\mathbb{R}$, so $z \in \overline{U_0} = U_0$. If $z = y \in V_0$, we have a contradiction, so assume that $z \neq y$. Since U_0 is open in $[x, y]$ we can find $u \in U_0$ so that $z < u$, which contradicts the fact that $z = \sup U_0$ (to be completely precise about this last point, $U_0 = [x, y] \cap U_1$ for an open set $U_1 \subset \mathbb{R}$, and we can find a neighbourhood $B(z, r)$ of $z \in U_1$ so small that $B(z, r) \subset U_0$; we now just take $u = z + r/2$).

On the other hand, one can show that if $I \subset \mathbb{R}$ is connected, then I is an interval (this is Exercise 2.4).

Lemma 6.7. *Let $X = U \cup V$ for disjoint open sets U and V, and let $Y \subset X$ be a subspace. If Y is connected, then $Y \subset U$ or $Y \subset V$.*

Proof. We will show the contrapositive of the statement, so assume that $Y \cap U \neq \emptyset$ and $Y \cap V \neq \emptyset$. Then

$$Y = Y \cap X = Y \cap (U \cup V) = (Y \cap U) \cup (Y \cap V)$$

is a separation of Y, since $Y \cap U$ and $Y \cap V$ are disjoint, non-empty and open in the subspace topology. Thus Y is not connected. □

Theorem 6.8. *Let $\{A_i\}_{i \in I}$ be a collection of connected subspaces of a topological space X with a common point $x \in X$; i.e. $x \in A_i$ for all $i \in I$. Then $\bigcup_{i \in I} A_i$ is connected.*

Proof. Suppose that $\bigcup_{i \in I} A_i = U \cup V$ for disjoint subsets U and V that are open in $\bigcup_{i \in I} A_i$ and let us show that either U or V must be empty. Assume without loss of generality that $x \in U$. By Lemma 6.7 we have for each i that either $A_i \subset U$ or $A_i \subset V$. Since $x \in A_i$ we must have $A_i \subset U$ for all $i \in I$. This implies that $\bigcup_{i \in I} A_i \subset U$, so V must be empty. □

Theorem 6.9. *Let $A \subset X$ be connected. If a subset $B \subset X$ satisfies $A \subset B \subset \overline{A}$, then B is also connected. In particular, $\overline{A}$ is connected when A is.*

Proof. Suppose that $B = U \cup V$ for disjoint subsets U and V that are open in B. Then by Lemma 6.7 we must have that $A \subset U$ or $A \subset V$, so assume without loss of generality that $A \subset U$. Then $B \subset \overline{A} \subset \overline{U}$ (where all closures are in the bigger space X).

By definition of the subspace topology, there are open sets U' and V' in X so that $U = B \cap U'$, $V = B \cap V'$, and

$$U = B \setminus V = B \setminus (B \cap V') \subset X \setminus V'.$$

The latter space is closed so $\overline{U} \subset X \setminus V' \subset X \setminus V$. Putting this together, $B \subset X \setminus V$ which means that $B \cap V = \emptyset$, so $V = \emptyset$, and so B is connected. □

Theorem 6.10. *Let $f : X \to Y$ be a continuous map between two topological spaces. If X is connected, then $f(X)$ is also connected.*

Proof. Suppose that $f(X) = U \cup V$ for disjoint subsets U and V that are open in $f(X)$. Then $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint open subsets of X with $X = f^{-1}(U) \cup f^{-1}(V)$. This means that either $f^{-1}(U)$ or $f^{-1}(V)$ is empty. Suppose that $f^{-1}(U)$ is empty. Then since $U \subset f(X)$ we must have $U = \emptyset$. □

Corollary 6.11. *Let X be a connected topological space, and let Y be any set. Suppose that $f : X \to Y$ is a locally constant map, meaning that every point $x \in X$ has a neighbourhood U so that $f|_U$ is constant. Then f is constant.*

Proof. Give Y the discrete topology. Then the condition that f is locally constant implies that f is continuous at every point, so f is continuous. Thus $f(X)$ is connected by Theorem 6.10, but $f(X)$ also has the discrete topology, so by Example 6.5 it consists of a single point which is the same as saying that f is constant. □

Corollary 6.12 (Intermediate value theorem)**.** *Let $f : X \to \mathbb{R}$ be continuous and assume that X is connected. If there is an $r \in \mathbb{R}$ and $x, y \in X$ so that $f(x) < r < f(y)$, then there is a $z \in X$ with $f(z) = r$.*

Proof. By Theorem 6.10, $f(X)$ is connected, thus an interval by Exercise 2.4. Since $f(x), f(y) \in f(X)$, we therefore have $r \in [f(x), f(y)] \subset f(X)$. □

Remark 6.13. Being connected is a topological property which can be used to define a simple topological invariant: if X is connected any Y is not, then X and Y are not homeomorphic.

Theorem 6.14. *If $\{X_i\}_{i \in I}$ is a family of topological spaces, then their product $\prod_{i \in I} X_i$ is connected if and only if every X_i is.*

Proof. Suppose that the product is connected. Recall that the projection $\pi_j : \prod_{i \in I} X_i \to X_j$ is continuous for every j, so every X_j is connected by Theorem 6.10.

Let us show the converse in the case where I is finite. The infinite case is left as Exercise 2.15. Moreover, we can reduce to the case $|I| = 2$ by induction since $(X_1 \times \cdots X_{n-1}) \times X_n \simeq X_1 \times \cdots X_n$, which is not difficult to show.

Thus, we are left to show that $X \times Y$ is connected when X and Y are. We will write $X \times Y$ as a union of connected spaces with a common point and use Theorem 6.8. For any point $x \in X$, we let $A_x = \{x\} \times Y$. Then A_x is the image of the map $Y \to X \times Y$ given by $y \mapsto (x, y)$, so A_x is connected by Theorem 6.10. Similarly one shows that $B_y = X \times \{y\}$ is connected for all $y \in Y$. By Theorem 6.8, $A_x \cup B_y$ is connected for all $y \in Y$ since (x, y) is contained in both A_x and B_y. Now clearly,

$$X \times Y = \bigcup_{y \in Y} A_x \cup B_y,$$

and all the sets on the right hand side have the common point $(x, y') \in A_x$, where $y' \in Y$ can be taken to be anything. Therefore, their union is connected by Theorem 6.8. □

Since we have seen that $\mathbb{R}$ is connected, we obtain the following.

Corollary 6.15. *Euclidian space $\mathbb{R}^n$ is connected for any $n \in \mathbb{N}$.*

Proof. It follows from Theorem 6.14 that $\mathbb{R}^n = \mathbb{R} \times \cdots \times \mathbb{R}$ is connected in the product topology, so it suffices to check that the product topology and the standard topology on $\mathbb{R}^n$ are the same; this may be done by applying Lemma 2.15 twice. The case $n = 2$ is Exercise 1.9. $\square$

Proposition 6.16. *The n-sphere S^n and the n-torus T^n are connected for all $n \in \mathbb{N}$.*

Proof. If we can show that S^1 is connected, then so is $T^n = S^1 \times \cdots \times S^1$ for all $n \geq 1$.

Let us show that S^n is connected. Recall from Proposition 5.11 that $S^n \setminus \{p\} \simeq \mathbb{R}^n$, where p is the north pole. It follows that $S^n \setminus \{p\}$ is connected. Now one can show that $\overline{S^n \setminus \{p\}} = S^n$ and so the result follows from Theorem 6.9. $\square$

Alternatively, one can show that $\mathbb{R}^{n+1} \setminus \{0\}$ is connected when $n \geq 1$ (Exercise 2.5); then the connectedness of S^n follows because S^n is the image of the continuous map $f : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}^{n+1}$ given by $f(x) = x/\|x\|$.

The following is an example of how to use connectedness as a topological invariant:

Proposition 6.17. *For any $n \in \mathbb{N}$, we have $S^n \not\simeq \mathbb{R}$.*

Proof. Suppose that we had a homeomorphism $f : S^n \to \mathbb{R}$. Then f would restrict to a homeomorphism if we removed the north pole p from S^n; that is, $f|_{S^n \setminus \{p\}} : S^n \setminus \{p\} \to \mathbb{R} \setminus \{f(p)\}$ is a homeomorphism. This implies that $\mathbb{R}^n \simeq S^n \setminus \{p\} \simeq \mathbb{R} \setminus \{f(p)\}$, but $\mathbb{R}^n$ is connected while $\mathbb{R} \setminus \{f(p)\}$ is not connected, so they can not be homeomorphic, and we obtain a contradiction. $\square$

Along the same lines, we mention the following result.

Theorem 6.18 (Brouwer's invariance of dimension)**.** *Consider two non-empty open sets $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$. If $U \simeq V$, then $n = m$.*

The proof uses methods from algebraic topology and will not be covered here; see e.g. [Hat02, Thm. 2.26]. A special case is contained in Exercise 2.6.

6.2 Paths and path-connectedness

Another natural notion of connectedness is obtained by requiring that all points in a space can be connected to each other; what exactly to mean by this is contained in the following definition.

Definition 6.19. Given two points x and y in a topological space X, a *path* (sv: *väg*) from x to y is a continuous map $\gamma : [0, 1] \to X$ so that $\gamma(0) = x$, $\gamma(1) = y$. If for any pair x, y in X there is a path from x to y, we say that X is *path-connected* (sv: *bågvis sammanhängande*).

Proposition 6.20. *A path-connected space is connected.*

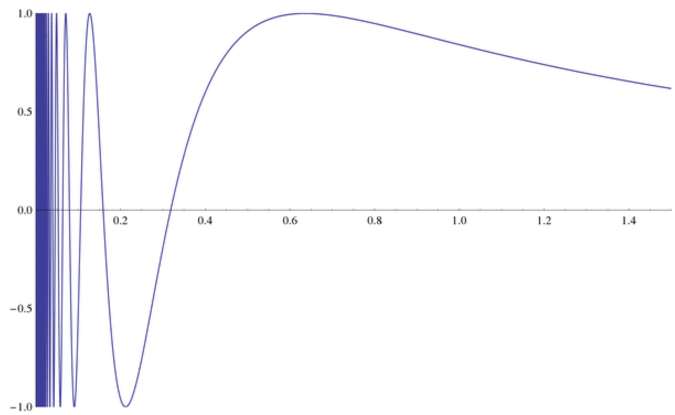

Figure 14: The topologist's sine curve.

Proof. We will prove the contrapositive, so let $X = U \cup V$ be a separation of X. Suppose that $\gamma : [0,1] \to X$ is any path. Then by Example 6.6 and Theorem 6.10 we see that $\gamma([0,1])$ is connected and by Lemma 6.7, $\gamma([0,1])$ is contained entirely in U or in V. This means that it is not possible to find paths from points in U to points in V. □

Example 6.21. For $x, y \in \mathbb{R}^n$, the path $\gamma : [0,1] \to \mathbb{R}^n$ defined by

$$\gamma(t) = (1-t)x + ty$$

is a path from x to y, so $\mathbb{R}^n$ is path-connected.

Example 6.22. A subset A of $\mathbb{R}^n$ is called convex if for any $x, y \in A$, the image of the straight line $\gamma(t) = (1-t)x + ty$ belongs to A. As in the previous example, it follows that convex subsets are path-connected and thus also connected.

Some examples of convex subsets are the upper half-plane

$$\{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n > 0\}$$

and any ball $B(x,r)$, $x \in \mathbb{R}^n$, $r > 0$.

Example 6.23. A connected space does not need to be path-connected. A counter-example is the so-called topologist's sine curve

$$S = \{(x,y) \in \mathbb{R}^2 \mid y = \sin(1/x), x > 0\} \cup \{(0,y) \mid -1 \leq y \leq 1\},$$

that is, the closure of the graph of $x \mapsto \sin(1/x)$ for $x > 0$; see Figure 14. For details, see [Mun00, §24].

On the other hand, the following result provides a sufficient condition for a connected space to be path-connected.

Definition 6.24. A topological space X is called *locally (path-)connected at* $x \in X$ if every neighbourhood of x contains a (path-)connected neighbourhood of x. If X is locally (path-)connected at each point $x \in X$, we say that X is *locally (path-)connected* (sv: *lokalt (bågvis) sammanhängande*).

Theorem 6.25. *If a topological space is connected and locally path-connected, then it is path-connected.*

To show this, it will be useful to have at our disposal the notion of connected components.

6.3 Connected components and path-connected components

Recall that the subset $(-2,-1) \cup (1,2) \subset \mathbb{R}$ was not connected. It does however have two natural subspaces, $(-2,-1)$ and $(1,2)$ which *are* connected. We will now see how to split every topological space into connected parts.

Proposition 6.26. *Let X be a topological space. Define a relation $\sim$ on X by declaring that $x \sim y$ if and only if there is a connected set $A \subset X$ so that $x, y \in A$. Then $\sim$ is an equivalence relation. The equivalence classes of $\sim$ are called the* connected components *(sv:* sammanhängande komponenter*) of X.*

Proof. We see that $x \sim x$ for every x, since $\{x\}$ is connected.

If $x \sim y$ there exists a connected set A with $x, y \in A$. Then clearly $y, x \in A$ so $y \sim x$.

If $x \sim y$ and $y \sim z$ we get connected sets A, B so that $x, y \in A$ and $y, z \in B$. Let $C = A \cup B$. Then $x, z \in C$, and C is connected by Theorem 6.8, so $x \sim z$. $\square$

Proposition 6.27. *Let $\{C_i\}_{i \in I}$ be the set of connected components of a topological space X. Then*

(i) $X = \bigcup_{i \in I} C_i$ *and the C_i are pairwise disjoint,*

(ii) if $Y \subset X$ is connected, then $Y \subset C_i$ for some $i \in I$,

(iii) $C_i \subset X$ is connected for each $i \in I$, and

(iv) C_i is closed for all $i \in I$.

Proof. The first part is trivial since it it always true for equivalence classes of an equivalence relation. Let $Y \subset X$ be connected, and let $x \in Y$. Then $y \in [x]$ for all other $y \in Y$ since Y is a connected set containing both x and y, so $Y \subset [x]$, and $[x]$ is one of the C_i by definition.

Similarly, fix $x \in C_i$. Then for every other $y \in C_i$ there a connected subset A_y so that $x, y \in A_y$. Then $A_y \subset C_i$ by (ii), and we now use our usual trick and find that $C_i = \bigcup_{y \in C_i} A_y$. Since all of the A_y contain x, we use Theorem 6.8 to conclude that C_i is connected.

Finally, we will show that C_i is closed by showing that $C_i = \overline{C_i}$. Once more, write $C_i = [x]$ for any $x \in C_i$. Let $y \in \overline{C_i}$. Then $\overline{C_i}$ is a subset containing both x and y, and $\overline{C_i}$ is connected by Theorem 6.9, so $y \in [x] = C_i$. $\square$

Example 6.28. It follows from Proposition 6.27 that connected components are open if there are only finitely many of them. This need not be the case though: we claim that the connected components of $\mathbb{Q}$ are the singleton sets $\{x\}$. Indeed, let X be any subset of $\mathbb{Q}$ containing at least two points, and suppose that $x, y \in X$, $x \neq y$. There there is an irrational number r, $x < r < y$, and

$$X = (X \cap (-\infty, r)) \cup (X \cap (r, \infty))$$

is a separation of X. We have already seen that the topology on $\mathbb{Q}$ is not the discrete one, so the connected component $\{x\}$ is not open for any x.

Remark 6.29. The number of connected components is a topological invariant $\mathbf{Top} \to \mathbb{Z} \cup \{\infty\}$. That is, if two topological spaces have a different number of connected components, then they can not be homeomorphic.

We now turn back to our study of path-connected spaces, creating an analogous construction of path-connected components. For this, it will first be useful to introduce the following two operations on paths. For any path $\gamma : [0,1] \to X$ in a topological space X, define the *reverse* of γ, denoted γ^{rev}, by $\gamma^{\mathrm{rev}}(t) = \gamma(1-t)$. Then γ^{rev} is continuous, and if γ is a path from x to y, then γ^{rev} is a path from y to x. Let $\gamma_1, \gamma_2 : [0,1] \to X$ be two paths so that $\gamma_1(1) = \gamma_2(0)$, so that γ_1 is a path from x to y, and γ_2 is a path from y to z. We then form a path from x to z as follows: define the *concatenation* (sv: *konkatenering*) $\gamma_1 \star \gamma_2 : [0,1] \to X$ by

$$\gamma_1 \star \gamma_2(t) = \begin{cases} \gamma_1(2t), & t \in [0, \frac{1}{2}], \\ \gamma_2(2t-1), & t \in [\frac{1}{2}, 1]. \end{cases}$$

We need to check that $\gamma_1 \star \gamma_2$ is actually continuous. This, however follows directly from the closed-set version of the pasting lemma, see Remark 3.10, applied to the two intervals $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$.

With this, we can define a relation on X by saying that $x \sim_{\mathrm{path}} y$ if there is a path from x to y.

Lemma 6.30. *The relation $\sim_{\mathrm{path}}$ is an equivalence relation. The equivalence classes of $\sim_{\mathrm{path}}$ are called the* path-connected components *(sv:* bågvis sammanhängande komponenter*) of X.*

Proof. The constant path $\gamma : [0,1] \to X$, $\gamma(t) = x$, is continuous for any given $x \in X$, so $x \sim_{\mathrm{path}} x$. The other conditions for an equivalence relation are obtained by using the operations on paths introduced above. □

We have the following analogue of Proposition 6.27 for path-connected components; its proof is similar and omitted.

Proposition 6.31. *The path-connected components are path-connected disjoint subspaces of X whose union is X. If a subspace is path-connected, it is contained in a path-connected component.*

Theorem 6.32. *If a topological space X is locally path-connected, then its connected components and path-connected components are the same.*

The special case where X consists of a single path-connected component is exactly Theorem 6.25.

Proof. Let C be a connected component of X, let $x \in C$, and let P be the path-connected component containing x. Since P is also connected by Proposition 6.20, Proposition 6.27 implies that $P \subset C$. We want to show that $P = C$; assume that $P \neq C$, and let Q denote the union of all the path-connected components of X that intersect C but are not equal to P. By the same argument, each of these path-connected components will necessarily be contained in C, so we can write $C = P \cup Q$. Since P and Q are disjoint non-empty sets, this would contradict the connectedness of C, if we can show that both P and Q are open. This is where we need the locally path-connectedness as X and we word the result as a lemma below. $\square$

Lemma 6.33. *If X is locally (path-)connected, then all its (path-)connected components are open.*

Proof. Let us only show the result for locally path-connected spaces and leave the other part of the claim as Exercise 2.10. Let P be a path-connected component, and let us show that $P = \operatorname{Int} P$, so let $x \in P$. Since X is locally path-connected, we can choose a path-connected neighbourhood U of x. By Proposition 6.31, $U \subset P$, so $x \in \operatorname{Int} P$. $\square$

Example 6.34. Any open subset of $\mathbb{R}^n$ is locally path-connected. Thus in particular, connectedness and path-connectedness are equivalent for open subsets in $\mathbb{R}^n$.

Remark 6.35. Let X be a topological space. Denote by $\pi_0(X)$ the set of path-connected components of X. We remark that the cardinality of $\pi_0(X)$ is a topological invariant: that is, if two topological spaces X and Y have different numbers of path-connected components, then they are not homeomorphic.

7 Compactness and sequential compactness

7.1 Compactness

Definition 7.1. Let $(X, \mathcal{T})$ be a topological space.

(i) A collection $\mathcal{U} \subset \mathcal{T}$ of open sets of called an *open cover* (sv: *öppen övertäckning*) of X if $X = \bigcup_{U \in \mathcal{U}} U$.

(ii) The space X is called *compact* (sv: *kompakt*) if *every* open cover $\mathcal{U}$ of X has a finite subcover, meaning that one can find finitely many open sets $U_1, \ldots, U_n \in \mathcal{U}$ so that $X = \bigcup_{i=1}^{n} U_i$.

We will often be concerned with compactness of subspaces of topological spaces. In each such case, the subspace in question is called compact if it is compact in the subspace topology.

For first-timers, the condition of being compact if often incorrectly read as "X has a finite open cover"; this, however, is always trivially true, since X has the finite open cover $\mathcal{U} = \{X\}$, and so has nothing to do with compactness. To show that something is compact, it is therefore essential that we consider *any* open cover $\mathcal{U}$ and show that we can find a finite subcover *of that*.

Example 7.2. Every finite topological space is compact, since there are only finitely many open sets. Thus, given any open cover $\mathcal{U}$, the open cover $\mathcal{U}$ is itself a finite subcover.

Example 7.3. The real line $\mathbb{R}$ is not compact: consider the open cover $\mathcal{U}$ consisting of the open sets $U_n = (-n, n)$, $n \in \mathbb{N}$. Clearly, it is impossible to choose finitely many such U_n and still have something that covers all of $\mathbb{R}$.

Example 7.4. The subspace $A = \{1/n \mid n \in \mathbb{N}\} \subset \mathbb{R}$ is not compact. One can see that $U_n = \{1/n\}$ is an open set in the subspace topology, so letting $\mathcal{U} = \{U_n \mid n \in \mathbb{N}\}$, we get an open cover of A. Clearly, we can not find a finite subcover, since any finite subcover would cover only finitely many points of the infinite set A.

Example 7.5. Let $X = A \cup \{0\}$, where A is the set from the previous example. We claim that X is compact. Let $\mathcal{U}$ be an arbitrary open cover of X. Then there is an open set $U \in \mathcal{U}$ so that $0 \in U$. By definition of the topology on $\mathbb{R}$, U will contain the points $1/n$ for all large enough n, say all $n > N$ for some N. Since $\mathcal{U}$ is an open cover, we can also find open sets $U_1, \dots, U_N \in \mathcal{U}$ so that $1/k \in U_k$ for all $k = 1, \dots, N$. We now see that the collection $U, U_1, \dots, U_N$ together form a finite subcover of X.

Example 7.6. The half-open interval $(0, 1] \subset \mathbb{R}$ is not compact since the open cover $\mathcal{U}$ consisting of open sets $U_n = (\frac{1}{n}, 1]$, $n \in \mathbb{N}$, does not have a finite subcover, by more or less the same argument as in Example 7.3. Similarly, $(0, 1)$ is not compact, since the sets $(\frac{1}{n}, 1 - \frac{1}{n})$ form an open cover with no finite subcover.

Example 7.7. The closed interval $[0, 1] \subset \mathbb{R}$ is compact. This is a special case of the Heine–Borel theorem, Theorem 7.25, which we show below.

In the following theorems, we will collect a number of properties of compact sets that we will use over and over again.

Theorem 7.8. *A closed subspace of a compact space is compact.*

Proof. Let $A \subset X$ be closed, and assume that X is compact. To show that A is compact, let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover of A. That is, every U_i is open in A in the subspace topology. By definition, we can find for every $i \in I$ open subsets V_i of X so that $U_i = A \cap V_i$. Since the U_i cover A, it follows that the family $\mathcal{V} = \{V_i\}_{i \in I} \cup \{A^c\}$ is an open cover of X; open because A was assumed to be closed. Since X is compact, there is a finite subcover $V_{i_1}, \dots, V_{i_n} \in \mathcal{V}$ of X. Going back, we see that $V_{i_1} \cap A, \dots, V_{i_n} \cap A \in \mathcal{U}$ form a finite subcover of A, which is what we wanted to prove. $\square$

Theorem 7.9. *A compact subspace of a Hausdorff space is closed.*

Proof. Assume that X is a Hausdorff space, and let $A \subset X$ be compact. We want to show that A^c is open, and we will use the usual trick of showing that every point in A^c has an open neighbourhood contained entirely in A^c, so that $A^c = \operatorname{Int} A^c$.

Let $x \in A^c$ be a fixed point (and notice that if $A^c = \emptyset$, there is little to prove). For every point $y \in A$, we can find disjoint neighbourhoods U_y and V_y of x and y respectively, since X is Hausdorff. Now the collection $\{A \cap V_y\}_{y \in A}$ is an open cover of A, and since A is compact, we can choose finitely many $y_1, \dots, y_n$ so that $\{A \cap V_{y_i}\}_{i=1,\dots,n}$ is a finite subcover. In particular, $A \subset V_{y_1} \cup \cdots \cup V_{y_n}$.

Let $U^x = U_{y_1} \cap \cdots \cap U_{y_n}$. Now U^x is open by (T3), and $U^x \subset A^c$: if $z \in U^x$, then $z \in V_{y_i}^c$ for every $i = 1, \dots, n$, so $z \in (V_{y_1} \cup \cdots \cup V_{y_n})^c \subset A^c$. □

Theorem 7.10. *Let X and Y be topological spaces, assume that X is compact, and let $f : X \to Y$ be a continuous map. Then the image $f(X) \subset Y$ is compact. If furthermore Y is Hausdorff, and f is a bijection, then f is a homeomorphism.*

Proof. Let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover of $f(X)$ and let us find a finite subcover. Define $V_i = f^{-1}(U_i)$ for every i. Then $\{V_i\}_{i \in I}$ is an open cover of X which has a finite subcover $\{V_{i_1}, \dots, V_{i_n}\}$ since X is compact. Now clearly, the corresponding collection $\{U_{i_1}, \dots, U_{i_n}\}$ is a finite subcover of $f(X)$.

Assume now that Y is Hausdorff and f is bijective. We have to show that f^{-1} is continuous, so let $U \subset X$ be open, and let us show that $f(U)$ is also open. To do so, note that U^c is closed and thus compact by Theorem 7.8. By the first part of the theorem, $f(U)^c = f(U^c)$ is also compact. By Theorem 7.9, this means that $f(U)^c$ is closed, so $f(U)$ is open. □

Corollary 7.11. *If $f : X \to Y$ is continuous and injective, X is compact, and Y is Hausdorff, then $f : X \to f(X)$ is a homeomorphism.*

Proof. This follows from the theorem above since $f(X)$ is Hausdorff by Exercise 1.11 and $f : X \to f(X)$ is a bijection. □

Corollary 7.12. *If $p : X \to Y$ is a quotient map, and X is compact, then so is Y.*

Proof. Quotient maps are continuous and surjective, so the claim follows from Theorem 7.10. □

Example 7.13. If we trust Example 7.7 which says that $[0, 1]$ is compact, it follows that S^1 is compact, since the map $f : [0, 1] \to S^1$ given by $f(x) = (\cos(2\pi x), \sin(2\pi x))$ is continuous and surjective. Alternatively, one could combine Example 5.21 and Corollary 7.12.

Example 7.14. A *simple closed curve* (sv: *enkel sluten kurva*) in a topological space X is a continuous injective map $f : S^1 \to X$. If X is Hausdorff, then $f : S^1 \to f(S^1)$ is a homeomorphism by Corollary 7.11.

Our next goal will be to see that any products of compact spaces are compact. To do so, the following result will be useful.

Lemma 7.15 (The tube lemma)**.** *Let X and Y be topological spaces where Y is compact. If N is an open set of $X \times Y$ which contains $\{x_0\} \times Y$ for some $x_0 \in X$, then N contains a "tube" $W \times Y$, where $W \subset X$ is a neighbourhood of x_0.*

Proof. Since N is open we can choose, by the definition of the product topology, for any $y \in Y$ an open neighbourhood $U_y \times V_y \subset N$ of (x_0, y). Since the map $Y \to X \times Y$ given by $y \mapsto (x_0, y)$ is continuous, its image $\{x_0\} \times Y$ is compact by Theorem 7.10. Therefore, since $\{U_y \times V_y\}_{y \in Y}$ is an open cover of $\{x_0\} \times Y$, we can find $y_1, \ldots, y_n \in Y$ so that $\{U_{y_1} \times V_{y_1}, \ldots, U_{y_n} \times V_{y_n}\}$ is a subcover of $\{x_0\} \times Y$. Now, let $W = \bigcap_{i=1}^n U_{y_i}$. Then W is open by (T3), and is a non-empty neighbourhood of x_0. By construction, $W \times Y = W \times (V_{y_1} \cup \cdots \cup V_{y_n}) \subset N$. □

Theorem 7.16. *Let $X_1, \ldots, X_n$ be topological spaces. Then $\prod_{i=1}^n X_i$ is compact if and only if X_i is compact for all $i = 1, \ldots, n$.*

Proof. If the product is compact, then so is every X_i, since X_i is the image of a continuous map; the projection.

For the converse, by induction it suffices to show that a product $X \times Y$ of two compact spaces is compact. Let $\mathcal{U}$ be an open cover of $X \times Y$. As in the proof of the tube lemma, for every $x \in X$ the space $\{x\} \times Y$ is compact. Therefore, its open cover $\{(\{x\} \times Y) \cap U \mid U \in \mathcal{U}\}$ has a finite subcover by sets

$$(\{x\} \times Y) \cap U_i^x,$$

where $i = 1, \ldots, n$, and all U_i^x are in $\mathcal{U}$. Let $N_x = \bigcup_{i=1}^n U_i^x$. Then by the tube lemma, there is a neighbourhood W_x of x so that $W_x \times Y \subset N_x$. Now the collection $\{W_x \mid x \in X\}$ is an open cover of X, and since X is compact, we can find finitely many elements $x_1, \ldots, x_m \in X$ so that $\{W_{x_1}, \ldots, W_{x_m}\}$ is a finite subcover. We claim that the finite collection

$$\{U_i^{x_j} \mid i = 1, \ldots, n,\ j = 1, \ldots, m\} \subset \mathcal{U}$$

covers X. To see this, let $(x, y) \in X \times Y$ be arbitrary. Then there is a $j \in \{1, \ldots, m\}$ so that $x \in W_{x_j}$, and $(x, y) \in N_{x_j}$, and then by definition of N_{x_j}, there is an $i \in \{1, \ldots, n\}$ so that $(x, y) \in U_i^{x_j}$. □

Remark 7.17. The same result holds true even if one allows infinite products; this general statement is known as Tikhonov's theorem. We refer to [Mun00, §37] for the details.

Example 7.18. If we trust Example 7.13 which said that S^1 is compact, it follows from Theorem 7.16 that the n-torus $T^n = S^1 \times \cdots \times S^1$ is also compact.

Before moving on, let us remark that one can also characterize compactness in terms of closed sets rather than open sets:

Definition 7.19. A collection of subsets $\mathcal{C} \subset \mathcal{P}(X)$ of a set X is said to have the *finite intersection property* (sv: *?*) if for every finite subcollection $\{C_1, \ldots, C_n\} \subset \mathcal{C}$, one has $\bigcap_{i=1}^n C_i \neq \emptyset$.

Proposition 7.20. *A topological space X is compact if and only if any collection $\mathcal{C}$ of closed subsets of X with the finite intersection property satisfies $\bigcap_{C\in\mathcal{C}} C \neq \emptyset$.*

Proof. Exercise. □

7.2 Sequential compactness

In this section, we introduce a different notion of compactness that might look more familiar.

Definition 7.21. A topological space is called *sequentially compact* (sv: *följdkompakt*) if every sequence in it has a convergent subsequence.

Theorem 7.22. *Let X be a topological space.*

(i) If X is first countable, then compactness of X implies sequential compactness.

(ii) If X is a metric space with the metric topology, then compactness and sequential compactness of X are equivalent.

Proof. Assume that X is first countable and compact. Let $\{x_n\}$ be any sequence and let us show that it has a convergent subsequence. Assume first that there is a point $x \in X$ with the property that for any neighbourhood U of x, there are infinitely many n so that $x_n \in U$. Let $\{B_i\}$ be a countable basis at x and let $U_k = \bigcap_{i=1}^k B_i$, which is a neighbourhood of x. Then $x_n \in U_k$ for infinitely many n, so in particular we can choose $x_{n_k} \in U_k$ for some increasing sequence n_k; we claim that $\{x_{n_k}\}$ converges to x. For any neighbourhood U there exists an $N \in \mathbb{N}$ so that $B_N \subset U$. It follows that for all k with $n_k > N$,

$$x_{n_k} \in U_{n_k} \subset U_N \subset B_N \subset U,$$

which says that $x_{n_k} \to x$.

Suppose now that no x has the property we used above; that is, suppose that for every $x \in X$ there is a neighbourhood U_x of x so that only finitely many x_n are in U_x. We will use compactness of X to arrive at a contradiction. The collection $\{U_x \mid x \in X\}$ is a cover of X, so by compactness we get finitely many points $y_1, \dots, y_n \in X$ so that $U_{y_1}, \dots, U_{y_n}$ cover X. This is impossible though since then, at least one of the U_{y_i} must contain infinitely many of the x_n.

Next we turn our attention to the metric case. First of all, recall that metric spaces are first-countable, so we only need to show that sequential compactness implies compactness. So, assume that (X, d) is a sequentially compact metric space.

Take any real number $r > 0$. Then the collection $\{B_d(x, r) \mid x \in X\}$ is an open cover which we claim has a finite subcover: if it didn't, we could define a sequence $\{x_n\}$ with the property that

$$x_{n+1} \notin \bigcup_{i=1}^n B_d(x_i, r).$$

This sequence has the property that $d(x_n, x_m) > r$ for all $n, m \in \mathbb{N}$, $n \neq m$, so it can have no convergent subsequence (recall the characterization of convergence in metric spaces from Proposition 4.19), which is our contradiction, so there is a finite subcover $\{B_d(x_i, r)\}_{i=1,\dots,n}$.

Now let $\mathcal{U} = \{U_i\}_{i\in I}$ be an arbitrary open cover of X. We claim that there is a real number $r > 0$ such that for every $x \in X$, $B_d(x, r) \subset U_i$ for some $i \in I$. This will complete the proof since then we could consider our finite subcover $\{B_d(x_\alpha, r)\}$ from before, and let i_α be so that $B_d(x_\alpha, r) \subset U_{i_\alpha}$. Then clearly, the finitely many U_{i_α} will cover X.

So suppose once more for a contradiction that no such real number $r > 0$ exists. That is, suppose that for every $r > 0$ there exists an $x \in X$ so that $B_d(x, r)$ is not a subset of U_i for any $i \in I$. In particular, for each $n \in \mathbb{N}$ we can choose x_n so that $B_d(x_n, 1/n)$ is not a subset of any U_i. Choose a convergent subsequence $\{x_{n_k}\}$ of $\{x_n\}$, i.e. $x_{n_k} \to x$ for some x.

Then $x \in U_j$ for some $j \in I$, and since U_j is open, we have $B_d(x, 1/N) \subset U_j$ for some $N \in \mathbb{N}$. Now choose a K so large that $d(x_{n_k}, x) < \frac{1}{2N}$ for all $k > K$. Let moreover $k > K$ be large enough that $n_k > 2N$. Then for every $y \in B_d(x_{n_k}, 1/n_k)$, we have

$$d(y, x) \leq d(y, x_{n_k}) + d(x_{n_k}, x) < \frac{1}{n_k} + \frac{1}{2N} < \frac{1}{2N} + \frac{1}{2N} = \frac{1}{N},$$

so $B_d(x_{n_k}, 1/n_k) \subset B_d(x, 1/N) \subset U_j$ which is a contradiction. □

7.3 Compactness in $\mathbb{R}^n$

We will now turn to the promised characterisation of (sequentially) compact subsets in $\mathbb{R}^n$.

Theorem 7.23. *A closed bounded interval $[a, b] \subset \mathbb{R}$ is compact.*

Proof. Let $\{U_i\}_{i\in I}$ be an open cover of $[a, b]$, and consider the set

$$M = \{x \in [a, b] \mid [a, x] \text{ is covered by finitely many } U_i\}.$$

We are done if we can show that $M = [a, b]$. Clearly, $a \in M$. Let $m = \sup M > a$. Clearly M is an interval, and we claim that $m = b$. Assume that $m < b$.

Since $m \in [a, b)$, there is a $j \in I$ with $m \in U_j$. Since U_j is open (in the subspace topology of the standard topology on $\mathbb{R}$), we get that $(m-\varepsilon, m+\varepsilon) \subset U_j$ and $m-\varepsilon \in M$ for some small enough $\varepsilon > 0$. This says that $[a, m - \varepsilon]$ is covered by finitely many U_i, so by adding U_j to this collection, we see that also $[a, m + \varepsilon/2]$ is covered by finitely many U_i. That is, $m + \varepsilon/2 \in M$, which contradicts the fact that $m = \sup M$. □

Definition 7.24. A subset $A \subset \mathbb{R}^n$ is called *bounded* (sv: *begränsad*) if $A \subset [-K, K]^n$ for some $K > 0$.

Theorem 7.25 (Heine–Borel)**.** *A set $A \subset \mathbb{R}^n$ is compact if and only if it is closed and bounded.*

Proof. If A is compact, then A is closed by Theorem 7.9. If A were not bounded, we could choose $x_k \in A$ with $d(x_k, 0) > k$ for all $k \in \mathbb{N}$. Now clearly, the collection $U_k = A \cap B(0, k)$, $k \in \mathbb{N}$, is an open cover of A but for all $k \in \mathbb{N}$ we see that $x_k \notin U_k$, so $\{U_k\}$ has no finite subcover, contradicting compactness, so A is bounded.

If A is closed and bounded, $A \subset [-K, K]^n$ for some $K > 0$. Now $[-K, K]$ is compact by Theorem 7.23, so $[-K, K]^n$ is compact by Theorem 7.16. Thus A is compact by Theorem 7.8 (notice that A is closed in the subspace topology on $[-K, K]^n$). □

Corollary 7.26. *If X is compact and $f : X \to \mathbb{R}$ is continuous, then there are points x_1 and x_2 with $f(x_1) = \sup f(X)$, $f(x_2) = \inf f(X)$.*

Proof. By Theorem 7.10, $f(X)$ is compact, so by the Heine–Borel theorem, $f(X)$ is closed and bounded. Thus $\sup f(X) < \infty$ and $\sup f(X) \in f(X)$, and similarly for inf. □

Corollary 7.27. *The n-sphere S^n is compact.*

Proof. Clearly, the n-sphere is bounded. Notice that we can describe the sphere as $S^n = (\|\cdot\|)^{-1}(\{1\})$, the pre-image of a closed set $\{1\}$ under the norm map $\|\cdot\| : \mathbb{R}^{n+1} \to \mathbb{R}$, which is continuous. Thus S^n is closed, and therefore compact by the Heine–Borel theorem. □

Example 7.28. The genus n surface from Example 5.25 is compact for every n, since the $4n$-gon X_n is closed and bounded, thus compact, and quotients of compact spaces are compact by Corollary 7.12.

Theorem 7.29 (Bolzano–Weierstrass)**.** *A set $A \subset \mathbb{R}^n$ is sequentially compact if and only if it is closed and bounded.*

Proof. This follows immediately by combining Theorem 7.22 and the Heine–Borel theorem. □

We also now have the necessary tools to prove a claim we gave in Example 5.17. Recall that we write D^n for the closed unit ball in $\mathbb{R}^n$.

Proposition 7.30. *The quotient $D^n/\partial D^n$ is homeomorphic to S^n.*

Proof. The strategy will be to construct a map $\widehat{\psi} : D^n/\partial D^n \to S^n$ explicitly and then apply the last part of Theorem 7.10.

First of all, notice that D^n is closed and bounded, thus compact by the Heine–Borel theorem. Therefore $D^n/\partial D^n$ is compact since quotients of compact spaces are compact. On the other hand, S^n is Hausdorff because it is a subspace of a Hausdorff space. It therefore suffices to construct $\widehat{\psi}$, and show that it is a continuous bijection.

Recall from Example 5.5 that there exists a homeomorphism $\varphi : B^n \to \mathbb{R}^n$ from the *open* ball $B^n = B(0, 1)$ to $\mathbb{R}^n$ and from the proof of Proposition 5.11 that the inverse of the stereographic projection provides a homeomorphism $g : \mathbb{R}^n \to S^n \setminus \{p\}$, where p

is the north pole. Then also $g \circ \varphi : B^n \to S^n \setminus \{p\}$ is a homeomorphism. Now define $\psi : D^n \to S^n$ by

$$\psi(x) = \begin{cases} (g \circ \varphi)(x), & \text{if } x \in B^n, \\ p, & \text{if } x \in \partial D^n. \end{cases}$$

See Figures 15–17 for an illustration of the map ψ. We claim that ψ is continuous, which will then complete the proof: since ψ is constant on ∂D^n, Lemma 5.18 provides us with a continuous map $\widehat{\psi} : D^n/\partial D^n \to S^n$, which is continuous if ψ is, and which is bijective by construction.

To show that ψ is continuous, we will show that it is continuous at each point. It is continuous on each point in B^n, since its restriction to B^n is continuous, so it suffices to show that it is continuous at every point in ∂D^n, so let $x \in \partial D^n$, so let $U \subset S^n$ be a neighbourhood of $f(x) = p$, and let us find a neighbourhood V of x so that $\psi(V) \subset U$. Since U is open, there is a k, $0 < k < 1$ so close to 1 that

$$\{y \in S^n \mid k < y_{n+1} \leq 1\} \subset U.$$

Notice that for $z \in B^n$, by the proof of Proposition 5.11, the $(n+1)$'st coordinate of

$$y = g \circ \varphi(z) = g\left(\frac{z}{1 - \|z\|}\right)$$

is exactly

$$y_{n+1} = 1 - t\left(\frac{z}{1 - \|z\|}\right) = 1 - \frac{2}{1 + (\|z\|/(1 - \|z\|))^2} = \frac{2\|z\| - 1}{2\|z\|^2 - 2\|z\| + 1}.$$

It follows that there is K, $0 < K < 1$ so close to 1 so that all $z \in D^n$ with $\|z\| > K$ satisfy $y_{n+1} > k$. Therefore, if we let $V = \{z \mid \|z\| > K\}$, we have

$$\psi(V) \subset \{y \in S^n \mid k < y_{n+1} \leq 1\} \subset U.$$

□

7.4 Local compactness and one-point compactification

In this section we will see how we can turn a Hausdorff space into a compact space by adding one more point, in a way that the open sets of the original space are still open. We will also see that the new space is Hausdorff itself, as long as it looks sufficiently much like a compact space close to points.

Definition 7.31. Let $(X, \mathcal{T})$ be a Hausdorff space. The *one-point compactification* (sv: *enpunktskompaktifiering*) of X is the space $\widehat{X} = X \cup \{\star\}$ obtained from X by adding a single point, called $\star$, equipped with the topology

$$\widehat{\mathcal{T}} = \mathcal{T} \cup \{(X \setminus K) \cup \{\star\} \mid K \subset X \text{ compact}\}.$$

The point $\star$ is often called the *point at infinity*.

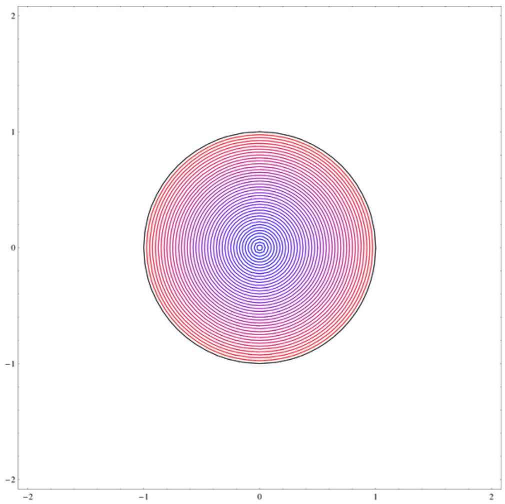

Figure 15: The closed unit ball D^2 and a number of contours in $B^2 \subset D^2$.

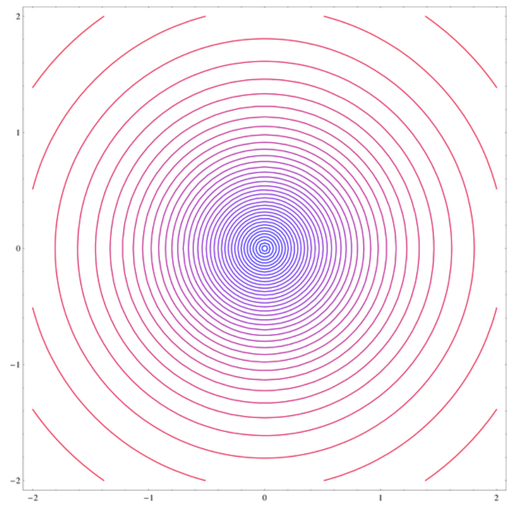

Figure 16: The result of applying $\varphi : B^2 \to \mathbb{R}^2$.

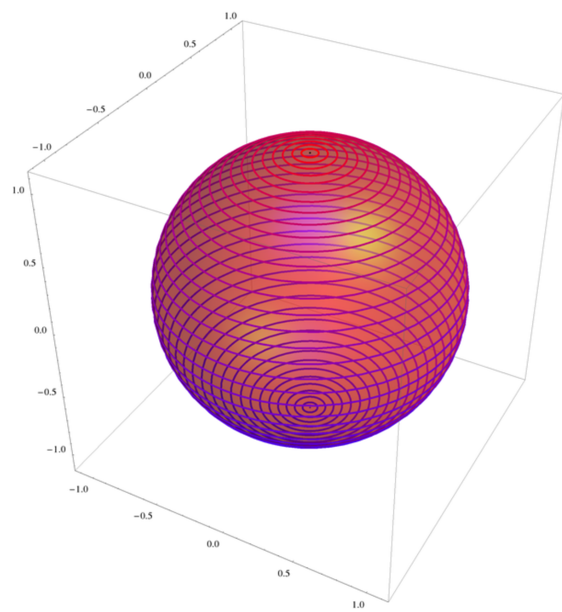

Figure 17: The contours in S^2 obtained by the map $g \circ \varphi$.

Proposition 7.32. *The collection $\widehat{\mathcal{T}}$ is actually a topology on $\widehat{X}$.*

Proof. Exercise 3.1. $\square$

Proposition 7.33. *The one-point compactification is compact.*

Proof. Let $\mathcal{U} = \{U_i\}_{i\in I}$ be an open cover of $\widehat{X}$ and let us show that it has a finite subcover. There is a $j \in I$ so that $\star \in U_j$. Clearly, $U_j \in \widehat{\mathcal{T}} \setminus \mathcal{T}$, so we see that $U_j = (X \setminus K) \cup \{\star\}$ for a compact set $K \subset X$.

Now clearly, $\{U_i \cap K\}_{i\in I}$ is an open cover of K, so by compactness there are $i_1, \dots, i_n$ so that $\{U_{i_1} \cap K, \dots, U_{i_n} \cap K\}$ cover K by compactness. In particular, $K \subset U_{i_1} \cup \cdots \cup U_{i_n}$. Now

$$\widehat{X} = X \setminus K \cup \{\star\} \cup K = U_j \cup K \subset U_j \cup U_{i_1} \cup \cdots \cup U_{i_n},$$

so we have found a finite subcover. $\square$

Definition 7.34. A topological space is called *locally compact* (sv: *lokalt kompakt*) if every point $x \in X$ has a neighbourhood contained in a compact subspace. I.e. if there exists a neighbourhood U of x and a compact set $K \subset X$ so that $x \in U \subset K \subset X$.

Proposition 7.35. *If X is a locally compact Hausdorff space, then its one-point compactification is Hausdorff.*

Proof. Let $x, y \in \widehat{X}$ be given, and assume that $x \neq y$. If $x, y \in X$, then we are done since X is Hausdorff and the open sets in $\widehat{X}$ include those that are open in X.

So, assume that $x \in X$ and $y = \star$. Using the definition of locally compactness, let $U \subset X$ be a neighbourhood at x, and let $K \subset X$ be compact, so that $x \in U \subset K \subset X$. It follows that $U \cap ((X \setminus K) \cup \{\star\}) = \emptyset$, and since $(X \setminus K) \cup \{\star\}$ is a neighbourhood of $\star$, we are done. $\square$

Before going into details on what the one-point compactification looks like, let us provide some examples of locally compact spaces.

Example 7.36. All compact spaces are locally compact.

Example 7.37. Euclidean space $\mathbb{R}^n$ is locally compact. For instance, for any x, $\overline{B(x,1)}$ is compact (by the Heine–Borel theorem) and contains $B(x,1)$.

Now as the inquisitive reader will have noticed, the word "locally" used in our definition of "locally compact" does not quite match up with how we used the word when talking about locally connected and locally path-connected spaces. The next result says that for Hausdorff spaces, the use of the word "local" will agree in these various cases.

Theorem 7.38. *Let X be a Hausdorff space. Then X is locally compact if and only if for every $x \in X$ and every neighbourhood U of x, there is a neighbourhood V of x so that $\overline{V}$ is compact and $\overline{V} \subset U$.*

Proof. Exercise 3.4. $\square$

We can use this to create a ton of locally compact spaces as subspaces of locally compact spaces. As for compact spaces, whenever we say that a subset of a topological space is locally compact, we mean that it is locally compact in the subspace topology.

Proposition 7.39. *If X is a locally compact space, and let $A \subset X$ be a subspace. Then A is locally compact if A is closed. If furthermore X is Hausdorff, then A is locally compact if A is open.*

Proof. Suppose that A is closed, and let $x \in A$. Let K and U be a compact subspace and neighbourhood of x respectively, with $x \in U \subset K \subset X$. Now $A \cap K$ is closed and thus compact by Theorem 7.8. Also, $U \cap A$ is a neighbourhood of x in A, and $x \in U \cap A \subset K \cap A \subset A$, so A is locally compact.

Assume now that X is Hausdorff, let A be open, and let $x \in A$. Then A is a neighbourhood of x, and by Theorem 7.38 we obtain a neighbourhood V of x so that $\overline{V}$ is compact, and $\overline{V} \subset A$, so A is locally compact. □

Equipped with a number of examples, we now look at concrete examples of one-point compactifications. First we show that one-point compactifications are in a certain sense unique.

Proposition 7.40. *Let X be a locally compact Hausdorff space. Suppose that Y has the properties that*

1. *X is a subspace of Y,*
2. *$Y \setminus X$ consists of a single point, and*
3. *Y is compact and Hausdorff.*

If Y' is another space with the same properties, then $Y \simeq Y'$.

Proof. Exercise 3.5. □

Proposition 7.41. *If Y is a compact Hausdorff space, then $\widehat{Y \setminus \{x\}} \simeq Y$ for any $x \in Y$.*

Proof. This follows from Proposition 7.40 since both $\widehat{Y \setminus \{x\}}$ and Y have the listed properties, where, in the notation of the proposition, $X = Y \setminus \{x\}$. □

Proposition 7.42. *We have $\widehat{\mathbb{R}^n} \simeq S^n$.*

Proof. This follows immediately from Proposition 7.41 and Proposition 5.11. □

One way to picture this result is shown in Figure 18, which also illustrates why we call $\star$ a "point at infinity".

Our next result says that under certain conditions, continuous maps between Hausdorff spaces extend to the one-point compactifications.

Definition 7.43. A continuous map $f : X \to Y$ is called *proper* (sv: *?*)proper map? if $f^{-1}(K)$ is compact, whenever $K \subset Y$ is compact.

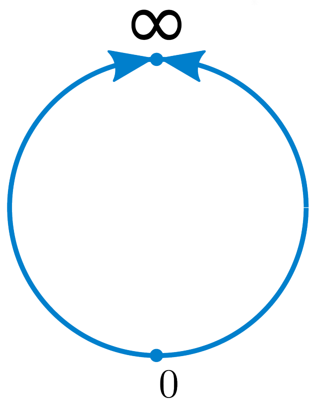

Figure 18: Illustration of $\widehat{\mathbb{R}} \simeq S^1$: we add a point at infinity and tie up the entire real line so it connects at this point.

Proposition 7.44. *Let X and Y be Hausdorff spaces, let $f : X \to Y$ be a proper map, and let $\widehat{X} = X \cup \{\star_X\}$ and $\widehat{Y} = Y \cup \{\star_Y\}$ denote the one-point compactifications. Then the map $\widehat{f} : \widehat{X} \to \widehat{Y}$ given by*

$$\widehat{f}(x) = \begin{cases} f(x), & \text{if } x \in X, \\ \star_Y, & \text{if } x = \star_X, \end{cases}$$

is continuous.

Proof. Let $U \subset \widehat{Y}$ be open. If $U \subset Y$, then $\widehat{f}^{-1}(U) = f^{-1}(U)$ is open since f is continuous. On the other hand, if $\star_Y \in U$, we know that $U = (Y \setminus K) \cup \{\star_Y\}$ for some compact set K in Y. Then

$$\widehat{f}^{-1}(U) = f^{-1}(Y \setminus K) \cup \{\star_X\} = (X \setminus f^{-1}(K)) \cup \{\star_X\}.$$

Since f is proper, $f^{-1}(K)$ is compact in X, so this says that $\widehat{f}^{-1}(U)$ is open. $\square$

We end this section with a discussion of an example of a space that is *not* locally compact. Consider the vector space

$$\mathbb{R}^\infty = \{x : \mathbb{N} \to \mathbb{R}\} = \{\{x_n\}_{n\in\mathbb{N}} \mid x_n \in \mathbb{R}\}$$

of all sequences of real numbers. For any $p \in [1, \infty)$ and $x \in \mathbb{R}^\infty$, define the p-norm of x by

$$\|x\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{1/p},$$

and the ∞-norm by

$$\|x\|_\infty = \sup_{n\in\mathbb{N}} \{|x_n|\}.$$

Now for any p, $1 \le p \le \infty$, notice that $\|\cdot\|_p$ actually defines a norm, meaning that $\|x\|_p \ge 0$ for all $x \in \mathbb{R}^\infty$, that $\|x\|_p = 0$ only if x is the constant sequence $x_n = 0$, that $\|rx\|_p = |r|\|x\|_p$ for all $r \in \mathbb{R}$, $x \in \mathbb{R}^\infty$, and that $\|x + y\|_p \leq \|x\|_p + \|y\|_p$ for all $x, y \in \mathbb{R}^\infty$. Put

$$\ell^p = \{x \in \mathbb{R}^\infty \mid \|x\|_p < \infty\}.$$

The norm can be used to define a metric on ℓ^p by $d_p(x, y) = \|x - y\|_p$, which in turn can be used to give ℓ^p the metric topology.

Proposition 7.45. *The topological space ℓ^p is not locally compact for any $1 \leq p \leq \infty$.*

Proof. Let U be any neighbourhood of the constant sequence $0 \in \ell^p$. We claim that there is no compact subspace containing U. Since U is open, there is an $\varepsilon > 0$ so that $B_{d_p}(0, \varepsilon) \subset U$. For each $n \in \mathbb{N}$, define a sequence $\delta_n = \{x_m\}_{m \in \mathbb{N}}$ by

$$x_m = \delta_{m,n} = \begin{cases} 1, & \text{if } m = n, \\ 0, & \text{otherwise.} \end{cases}$$

Then for any $n \in \mathbb{N}$, we have $\|\delta_n\|_p = 1$ for all p, $1 \leq p \leq \infty$, so $\delta_n \in \ell^p$.

Put $y_n = \frac{\varepsilon}{2}\delta_n$. Then for all n and p, $\|y_n\|_p = \frac{\varepsilon}{2}$, so $y_n \in B_{d_p}(0, \varepsilon)$. We claim that the sequence of sequences $\{y_n\}$ has no convergent subsequence; this then means that $B_{d_p}(0, \varepsilon)$ is not contained in any sequentially compact space, and we obtain the proposition from Theorem 7.22.

To see that $\{y_n\}$ has no convergent subsequence, notice that for $n, m \in \mathbb{N}$, $n \neq m$, we have

$$d_p(y_n, y_m) = \frac{\varepsilon}{2} d_p(\delta_n, \delta_m) = \frac{\varepsilon}{2}\|\delta_n - \delta_m\|_p = \begin{cases} \frac{\varepsilon}{2} 2^{1/p}, & p \in [1, \infty), \\ \frac{\varepsilon}{2}, & p = \infty. \end{cases}$$

In either case, $d_p(y_n, y_m) \geq \frac{\varepsilon}{2}$, so $\{y_n\}$ has no convergent subsequence. □

8 Separation and countability axioms

We encountered the first separation axioms in Section 4.2; in this section we introduce further notions of separations. We have already seen how the property of being T_2 allows for many useful results, and in the same spirit we will see how having more fine-grained separation allows for further characterisation of topological spaces.

8.1 Separation – part 2

Definition 8.1. A topological space is called *regular* (sv: *reguljärt*) if for every closed set $F \subset X$ and any point $x \in X \setminus F$ there exist open sets $U_x, U_F \subset X$ so that $x \in U_x$, $F \subset U_F$, and $U_x \cap U_F = \emptyset$.

A regular T_1-space is called T_3.

Notice that T_3-spaces are Hausdorff by Proposition 4.9.

Proposition 8.2. *Metric spaces are regular.*

Proof. Let (X, d) be a metric space, let $F \subset X$ be closed, and let $x \in X \setminus F$. Since $X \setminus F$ is open, there is an $\varepsilon > 0$ so that $B_d(x, \varepsilon) \subset X \setminus F$. Now let

$$V = \bigcup_{y \in F} B(y, \varepsilon/2).$$

Then V is open and $F \subset V$. We claim that $V \cap B(x, \varepsilon/2) = \emptyset$ which completes the proof. Assume that $z \in V \cap B(x, \varepsilon/2)$. Then there is a $y \in F$ with $d(z, y) < \varepsilon/2$. Since also $d(x, z) < \varepsilon/2$, the triangle inequality implies that

$$d(x, y) \leq d(x, z) + d(y, z) < \varepsilon,$$

which is impossible by definition of ε. □

Example 8.3. Consider the set $\mathbb{R}_K$ from Example 2.17. Then $\mathbb{R}_K$ is Hausdorff since $\mathbb{R}$ is Hausdorff, and since the topology on $\mathbb{R}_K$ is finer than the standard topology. Now K is closed in $\mathbb{R}_K$ by definition of the K-topology but it is impossible to separate 0 and K with disjoint open sets: assume that we could, and let U and V be the corresponding neighbourhoods of 0 and K respectively. Choose a basis element B with $0 \in B \subset U$. Now B must be of the form $(a, b) \setminus K$ since all intervals around 0 contain elements from K. Now take n so large that $1/n \in (a, b)$ and choose a basis element B' with $1/n \in B \subset V$. Then B must be an interval, and clearly this interval intersects $(a, b) \setminus K$, so U and V intersect.

At this point it is worth mentioning the existence of [SS70], the standard reference for answers to questions of the form "What topological space has property A but not property B?".

Definition 8.4. A topological space X is called *normal* (sv: *normalt*) if for all disjoint closed sets $F, G \subset X$ there are open sets U_F, U_G with $F \subset U_F$, $G \subset U_G$ and $U_F \cap U_G = \emptyset$. A T_1-space which is normal is called T_4.

Proposition 8.5. *Metric spaces are normal.*

Proof. Exercise. □

Proposition 8.6. *Compact Hausdorff spaces are normal.*

Proof. Exercise 3.3. □

Example 8.7. As before, Proposition 4.9 implies that T_4-spaces are T_3, but just as we saw above that a T_2-space need not be T_3, a T_3-space need not be T_4: an example of a T_3-space which is not T_4 is the so-called *Sorgenfrey plane* $\mathbb{R}_l \times \mathbb{R}_l$, where $\mathbb{R}_l$ was defined in Example 2.16. For details, see [Mun00, §31, Example 3].

Lemma 8.8. *Let X be T_1. Then*

(i) X is T_3 if and only if for each $x \in X$ and every neighbourhood U of x, there is a neighbourhood V of x with $x \in \overline{V} \subset U$, and

(ii) X *is* T_4 *if and only if for every closed set* $F \subset X$ *and every open* $U \subset X$ *with* $F \subset U$ *there is an open set* $V \subset X$ *with* $F \subset V \subset \overline{V} \subset U$*.*

Proof. Assume that X is T_3, let $x \in X$, and let U be a neighbourhood of x. Then $F = X \setminus U$ is closed and we can find open disjoint subsets $V, W \subset X$ so that $x \in V$ and $F \subset W$. We claim that $\overline{V} \cap F = \emptyset$ from which it follows that $\overline{V} \subset U$. If $y \in F$, then W is a neighbourhood of y which does not intersect V so $y \notin \partial V$, and $y \notin V$ so $y \notin \overline{V}$.

For the converse, let $x \in X$ and let $F \subset X$ be closed with $x \notin F$. Then $X \setminus F$ is a neighbourhood of x, so we can find a neighbourhood V of x with $x \in V \subset \overline{V} \subset X \setminus F$. That is, $X \setminus \overline{V}$ is open, contains F, and is disjoint from V.

For the second part of the theorem, one uses the same argument with x replaced by a general closed set. □

Theorem 8.9. *A subspace of a Hausdorff-space is Hausdorff, and a product of Hausdorff spaces is Hausdorff. A subspace of a* T_3*-space is* T_3*, and a product of* T_3*-spaces is* T_3*.*

Proof. The first part of the Theorem was Exercise 1.11.

Likewise, in Exercise 1.12 it is claimed that the product of two Hausdorff spaces is Hausdorff. Let us include a proof of the general case here: Let $\{X_i\}_{i \in I}$ be Hausdorff spaces, and let $x, y \in \prod_{i \in I} X_i$, $x \neq y$. Now for some $i \in I$, $x_i \neq y_i$, so choose U and V, neighbourhoods of x_i and y_i respectively with $U \cap V = \emptyset$. Then $\pi_i^{-1}(U)$ and $\pi_i^{-1}(V)$ are disjoint neighbourhoods of x and y respectively.

Let $Y \subset X$ be a subset of a T_3-space X. By the first part, Y is Hausdorff so in particular Y is T_1. Let $y \in Y$ be a point, and let $F \subset Y$ be closed in Y. Let $\overline{F}$ be the closure of F in X. Then $\overline{F} \cap Y = F$ so $y \notin \overline{F}$. By the T_3-property for X, we get open sets U_y, $U_{\overline{F}}$ with $y \in U_y$, $\overline{F} \subset U_{\overline{F}}$ and $U_y \cap U_{\overline{F}} = \emptyset$. Now the sets $Y \cap U_y$ and $Y \cap U_{\overline{F}}$ do the job; they are open in Y, disjoint, $y \in Y \cap U_y$ and $F \subset Y \cap U_{\overline{F}}$.

Let $\{X_i\}_{i \in I}$ be T_3-spaces and let $X = \prod_{i \in I} X_i$. As before, X is T_1 since X is T_2. Let $x = (x_i)_{i \in I} \in X$ be a point. We will use Lemma 8.8 to show that X is T_3, so let U be any neighbourhood of x. By definition of the product topology we can find open sets U_i in X_i so that $x \in \prod_{i \in I} U_i \subset U$, and so that $U_i = X_i$ for all but finitely many $i \in I$. Since each X_i is T_3, Lemma 8.8 provides us with neighbourhoods V_i of x_i so that $x_i \in V_i \subset \overline{V_i} \subset U_i$; if $U_i = X_i$ we simply take $V_i = X_i$. That is, $V_i = X_i$ for all but finitely many i so that $V = \prod_{i \in I} V_i$ is open in X, and by Proposition 4.7, we have $\overline{V} = \prod_{i \in I} \overline{V_i}$. Altogether we see that $x \in V \subset \overline{V} \subset U$, so X is T_3. □

8.2 Second countability

Definition 8.10. A topological space X is called *second-countable* (sv: *?*) if the topology on X has a countable basis.

Notice that a second-countable space is always first-countable.

Example 8.11. Euclidean space $\mathbb{R}^n$ is second-countable (Exercise 3.7).

Theorem 8.12. *Let* X *be second-countable. Then* *(i) every open cover of X has a countable subcover, and*

(ii) there is a countable dense subset of X.

A general space which has the property in (i) is called *Lindelöf*, and a space with the property in (ii) is called *separable* (sv: *separabelt*). The theorem then says that a second-countable space is Lindelöf and separable.

Proof. Let $\{B_n\}_{n\in\mathbb{N}}$ be a countable basis for the topology on X.

Let $\mathcal{U}$ be an open cover of X, and construct a countable cover as follows: for every $n \in \mathbb{N}$, we put $U_n = \emptyset$ if B_n is not contained in any $U \in \mathcal{U}$, and otherwise we let $U_n = U$ for some U with $B_n \subset U$. We need to show that $\{U_n\}$ cover X. So, let $x \in X$. Then there is a $U \in \mathcal{U}$ so that $x \in U$, and since U is open, there is a basis element B_n with $x \in B_n \subset U$. Now $x \in B_n \subset U_n$, so $x \in \bigcup_{n\in\mathbb{N}} U_n$, which means that the $\{U_n\}$ cover X.

For the second part, choose $x_n \in B_n$ for every $n \in \mathbb{N}$. For each $x \in X \setminus \{x_n\}_{n\in\mathbb{N}}$, and for any neighbourhood U of x, there is an n with $x \in B_n \subset U$. this implies that $x \in \overline{\{x_n\}_{n\in\mathbb{N}}}$, and since x was arbitrary, $\overline{\{x_n\}_{n\in\mathbb{N}}} = X$. $\square$

Example 8.13. Let X be an uncountable set with the discrete topology. Then $\{\{x\} \mid x \in X\}$ is an open cover of X which has no countable subcover, so X is not second-countable.

Theorem 8.14. *A second-countable T_3-space is normal (and thus T_4).*

Proof. Let X be a T_3-space with a countable basis $\{B_n\}_{n\in\mathbb{N}}$, and let F and G be closed in X. Since X is T_3, every $x \in F$ has a neighbourhood U_x which is disjoint from G. By Lemma 8.8, we can also find a neighbourhood V_x of x with $\overline{V_x} \subset U_x$, and finally we can find a basis element B_n so that $x \in B_n \subset V_x$. Carrying out this procedure for every $x \in F$, we obtain a countable family of basis elements $\{B_k^F\}_{k\in\mathbb{N}}$ that covers F and whose closures do not intersect G. Now, let $\tilde{U}^F = \bigcup_{k\in\mathbb{N}} B_k^F$.

By doing the same for all points in G, we find a countable family $\{B_k^G\}_{k\in\mathbb{N}}$ that covers G and such that the closure of each basis element does not intersect F, and we let $\tilde{U}^G = \bigcup_{k\in\mathbb{N}} B_k^G$. Now $\tilde{U}^F$ and $\tilde{U}^G$ are open and contain F and G respectively, but they need not be disjoint.

What we do instead is essentially remove all the problematic points from $\tilde{U}^F$ and $\tilde{U}^G$ as follows: for every given $n \in \mathbb{N}$, define

$$\widehat{B}_n^F = B_n^F \setminus \bigcup_{k=1}^{n} \overline{B_k^G}, \quad \widehat{B}_n^G = B_n^G \setminus \bigcup_{k=1}^{n} \overline{B_k^F}$$

Then $\widehat{B}_n^F$ and $\widehat{B}_n^G$ are open for all n since we remove from an open set something closed (and in general, such a difference can be written as the intersection of two open sets). Let

$$U_F = \bigcup_{n\in\mathbb{N}} \widehat{B}_n^F, \quad U_G = \bigcup_{n\in\mathbb{N}} \widehat{B}_n^G.$$

Then the sets U_F and U_G are open, and we claim that $F \subset U_F$, $G \subset U_G$, and $U_F \cap U_G = \emptyset$.

If $x \in F$, then $x \in B_n^F$ for some n. Since none of the $\overline{B_k^G}$ intersect F, we know that x does not belong to any of these, so it follows that $x \in \widehat{B}_n^F \subset U_F$. Now $G \subset U_G$ by the same logic.

To see that U_F and U_G are disjoint, let $x \in U_F \cap U_G$. Then $x \in \widehat{B}_n^F \cap \widehat{B}_m^G$ for some n and m. We see that this is impossible by definition of $\widehat{B}_n^F$ and $\widehat{B}_m^G$ by considering the two cases $n \leq m$ and $m \leq n$. □

8.3 Urysohn's lemma

By definition, a T_4-space is a space where disjoint closed sets can be separated by disjoint open sets containing the closed sets. In this section, we mention Urysohn's lemma, which says that disjoint closed sets can be separated by continuous functions, in a very concrete sense.

Lemma 8.15 (Urysohn's lemma)**.** *Let X be a T_4-space, let F and G be closed disjoint subsets, and let $a, b \in \mathbb{R}$ be real numbers with $a \leq b$. Then there is a continuous function $f : X \to [a, b]$ so that $f(F) = \{a\}$, $f(G) = \{b\}$.*

The proof is rather involved and unlike all other results that we have encountered so far, it is not sufficient to simply juggle definitions. Instead of giving a proof, which can be found in [Mun00, §33], we will provide an example of its power in the next section.

Let us end this section with a different application of Urysohn's lemma. Notice that so far, most of the concrete examples of topological spaces that we have considered have all been metric spaces. Likewise we know that any given set can be given both a topology and a metric, so there is a natural question: given a topological space $(X, \mathcal{T})$, is there a metric d on X so that $\mathcal{T}$ is the metric topology? If so, we say that X is *metrisable* (sv: *metriserbart*).

Example 8.16. All metric spaces are metrisable.

Example 8.17. All products of metrisable spaces are metrisable by Exercise 1.16.

Example 8.18. Any discrete topological space is metrisable by Example 2.23.

If a topological space X is metrisable, it has all the topological properties that general metric spaces have. For instance, all metrisable spaces are normal by Proposition 8.5. Therefore a space which is not normal is also not metrisable, and such spaces exist.

Now Urysohn's metrisation theorem provide a sufficient condition for a topological space to be metrisable. A proof can be found in [Mun00, §34].

Theorem 8.19 (Urysohn's metrisation theorem)**.** *All second-countable T_3-spaces are metrisable.*

Notice that the converse is not true: an uncountable metric space with the discrete metric (Example 2.23) is not second-countable (Example 8.13).

9 Manifolds

The concept of a manifold is central in all of differential geometry and mathematical physics; roughly, a manifold is a topological space which locally looks like $\mathbb{R}^n$. Another way of viewing it is that a manifold is something which is obtained by gluing together copies of $\mathbb{R}^n$. As such, its usefulness in for instance geometry comes from the fact that we can transfer everything we know about calculus on $\mathbb{R}^n$ to this much more general family of topological spaces, as long as one ensures that the gluing is sufficiently compatible with calculus. Now, we will not be discussing calculus here but rather take a look at manifolds from a purely topological point of view.

9.1 Topological manifolds

Definition 9.1. A topological space X is called *locally Euclidean* (sv: *lokalt euklidisk*) if there is an $n \in \mathbb{N}$ so that every point in X has a neighbourhood which is homeomorphic to $\mathbb{R}^n$.

Definition 9.2. An *n-dimensional manifold* (sv: *mångfald*) or simply an *n-manifold* is a locally Euclidean second-countable Hausdorff space. The n refers to the n of Definition 9.1.

Really what we have defined above is a *topological manifold*; since this is the only kind of manifold we will encounter, we will simply call them "manifolds".

Example 9.3. Euclidean space $\mathbb{R}^n$ is an n-manifold since $\mathbb{R}^n$ itself is a neighbourhood of all of its points.

Example 9.4. The n-sphere S^n is an n-manifold. We know by now that a subspace of a Hausdorff space is Hausdorff, and it is not difficult to see that a subspace of a second-countable space is itself second-countable, so we only need to see that S^n is locally Euclidean.

If $x \in S^n$ is a point different from the north pole $p = (0, \ldots, 0, 1)$, then $S^n \setminus \{p\}$ is a neighbourhood of x which is homeomorphic to $\mathbb{R}^n$ by Proposition 5.11. If $x = p$, let q denote the south pole. Then $S^n \setminus \{q\}$ is a neighbourhood of x which is homeomorphic to $\mathbb{R}^n$ by Remark 5.12.

Lemma 9.5. *The product of an n-manifold and an m-manifold is an $(n+m)$-manifold.*

Proof. Exercise 4.1. $\square$

Example 9.6. The n-torus T^n is an n-manifold by Lemma 9.5 and Example 9.4.

Example 9.7. The genus g surfaces Σ_g from Example 5.25 are 2-manifolds. Our definition of Σ_g is unprecise enough that this is slightly painful to prove; it should, however, be a very reasonable claim, given Figures 11-13.

9.2 Embeddings of manifolds

Notice that by definition, S^n can be embedded in $\mathbb{R}^{n+1}$. Similarly, T^n can be embedded in $\mathbb{R}^{2n}$, and Figures 11-13 suggest that Σ_g can be embedded in $\mathbb{R}^3$.

In this section we will see how to use Urysohn's lemma to show the following result.

Theorem 9.8. *Any compact m-manifold can be embedded in $\mathbb{R}^N$ for some $N \in \mathbb{N}$.*

Definition 9.9. Let X be a topological space and $f : X \to \mathbb{R}$ a function. The *support* (sv: *stöd*) of f is the set

$$\operatorname{supp}(f) = \overline{\{x \mid f(x) \neq 0\}}.$$

Definition 9.10. Let X be a topological space, and let $\{U_1, \dots, U_n\}$ be an open cover of X. A family $\{\varphi_1, \dots, \varphi_n\}$ of continuous functions $\varphi_i : X \to [0,1]$ is called a *partition of unity* (sv: *partition av enheten*) dominated by $\{U_i\}$ if

- $\operatorname{supp}(\varphi_i) \subset U_i$ for $i = 1, \dots, n$, and
- $\sum_{i=1}^n \varphi_i(x) = 1$ for all $x \in X$.

Theorem 9.11. *Let X be a T_4-space, and let $\{U_1, \dots, U_n\}$ be a finite open cover. Then there exists a partition of unity dominated by $\{U_1, \dots, U_n\}$.*

Proof. We first show that we can find an open cover $\{V_1, \dots, V_n\}$ so that $\overline{V_i} \subset U_i$ for all i. Consider the set $A_1 = X \setminus (U_2 \cup \dots \cup U_n)$. This is clearly closed, and $A_1 \subset U_1$ since $\{U_i\}$ is a cover. Since X is T_4, by Theorem 8.8 we obtain an open set V_1 so that $A_1 \subset V_1 \subset \overline{V_1} \subset U_1$, and in particular $\{V_1, U_2, \dots, U_n\}$ is still an open cover. We proceed now by finite induction: suppose that we have constructed open sets V_i, $i < k$, so that $\overline{V_i} \subset U_i$, and so that $\{V_1, \dots, V_{k-1}, U_k, \dots, U_n\}$ covers X. Then let

$$A_k = X \setminus (V_1 \cup \dots \cup V_{k-1} \cup U_{k+1} \cup \dots \cup U_n).$$

Then $A_k \subset U_k$, and we find as above an open set V_k with $A_k \subset V_k \subset \overline{V_k} \subset U_k$ so that $\{V_1, \dots, V_k, U_{k+1}, \dots, U_n\}$ covers X.

Now go through the same procedure again to obtain an open cover $\{W_1, \dots, W_n\}$ with $\overline{W_i} \subset V_i$ for all i. Applying Urysohn's lemma for each $i = 1, \dots, n$, we find continuous functions $\psi_i : X \to [0,1]$ so that $f(X \setminus V_i) = \{0\}$ and $f(\overline{W_i}) = \{1\}$. It follows that

$$\operatorname{supp}(\psi_i) \subset \overline{V_i} \subset U_i.$$

Since $\{W_i\}$ is a cover of X, it follows that $\psi(x) = \sum_{i=1}^n \psi_i(x) > 0$ for all x. Now define $\varphi_i : X \to [0,1]$ by

$$\varphi_i(x) = \frac{\psi_i(x)}{\psi(x)}.$$

We then have $\operatorname{supp}(\varphi_i) = \operatorname{supp}(\psi_i) \subset U_i$, and for every $x \in X$, we have

$$\sum_{i=1}^n \varphi_i(x) = \frac{1}{\psi(x)} \sum_{i=1}^n \psi_i(x) = 1,$$

so $\{\varphi_i\}$ is a partition of unity dominated by $\{U_1, \dots, U_n\}$. $\square$

Proof of Theorem 9.8. Let X be a compact m-manifold, and choose for every $x \in X$ a neighbourhood U_x of x so that $U_x \simeq \mathbb{R}^m$. These will cover X, so since X is compact, we obtain a finite open cover $\{U_1, \dots, U_n\}$ together with homeomorphisms $g_i : U_i \to \mathbb{R}^m$ for every i. Since X is compact and Hausdorff, X is T_4 by Proposition 8.6, so by Theorem 9.11 we can find a partition of unity $\{\varphi_i\}$ dominated by $\{U_i\}$. Let $A_i = \operatorname{supp}(\varphi_i)$, note that $X = U_i \cup (X \setminus A_i)$, and define for each $i = 1, \dots, n$ a function $h_i : X \to \mathbb{R}^m$ by

$$h_i(x) = \begin{cases} \varphi_i(x) g_i(x), & \text{for } x \in U_i \\ 0, & \text{for } x \in X \setminus A_i. \end{cases}$$

Notice that h_i is well-defined since $\varphi_i(x) g_i(x) = 0$ for $x \in X \setminus A_i$, and h_i is continuous by Lemma 3.9; here one has to check that $x \mapsto \varphi_i(x) g_i(x)$ is continuous on U_i which can be seen by Theorem 4.24. The desired embedding will be the map

$$F : X \to \underbrace{\mathbb{R} \times \cdots \times \mathbb{R}}_{n \text{ factors}} \times \underbrace{\mathbb{R}^m \times \cdots \times \mathbb{R}^m}_{n \text{ factors}} \simeq \mathbb{R}^{(m+1)n}$$

given by

$$F(x) = (\varphi_1(x), \dots, \varphi_n(x), h_1(x), \dots, h_n(x))$$

Now F is continuous since the φ_i and h_i are, so since X is compact, it follows from Corollary 7.11 that F is an embedding if we can show that F is injective.

Suppose that $F(x) = F(y)$. Then $\varphi_i(x) = \varphi_i(y)$ and $h_i(x) = h_i(y)$ for all i. Since $\sum_{i=1}^n \varphi_i(x) = 1$, there is an i with $\varphi_i(x) > 0$, so $\varphi_i(y) > 0$ as well, which implies that $x, y \in \operatorname{supp}(\varphi_i) \subset U_i$. Now

$$\varphi_i(x) g_i(x) = h_i(x) = h_i(y) = \varphi_i(y) g_i(y),$$

so we must also have $g_i(x) = g_i(y)$. Since each g_i was a homeomorphism, this implies that $x = y$. $\square$

In fact, it turns out that the condition of Theorem 9.8 that the manifold is compact is not necessary. Moreover, one could ask how small it is possible to choose N in the theorem; in the proof we saw that an m-manifold X can be embedded in $\mathbb{R}^{(m+1)n}$, where n is the cardinality of an open cover of X whose constituent open sets are homeomorphic to $\mathbb{R}^m$. As the examples in the beginning of this section illustrate however, we should be able to do better: S^m is an m-manifold which can be covered by 2 such open sets, so the theorem provides us with an embedding $S^m \to \mathbb{R}^{2(m+1)}$, but we also know that there is also an embedding $S^m \to \mathbb{R}^{m+1}$.

The following result extends Theorem 9.8 to the non-compact case and provides an explicit bound on the required dimension. We do not include a proof and refer instead to [Mun00, §50, Exercises 6–7].

Theorem 9.12. *Any m-manifold can be embedded in $\mathbb{R}^{2m+1}$.*

Recall that manifolds are assumed to be both Hausdorff and second-countable. Since we argued that manifolds are natural objects in geometry, we should provide some motivation for these requirements. Now as we have seen plenty of times, the property of being Hausdorff is necessary to do any kind of calculus – for instance, without this condition, one would have manifolds with convergent sequences but no unique limit (compare with Proposition 4.20).

It is less clear though, why we require manifolds to be second-countable, but it turns out that if we did not add this condition, Theorem 9.12 would be false; a counter-example is the so-called long line – see [Mun00, §24, Exercise 12]. Thus insofar that one considers embedding into Euclidean space to be a sufficient amount of motivation, second-countability is sufficient.

9.3 Paracompactness

As it turns out, one of the most useful tools for studying manifolds are the partitions of unity that we encountered in the previous section. In the proof of Theorem 9.8 – where they played an essential role – we saw that these exist for compact manifolds but many interesting manifolds are not compact; second-countability provides us with something almost as good. Here, we will illustrate how, referring to [Mun00] for most of the proofs.

Definition 9.13. Let X be a topological space. A collection $\mathcal{U}$ of subsets of X is called *locally finite* (sv: *lokalt ändlig*) if every point of X has a neighbourhood that intersects only finitely many elements of $\mathcal{U}$.

Definition 9.14. A topological space X is called *paracompact* (sv: *parakompakt*) if every open cover has a locally finite subcover.

Notice that a finite cover is always locally finite. Thus in particular, all compact spaces are paracompact. The next result says that for paracompact Hausdorff spaces, we always have a locally finite version of partitions of unity. It turns out that such spaces are normal [Mun00, Thm. 41.1] and so the proof is almost identical to that of Theorem 9.11; see [Mun00, Thm. 41.7].

Theorem 9.15. *Let X be a paracompact Hausdorff space, and let $\{U_i\}_{i\in I}$ be an open cover of X. Then there exists a partition of unity dominated by $\{U_i\}_{i\in I}$; that is, there exists a family $\{\varphi_i\}_{i\in I}$ of continuous functions $\varphi_i : X \to [0,1]$ so that*

(i) $\operatorname{supp}(\varphi_i) \subset U_i$ *for all* $i \in I$,

(ii) $\{\operatorname{supp}(\varphi_i)\}_{i\in I}$ *is locally finite, and*

(iii) $\sum_{i\in I} \varphi_i(x) = 1$ *for every* $x \in X$.

Notice here that the sum appearing in (iii) makes sense because of the locally finiteness from (ii). We end our discussion by noting that in the context of manifolds, paracompactness and second-countability is almost the same thing.

Theorem 9.16. *Let X be a locally Euclidean Hausdorff space. Then X is second-countable if and only if X is paracompact and has countably many connected components.*

10 Introduction to homotopy theory

In this final section of this note, we will introduce a powerful topological invariant. In doing so, we tread slightly into the realm of algebraic topology.

10.1 Homotopy

Definition 10.1. Let X and Y be topological spaces, and let $f, g : X \to Y$ be continuous maps. We say that f is *homotopic* (sv: *homotop*) to g if there exists a continuous map $F : X \times [0,1] \to Y$ so that

$$F(x, 0) = f(x) \quad \text{and} \quad F(x, 1) = g(x)$$

for all $x \in X$. The map F is called a *homotopy* (sv: *homotopi*) from f to g, and we write $f \sim g$. If $f \sim g$ where g is a constant map, we say that f is *null-homotopic* (sv: *nollhomotop*)

We will primarily be interested in the special case where the maps f and g are paths that start and end at the same point. In this case, we will furthermore require that the homotopy fixes the two end-points of the paths:

Definition 10.2. Two $\gamma, \gamma' : [0,1] \to X$ be two paths from x to y in a topological space X. We say that γ is *path homotopic* (sv: *väghomotop*) to γ' if there is a homotopy $F : [0,1] \times [0,1] \to X$ from γ to γ' so that

$$F(0, t) = x, \quad F(1, t) = y$$

for all $t \in [0,1]$. The map F is called a *path homotopy* (sv: *väghomotopi*), and we write $\gamma \sim_p \gamma'$. See Figure 19.

Lemma 10.3. *Homotopy $\sim$ and path homotopy $\sim_p$ are equivalence relations.*

Proof. Let $f, g, h : X \to Y$ be continuous maps.

To see reflexivity, define $F : X \times [0,1] \to Y$ by $F(x,t) = f(x)$. Then F is continuous and $F(x,1) = F(x,0) = f(x)$ for all x, so F is a homotopy from f to f, and $f \sim f$. If f is a path, then F is a path homotopy, so $f \sim_p f$.

For symmetry, suppose that $f \sim g$. Then there is a homotopy $F : X \times [0,1] \to Y$ from f to g. Define $G(x,t) = F(x, 1-t)$. Then G is continuous since it is a composition of continuous functions, and G is a homotopy from g to f, so $g \sim f$. If f and g are paths, then G is a path homotopy, so $f \sim_p g$ implies that $g \sim_p f$.

Finally, for transitivity, if $f \sim g$ and $g \sim h$, let F be a homotopy from f to g, and let G be a homotopy from g to h. Define a function $H : X \times [0,1] \to Y$ by

$$H(x,t) = \begin{cases} F(x, 2t), & \text{if } t \in [0, \frac{1}{2}], \\ G(x, 2t-1), & \text{if } t \in [\frac{1}{2}, 1]. \end{cases}$$

Then H is continuous by Remark 3.10, and H is a homotopy from f to h, so $f \sim h$. If F and G are path homotopies, then so is H. $\square$

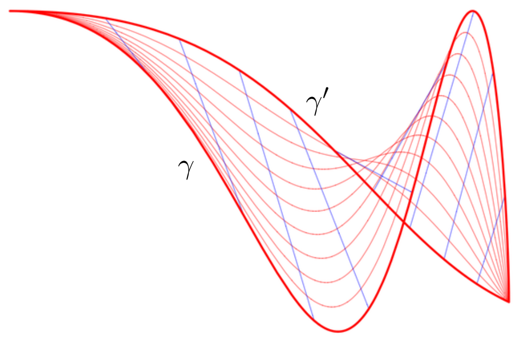

Figure 19: Two homotopic paths γ and γ' in $\mathbb{R}^2$ as well as a path homotopy F between them. That is, we picture $F(s,0) = \gamma(s)$, $F(s,1) = \gamma'(s)$, the paths $F(s, \frac{1}{10}), F(s, \frac{2}{10}), \dots, F(s, \frac{9}{10})$ (in red) and the paths $F(\frac{1}{10}, t), F(\frac{2}{10}, t), \dots, F(\frac{9}{10}, t)$ (in blue).

Example 10.4. Let $f, g : X \to \mathbb{R}^n$ be two continuous functions. Then the map $F : X \times [0,1] \to \mathbb{R}^n$ given by

$$F(x,t) = (1-t)f(x) + tg(x)$$

is a homotopy from f to g. That is, all functions into $\mathbb{R}^n$ are homotopic. In other words, there is only one homotopy equivalence class.

Likewise, if γ and γ' are paths from x to y in $\mathbb{R}^n$, then γ and γ' are homotopic: there is only a single equivalence class of path homotopy. Indeed, the path homotopy illustrated in Figure 19 is obtained in exactly this way.

In the special case where $x = y$, this means that all paths are null-homotopic.

Example 10.5. Let γ and γ' be the paths from $(0,1)$ to $(0,-1)$ given by

$$\gamma(t) = (\cos(\pi t), \sin(\pi t)), \quad \gamma'(t) = (\cos(\pi t), -\sin(\pi t)).$$

Then γ and γ' are path homotopic as paths in $\mathbb{R}^2$ by the previous example, but they are *not* path homotopic as paths in $\mathbb{R}^2 \setminus \{(0,0)\}$. This is a non-trivial fact though (and can be seen as a consequence of Exercise 4.9), but for instance, the homotopy from the previous example does not work since

$$F(\tfrac{1}{2}, \tfrac{1}{2}) = \tfrac{1}{2}(\gamma(\tfrac{1}{2}) + \gamma'(\tfrac{1}{2})) = (0,0).$$

If γ is a path, denote by $[\gamma]$ its path homotopy equivalence class or in short, its *homotopy class* (sv: *homotopiklass*). Recall from Section 6.3 the definitions of concatenation of paths and the reverse of a path.

Proposition 10.6. *Let γ be a path from x to y in some space X, and let γ' be a path from y to z. Then the operation*

$$[\gamma] \star [\gamma'] = [\gamma \star \gamma']$$

is well-defined.

Proof. Suppose that F is a path homotopy from γ to some other curve $\tilde{\gamma}$ and that G is a path homotopy from γ' to $\widetilde{\gamma'}$. The claim that the operation is well-defined is then the claim that $\gamma \star \gamma' \sim_p \tilde{\gamma} \star \widetilde{\gamma'}$. Define $H : [0,1] \times [0,1] \to X$ by

$$H(s,t) = \begin{cases} F(2s,t), & \text{if } s \in [0, \frac{1}{2}], \\ G(2s-1,t), & \text{if } s \in [\frac{1}{2}, 1]. \end{cases}$$

Then H is continuous by Remark 3.10 and it is easy to check that H is a path homotopy from $\gamma \star \gamma'$ to $\tilde{\gamma} \star \widetilde{\gamma'}$. □

For a point $x \in X$ in a topological space, let $e_x : [0,1] \to X$ denote the constant path $e_x(t) = x$, for $t \in [0,1]$.

Theorem 10.7. *The operation $\star$ has the following properties for all paths γ, γ', and γ'' in a topological space X:*

(i) $[\gamma] \star ([\gamma'] \star [\gamma'']) = ([\gamma] \star [\gamma']) \star [\gamma'']$ *when one (and thus both) are defined,*

(ii) $[\gamma] \star [e_y] = [e_x] \star [\gamma] = [\gamma]$, *if γ is a path from x to y, and*

(iii) $[\gamma] \star [\gamma^{\mathrm{rev}}] = [e_x]$, $[\gamma^{\mathrm{rev}}] \star [\gamma] = [e_y]$, *if γ is a path from x to y.*

Proof. We begin by showing that the homotopy class of a curve γ from x to y does not depend on its parametrisation. To be precise, let $\varphi : [0,1] \to [0,1]$ be any continuous map with $\varphi(0) = 0$, $\varphi(1) = 1$. Then $\gamma \circ \varphi$ is a path from x to y, and we claim that $\gamma \sim_p \gamma \circ \varphi$. To see this, let $F : [0,1] \times [0,1] \to X$ be the map

$$F(s,t) = \gamma(t\varphi(s) + (1-t)s).$$

Then F is continuous, $F(s,0) = \gamma(s)$, $F(s,1) = \gamma \circ \varphi(s)$, $F(0,t) = \gamma(0) = x$, and $F(1,t) = \gamma(1) = y$, so F is a homotopy from γ to $\gamma \circ \varphi$.

Now we can show each of the first two cases of the theorem by picking φ appropriately. Let us begin, for instance, by showing (ii). We have to show that $\gamma \star e_y \sim_p \gamma$, and that $e_x \star \gamma \sim_p \gamma$. By definition,

$$(\gamma \star e_y)(s) = \begin{cases} \gamma(2s), & s \in [0, \frac{1}{2}], \\ e_y(2s-1), & s \in [\frac{1}{2}, 1], \end{cases} = \begin{cases} \gamma(2s), & s \in [0, \frac{1}{2}], \\ y, & s \in [\frac{1}{2}, 1], \end{cases} = \begin{cases} \gamma(2s), & s \in [0, \frac{1}{2}], \\ \gamma(1), & s \in [\frac{1}{2}, 1]. \end{cases}$$

That is, $(\gamma \star e_y)(s) = \gamma(\varphi_1(s))$, where $\varphi_1 : [0,1] \to [0,1]$ is first map illustrated in Figure 20. Thus $\gamma \star e_y = \gamma \circ \varphi_1 \sim_p \gamma$.

Similarly, $e_x \star \gamma = \gamma \circ \varphi_2$, which completes the proof of (ii). For (i), one finds that $\gamma \star (\gamma' \star \gamma'') = ((\gamma \star \gamma') \star \gamma'') \circ \varphi_3$.

For (iii) we give a homotopy explicitly. Let us show that $\gamma \star \gamma^{\mathrm{rev}} \sim_p e_x$. For $t \in [0,1]$, define a path $\gamma_t : [0,1] \to X$ by $\gamma(s) = \gamma(ts)$, and define $G : [0,1] \times [0,1] \to X$ by

$$G(s,t) = (\gamma_t \star \gamma_t^{\mathrm{rev}})(s).$$

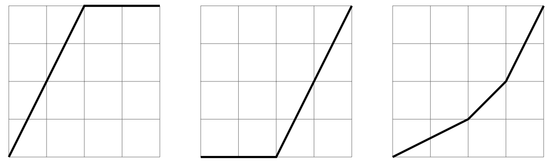

Figure 20: Graphs of the functions φ_1, φ_2, and φ_3 respectively.

That G is continuous follows once again from an argument using Remark 3.10, and we see that G is a homotopy from e_x to $\gamma \star \gamma^{\mathrm{rev}}$ since

$$\begin{aligned}
G(s,0) &= (\gamma_0 \star \gamma_0^{\mathrm{rev}})(s) = \gamma(0) = x = e_x(s),\\
G(s,1) &= (\gamma_1 \star \gamma_1^{\mathrm{rev}})(s) = (\gamma \star \gamma^{\mathrm{rev}})(s),\\
G(0,t) &= \gamma_t(0) = \gamma(0) = x,\\
G(1,t) &= \gamma_t^{\mathrm{rev}}(1) = \gamma(0) = x,
\end{aligned}$$

for every s and t. That $\gamma^{\mathrm{rev}} \star \gamma \sim_p e_y$ follows by an analogous argument. □

10.2 The fundamental group

The idea in this section will be to use the operation $\star$ on path homotopy classes to associate an algebraic structure to any pair (X, x) for X a topological space and $x \in X$. Moreover, when X is path-connected, this structure will form a powerful topological invariant.

If γ is a path from x to x, we say that γ is a *loop* (sv: *ögla*) based at x.

Definition 10.8. Let X be a topological space, and let $x \in X$. Then the *fundamental group* (sv: *fundamentalgrupp*) $\pi_1(X, x)$ is the set of all path homotopy classes of loops based at x.

To make sense of the terminology, let us recall a few basic notions from abstract algebra.

Definition 10.9. A *group* (sv: *grupp*) is a set G with an operation $G \times G \to G$, denoted $(g, h) \mapsto g \cdot h$, an element $e \in G$ called a unit, and a bijection $G \to G$ denoted $x \mapsto x^{-1}$ called the inverse, so that

- $g \cdot (h \cdot k) = (g \cdot h) \cdot k$ for all $g, h, k \in G$,
- $e \cdot g = g = g \cdot e$ for all $g \in G$, and
- $g \cdot g^{-1} = g^{-1} \cdot g = e$ for all $g \in G$.

If G and H are groups, then a map $\varphi : G \to H$ is called a *homomorphism* (sv: *homomorfi*) if $\varphi(g \cdot h) = \varphi(g) \cdot \varphi(h)$ for all $g, h \in G$. A bijective group homomorphism is called an *isomorphism* (sv: *isomorfi*).

Example 10.10. The one-point set $\{e\}$ is a group under the operation $(e, e) \mapsto e$. This group is called the *trivial group*.

Example 10.11. The integers form a group under the operation $(g, h) \mapsto g + h$. The unit is $0 \in \mathbb{Z}$, and if $n \in \mathbb{Z}$, then the inverse of n is $-n$.

Example 10.12. The set $\mathbb{R} \setminus \{0\}$ is a group with operation $(g, h) \mapsto gh$. The unit is 1, and the inverse of $x \in \mathbb{R} \setminus \{0\}$ is $1/x$.

Example 10.13. The set $\mathrm{GL}(n, \mathbb{R})$ of invertible $(n \times n)$-matrices with entries in $\mathbb{R}$ is a group under matrix multiplication. The unit is the unit matrix.

Proposition 10.14. *The fundamental group $\pi_1(X, x)$ is a group under the operation $\star$ on homotopy classes of loops for any topological space X and any $x \in X$.*

Proof. This follows immediately from Theorem 10.7. $\square$

Example 10.15. In Example 10.4 we saw that any two given paths in $\mathbb{R}^n$ between the same points were homotopic. This in particular implies that any loop based at a point $x \in \mathbb{R}^n$ is null-homotopic; that is, homotopic to e_x. In other words,

$$\pi_1(\mathbb{R}^n, x) = \{[e_x]\},$$

the trivial group, for all $x \in \mathbb{R}^n$.

As the next thing, let us see how $\pi_1(X, x)$ depends on x.

Theorem 10.16. *Let X be a topological space, and let α be a path from x to y in X. Define a map $\widehat{\alpha} : \pi_1(X, x) \to \pi_1(X, y)$ by*

$$\widehat{\alpha}([\gamma]) = [\alpha^{\mathrm{rev}}] \star [\gamma] \star [\alpha].$$

Then $\widehat{\alpha}$ is well-defined and an isomorphism.

Proof. That $\widehat{\alpha}$ is well-defined means that $\widehat{\alpha}([\gamma]) = \widehat{\alpha}([\gamma'])$ whenever $[\gamma] = [\gamma']$, i.e. whenever $\gamma \sim_p \gamma'$. And indeed, if $F : [0, 1] \times [0, 1] \to X$ is a path homotopy from γ to γ', then $G : [0, 1] \times [0, 1] \to X$, defined by

$$G(s, t) = (\alpha^{\mathrm{rev}} \star F(\cdot, t) \star \alpha)(s)$$

is a path homotopy from $\alpha^{\mathrm{rev}} \star \gamma \star \alpha$ to $\alpha^{\mathrm{rev}} \star \gamma' \star \alpha$, so $\widehat{\alpha}$ is well-defined.

To see that $\widehat{\alpha}$ is an homomorphism, notice that for any $[\gamma], [\gamma'] \in \pi_1(X, x)$, we have

$$\begin{aligned} \widehat{\alpha}([\gamma]) \star \widehat{\alpha}([\gamma']) &= [\alpha^{\mathrm{rev}}] \star [\gamma] \star [\alpha] \star [\alpha^{\mathrm{rev}}] \star [\gamma'] \star [\alpha] \\ &= [\alpha^{\mathrm{rev}}] \star ([\gamma] \star [\gamma']) \star [\alpha] = \widehat{\alpha}([\gamma] \star [\gamma']). \end{aligned}$$

To see that $\widehat{\alpha}$ is a bijection, notice that $\widehat{\alpha^{\mathrm{rev}}} \circ \widehat{\alpha}$ is the identity on $\pi_1(X, x)$ since for any $[\gamma] \in \pi_1(X, x)$, we have

$$(\widehat{\alpha^{\mathrm{rev}}} \circ \widehat{\alpha})[\gamma] = \widehat{\alpha^{\mathrm{rev}}}([\alpha^{\mathrm{rev}}] \star [\gamma] \star [\alpha]) = [\alpha] \star [\alpha^{\mathrm{rev}}] \star [\gamma] \star [\alpha] \star [\alpha^{\mathrm{rev}}] = [\gamma],$$

and $\widehat{\alpha} \circ \widehat{\alpha^{\mathrm{rev}}}$ is the identity on $\pi_1(X, y)$ by the same reasoning, so $\widehat{\alpha}$ is a bijection and thus a group isomorphism. $\square$

Corollary 10.17. *If X is a path-connected topological space, then $\pi_1(X, x)$ is independent of $x \in X$ up to isomorphism.*

Because of this result, one often writes $\pi_1(X) = \pi_1(X, x)$ for any $x \in X$ when X is path-connected. It is then understood that the equality is really up to isomorphism.

Definition 10.18. A topological space X is called *simply-connected* (sv: *enkelt sammanhängande*) if it is path-connected and $\pi_1(X)$ consists of a single element.

Example 10.19. By Example 10.15, $\mathbb{R}^n$ is simply-connected.

The next result says that for path-connected spaces, π_1 is a topological invariant. Even when the spaces in question are not path-connected, one obtains a topological invariant by considering the collection of groups $\pi_1(X, x_i)$ up to isomorphism, where each of the x_i belongs to a different path-component of X.

As preparation, suppose that $f : X \to Y$ is a continuous map, and let $x \in X$. Define a map

$$f_* : \pi_1(X, x) \to \pi_1(Y, f(x))$$

by

$$f_*([\gamma]) = [f \circ \gamma].$$

Theorem 10.20. *Let $f : X \to Y$ and $g : Y \to Z$ be continuous maps, and let $x \in X$. Then*

(i) *$f_* : \pi_1(X, x) \to \pi_1(Y, f(x))$ is a well-defined homomorphism,*

(ii) *$(g \circ f)_* = g_* \circ f_*$, and if $\mathrm{id} : X \to X$ denotes the identity, then $\mathrm{id}_* : \pi_1(X, x) \to \pi_1(X, x)$ is the identity on $\pi_1(X, x)$.*

(iii) *Finally, if f is a homeomorphism, then f_* is an isomorphism.*

Proof. That f_* is well-defined means that $f \circ \gamma \sim_p f \circ \gamma'$ whenever $\gamma \sim_p \gamma'$. This is the case since if F is a homotopy from γ to γ', then $f \circ F$ is a homotopy from $f \circ \gamma$ to $f \circ \gamma'$.

To see that f_* is a homomorphism, let $[\gamma], [\gamma'] \in \pi_1(X, x)$ be arbitrary homotopy classes. We first notice that by definition of concatenation, we have

$$f \circ (\gamma \star \gamma') = (f \circ \gamma) \star (f \circ \gamma'),$$

from which it follows that

$$\begin{aligned} f_*([\gamma] \star [\gamma']) &= f_*([\gamma \star \gamma']) = [f \circ (\gamma \star \gamma')] = [(f \circ \gamma) \star (f \circ \gamma')] \\ &= [f \circ \gamma] \star [f \circ \gamma'] = f_*([\gamma]) \star f_*([\gamma']), \end{aligned}$$

so f_* is a homomorphism, which shows (i).

Similarly,

$$(g_* \circ f_*)([\gamma]) = g_*([f \circ \gamma]) = [g \circ f \circ \gamma] = (g \circ f)_*([\gamma]),$$

which shows the first part of (ii). The last part of (ii) is obvious.

Finally, (iii) follows from (ii) as it follows that $(f^{-1})_*$ satisfies that both $f_* \circ (f^{-1})_*$ and $(f^{-1})_* \circ f_*$ are the identity homomorphisms. Thus f_* is a bijection and therefore an isomorphism. $\square$

If G and H are two groups, then their Cartesian product $G \times H$ is a group with the group operation

$$(g, h) \cdot (g', h') = (g \cdot g', h \cdot h').$$

Proposition 10.21. *Let X and Y be topological spaces, and let $x \in X$, $y \in Y$. Then $\pi_1(X \times Y, (x, y))$ is isomorphic to $\pi_1(X, x) \times \pi_1(Y, y)$.*

Proof. Exercise 4.8. □

10.3 Covering spaces and fundamental groups of spheres

The main result of this section is a calculation of $\pi_1(S^n)$ for all $n \geq 1$.

Theorem 10.22. *We have $\pi_1(S^1) = \mathbb{Z}$, but S^n is simply-connected for $n \geq 2$.*

To prove the case $n = 1$, it will be convenient to have at our disposal some basic results about covering spaces. Before going into any detail about these, let us consider the "easy" part of the claim.

Proof of Theorem 10.22 for $n \geq 2$. Let γ be a loop in S^n, based at some point $\gamma(0)$, and let us show that γ is null-homotopic. If there is a point p not in the image of γ, we can view γ as a loop in $S^n \setminus \{p\}$, which is homeomorphic to $\mathbb{R}^n$ by Remark 5.12. Since $\mathbb{R}^n$ is simply-connected, this tells us that γ is null-homotopic as a loop in $S^n \setminus \{p\}$ through some homotopy $[0, 1] \times [0, 1] \to S^n \setminus \{p\}$. By composition with the inclusion, this gives us a homotopy $[0, 1] \times [0, 1] \to S^n$, so γ is also null-homotopic as a loop in S^n.

This shows the claim in the case where $\gamma([0, 1]) \neq S^n$. Now, let p be a point in S^n distinct from $\gamma(0)$. We will show how to make a path homotopy from γ to some other loop, denoted γ_k below, whose image does not contain p. This other loop will then be null-homotopic by the first part of the proof, so γ will be as well.

Let U be any neighbourhood of p which does not contain $\gamma(0)$. After possibly having to pass to a smaller neighbourhood we can assume that U is homeomorphic to an open ball in $\mathbb{R}^n$. Now $\gamma^{-1}(U) \subset (0, 1) \subset [0, 1]$ is an open set and therefore a union of open disjoint intervals (a_i, b_i), $i \in I$. Since $\{p\}$ is closed in S^n, $\gamma^{-1}(\{p\})$ is closed in $[0, 1]$, and since $[0, 1]$ is compact, so is $\gamma^{-1}(\{p\})$. Since $\{(a_i, b_i)\}_{i \in I}$ is an open cover of $\gamma^{-1}(\{p\})$, this compactness implies that we can find finitely many intervals $(a_1, b_1), \dots, (a_k, b_k)$ that cover $\gamma^{-1}(\{p\})$. We will now cook up the desired homotopy for each of these finitely many intervals.

Since $(a_1, b_1) \subset \gamma^{-1}(U)$ with $a_1, b_1 \notin \gamma^{-1}(U)$, we get that $\gamma([a_1, b_1]) \subset \overline{U}$ and $\gamma(a_1), \gamma(b_1) \in \partial U$. Now take any path $\widetilde{\gamma_1}$ in $\overline{U}$ from $\gamma(a_1)$ to $\gamma(b_1)$ which does not go through p. Since U was assumed to be homeomorphic to a ball, $\gamma|_{[a_1, b_1]}$ is path homotopic to $\widetilde{\gamma_1}$ (ignoring the minor detail that the paths in question have to be defined on $[0, 1]$) and this path homotopy extends to a path homotopy from γ to some loop γ_1 with the property that $p \notin \gamma_1([a_1, b_1])$ and so that γ_1 agrees with γ on the complement of $[a_1, b_1]$ in $[0, 1]$. We now iterate this procedure to obtain for each $j = 1, \dots, k$ a loop γ_j so that $\gamma \sim_p \gamma_j$ and $p \notin \gamma_j([a_1, b_1] \cup \dots \cup [a_j, b_j])$. Then $p \notin \gamma_k([0, 1])$ and $\gamma \sim \gamma_k$, so we are done. □

Before moving on to the case $n = 1$, let us notice the following non-trivial corollary of the theorem.

Corollary 10.23. *We have $S^n \simeq T^m$ if and only if $n = m = 1$.*

Proof. By Proposition 10.21 and Theorem 10.22, $\pi_1(T^m) = \pi_1(S^1 \times \cdots \times S^1)$ is the product of m copies of $\mathbb{Z}$. Since $\mathbb{Z}^m$ is not isomorphic to $\mathbb{Z}$ if $m > 1$, $\pi_1(T^m)$ can only be isomorphic to $\pi_1(S^n)$ if $n = m = 1$. □

Definition 10.24. Let B be a topological space. A *covering space* (sv: *?*) of B is a topological space E and a continuous surjective map $p : E \to B$, called a *covering map* (sv: *?*), so that each point $b \in B$ has an open neighbourhood U with the property that $p^{-1}(U)$ is a disjoint union of open sets in E, each of which is mapped homeomorphically to U by p. See Figure 21.

Example 10.25. The real line $\mathbb{R}$ is a covering space of S^1 with covering map $p : \mathbb{R} \to S^1$ given by $p(x) = (\cos(2\pi x), \sin(2\pi x))$.

Definition 10.26. Let $p : E \to B$ be a covering map, and let $f : X \to B$ be a continuous map. A map $\tilde{f} : X \to E$ is called a *lifting* (sv: *löft*) of f if $f = p \circ \tilde{f}$.

The two following lemmas will be proven in Section 10.4 below.

Lemma 10.27 (Path lifting lemma)**.** *Let $p : E \to B$ be a covering map, let $b \in B$, and let $e \in E$ with $p(e) = b$. Then any path $\gamma : [0,1] \to B$ with $\gamma(0) = b$ has a unique lifting $\tilde{\gamma} : [0,1] \to E$ so that $\tilde{\gamma}(0) = e$.*

Lemma 10.28 (Homotopy lifting lemma)**.** *Let $p : E \to B$ be a covering map, and let $p(e) = b$ as above. Let $F : [0,1] \times [0,1] \to B$ be a homotopy with $F(0,0) = b$. Then there is a unique lifting $\tilde{F} : [0,1] \times [0,1] \to E$ so that $\tilde{F}(0,0) = e$. If F is a path homotopy then so is $\tilde{F}$.*

Now as above, let $p : E \to B$ be a covering map, let $b \in B$, and choose $e \in E$ with $p(e) = b$. Let $[\gamma] \in \pi_1(B, b)$ be a homotopy class of a path γ, and let $\tilde{\gamma}$ be the unique lifting from Lemma 10.27, with $\tilde{\gamma}(0) = e$. Define a map, called the *lifting correspondence* (sv: *?*),

$$\varphi_e : \pi_1(B, b) \to p^{-1}(\{b\})$$

by $\varphi_e([\gamma]) = \tilde{\gamma}(1)$. To see that this is well-defined, assume that $\gamma \sim_p \gamma'$, and let F denote a path homotopy from γ to γ'. Then the unique lifting $\tilde{F}$ from Lemma 10.28 is a path homotopy between the unique lifts $\tilde{\gamma}$ and $\widetilde{\gamma'}$, so in particular $\tilde{\gamma}(1) = \widetilde{\gamma'}(1)$.

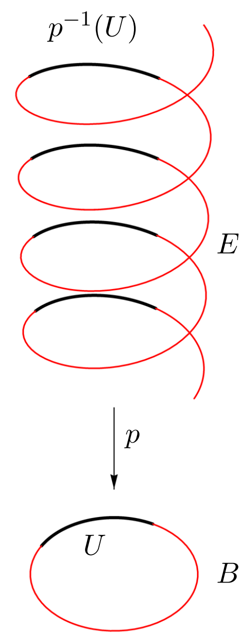

Figure 21: A covering map $p : E \to B$.

Proposition 10.29. *Let $p : E \to B$, $p(e) = b$, be as above.*
If E is path-connected then φ_e is surjective. If E is simply-connected, then φ_e is bijective.

Proof. Assume first that E is path-connected and let $q \in p^{-1}(\{b\})$. Choose any path $\tilde{\gamma}$ from e to q, and let $\gamma = p \circ \tilde{\gamma}$. Then $\tilde{\gamma}$ is a lift of γ by construction, and $\varphi_e([\gamma]) = q$, so φ_e is surjective.

Assume now that E is simply-connected, suppose that $\varphi_e([\gamma]) = \varphi_e([\gamma'])$ for two homotopy classes $[\gamma], [\gamma'] \in \pi_1(B, b)$, and let us show that $[\gamma] = [\gamma']$. Let $\tilde{\gamma}$ and $\widetilde{\gamma'}$ denote the corresponding lifts, so that $\tilde{\gamma}(1) = \widetilde{\gamma'}(1)$ by assumption. Then $\tilde{\gamma} \star \widetilde{\gamma'}^{\mathrm{rev}}$ is a loop based at e and thus path homotopic to the constant map since E is simply-connected. This implies that

$$[\tilde{\gamma}] = [\tilde{\gamma}] \star [\widetilde{\gamma'}^{\mathrm{rev}}] \star [\widetilde{\gamma'}] = [\tilde{\gamma} \star \widetilde{\gamma'}^{\mathrm{rev}}] \star [\widetilde{\gamma'}] = [\widetilde{\gamma'}],$$

so there is a path homotopy $\tilde{F}$ from $\tilde{\gamma}$ to $\widetilde{\gamma'}$. Then $p \circ \tilde{F}$ is a path homotopy from $p \circ \tilde{\gamma} = \gamma$ to $p \circ \widetilde{\gamma'} = \gamma'$, or in other words, $[\gamma] = [\gamma']$. $\square$

We are now in a position to prove our main result.

Proof of Theorem 10.22 for $n = 1$. Let $p : \mathbb{R} \to S^1$ be the covering map from Example 10.25, let $b = (1, 0) \in S^1$, and let $e = 0$. In this case, since $p^{-1}(\{b\}) = \mathbb{Z}$, and since $\mathbb{R}$ is simply-connected, Proposition 10.29 implies that $\varphi_e : \pi_1(S^1, b) \to \mathbb{Z}$ is bijective, and we claim that it is a homomorphism.

Let $m \in \mathbb{Z}$, and let $\gamma_m : [0, 1] \to S^1$ be the loop given by

$$\gamma_m(t) = (\cos(2\pi m t), \sin(2\pi m t)).$$

This loop lifts to $\widetilde{\gamma_m} : [0, 1] \to \mathbb{R}$ given by $\widetilde{\gamma_m}(t) = mt$. This tells us that $\varphi_e([\gamma_m]) = m$. Since φ_e was a projection, each loop γ based at b will belong to $[\gamma_m]$ for some $m \in \mathbb{Z}$, so to show that φ_e is a homomorphism, it suffices to show that

$$\varphi_e([\gamma_m \star \gamma_n]) = m + n = \varphi_e([\gamma_m]) + \varphi_e([\gamma_n]),$$

for all $m, n \in \mathbb{Z}$. Let $\widetilde{\gamma_n^m} = m + \widetilde{\gamma_n}$ be the lifting of γ_n starting at m and ending at $m + n$. Then $\widetilde{\gamma_m} \star \widetilde{\gamma_n^m}$ is a path from 0 to $m + n$ and $p \circ (\widetilde{\gamma_m} \star \widetilde{\gamma_n^m}) = \gamma_m \star \gamma_n$, so by definition of φ_e, this tells us that $\varphi_e([\gamma_m \star \gamma_n]) = m + n$. $\square$

Up until this point, it is not clear how useful π_1 actually is as an invariant: indeed, most of the calculations of fundamental groups that we have done turned out to yield trivial groups, and, for instance, we can not use the fundamental group to tell apart the spaces $\mathbb{R}^n$ and S^n for $n \geq 2$, even though one can show by elementary means that these spaces are non-homeomorphic.

Restricting to the class of manifolds considered in Section 9 one can say quite a bit more. It is not too difficult to prove, for instance, that if a 2-manifold is compact and simply-connected, then it is homeomorphic to S^2. That is, the fundamental group can detect S^2 among all compact 2-manifolds. The same statement holds true for 3-manifolds, although it is currently significantly harder to prove.

Theorem 10.30 (The Poincaré conjecture)**.** *If X is a simply-connected compact 3-manifold, then $X \simeq S^3$.*

Conjectured by Henri Poincaré in 1904, and first proven by Grigori Perelman in 2003, at the time of writing this is the only solved out of the seven so-called Millennium Prize Problems[4].

10.4 Proofs of lifting lemmas

We now turn to the proofs of Lemmas 10.27 and 10.28 where we will need a technical result on subdivisions of $[0,1]$ and $[0,1] \times [0,1]$. We word it here for general compact metric spaces.

Let (X, d) be a metric space, and let A be a non-empty subset. Then for every $x \in X$, we define the distance from x to A by

$$d(x, A) = \inf\{d(x, a) \mid a \in A\}.$$

It is easy to see that for fixed A, the function $x \mapsto d(x, A)$ is continuous. If moreover A is bounded in the sense that the set $\{d(a_1, a_2) \mid a_1, a_2 \in A\} \subset \mathbb{R}$ is bounded, we define the *diameter* (sv: *diameter*) of A to be

$$\operatorname{diam}(A) = \sup\{d(a_1, a_2) \mid a_1, a_2 \in A\} \in \mathbb{R}.$$

Lemma 10.31 (Lebesgue's number lemma)**.** *Let $\mathcal{U}$ be an open cover of a compact metric space (X, d). Then there is a $\delta > 0$, called a* Lebesgue number*, so that for every subset A of diameter less than δ, there is a $U \in \mathcal{U}$ with $A \subset U$.*

Proof. First of all, if $X \in \mathcal{U}$, we are done since any $\delta > 0$ does the job, so assume that $X \notin \mathcal{U}$.

Now use the compactness of X to take an finite subcover $\{U_1, \dots, U_n\} \subset \mathcal{U}$. Let $C_i = X \setminus U_i \neq \emptyset$ for $i = 1, \dots, n$, and define a function $f : X \to \mathbb{R}$ by

$$f(x) = \frac{1}{n} \sum_{i=1}^{n} d(x, C_i).$$

We claim that $f(x) > 0$ for all $x \in X$. To see this, let $x \in X$ and choose an i so that $x \in U_i$. Since U_i is open, we can find an $\varepsilon > 0$ so that $B(x, \varepsilon) \subset U_i$. Then $d(x, C_i) \geq \varepsilon$, so $f(x) \geq \varepsilon/n > 0$. Since f is continuous, it follows from Corollary 7.26 that $f(X)$ has a minimum $\delta > 0$, and we now claim that this number is our desired Lebesgue number.

Let $A \subset X$ have diameter less than δ, and let $a \in A$. Then $A \subset B_d(a, \delta)$, and by taking $m \in \{1, \dots, n\}$ so that $d(a, C_m)$ is maximal, we have

$$\delta \leq f(a) \leq d(a, C_m),$$

so $A \subset B_d(a, \delta) \subset X \setminus C_m = U_m$. □

[4]See https://en.wikipedia.org/wiki/Millennium_Prize_Problems.

We can now prove the two lemmas.

Proof of Lemma 10.27. Choose an open cover $\mathcal{U}$ of B by sets $U \in \mathcal{U}$ with the property that $p^{-1}(U)$ is a disjoint union of open sets in E, each of which is mapped homeomorphically to U by p.

Now $\{\gamma^{-1}(U) \mid U \in \mathcal{U}\}$ is an open cover of the compact set $[0, 1]$. By Lemma 10.31, there is a subdivision $0 = s_0 < s_1 < \cdots < s_n = 1$ of $[0, 1]$ so that for every $i = 1, \ldots, n$, we have $[s_{i-1}, s_i] \subset \gamma^{-1}(U)$ for some $U \in \mathcal{U}$. That is, $\gamma([s_{i-1}, s_i]) \subset U$. We will now construct the lifting $\tilde{\gamma}$ on each of these smaller intervals through a finite induction.

Let $\tilde{\gamma}(0) = e$ and assume that the lifting $\tilde{\gamma}$ has been defined on $[0, s_i]$. We then define $\tilde{\gamma}$ on $[s_i, s_{i+1}]$ as follows: let $U \in \mathcal{U}$ be the open set so that $\gamma([s_i, s_{i+1}]) \subset U$. Then $\tilde{\gamma}(s_i) \in p^{-1}(U)$, so we can choose $V \subset E$ so that $p|_V : V \to U$ is a homeomorphism and $\tilde{\gamma}(s_i) \in V$. Now, for any $s \in [s_i, s_{i+1}]$, we can define

$$\tilde{\gamma}(s) = (p|_V)^{-1}(\gamma(s)).$$

Then $\tilde{\gamma}$ is continuous on $[s_i, s_{i+1}]$, agrees with $\tilde{\gamma}$ on the one-point set $\{s_i\}$, and so defines a function $\tilde{\gamma} : [0, s_{i+1}] \to E$ which is continuous by the pasting lemma, Remark 3.10. Moreover, $\tilde{\gamma}$ is constructed to satisfy $\gamma = p \circ \tilde{\gamma}$ where it is defined, so repeating this procedure n times we obtain our desired lift.

It remains to prove that $\tilde{\gamma}$ is unique. Suppose that $\tilde{\gamma}'$ is another lifting of γ with $\tilde{\gamma}'(0) = e$, and suppose that $\tilde{\gamma} = \tilde{\gamma}'$ on $[0, s_i]$ (which we know is true for $i = 0$). We then claim that the lifts agree on $[0, s_{i+1}]$ as well, and therefore $\tilde{\gamma} = \tilde{\gamma}'$ on all of $[0, 1]$, so let $s \in [s_i, s_{i+1}]$, and let V be as above so that $\tilde{\gamma}(s_i), \tilde{\gamma}'(s_i) \in V$.

Since $[s_i, s_{i+1}]$ is connected, so is the set $\tilde{\gamma}'([s_i, s_{i+1}])$ by Theorem 6.10, and therefore it must be contained in a single connected component by Proposition 6.27. Now, since the open sets making up $p^{-1}(U)$ are disjoint, this implies that $\tilde{\gamma}'([s_i, s_{i+1}]) \subset V$, so in particular $\tilde{\gamma}'(s) \in V$. Now, since $p|_V : V \to U$ is bijective, there is only one point in V that is mapped to $\gamma(s)$ under p, namely $(p|_V)^{-1}(\gamma(s))$, so since $\tilde{\gamma}'$ is a lifting of γ, it follows that

$$\tilde{\gamma}'(s) = (p|_V)^{-1}(\gamma(s)) = \tilde{\gamma}(s).$$

□

Proof of Lemma 10.28. As in the proof above, we will define a lift $\tilde{F}$ in a step-by-step fashion, so let $\mathcal{U}$ denote an open cover of B as above. Begin by letting $\tilde{F}(0, 0) = e$, and use Lemma 10.27 to uniquely define $\tilde{F}$ on $[0, 1] \times \{0\}$ and $\{0\} \times [0, 1]$, lifting F on these subsets.

As above, we can now consider $\{F^{-1}(U) \mid U \in \mathcal{U}\}$ and conclude by Lemma 10.31 that there exist subdivisions $0 = s_0 < s_1 < \cdots < s_m = 1$ and $0 = t_0 < t_1 < \cdots < t_n = 1$ so that for each rectangle

$$R_{i,j} = [s_{i-1}, s_i] \times [t_{j-1}, t_j] \subset [0, 1] \times [0, 1]$$

there is a $U \in \mathcal{U}$ with $F(R_{i,j}) \subset U$. We now define $\tilde{F}$ on the rectangles $R_{i,j}$ in the order

$$R_{1,1}, R_{2,1}, \ldots, R_{m,1}, R_{1,2}, R_{2,2}, \ldots, R_{m,2}, \ldots, R_{1,n}, R_{2,n}, \ldots, R_{m,n}.$$

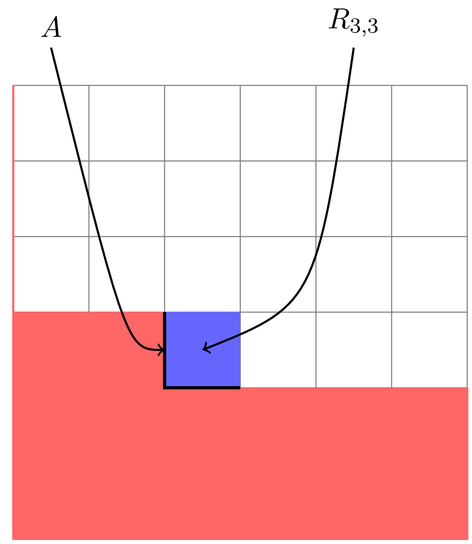

Figure 22: The sets involved in the proof of Lemma 10.28. The red part indicates where $\tilde{F}$ has been defined, and the blue part where we are in the process of defining $\tilde{F}$.

Assume that the lifting $\tilde{F}$ has been defined on all rectangles up to a certain point, and let us define $\tilde{F}$ on the next rectangle, $R_{i,j}$, say. In particular, $\tilde{F}$ has been defined on A: the union of the left and bottom edge of $R_{i,j}$, which is a connected set (see Figure 22). By the exact same logic as in the previous proof, this implies that $\tilde{F}(A) \subset V$, where $V \subset E$ is so that $p|_V : V \to U$ is a homeomorphism, and where $U \in \mathcal{U}$ is so that $F(R_{i,j}) \subset U$. This means that we can extend $\tilde{F}$ to $R_{i,j}$ by letting

$$\tilde{F}(x) = (p|_V)^{-1}(F(x)).$$

Proceeding like this for all rectangles, we define $\tilde{F}$ on all of $[0,1] \times [0,1]$. Then $\tilde{F}$ is continuous by the pasting lemma and a lifting of F by construction. That $\tilde{F}$ is the unique lifting with $\tilde{F}(0,0) = e$ follows by the same logic as in the proof of Lemma 10.27 above.

It remains to show that if F is a path homotopy, then so is $\tilde{F}$. So, assume that $F(\{0\} \times [0,1]) = \{b\}$. Then $\tilde{F}(\{0\} \times [0,1]) \subset p^{-1}(\{b\})$. Now $\{0\} < \times [0,1]$ is connected, so its image under $\tilde{F}$ is connected, and on the other hand, $p^{-1}(\{b\})$ is discrete so its connected components are points, which means that $\tilde{F}$ is constant on $\{0\} \times [0,1]$. Similarly, if F is constant on $\{1\} \times [0,1]$, one uses connectedness to argue that $\tilde{F}$ is as well. $\square$

Exercises

The exercises are split into four sets, corresponding to the four exercise sessions held as part of the course. Many of the exercises are collected from previous iterations of the course, and these in turn may originate from [Mun00]. A few have been inspired by exercises in [DM07] and [Hat02].

A solution manual is available upon request.

Set #1

1.1 Define a relation on $\mathbb{R}$ by

$$C = \{(x, y) \mid x - y \in \mathbb{Z}\}.$$

Show that C is an equivalence relation and describe the set of equivalence classes of C.

1.2 Describe all possible topologies on the set $X = \{a, b, c\}$.

1.3 Let X be a set, and let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two different topologies on X. When is the identity map $\mathrm{id} : X \to X$ given by $\mathrm{id}(x) = x$ a continuous map from $(X, \mathcal{T}_1)$ to $(X, \mathcal{T}_2)$?

1.4 Show that the subspace topology $\mathcal{T}_Y$ is the smallest (meaning coarsest) topology on $Y \subset X$ for which the inclusion $\iota : Y \to X$ is a continuous map.

1.5 Let $Y \subset X$ be an open (closed) subset of a topological space X. Show that a set $U \subset Y$ is open (closed) in the subspace topology on Y if and only if U is open (closed) in X.

1.6 Let X be a topological space, and let A and B be subspaces so that $A \subset B \subset X$. Then A can be endowed with the subspace topology $\mathcal{T}_A$ from X, or the subspace topology $(\mathcal{T}_B)_A$ from B (which in turn has the subspace topology coming from X). Show that $\mathcal{T}_A = (\mathcal{T}_B)_A$.

1.7 Prove Lemma 3.6.

1.8 (a) Describe the open sets in the poset topology on $\{a, b, c, d\}$ defined by the relations $a \preceq b \preceq c$ and $a \preceq d$.

(b) Describe the open sets in the poset topology on $(\mathbb{R}, \leq)$.

1.9 The Euclidean space $\mathbb{R}^2$ can be identified with the Cartesian product $\mathbb{R} \times \mathbb{R}$. Use Lemma 2.15 to show that the standard topology on $\mathbb{R}^2$ equals the product topology on $\mathbb{R} \times \mathbb{R}$ (where each $\mathbb{R}$ has the standard topology).

1.10 Show that metric spaces are always Hausdorff.

1.11 Show that if X is Hausdorff, then so is any subset $Y \subset X$ with the subspace topology.

1.12 Show that the product of two Hausdorff spaces is Hausdorff.

1.13 Show that a topological space X is Hausdorff if and only if the diagonal

$$\Delta = \{(x, x) \in X \times X \mid x \in X\} \subset X \times X$$

is closed in the product topology on $X \times X$.

1.14 Let (X, d) be a metric space, and let

$$\mathcal{B} = \{B_d(x, 1/n) \mid x \in X, n \in \mathbb{N}\}.$$

Show that $\mathcal{B}$ is a basis for the metric topology on X.

1.15 Let $(Y, \preceq)$ be a totally ordered set made into a topological space with the order topology.

(*a*) Show that for any two distinct points $x, y \in Y$, $x < y$, there are disjoint neighbourhoods, U and V, of x and y respectively, so that $u < v$ for all $u \in U, v \in V$. Conclude that Y is Hausdorff.

(*b*) Let X be any topological space, and let $f, g : X \to Y$ be two continuous functions. Show that the set $\{x \mid f(x) \preceq g(x)\}$ is closed in X.

1.16 Let (X_1, d_1) and (X_2, d_2) be metric spaces. Define a metric on $X_1 \times X_2$ by

$$d((x_1, x_2), (y_1, y_2)) = \max(d_1(x_1, y_1), d_2(x_2, y_2)).$$

Show that the metric topology on $X_1 \times X_2$ induced by d is the product topology, where X_1 and X_2 have the metric topologies from d_1 and d_2 respectively.

1.17 Let X, Y, Z be topological spaces and consider a function $F : X \times Y \to Z$. We say that F is *continuous in each variable* if for each $y_0 \in Y$ the function $h : X \to Z$ defined by $h(x) = F(x, y_0)$ is continuous, *and* if for each $x_0 \in X$ the function $g : Y \to Z$ defined by $g(y) = F(x_0, y)$ is continuous. Show that if F is continuous, then F is continuous in each variable.

1.18 (*a*) A poset topology is T_0. When is it T_1?

(*b*) If X is a T_0-space with finitely many elements. Then we can define a relation

$$x \preceq y \Leftrightarrow y \in \bigcap_{U \subset X \text{ open}, x \in U} U.$$

Show that $\preceq$ is a partial order. What is the poset topology on $(X, \preceq)$?

Set #2

2.1 Show that a topological space X is connected if and only if the following condition holds: if $X = C \cup D$ where C and D are disjoint closed subsets of X, then either $C = \emptyset$ or $D = \emptyset$.

2.2 Let $X = \{a, b, c\}$ have the topology $\{\emptyset, \{a, b\}, \{c\}, \{a, b, c\}\}$. Is X connected? Path-connected?

2.3 Consider the topology on $\mathbb{R}$ generated by the basis $\{(a, \infty) \mid a \in \mathbb{R}\}$, and let $x_0 \in \mathbb{R}$.

(a) What is $\{x_0\}'$, the set of limit points of $\{x_0\}$?

(b) What is the closure $\overline{\{x_0\}}$?

(c) Is $\mathbb{R}$ Hausdorff in this topology?

2.4 Show that the connected subsets of $\mathbb{R}$ are exactly the intervals.

2.5 Show that $\mathbb{R}^n \setminus \{0\}$ is connected when $n \geq 2$.

2.6 Show that $\mathbb{R}^n \not\cong \mathbb{R}$ when $n \geq 1$.

2.7 Show that the concatenation $\gamma_1 \star \gamma_2$ is continuous by using Theorem 4.24 instead of the pasting lemma.

2.8 Show that S^n is path-connected for every $n > 0$.

2.9 Let $p : X \to Y$ be a quotient map. Show that if X is locally connected then so is Y.

2.10 Show that the connected components of a locally connected space are open.

2.11 Let $\{A_n\}_{n \in \mathbb{N}}$ be a family of connected subspaces of a topological space X so that $A_n \cap A_{n+1} \neq \emptyset$ for every $n \in \mathbb{N}$. Show that $\bigcup_{n \in \mathbb{N}} A_n$ is connected.

2.12 A space is called *totally disconnected* if its only non-empty connected subspaces are one-point sets. Show that if X has the discrete topology, then X is totally disconnected. Does the converse hold?

2.13 Let $f : S^1 \to \mathbb{R}$ be continuous. Show that there is a point $x \in S^1$ so that $f(x) = f(-x)$.

2.14 Let $f : [0, 1] \to [0, 1]$ be continuous. Show that f has a fixed point, i.e. that there is a point $x \in [0, 1]$ so that $f(x) = x$.

2.15 Show Theorem 6.14 in the case where I is infinite. Inspiration can be found in [Mun00, Ex. 23.7].

Set #3

3.1 Let X be a topological space.

(a) Show that if $K_1, \dots, K_n$ are compact subspaces of X, then $K_1 \cup \dots \cup K_n$ is compact.

(b) Suppose that X is Hausdorff. Show that if $\{K_i\}_{i \in I}$ is a family of compact subspaces of X, then $\bigcap_{i \in I} K_i$ is compact.

(c) Prove Proposition 7.32.

3.2 Let $A \subset X$ be a subspace of a topological space $(X, \mathcal{T})$. Show that the following are equivalent:

- A is compact (in the subspace topology)
- For every collection of open sets $\mathcal{U} \subset \mathcal{T}$ so that $A \subset \bigcup_{U \in \mathcal{U}} U$ there exists a finite subcollection $U_1, \dots, U_n \in \mathcal{U}$ so that $A \subset U_1 \cup \dots \cup U_n$.

3.3 Show that compact Hausdorff spaces are normal.

3.4 Prove Theorem 7.38 (Hint: Use the one-point compactification.)

3.5 Prove Proposition 7.40.

3.6 Show that if X is T_3 and $C \subset X$ a closed subset, then the quotient space X/C is Hausdorff.

3.7 Show that $\{B(x, r) \mid x \in \mathbb{Q}^n, r \in \mathbb{Q}_{>0}\}$ is a basis, and that it generates the standard topology on $\mathbb{R}^n$. Conclude that $\mathbb{R}^n$ is second-countable.

3.8 Let Y be a compact space, and let X be any topological space.

(a) Show the canonical projection map $\pi : X \times Y \to X$ is closed, i.e. that images of closed sets are closed.

(b) Suppose moreover that Y is Hausdorff, and let $f : X \to Y$ be a map. Show that f is continuous if and only if its graph

$$G_f = \{(x, f(x)) \mid x \in X\} \subset X \times Y$$

is closed.

3.9 Let $(X, \preceq)$ be a totally ordered set with the order topology, and assume that every closed interval $[a, b]$ is compact. Show that X has the *least-upper-bound property*; that is, show that every non-empty subset of X which is bounded from above has a least upper bound in X.

3.10 Let X be a locally compact Hausdorff space, and let $U_1, U_2, U_3, \dots$ be open dense subsets of X. Show that $\bigcap_{n \in \mathbb{N}} U_n$ is dense; a result known as the Baire category theorem. Hints:

(a) Let B_0 be any non-empty open subset of X. Construct open sets $B_1, B_2, \ldots$ so that
$$\overline{B_n} \subset U_n \cap B_{n-1}$$
for $n \geq 1$ and so that $\overline{B_n}$ is compact for all n.

(b) Let $K_1, K_2, \ldots$ be non-empty compact subsets of a topological space. Show that if we have inclusions $K_1 \supset K_2 \supset \cdots$, then
$$\bigcap_{n \in \mathbb{N}} K_n \neq \emptyset.$$
This result is known as Cantor's intersection theorem.

What happens if we replace the countable family $\{U_n\}_{n \in \mathbb{N}}$ with an arbitrary family of open dense subsets?

3.11 Let X be a locally compact Hausdorff space, and let $\{F_n\}_{n \in \mathbb{N}}$ be a countable family of closed subsets of X. Show that if $\mathrm{Int}(F_m) = \emptyset$ for every $m \in \mathbb{N}$, then $\mathrm{Int}\left(\bigcup_{n \in \mathbb{N}} F_n\right) = \emptyset$. (Hint: use Exercise 3.10.)

Set #4

4.1 Prove Lemma 9.5.

4.2 Use Theorem 6.18 to show that if X is both an n-manifold and an m-manifold, then $m = n$.

4.3 A subspace $A \subset \mathbb{R}^n$ is called *star-shaped* if there exists an element $a \in A$ so that for any $x \in A$, the line segment from a to x is contained in A. Show that star-shaped sets are simply-connected.

4.4 Let $x \in \mathbb{Q}$. Find $\pi_1(\mathbb{Q}, x)$. Is $\mathbb{Q}$ simply-connected?

4.5 Let α be a path from x to y, and let β be a path from y to z in some topological space. Show that $\widehat{\alpha \star \beta} = \widehat{\beta} \circ \widehat{\alpha}$, where $\widehat{\cdot}$ is as in Theorem 10.16.

4.6 Show that if α and β are path homotopic, then $\widehat{\alpha} = \widehat{\beta}$.

4.7 Let $f : X \to Y$ be a continuous map, and let α be a path in X. Show that $f_* \circ \widehat{\alpha} = \widehat{f \circ \alpha} \circ f_*$.

4.8 Prove Proposition 10.21.

4.9 Let $n \geq 2$. Show that $\mathbb{R}^n \setminus \{0\}$ is homeomorphic to $S^{n-1} \times (0, \infty)$. Is $\mathbb{R}^n \setminus \{0\}$ simply-connected?

4.10 Define an equivalence relation $\sim$ on S^n by saying that $x \sim y$ if $x = \pm y$. Let $\mathbb{R}\mathrm{P}^n = S^n / \sim$ denote the quotient space, called the *real projective space* (sv: *reella projektiva rummet*). Show that for $n \geq 2$, we have $\pi_1(\mathbb{R}\mathrm{P}^n) = \mathbb{Z}_2$, the (only) group containing two elements. (Hint: show that the projection map $p : S^n \to \mathbb{R}\mathrm{P}^n$ is a covering map.)

4.11 Let $A \subset X$ be a subspace of a topological space, and let $r : X \to A$ be a continuous map so that $r|_A = \mathrm{id}_A$ (such an r is called a *retraction*). Let $a \in A$.

(a) Show that $r_* : \pi_1(X, a) \to \pi_1(A, a)$ is surjective.

(b) Suppose moreover that there is a homotopy $F : X \times [0, 1] \to X$ from id_X to r with the property that $F(x, t) = x$ for all $x \in A$ (such an F is called a *deformation retraction*). Show that r_* is an isomorphism.

(c) Recall that D^2 denotes the unit disk in $\mathbb{R}^3$. Show that any continuous map $h : D^2 \to D^2$ has a fixed point, i.e. a point $x \in D^2$ so that $h(x) = x$. (Hint: assume that h has no fixed points, and use h to construct a retraction $r : D^2 \to S^1$ to arrive at a contradiction with (a) above.)

This is a special case of Brouwer's fixed-point theorem which says that in general, any continuous map $A \to A$, where $A \subset \mathbb{R}^n$ is compact and convex, has a fixed point. The case $A = [0, 1]$ was covered in Exercise 2.14.[5]

[5] See https://en.wikipedia.org/wiki/Brouwer_fixed-point_theorem#Illustrations for a couple of curious "real world applications" of this theorem.

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Dictionary

English	Swedish
algebraic topology	algebraisk topologi
anti-symmetric	antisymmetrisk
ball	boll
basis	bas
binary relation	binär relation
boundary	rand
bounded	begränsad
Cartesian product	kartesisk produkt
category theory	kategoriteori
closed	sluten
closure	slutet hölje
coarse	grov
compact	kompakt
complement	komplement
concatenation	konkatenering
connected component	sammanhängande komponent
connected	sammanhängande
continuous	kontinuerlig
convergent	konvergent
cover	övertäckning
dense	tät
diameter	diameter
difference	differens
discrete metric	den diskreta metriken
discrete topology	den diskreta topologin
disjoint	disjunkt
distance	avstånd
equivalence class	ekvivalensklass
equivalence relation	ekvivalensrelation
fine	fin
finite intersection property	?
first-countable	?
fundamental group	fundamentalgrupp
general topology	allmän topologi
group	grupp
homeomorphic	homeomorfa
homeomorphism	homeomorfi
homomorphism	homomorfi
homotopic	homotop
homotopy class	homotopiklass

homotopy	homotopi
interior	inre
intersection	snitt
isomorphism	isomorfi
Klein bottle	Kleinflaska
lifting correspondence	?
lifting	löft
limit point	gränspunkt
locally (path-)connected	lokalt (bågvis) sammanhängande
locally compact	lokalt kompakt
locally Euclidean	lokalt euklidisk
locally finite	lokalt ändlig
loop	ögla
manifold	mångfald
metric	metrik
metric space	metriskt rum
metrisable	metriserbart
neighbourhood	omgivning
normal	normalt
null-homotopic	nollhomotop
one-point compactification	enpunktskompaktifiering
open	öppen
order topology	?
paracompact	parakompakt
partial order	partiell ordning
partition of unity	partition av enheten
path-connected	bågvis sammanhängande
path-connected component	bågvis sammanhängande komponent
path homotopic	väghomotop
path homotopy	väghomotopi
path	väg
point-set topology	punktmängdstopologi
poset	pomängd
preimage	urbild
product topology	produkttopologin
proper map	?
proper subset	äkta delmängd
quotient map	kvotavbildning
quotient topology	kvottopologin
real projective plane	reella projektiva planet
reflexive	reflexiv
regular	reguljärt
relation	relation
relative topology	relativ topologi
second-countable	?

separable	separabelt
separation	separation
sequence	följd
sequentially compact	följdkompakt
set	mängd
set theory	mängdteori
simple closed curve	enkel sluten kurva
structure	struktur
subbasis	delbas
subsequence	delföljd
subset	delmängd
subspace topology	underrumstopologi
support	stöd
symmetric	symmetrisk
topological space	topologiskt rum
topology generated by	topologin genererad av
topology	topologi
total	total
transitive	transitiv
trivial topology	den triviala toplogin

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