# Differential algebra on lattice Green functions and Calabi-Yau operators (unabridged version) 

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#### Abstract

. We revisit miscellaneous linear differential operators mostly associated with lattice Green functions in arbitrary dimensions, but also Calabi-Yau operators and order-seven operators corresponding to exceptional differential Galois groups. We show that these irreducible operators are not only globally nilpotent, but are such that they are homomorphic to their (formal) adjoints. Considering these operators, or, sometimes, equivalent operators, we show that they are also such that, either their symmetric square or their exterior square, have a rational solution. This is a general result: an irreducible linear differential operator homomorphic to its (formal) adjoint is necessarily such that either its symmetric square, or its exterior square has a rational solution, and this situation corresponds to the occurrence of a special differential Galois group. We thus define the notion of being "Special Geometry" for a linear differential operator if it is irreducible, globally nilpotent, and such that it is homomorphic to its (formal) adjoint. Since many Derived From Geometry $n$-fold integrals ("Periods") occurring in physics, are seen to be diagonals of rational functions, we address several examples of (minimal order) operators annihilating diagonals of rational functions, and remark that they also seem to be, systematically, associated with irreducible factors homomorphic to their adjoint.


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[^0]
## 1. Introduction

When one considers all the irreducible factors of the globally nilpotent linear differential operators encountered in the study of $n$-folds integrals of the Ising class [1] (or the one's displayed by other authors in an enumerative combinatorics framework [2, 3, or in a Calabi-Yau framework [4, 5, (6), one finds out that their differential Galois groups are not the $S L(N, \mathbb{C})$, or extensions of $S L(N, \mathbb{C})$, groups one could expect generically, but selected $S O(N), S p(N, \mathbb{C}), G_{2}, \ldots$ differential Galois groups 7].

Along this line it is worth recalling that globally nilpotent linear differential operators associated with generic ${ }_{n} F_{n-1}$ hypergeometric functions with rational parameter $\mathbb{C}$, have $S L(N, \mathbb{C}$ ) (or extensions of $S L(N, \mathbb{C})$ ) differential Galois groups. For instance the ${ }_{3} F_{2}$ hypergeometric function

$$
\begin{equation*}
{ }_{3} F_{2}\left(\left[\frac{191}{479}, \frac{359}{311}, \frac{503}{89}\right],\left[\frac{521}{151}, \frac{401}{67}\right], x\right), \tag{1}
\end{equation*}
$$

has a $S L(3, \mathbb{C})$ differential Galois group $\#$. In contrast, in the simplest examples, the emergence of "selected" differential Galois groups can be seen very explicitely [10], and understood (from a physicist's viewpoint) as the emergence of some "invariant". As far as the $S O(3, \mathbb{C})$ group is concerned $\sharp$, let us consider the non-Fuchsian $\dagger$ operator $(\theta=x \cdot d / d x)$ :

$$
\begin{equation*}
2 \theta \cdot(3 \theta-2) \cdot(3 \theta-4)-9 x \cdot(2 \theta+1) \tag{2}
\end{equation*}
$$

with the three ${ }_{1} F_{2}$ hypergeometric solutions

$$
{ }_{1} F_{2}\left(\left[\frac{1}{2}\right],\left[-\frac{1}{3}, \frac{1}{3}\right], x\right), \quad x^{2 / 3} \cdot{ }_{1} F_{2}\left(\left[\frac{7}{6}\right],\left[\frac{1}{3}, \frac{5}{3}\right], x\right), \quad x^{4 / 3} \cdot{ }_{1} F_{2}\left(\left[\frac{11}{6}\right],\left[\frac{5}{3}, \frac{7}{3}\right], x\right) .
$$

If $f$ denotes a solution of this operator (in the above closed form or as a formal solution at the origin or at $\infty$ ), one has the following quadratic relation $Q\left(f, f^{\prime}, f^{\prime \prime}\right)=$ const., where:

$$
\begin{gathered}
Q\left(X_{0}, X_{1}, X_{2}\right)=9 \cdot(36 x+5) \cdot x^{2} \cdot X_{2}^{2}-324 \cdot x^{2} \cdot X_{2} \cdot X_{1}-648 x^{2} \cdot X_{2} \cdot X_{0} \\
\quad+(81 x-5) \cdot X_{1}^{2}+9 \cdot(36 x-5) \cdot X_{0} \cdot X_{1}+9 \cdot(36 x-5) \cdot X_{0}^{2}
\end{gathered}
$$

The constant depends on the linear combination of solutions used. For instance, with the first ${ }_{1} F_{2}$ hypergeometric solution one has $Q\left(f, f^{\prime}, f^{\prime \prime}\right)=225 / 4$, while with the two other ${ }_{1} F_{2}$ solutions it reads $Q\left(f, f^{\prime}, f^{\prime \prime}\right)=0$. In other words $Q$ is a first integral.

The emergence of such "special" differential Galois groups in so many domains of theoretical physics is clearly something we need to understand better.

We have provided a large number of linear ODEs on various problems of lattice statistical mechanics, in particular for the magnetic susceptibility of the twodimensional Ising model [8, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20. These linear ODEs factorize into many factors of order ranging from one, to 12 (for $\chi^{(5)}$ ) and even 23 (for $\left.\chi^{(6)}\right)$. As far as the factors of smallest orders (two, three and four) are concerned, one can verify that all these linear differential operators are homomorphic to their adjoint.

ब Their corresponding linear differential operators are necessarily globally nilpotent 8].
$\ddagger$ One shows that there are no rational solutions of symmetric powers in degree $2,3,4,6,8,9,12$, using an algorithm in M. van Hoeij et al. (9]
$\sharp$ This operator is actually homomorphic to its adjoint (see below) with non-trivial order-two intertwiners.
$\dagger$ This operator has an irregular singularity at infinity. At $x=\infty$ the solutions behave like: $t \cdot\left(1+77 / 72 t^{2}+\cdots\right)$, and $\exp (-2 / t) / t \cdot(1+13 / 36 t+\cdots)$ where $t= \pm 1 / \sqrt{x}$.

Furthermore, one remarks experimentally, that their exterior square or symmetric square, either have a rational solution, or are of an order smaller than the order one would expect generically. Quite often these differential operators are simply conjugated to their adjoint, i.e. the intertwiner between the operator and its adjoint, is just an order-zero operator, namely a function. In that case they can easily be recast into self-adjoint operators. A large set of linear differential operators conjugated to their adjoint, can be found in the very large list of Calabi-Yau order-four operators obtained by Almkvist et al. [5], or displayed by Batyrev and van Straten [4], or some simple order-three operators displayed in a paper by Golyshev [21, 22] (see also Sanabria Malagon (23]).

Throughout this paper we will see examples of irreducible operators where these two differential algebra properties occur simultaneously. On the one hand, these operators are homomorphic to their adjoint, and on the other hand, their symmetric or exterior square have a rationa $\dagger$ solution. These simultaneous properties correspond to special differential Galois groups. In fact, these properties are equivalent

We will, in this paper, have a learn-by-example approach of all these concepts. In this respect, we will display, for pedagogical reasons, a set of enumerative combinatorics examples corresponding to miscellaneous lattice Green functions [2, 3, 25, 26, 27] as well as Calabi-Yau examples, together with order-seven operators 29, 30, associated with differential Galois exceptional groups. We will show that these lattice Green operators, Calabi-Yau operators and order-seven operators associated with exceptional groups, are a perfect illustration of differential operators with selected differential algebra structures: they are homomorphic to their adjoint, and, either their symmetric, or exterior, powers (most of the time squares) have a rational solution, or the previous symmetric, or exterior, powers of some equivalent operators have a rational solution. This situation corresponds to the emergence of selected differential Galois groups, a situation we could call "Special Geometry". Among the Derived From Geometry $n$-fold integrals ("Periods") occurring in physics, we have seen that they are quite often diagonals of rational functions [18, 19. We will also address in this paper examples of (minimal order) operators annihilating diagonals of rational functions, and will remark that they also seem to be associated with irreducible factors homomorphic to their adjoint.

## 2. Adjoint of differential operators and invertible homomorphisms of an operator with its adjoint

In the next section, examples of linear differential operators corresponding to lattice Green functions on various lattices are displayed according to their order $N$ and their complexity. We focus on the differential algebra structures of these linear differential operators, in particular with respect to an important "duality" with amounts to performing the adjoint, or, more precisely (see 2.1 in 31), the "formal adjoint" of
$\dagger$ For hyperexponential solutions [24] (command expsols in DEtools), i.e. $N$-th root of rational solutions, one must consider homomorphisms up to algebraic extensions.
$\ddagger$ In a Tannakian formulation, one could say that the homomorphisms of an operator $L_{1}$ with another operator $L_{2}$ are isomorphic to the product $\operatorname{Hom}\left(L_{1}, L_{2}\right) \simeq L_{1} \otimes L_{2}^{\star}$, giving, in the case of the homomorphisms of an operator $L$ with its adjoint $L^{\star}, \operatorname{Hom}\left(L, L^{\star}\right) \simeq L \otimes\left(L^{\star}\right)^{\star} \simeq L \otimes L$, which is isomorphic to the direct sum $L \otimes L \simeq \operatorname{Ext}^{2}(L) \oplus \operatorname{Sym}^{2}(L)$.
the operator ( $D_{x}$ in the whole paper denotes the derivative $d / d x$ ):

$$
\begin{equation*}
L=\sum_{n=0}^{N} a_{n}(x) \cdot D_{x}^{n} \quad \longrightarrow \quad \operatorname{adjoint}(L)=(-1)^{N} \cdot \sum_{n=0}^{N}(-1)^{n} \cdot D_{x}^{n} \cdot a_{n}(x), \tag{3}
\end{equation*}
$$

that is:

$$
\begin{equation*}
\sum_{n=0}^{N} a_{n}(x) \cdot \frac{d^{n} f(x)}{d x^{n}}=0 \quad \longrightarrow \quad(-1)^{N} \cdot \sum_{n=0}^{N}(-1)^{n} \cdot \frac{d^{n}\left(a_{n}(x) \cdot f(x)\right)}{d x^{n}}=0 \tag{4}
\end{equation*}
$$

Since exterior powers play a key role in this paper, it is worth recalling that the adjoint of an $N$-th order operator $L_{N}$ is nothing but the $(N-1)$-th exterior powen $\dagger$ of this operator, up to a factor that is the Wronskian $W\left(L_{N}\right)$ of the operator:

$$
\begin{equation*}
W\left(L_{N}\right) \cdot \operatorname{adjoint}\left(L_{N}\right)=\operatorname{Ext}^{N-1}\left(L_{N}\right) \cdot a_{N}(x) \cdot W\left(L_{N}\right) \tag{5}
\end{equation*}
$$

### 2.1. Homomorphisms of an operator with its adjoint

Recall that two operators $L$ and $\tilde{L}$, of the same order, are called homomorphic (see [32, 31]) when there exist two operators $T$ and $S$ of smaller order than the one of $L$ and $\tilde{L}$, such that

$$
\begin{equation*}
\tilde{L} \cdot T=S \cdot L \tag{6}
\end{equation*}
$$

The intertwiner $T$ maps the solutions of $L$ into the solutions of $\tilde{L}$. When $T$ and $L$ have no common right-factor (or equivalently when $S$ and $\tilde{L}$ has no common left factor), for example when $L$ is irreducible, one can show that this map is bijective. When (6) holds, and $T$ and $L$ have no common right-factor, one says that $L$ and $\tilde{L}$ are equivalent. Thus, one also has intertwiners $\tilde{T}, \tilde{S}$ such that

$$
\begin{equation*}
L \cdot \tilde{T}=\tilde{S} \cdot \tilde{L} \tag{7}
\end{equation*}
$$

We say that $L$ is self-adjoint when $L=\operatorname{adjoint}(L)$. We say that $L$ is conjugated with its adjoint when there exists a rational, or $N$-th root of rational, function $f$ such that $L \cdot f=f \cdot \operatorname{adjoint}(L)$, i.e. $L \cdot f$ is self-adjoint. More generally, a differential operator $L$ is homomorphic to its adjoint (in the above sense) when there exists an (intertwiner) operator $T$ (of order les $\sharp$ than that of $L$ ) such that

$$
\begin{equation*}
L \cdot T=\operatorname{adjoint}(T) \cdot \operatorname{adjoint}(L) \tag{8}
\end{equation*}
$$

Again, this means that the operator $L \cdot T$ is self-adjoint.
The typical situation we encounter in physics is such that the differential operators are of a quite large order and factorize into many factors of various orders (see the minimal order operators [11, 13] annihilating the $\chi^{(n)}$ 's). For these large order differential operators, we will systematically factorize the operator. The
$\dagger$ In maple the exterior power is normalised to be a monic operator (the head polynomial is normalised to 1).
§ The intertwiner $T$ is given by the command Homomorphisms $(L, \tilde{L})$ of the DEtools package in Maple 33].
$\#$ Note that the constraint on the order rules out the "tautological" intertwining relation, satisfied by any operator, like $L \cdot \operatorname{adjoint}(T)=T \cdot \operatorname{adjoint}(L)$ with $T=L$.
【 It is easy to show, in the case of an homomorphism of an operator $L$ with its adjoint, that the intertwiner on the right-hand-side of (8) is necessarily equal to the adjoint of the intertwiner on the left-hand-side. Actually, from the equivalence $L \cdot T=S \cdot \operatorname{adjoint}(L)$, taking adjoint on both sides gives $\operatorname{adjoint}(T) \cdot \operatorname{adjoint}(L)=L \cdot \operatorname{adjoint}(S)$. For irreducible $L$, the intertwiner is unique, so $S=\operatorname{adjoint}(T)$.
interesting concept amounts to seeing if each irreducible factor in the factorization, is homomorphic to its adjoint.

We end this section with two comments. For irreducible $L$, one deduces, from (6) and (7), the equality

$$
\begin{equation*}
L \cdot \tilde{T} \cdot T=\tilde{S} \cdot S \cdot L \tag{9}
\end{equation*}
$$

so the rest of the right division of $\tilde{T} \cdot T$ by $L$ is a constant. When $\tilde{L}$ is the adjoint of $L$, we will see, in the sequel, that this relation on the intertwiners $T$ and $\tilde{T}$ makes a remarkable "decomposition" of $L$ emerge. The second comment is on the homomorphisms of an operator with its adjoint in the reducible case. For two reducible differential operators, $L$ and $\tilde{L}$, of the same order, the relation (6) may hold but may not be an equivalence of operators. In the case of a reducible 34] operator having the unique factorization $L=L_{n} \cdot L_{p}$, with $n \neq p$, one can show that the homomorphism with the adjoint just reduces to a homomorphism of the right factor $L_{p}$ with its adjoint. The corresponding rational solution for the symmetric or exterior square is precisely the rational solution induced by the right factor, since $\operatorname{Sym}^{2}\left(L_{p}\right)$ (resp. Ext ${ }^{2}\left(L_{p}\right)$ ) is a right-factor of $\operatorname{Sym}^{2}\left(L_{n} \cdot L_{p}\right)$ (resp. $\operatorname{Ext}^{2}\left(L_{n} \cdot L_{p}\right)$ ).

In the sequel, when studying homomorphism of an operator with its adjoint, we will restrict to irreducible operators.

## 3. Special ODEs from lattice statistical mechanics and enumerative combinatorics: lattice Green functions

We are going to display a set of miscellaneous examples of linear differential operators corresponding to lattice Green functions on various lattices. We will denote these lattice Green operators $G_{n}^{l a t t}$, where $n$ is the order of the operators, and where latt refers to the lattice one considers.

### 3.1. Special lattice Green ODEs: body-centered cubic lattice and simple cubic lattice

One of the simplest example of lattice Green function corresponds to the order-three operator, for the body-centered cubic lattice, given in equation (19) of [3], which reads

$$
\begin{equation*}
G_{3}^{b c c}=\theta^{3}-64 \cdot x \cdot(2 \theta+1)^{3} \tag{10}
\end{equation*}
$$

This order-three operator has the ${ }_{3} F_{2}$ hypergeometric solution

$$
\begin{equation*}
{ }_{3} F_{2}\left(\left[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right],[1,1] ; 512 x\right)=\left({ }_{2} F_{1}\left(\left[\frac{1}{4}, \frac{1}{4}\right],[1] ; 512 x\right)\right)^{2} \tag{11}
\end{equation*}
$$

where the ${ }_{2} F_{1}$ hypergeometric function is a series with integer coefficients:

$$
{ }_{2} F_{1}\left(\left[\frac{1}{4}, \frac{1}{4}\right],[1] ; 512 x\right)=1+32 x+6400 x^{2}+1843200 x^{3}+623001600 x^{4}+\cdots
$$

The operator (10) is conjugated to its adjoint: $x \cdot \operatorname{adjoint}\left(G_{3}^{b c c}\right)=G_{3}^{b c c} \cdot x$. The symmetric square of $G_{3}^{b c c}$ is of order five (instead of the order six one could expect generically for order-three operators).

The most well-known example of lattice Green function has been obtained [35] for the simple cubic lattice. The lattice Green function corresponds to the order-three

[^1]operator (see equation (19) in [3])
\[

$$
\begin{align*}
G_{3}^{s c}=\quad \theta^{3} & -2 x \cdot\left(10 \theta^{2}+10 \theta+3\right) \cdot(2 \theta+1)  \tag{12}\\
& +18 x^{2} \cdot(2 \theta+3)(2 \theta+2)(2 \theta+1)
\end{align*}
$$
\]

This order-three operator (12), when divided by $x$ on the left, is exactly self-adjoint. The symmetric square of $G_{3}^{s c}$ is of order five (instead of the generic order six).

The solution of (12), which corresponds to a series expansion with integer coefficients, is the Hadamard product of $(1-4 x)^{-1 / 2}$ with a Heun function, and is also the square of another Heun function which can also be written in terms of ${ }_{2} F_{1}$ hypergeometric functions with two possible algebraic pullbacks:

$$
\begin{align*}
& \text { Heun } G\left(9, \frac{3}{4}, \frac{1}{4}, \frac{3}{4}, 1, \frac{1}{2} ; 36 x\right)^{2}=\operatorname{Heun} G\left(\frac{1}{9}, \frac{1}{12}, \frac{1}{4}, \frac{3}{4}, 1, \frac{1}{2} ; 4 x\right)^{2} \\
& =(1-4 x)^{-1 / 2} \star \operatorname{Heun} G(1 / 9,1 / 3,1,1,1,1 ; x)=C_{ \pm}^{1 / 2} \cdot{ }_{2} F_{1}\left(\left[\frac{1}{6}, \frac{1}{3}\right],[1] ; P_{ \pm}\right)^{2} \\
& =1+6 x+90 x^{2}+1860 x^{3}+44730 x^{4}+1172556 x^{5}+\cdots \tag{13}
\end{align*}
$$

where the algebraic pull-backs $P_{ \pm}$and algebraic prefactors $C_{ \pm}$read:

$$
\begin{align*}
& P_{ \pm}=54 \cdot x \cdot\left(1-27 x+108 x^{2} \pm(1-9 x) \cdot((1-36 x) \cdot(1-4 x))^{1 / 2}\right) \\
& C_{ \pm}=-18 x+\frac{5}{2} \pm \frac{3}{2} \cdot((1-36 x) \cdot(1-4 x))^{1 / 2} \tag{14}
\end{align*}
$$

The fact that these selected Heun functions (13) correspond to modular forms [20] can be seen on the relation between the two algebraic pullbacks, $y=P_{+}$and $z=P_{-}$, namely the genus-zero modular curvet:

$$
\begin{array}{r}
4 \cdot y^{3} z^{3}-12 y^{2} z^{2} \cdot(z+y)+3 y z \cdot\left(4 y^{2} 4 z^{2}-127 y z\right) \\
-4 \cdot(y+z) \cdot\left(y^{2}+z^{2}+83 y z\right)+432 \cdot y z=0 \tag{15}
\end{array}
$$

Remark: If one compares two Heun functions with the same singular points and the same critical exponents, which just differ by their accessory parameter, namely $\operatorname{HeunG}(9,3 / 4,1 / 4,3 / 4,1,1 / 2,36 x)$ and $\operatorname{HeunG}(9,-3 / 4,1 / 4,3 / 4,1,1 / 2,36 x)$, one sees that the first one corresponds to a modular form and to series with integer coefficients, while the second one is not even a globally bounded series [18, 19]. These two Heun functions $\operatorname{Heun} G(9, \pm 3 / 4,1 / 4,3 / 4,1,1 / 2,36 x)$ are solutions of order-two linear differential operators

$$
\begin{equation*}
H_{2}^{( \pm)}=\theta^{2}-x \cdot\left(40 \theta^{2}+20 \theta \pm 3\right)+9 \cdot x^{2} \cdot(4 \theta+3) \cdot(4 \theta+1) \tag{16}
\end{equation*}
$$

which are, both, conjugated to their adjoint:

$$
f(x) \cdot \operatorname{adjoint}\left(H_{2}^{( \pm)}\right)=H_{2}^{( \pm)} \cdot f(x) \quad \text { with: } \quad f(x)=x \cdot((1-36 x) \cdot(1-4 x))^{1 / 2}
$$

[^2]
### 3.2. Special lattice Green ODEs: face-centered cubic lattice

A third order linear differential operator corresponds to the lattice Green function of the face-centered cubic lattice (see equation (19) in Guttmann's paper [3]):

$$
\begin{align*}
G_{3}^{f c c}=\theta^{3}- & 2 x \cdot \theta \cdot(\theta+1) \cdot(2 \theta+1)-16 x^{2} \cdot(\theta+1) \cdot\left(5 \theta^{2}+10 \theta+6\right) \\
& -96 x^{3} \cdot(\theta+1) \cdot(\theta+2) \cdot(2 \theta+3) \tag{17}
\end{align*}
$$

where $\theta$ is the homogeneous derivative: $\theta=x \cdot d / d x$. This operator, once divided by $x$, is exactly self-adjoint: $\quad 1 / x \cdot G_{3}^{f c c}=\operatorname{adjoint}\left(1 / x \cdot G_{3}^{f c c}\right)$. The symmetric square of $G_{3}^{f c c}$ is of order five (instead of the order six one could expect for generic order-three operators).

Let us introduce, instead of $G_{3}^{f c c}$, the equivalent operator $\tilde{G}_{3}^{f c c}$ such that

$$
\begin{equation*}
S_{1}^{f c c} \cdot G_{3}^{f c c}=\tilde{G}_{3}^{f c c} \cdot D_{x} \tag{18}
\end{equation*}
$$

where the order-one intertwiner $S_{1}^{f c c}$ reads up to a factor

$$
\begin{equation*}
D_{x}-\frac{d \ln (\rho(x))}{d x}, \tag{19}
\end{equation*}
$$

where the Wronskian $\rho(x)$ is a rational function:

$$
\begin{equation*}
\rho(x)=\frac{6 x+1}{x \cdot(4 x+1)^{2} \cdot(12 x-1)} . \tag{20}
\end{equation*}
$$

We find that the symmetric square of the equivalent operator $\tilde{G}_{3}^{f c c}$ has a rational solution $r(x)$ :

$$
\begin{equation*}
r(x)=\frac{1}{x^{2} \cdot(4 x+1)^{2}(12 x-1)} \tag{21}
\end{equation*}
$$

More precisely, the symmetric square of the equivalent operator $\tilde{G}_{3}^{f c c}$ is the direct sum§ of an order-one operator and an order-five operator:

$$
\begin{equation*}
\operatorname{Sym}^{2}\left(\tilde{G}_{3}^{f c c}\right)=M_{1} \oplus M_{5} \quad \text { where: } \quad M_{1}=D_{x}-\frac{d \ln (r(x))}{d x} \tag{22}
\end{equation*}
$$

The Wronskian of $G_{3}^{f c c}$ is the square root of a rational function. The differential Galois group is not the generic $S L(3, \mathbb{C})$ one could expect for a generic order-three operator, but is equal to the orthogonal group $O(3, \mathbb{C})$ : the rational solution (21) of $\operatorname{Sym}^{2}\left(\tilde{G}_{3}^{f c c}\right)$, comes from an invariant of degree 2 for the differential Galois group.

In fact the operator (17) happens to be the symmetric square of an order-two operator :

$$
\begin{equation*}
\theta^{2}-2 x \cdot \theta \cdot(4 \theta+1)-24 x^{2} \cdot(\theta+1)(2 \theta+1) \tag{23}
\end{equation*}
$$

From that last remark, one immediately deduces that the differential Galois group must be the differential Galois group of an order-two operator, generically $S L(2, \mathbb{C})$. Indeed, $O(3, \mathbb{C})$ is a symmetric square of $S L(2, \mathbb{C})$ (see [37]. It is shown in 37 that a third order operator has a symmetric square of order five (instead of six) if, and only if, it is the symmetric square of a second order operator.

[^3]
### 3.3. Special lattice Green ODEs: diamond lattice

Another example can be found in Guttmann and Prellberg [3, 25, and corresponds to an order-three operator which has the following ${ }_{3} F_{2}$ solution

$$
\begin{equation*}
\frac{1}{\left(4-x^{2}\right)^{3}} \cdot{ }_{3} F_{2}\left(\left[\frac{1}{3}, \frac{1}{2}, \frac{2}{3}\right],[1,1], \frac{27 x^{4}}{\left(4-x^{2}\right)^{3}}\right) \tag{24}
\end{equation*}
$$

associated with the Green function of the diamond lattice. This order-three linear differential operator reads:

$$
\begin{align*}
G_{3}^{\text {diam }}= & 64 \cdot \theta^{3}-16 x^{2} \cdot\left(7 \theta^{3}+27 \theta^{2}+42 \theta+24\right) \\
& +12 x^{4} \cdot\left(5 \theta^{3}+42 \theta^{2}+124 \theta+128\right)  \tag{25}\\
& \quad-x^{6} \cdot\left(13 \theta^{3}+171 \theta^{2}+762 \theta+1152\right)+x^{8} \cdot(\theta+6)^{3}
\end{align*}
$$

which can be seen to be conjugated to its adjoint:

$$
\frac{\left(x^{2}-4\right)^{2}}{x} \cdot G_{3}^{\text {diam }}=\operatorname{adjoint}\left(G_{3}^{\text {diam }}\right) \cdot \frac{\left(x^{2}-4\right)^{2}}{x}
$$

The symmetric square of $G_{3}^{\text {diam }}$ is of order five (instead of the order six one could expect for generic order-three operators).

Let us introduce, instead of $G_{3}^{\text {diam }}$, the equivalent operator $\tilde{G}_{3}^{\text {diam }}$

$$
\begin{equation*}
S_{1}^{\text {diam }} \cdot G_{3}^{\text {diam }}=\tilde{G}_{3}^{\text {diam }} \cdot D_{x} \tag{26}
\end{equation*}
$$

where the order-one intertwiner $S_{1}^{\text {diam }}$ reads

$$
\begin{equation*}
S_{1}^{d i a m}=-\frac{\left(x^{2}-4\right)^{2}}{x} \cdot r(x) \cdot\left(D_{x}-\frac{1}{\rho(x)} \cdot \frac{d \rho(x)}{d x}\right) \tag{27}
\end{equation*}
$$

where $r(x)$ and $\rho(x)$ are rational functions:

$$
\begin{align*}
& r(x)=\frac{1}{(x-2)^{5}(x-1)(x+1)(x+2)^{5} \cdot x^{2}}, \quad \text { and: }  \tag{28}\\
& \rho(x)=\left(3 x^{3}-8 x+4\right)\left(3 x^{3}-8 x-4\right) \cdot x^{2} \tag{29}
\end{align*}
$$

Again, the symmetric square of the equivalent operator $\tilde{G}_{3}^{\text {diam }}$ has the rational solution $r(x)$.

The Wronskian of (25) is the square root of a rational function. The differential Galois group is again $O(3, \mathbb{C})$.

As in the previous example, $G_{3}^{\text {diam }}$ is the symmetric square of an order-two operator:

$$
\begin{align*}
16 \cdot \theta^{2}- & 12 x^{2} \cdot\left(2 \theta^{2}+3 \theta+2\right) \\
& +3 x^{4} \cdot\left(3 \theta^{2}+11 \theta+12\right)-x^{6} \cdot(\theta+3)^{2} \tag{30}
\end{align*}
$$

### 3.4. Order-three operators conjugated to their adjoint

In fact, these previous results, for the bcc, sc, fcc, diamond lattices, can be seen as the consequence of the following general result on order-three linear differential operators (without any loss of generality we restrict to monic operators)

$$
\begin{equation*}
L_{3}=D_{x}^{3}+a_{2}(x) \cdot D_{x}^{2}+a_{1}(x) \cdot D_{x}+a_{0}(x) . \tag{31}
\end{equation*}
$$

Any (monic) order-three operator which is conjugated to its adjoint, namely $L_{3} \cdot f(x)=f(x) \cdot \operatorname{adjoint}\left(L_{3}\right)$, is the symmetric square of an order-two operator

$$
\begin{align*}
& L_{2}=D_{x}^{2}+b_{1}(x) \cdot D_{x}+b_{0}(x), \quad \text { with: } \quad b_{1}(x)=-\frac{1}{2} \frac{1}{f(x)} \frac{d f(x)}{d x}  \tag{32}\\
& \text { where: } \quad b_{0}(x)=\frac{a_{1}(x)}{4}+\frac{1}{8} \frac{1}{f(x)} \cdot \frac{d^{2} f(x)}{d x^{2}}-\frac{1}{4} \cdot\left(\frac{1}{f(x)} \cdot \frac{d f(x)}{d x}\right)^{2} .
\end{align*}
$$

Note that one necessarily has $a_{2}(x)=3 b_{1}(x)$. The Wronskian of $L_{3}$ is necessarily equal to $f(x)^{3 / 2}$, and the order-two operator (32) is conjugated to its adjoint by a function:

$$
\begin{equation*}
f(x)^{1 / 2} \cdot \operatorname{adjoint}\left(L_{2}\right)=L_{2} \cdot f(x)^{1 / 2} \tag{33}
\end{equation*}
$$

The symmetric square of such an order-three operator $L_{3}$, conjugated to its adjoint, is of order five (in contrast to order six for symmetric squares of generic order-three operators).

### 3.5. Special lattice Green ODEs: $4 D$ face-centered cubic lattice

A slightly more involved example, corresponding to the four-dimensional face-centered cubic lattice Green function, can be found in paragraph 2.5 of Guttmann's paper 3] (it is also ODE number 366 in the list of Almkvist et al. [5]). This order-four linear differential operator

$$
\begin{align*}
G_{4}^{4 D f c c}= & \theta^{4}+x \cdot\left(39 \cdot \theta^{4}-30 \cdot \theta^{3}-19 \cdot \theta^{2}-4 \theta\right) \\
& +2 x^{2} \cdot\left(16 \cdot \theta^{4}-1070 \cdot \theta^{3}-1057 \cdot \theta^{2}-676 \theta-192\right) \\
& -36 x^{3} \cdot\left(171 \cdot \theta^{3}+566 \cdot \theta^{2}+600 \theta+316\right) \cdot(3 \theta+2) \\
& -2^{5} 3^{3} x^{4} \cdot\left(384 \cdot \theta^{4}+1542 \cdot \theta^{3}+2635 \cdot \theta^{2}+2173 \theta+702\right) \\
& -2^{6} 3^{3} x^{5} \cdot\left(1393 \cdot \theta^{3}+5571 \cdot \theta^{2}+8378 \theta+4584\right) \cdot(\theta+1) \\
& -2^{10} 3^{5} x^{6} \cdot\left(31 \cdot \theta^{2}+105 \theta+98\right) \cdot(\theta+1) \cdot(\theta+2)  \tag{34}\\
& -2^{12} 3^{7} x^{7} \cdot(\theta+1) \cdot(\theta+2)^{2} \cdot(\theta+3) \\
= & x^{4} \cdot(1+3 x)(1+4 x)(1+8 x)(1+12 x)(1+18 x)^{2}(1-24 x) \cdot D_{x}^{4} \quad+\cdots
\end{align*}
$$

can be seen to be conjugated to its adjoint by a function $f^{4 D f c c}$ : $G_{4}^{4 D f c c} \cdot f^{4 D f c c}=f^{4 D f c c} \cdot \operatorname{adjoint}\left(G_{4}^{4 D f c c}\right), \quad$ with: $\quad f^{4 D f c c}=x \cdot(1+18 x)^{3}$.
The exterior square of operator (34) is an irreducible order-five operator (not order-six as could be expected): one easily checks that the "Calabi-Yau condition" (see [6] and (94) below) is actually satisfied for operator (34). If one considers an operator $\tilde{G}_{4}^{4 D f c c}$, non-trivially homomorphic [32, 31] to $G_{4}^{4 D f c c}$, its exterior square is, now, an operator of (the generic) order six, and it has a rational solution. For instance, if we consider the operator $\tilde{G}_{4}^{4 D f c c}$ equivalent to $G_{4}^{4 D f c c}$

$$
\begin{equation*}
S_{1}^{4 D f c c} \cdot G_{4}^{4 D f c c}=\tilde{G}_{4}^{4 D f c c} \cdot D_{x} \tag{35}
\end{equation*}
$$

where

$$
\begin{gather*}
S_{1}^{4 D f c c}=-\frac{r(x)}{(1+18 x)^{3} \cdot x} \cdot\left(D_{x}-\frac{d \ln (\rho(x))}{d x}\right), \quad \text { with }  \tag{36}\\
r(x)=\frac{18 x+1}{x^{3} \cdot(3 x+1)(4 x+1)(8 x+1)(12 x+1)(24 x-1)}, \quad \text { and } \\
\rho(x)=\left(1119744 x^{5}+508032 x^{4}+82512 x^{3}+6318 x^{2}+237 x+4\right) \cdot x,
\end{gather*}
$$

we find that the exterior square of $\tilde{G}_{4}^{4 D f c c}$ has the rational solution $r(x)$.
This is a situation we will encounter many times: switching to an equivalent operator "desingularizes" the drop of order of the exterior (resp. symmetric) $n$-th power of an operator, into a situation of emergence of a rational solution for that $n$-th power.

The Wronskian of $G_{4}^{4 D f c c}$ is a rational function. As the exterior square of $\tilde{G}_{4}^{4 D f c c}$ has a rational solution, the differential Galois group is included in the symplectic group $S p(4, \mathbb{C})$. Moreover, its symmetric square being irreducible, theorems A. 5 et A. 7 of Beukers et al. 38 show that the differential Galois group is exactly $S p(4, \mathbb{C})$.

### 3.6. Another version of the order four operator for $4 D$ fcc lattice

Another version of $G_{4}^{4 D f c c}$, can be found in the unpublished paper of D. Broadhurst (see equation (68) in [39]) and corresponds to the order-four operator

$$
\begin{equation*}
G_{4}^{4 D}=(2 \theta)^{4}+\sum_{j=1}^{6}(-1)^{j} \cdot P_{j}(2 \theta+j)=h_{4} \cdot D_{x}^{4}+\cdots \tag{37}
\end{equation*}
$$

with

$$
\begin{aligned}
& P_{1}(u)=105 u^{4}+166 u^{2}+17, \quad P_{2}(u)=2 \cdot\left(2095 u^{4}+2912 u^{2}+432\right), \\
& P_{3}(u)=72 \cdot\left(1155 u^{4}-892 u^{2}+577\right), \quad P_{4}(u)=864 \cdot\left(1011 u^{4}-5059 u^{2}+4900\right), \\
& P_{5}(u)=75600 \cdot\left(u^{2}-9\right)\left(61 u^{2}-145\right), \quad P_{6}(u)=9525600 \cdot\left(u^{2}-4\right)\left(u^{2}-16\right),
\end{aligned}
$$

and where the head polynomial $h_{4}$ reads:

$$
h_{4}=16 x^{4} \cdot(1-6 x)(1-10 x)(1-14 x)(1-15 x)(1-18 x)(1-42 x) .
$$

Operator $G_{4}^{4 D}$ is a globally nilpotent operator, as can be seen on the Jordan form reduction of its $p$-curvature [8, 40].

This irreducible order-four operator $G_{4}^{4 D}$ is conjugated to its adjoint (or exactly self-adjoint with a different normalization):

$$
\begin{equation*}
G_{4}^{4 D} \cdot x=x \cdot \operatorname{adjoint}\left(G_{4}^{4 D}\right) \tag{38}
\end{equation*}
$$

Its exterior square is an irreducible order-five linear differential operator (not ordersix as could be expected): the "order-five Calabi-Yau condition" (see (94) below) is satisfied for (37). Similarly to (35) and (44), one can switch, by operator equivalence, to an operator $\tilde{G}_{4}^{4 D}$ such that its exterior square has a rational solution:

$$
\begin{equation*}
S_{1}^{4 D f c c} \cdot G_{4}^{4 D}=\tilde{G}_{4}^{4 D} \cdot D_{x} \tag{39}
\end{equation*}
$$

where the order-one intertwiner $S_{1}^{4 D}$ reads

$$
\begin{equation*}
S_{1}^{4 D}=-\frac{r(x)}{x} \cdot\left(D_{x}-\frac{d \ln (\rho(x))}{d x}\right), \tag{40}
\end{equation*}
$$

with

$$
\rho(x)=\left(63504000 x^{5}-17388000 x^{4}+1644948 x^{3}-64578 x^{2}+950 x-3\right) \cdot x
$$

and where $r(x)$ is the rational function:

$$
\begin{equation*}
r(x)=\frac{1}{x^{3} \cdot(6 x-1)(10 x-1)(14 x-1)(15 x-1)(18 x-1)(42 x-1)} . \tag{41}
\end{equation*}
$$

- Corresponding to a change of variable: $F(x) \rightarrow F(x /(1-18 x)) /(1-18 x)$.

As above, the exterior square of the equivalent operator $\tilde{G}_{4}^{4 D}$ has (obviously) the rational solution $r(x)$ : the differential Galois group of $G_{4}^{4 D}$ is not $S L(4, \mathbb{C})$ but is included in the symplectic group $S p(4, \mathbb{C})$.

One can get, directly, a rational solution if one switches to the linear differential system associated with $G_{4}^{4 D}$, calculated the exterior square of that system and seeks for the rational solution of that exterior square system.

One gets, that way, the rational solution of the exterior square of the companion system of $G_{4}^{4 D}$ :

$$
\begin{equation*}
\left[0, \quad 0, \quad r(x),-r(x), x^{2} \cdot q_{5} \cdot r(x)^{2},-x \cdot q_{6} \cdot r(x)^{2}\right] \tag{42}
\end{equation*}
$$

where:

$$
\begin{aligned}
& q_{5}=85730400 x^{6}-36892800 x^{5}+6114528 x^{4}-498960 x^{3}+20950 x^{2}-420 x+3 \\
& q_{6}=190512000 x^{6}-72198000 x^{5}+10262808 x^{4}-690804 x^{3}+22406 x^{2}-304 x+1
\end{aligned}
$$

At first sight, for a physicist, performing calculations on a linear differential operator, like $\tilde{G}_{4}^{4 D}$, and finding that its exterior square has the rational solution (41), seems simpler, and more natural, than introducing the companion system. As far as practical calculations are concerned, the calculations on the linear differential systems turn out to be drastically more efficient, and allow to handle symmetric and exterior powers constructions on larger examples.

### 3.7. Special lattice Green ODEs: 5D staircase polygons

Another example of Guttmann and Prellberg [3, 25], corresponding to the generating function of the five-dimensional staircase polygons, is the order-four operator

$$
\begin{align*}
G_{4}^{5 D}= & \theta^{4}-x \cdot\left(35 \theta^{4}+70 \theta^{3}+63 \theta^{2}+28 \theta+5\right)  \tag{43}\\
& +x^{2} \cdot\left(259 \theta^{2}+518 \theta+285\right) \cdot(\theta+1)^{2}-225 x^{3} \cdot(\theta+1)^{2} \cdot(\theta+2)^{2} \\
& =x^{4} \cdot\left(1+35 x+259 x^{2}-225 x^{3}\right) \cdot D_{x}^{4}+\cdots
\end{align*}
$$

which can be seen to be conjugated to its adjoint:

$$
G_{4}^{5 D} \cdot x=x \cdot \operatorname{adjoint}\left(G_{4}^{5 D}\right)
$$

The exterior square operator of the order-four operator (43) is an irreducible order-five operator (not order-six as could be expected): the "order-five Calabi-Yau condition" (see (94) below) is satisfied for (43). Let us introduce, instead of $G_{4}^{5 D}$, the equivalent operator $\tilde{G}_{4}^{5 D}$ corresponding to the intertwining relation

$$
\begin{equation*}
S_{1}^{5 D} \cdot G_{4}^{5 D}=\tilde{G}_{4}^{5 D} \cdot D_{x} \tag{44}
\end{equation*}
$$

where the order-one intertwiner $S_{1}^{5 D}$ reads

$$
\begin{equation*}
S_{1}^{5 D}=-\frac{r(x)}{x} \cdot\left(D_{x}-\frac{d \ln ((60 x+1)(3 x-1) x)}{d x}\right) \tag{45}
\end{equation*}
$$

and where $r(x)$ is the rational function:

$$
\begin{equation*}
r(x)=\frac{1}{\left(225 x^{3}-259 x^{2}-35 x-1\right) \cdot x^{3}} . \tag{46}
\end{equation*}
$$

$\dagger$ In order to do these calculations download the three Maple Tools files TensorConstructions.m and IntegrableConnections.m in the web page 41. Using DEtools, you will need to use, on the ordersix operator $G_{4}^{4 D}$ written in a monic way, the command companion-system, then the command exterior-power-system( , 2), and, finally, the command RationalSolutions([],[x]).

We find, again, that the exterior square of the equivalent operator $\tilde{G}_{4}^{5 D}$ has the rational solution $r(x)$. The Wronskian of $G_{4}^{5 D}$ is a rational function. The differential Galois group is, again (see A. 5 et A. 7 in appendix A of [38]), the symplectic group $S p(4, \mathbb{C})$.
3.8. Order-six operator by Broadhurst and Koutschan: the lattice Green function of the five-dimensional fcc lattice

A more involved example of order-six, can be found in Koutschan's paper [27] and in an unpublished paper of D. Broadhurst (see equation (74) in 39) and corresponds to a five-dimensional fcc lattice

$$
\begin{equation*}
G_{6}^{5 D f c c}=3^{4} \cdot \theta^{5} \cdot(\theta-1)+\sum_{j=1}^{12} x^{j} \cdot Q_{j}(\theta)=h_{6} \cdot D_{x}^{6}+\cdots, \tag{47}
\end{equation*}
$$

where the polynomials $Q_{j}$ are degree-six polynomials with integer coefficients, and where the head polynomial $h_{6}$ reads:

$$
h_{6}=x^{6} \cdot \lambda(x) \cdot p_{6}
$$

with: $\quad \lambda(x)=(1-4 x)(1-8 x)(1+16 x)(1-16 x)(1-48 x)(3-16 x)$,
and: $\quad p_{6}=916586496 x^{6}-571981824 x^{5}+67242496 x^{4}-8372096 x^{3}$

$$
\begin{equation*}
+315096 x^{2}-6840 x+27 \tag{48}
\end{equation*}
$$

This order-six linear differential operator has, at the origin $x=0$, two independent analytic solutions (no logarithms, it is not MUM $\boldsymbol{1 9}$ ). One can build, from these two solutions, a one-parameter family of analytic solutions:

$$
\begin{aligned}
& 1+8 \cdot x \cdot c+\frac{8}{3} \cdot(41 \cdot c-2) \cdot x^{2} \\
& \quad+\frac{32}{27} \cdot(1933 \cdot c-286) \cdot x^{3} \\
&+\frac{8}{27} \cdot(183136 \cdot c-25537) \cdot x^{4}+\frac{256}{2025} \cdot(12082067 \cdot c-1788992) \cdot x^{5}+\cdots
\end{aligned}
$$

which, for $c=1$, (and only this value) becomes a series with integen coefficients:

$$
\begin{gathered}
1+8 \cdot x+104 \cdot x^{2}+1952 \cdot x^{3}+46696 \cdot x^{4}+1301248 \cdot x^{5}+40047584 \cdot x^{6} \\
+1319992192 \cdot x^{7}+45737941096 \cdot x^{8}+1646328483008 \cdot x^{9}+\cdots
\end{gathered}
$$

The question of the integrality of such D-finite series, emerging from physics, is addressed in previous papers [18, 19].

Remark: The other unique independent no-log series starting with $x$ reads:

$$
\begin{gathered}
z_{0}(x)=\quad x+\frac{41}{3} \cdot x^{2}+\frac{7732}{27} \cdot x^{3}+\frac{183136}{27} \cdot x^{4}+\frac{386626144}{2025} \cdot x^{5} \\
+\frac{106836145888}{18225} \cdot x^{6}+\frac{172725353100416}{893025} \cdot x^{7}+\cdots
\end{gathered}
$$

It is not a globally bounded series [18, 19, i.e. it is not a series that can be recast into a series with integer coefficients after a rescaling of the variable.
$\ddagger$ The integrality of these coefficients has been checked with 2000 coefficients, and the coefficients $c_{c \cdot 10000} \cdot x^{c \cdot 10000}$ coefficients, for $c=1,2,3,4$, have also been seen to be integers.

This order-six linear differential operator is globally nilpotent [8, 40, the Jordan reduction of its $p$-curvature [8, 40] reading:

$$
J_{6}=\left[\begin{array}{llllll}
0 & 0 & 0 & 0 & 0 & 0  \tag{49}\\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]
$$

We found that the order-six operator $G_{6}^{5 D f c c}$ is non-trivially homomorphic to its adjoint, with a simple order-one intertwiner

$$
\begin{equation*}
G_{6}^{5 D f c c} \cdot T_{1}^{5 D f c c}=\operatorname{adjoint}\left(T_{1}^{5 D f c c}\right) \cdot \operatorname{adjoint}\left(G_{6}^{5 D f c c}\right), \tag{50}
\end{equation*}
$$

with:

$$
\begin{align*}
& T_{1}^{5 D f c c}=x^{2} \cdot p_{2} \cdot p_{6} \cdot\left(D_{x}-\frac{1}{2} \cdot \frac{d \ln (R(x))}{d x}\right), \\
& R(x)=\frac{p_{2}^{5}}{x^{4} \cdot p_{6}^{4}} \quad \text { with } \quad p_{2}=1152 x^{2}-56 x-3 \tag{51}
\end{align*}
$$

Introducing

$$
\begin{equation*}
\rho(x)=\frac{p_{2}{ }^{6}}{p_{6}{ }^{3} \cdot x^{2}}, \tag{52}
\end{equation*}
$$

the previous order-one intertwiner $T_{1}^{5 D f c c}$, can be seen as the product of the rational function $\rho(x)$, and of a self-adjoint order-one operator $Y_{1}^{s}$ :

$$
\begin{equation*}
T_{1}^{5 D f c c}=\rho(x) \cdot Y_{1}^{s}, \quad Y_{1}^{s}=\frac{1}{R(x)} \cdot\left(D_{x}-\frac{1}{2} \cdot \frac{d \ln (R(x))}{d x}\right) \tag{53}
\end{equation*}
$$

The other intertwining relation is a bit more involved since the intertwiner is an order-five linear differential operator $S_{5}^{5 D f c c}$

$$
\begin{equation*}
\operatorname{adjoint}\left(S_{5}^{5 D f c c}\right) \cdot G_{6}^{5 D f c c}=\operatorname{adjoint}\left(G_{6}^{5 D f c c}\right) \cdot S_{5}^{5 D f c c}, \tag{54}
\end{equation*}
$$

where

$$
S_{5}^{5 D f c c}=\frac{x^{2} \cdot \lambda(x) \cdot p_{2}^{5}}{p_{6}^{3}} \cdot\left(D_{x}^{5}-\frac{1}{2} \cdot \frac{d \ln (\mu(x))}{d x} \cdot D_{x}^{4}+\cdots\right)
$$

with $\lambda(x)$ as above in (48), and:

$$
\mu(x)=-\frac{p_{2}^{5}}{\lambda(x)^{5} \cdot x^{20}}
$$

Quite remarkably, introducing the same function $\rho(x)$ as for $T_{1}^{5 D f c c}$ (see (52)), the previous order-five intertwiner $S_{5}^{5 D f c c}$, can be seen as the product $S_{5}^{5 D f c c}=$ $\rho(x) \cdot Y_{5}^{s}$, of the rational function $\rho(x)$ (see (52)) and of a self-adjoint order-five operator

$$
\begin{equation*}
Y_{5}^{s}=\frac{x^{4} \cdot \lambda(x)}{p_{2}} \cdot\left(D_{x}^{5}-\frac{1}{2} \cdot \frac{d \ln (\mu(x))}{d x} \cdot D_{x}^{4}+\cdots\right) . \tag{55}
\end{equation*}
$$

The self-adjoint order-five irreducible operator $Y_{5}^{s}$ has a solution analytic at $x=0$, (the other solution have log terms), which has the following expansion

$$
\begin{align*}
1+8 x & +102 x^{2}+\frac{487192}{243} x^{3}+\frac{86597215}{1944} x^{4}+\frac{22841991292}{16875} x^{5} \\
& +\frac{1874527149707741}{49207500} x^{6}+\frac{40302470144568331141}{29536801875} x^{7}+\cdots \tag{56}
\end{align*}
$$

This solution-series is not globally bounded [18, 19]. The examination of the formal series solutions at $x=0$ corresponds to a MUM structure.

The self-adjoint order-five irreducible operator $Y_{5}^{s}$ is such that its symmetric square is of order 14 instead of the order 15 expected generically (its exterior square is of order 10 as it should, with no rational solution).

The Wronskian of this order-six linear differential operator $G_{6}^{5 D f c c}$ is the square root of a rational function:

$$
W\left(G_{6}^{5 D f c c}\right)=\left(\frac{p_{6}^{2}}{x^{28} \cdot \lambda(x)^{7}}\right)^{1 / 2}
$$

The previous homomorphisms of the order-six operator $G_{6}^{5 D f c c}$ with its adjoint, namely (50) and (54), can be simply rewritten in terms of the self-adjoint operators $Y_{1}^{s}$ and $Y_{5}^{s}$ :

$$
\begin{align*}
& G_{6}^{5 D f c c} \cdot \rho(x) \cdot Y_{1}^{s}=Y_{1}^{s} \cdot \rho(x) \cdot \operatorname{adjoint}\left(G_{6}^{5 D f c c}\right),  \tag{57}\\
& Y_{5}^{s} \cdot \rho(x) \cdot G_{6}^{5 D f c c}=\operatorname{adjoint}\left(G_{6}^{5 D f c c}\right) \cdot \rho(x) \cdot Y_{5}^{s} \tag{58}
\end{align*}
$$

From these two intertwining relations it is straightforward to see that an operator of the form

$$
\begin{equation*}
\Omega_{6}=Y_{1}^{s} \cdot \rho(x) \cdot Y_{5}^{s}+\frac{\alpha}{\rho(x)}, \tag{59}
\end{equation*}
$$

satisfies the same intertwining relations (57) and (58), as $G_{6}^{5 D f c c}$. It is, in fact, a straightforward calculation to see that the order-six operator $G_{6}^{5 D f c c}$ is actually of the form (59) with $\alpha=-192$ :

$$
\begin{equation*}
G_{6}^{5 D f c c}=Y_{1}^{s} \cdot \rho(x) \cdot Y_{5}^{s}-\frac{192}{\rho(x)} \tag{60}
\end{equation*}
$$

Recalling section 2, and, more precisely, the fact that the right division of $\tilde{T} \cdot T$ by $L$ is a constant (see (9)), one can rewrite (60) as:

$$
\begin{equation*}
\rho(x) \cdot Y_{1}^{s} \cdot \rho(x) \cdot Y_{5}^{s}=192+\rho(x) \cdot G_{6}^{5 D f c c} \tag{61}
\end{equation*}
$$

In other words, the two intertwiners $\rho(x) \cdot Y_{1}^{s}$ and $\rho(x) \cdot Y_{5}^{s}$ are inverse of each other modulo the operator $\rho(x) \cdot G_{6}^{5 D f c c}$.

The operator $Y_{5}^{s}$ is not globally nilpotent [8], however we see the emergence of some structure: the Jordan form of the $p$-curvature of this self-adjoint operator $Y_{5}^{s}$ reads

$$
\left[\begin{array}{ccccc}
0 & 0 & 0 & 0 & 0  \tag{62}\\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & R_{1}^{p} & 0 & R_{2}^{p} & 0
\end{array}\right]
$$

$\S$ Using the identity $\operatorname{adjoint}(\Omega+f(x))=\operatorname{adjoint}(\Omega)+f(x)$ valid for any even order operator $\Omega$, and for any function $f(x)$.
its characteristic polynomial reading $x \cdot\left(x^{4}-R_{2}^{p} \cdot x^{2}-R_{1}^{p}\right)$.
The exterior square $\operatorname{Ext}^{2}\left(G_{6}^{5 D f c c}\right)$ is an order-fifteen linear differential operator which does not have a rational solution (or a hyperexponentional solution, see chapter 4 of [31] and 42]), thus excluding a symplectic structure with a $S p(6, \mathbb{C})$ differential Galois group.

In contrast, its symmetric square $S y m^{2}\left(G_{6}^{5 D f c c}\right)$, which does not have a rational solution, is of order 20 instead of the generic order 21 . In fact, the associated differential system does have a rational solution. Again, we see that the operator representation is not "intrinsic" enough, differential systems are more "intrinsic" from a practical viewpoint. The emergence (for the system) of a rational solution for the symmetric square means that the differential Galois group is included in the orthogonal group $O(6, \mathbb{C})$.

From that viewpoint, the order-six operator $G_{6}^{5 D f c c}$ seems to contradict an "experimental" principl $\ddagger$ that orthogonal groups occur from odd order operators, and symplectic groups occur from even order operators. In fact the exceptional character of this even order operator comes from this decomposition (60) in terms of odd order intertwiners (see (57) and (58)).

The $\log$ structure of the solutions is exactly the same as the one of a symmetric square of an order-three operator, $S y m^{2}\left(L_{3}\right)$, which might suggest that the differential Galois group would be the differential Galois group of a MUM order-three operator (generically $S L(3, \mathbb{C})$ ).

### 3.8.1. System representation of $G_{6}^{5 D f c c}$

The following calculations are performed using the "system representation" as in the previous section. One get§ the rational solution of the symmetric square of the companion system:

$$
\begin{aligned}
& {\left[\begin{array}{ccc}
c_{1}, & c_{2}, & c_{3},
\end{array} \cdots, c_{21}\right]=} \\
& =\left[0,0,0,0, \frac{2 \cdot Q_{5}}{\delta},-\frac{20 x^{3} \cdot Q_{6}}{\delta^{2}}, 0,0,-\frac{2 \cdot Q_{5}}{\delta}, \frac{12 x^{3} Q_{6}}{\delta^{2}},-\frac{2 x^{6} Q_{11}}{\delta^{3}}, \frac{Q_{5}}{\delta},\right. \\
& \quad-\frac{4 x^{7} \cdot Q_{6}}{\delta^{2}}, \frac{2 x^{6} Q_{14}}{\delta^{3}}, \frac{6 x^{9} Q_{15}}{\delta^{4}}, \frac{x^{6} Q_{16}}{\delta^{3}}, \frac{-2 x^{9} Q_{17}}{\delta^{4}}, \frac{8 x^{12} Q_{18}}{\delta^{5}}, \frac{x^{12} Q_{19}}{\delta^{5}}, \\
& \left.\quad \frac{-8 x^{15} Q_{20}}{\delta^{6}}, \frac{4 x^{18} Q_{21}}{\delta^{6}}\right],
\end{aligned}
$$

where

$$
\begin{equation*}
c_{1}=c_{2}=c_{3}=c_{4}=c_{7}=c_{8}=0, \quad c_{5}=c_{9}=c_{12} \tag{64}
\end{equation*}
$$

and where, recalling $p_{2}$ in (51) and $\lambda$ in (48)

$$
\begin{aligned}
& \delta=-x^{4} \cdot \lambda(x), \quad Q_{5}=p_{2}, \\
& Q_{6}= \\
& \\
& \\
& \hline 3495514624 x^{8}-8191475712 x^{7}+1552941056 x^{6}-94273536 x^{5} \\
& -3440640 x^{4}+498624 x^{3}-3632 x^{2}-609 x+9,
\end{aligned}
$$

【 In fact, an argument of Katz 7 enables, in principle, to see whether the differential Galois group is included in $O(6, \mathbb{C})$ or actually equal to $O(6, \mathbb{C})$. This argument is difficult to work out here. $\ddagger$ See Katz's book [7] and most of the explicit examples known in the literature.
$\S$ In order to do these calculations on the linear differential systems, download the Maple Tools files TensorConstructions.m and IntegrableConnections.m in the web page 41]. Using DEtools, you will need to use, on the order-six operator $G_{6}^{5 D f c c}$, the command companion-system, then the command symmetric-power-system $(, 2))$ and finally the command RationalSolutions( []$,[\mathrm{x}])$.
the other $Q_{n}$ 's being much larger polynomials.
If one wants to stick with an operator description, similarly to (35) or (44), one can switch, by operator equivalenct, to an operator such that its symmetric square is of the generic order 21 and has a rational solution.

The denominator of the monic order-twenty operator $\operatorname{Sym}^{2}\left(G_{6}^{5 D f c c}\right)$ is of the form $x^{16} \cdot \lambda(x)^{5} \cdot p_{278}$, where $p_{278}$ is a polynomial of degree 278 in $x$.

Let us introduce, for $n \geq 2$, an equivalent operator $G_{6}^{(n)}$, corresponding to an intertwining by $D_{x}^{n}$

$$
\begin{equation*}
S_{2}^{(n)} \cdot G_{6}^{5 D f c c}=G_{6}^{(n)} \cdot D_{x}^{n} \tag{65}
\end{equation*}
$$

For $n=2$, the symmetric square of $G_{6}^{(n)}$ has the rational solution

$$
\begin{equation*}
\frac{p_{2}}{x^{4} \cdot \lambda(x)}=\frac{1152 x^{2}-56 x-3}{x^{4} \cdot(16 x+1)(8 x-1)(4 x-1)(16 x-1)(48 x-1)(16 x-3)}, \tag{66}
\end{equation*}
$$

which is nothing but $c_{5} / 2$ in (63). For the symmetric square of the other $\tilde{G}_{6}^{(n)}$,s one finds, respectively for $n=3, n=4$ and $n=5$, the rational solutions $c_{16}, c_{19}$ and $c_{21}$ in (63). More generally the rational solution reads:

$$
\begin{equation*}
\frac{P_{12 n-22}(x) \cdot x^{8 n-12}}{x^{2 n} \cdot \delta^{2 n-3}} \tag{67}
\end{equation*}
$$

where $P_{m}(x)$ is a polynomial of degree $m$ in $x$.
Getting (or even only checking) the rational solution (66) for the symmetric square of the equivalent operator (65), paradoxically, corresponds to massive calculations compared to obtaining the rational solution on the symmetric square of the companion system (see (63)).

### 3.9. Koutschan's order-eight operator: the lattice Green function of the six-dimensional fcc lattice

A slightly more spectaculart example of order-eight, $G_{8}^{6 D f c c}$, has been found by Koutschan [27] for a six-dimensional face-centered cubic lattice. Its exterior square is of order 27. The irreducibility of this order-eight operator is hard to check $\oint$. One finds again, at the origin $x=0$, that there are two independent analytical solutions (no logarithms). Since the order-eight operator $G_{8}^{6 D f c c}$ has two analytical solutions, it cannot be MUM [19] at $x=0$.

A linear combination of these solutions is globally bounded [18, 19]. It is such that, after rescaling, it can be recast into a series with integer coefficients [18, 19]:

$$
\begin{align*}
1 & +60 x^{2}+960 x^{3}+30780 x^{4}+996480 x^{5}+36560400 x^{6}+1430553600 x^{7} \\
& +59089923900 x^{8}+2543035488000 x^{9}+113129280527760 x^{10}+\cdots \tag{68}
\end{align*}
$$

In order to fix the normalization, this order-eight operator will be analyzed in a monic form: $G_{8}^{6 D f c c}=D_{x}^{8}+\cdots$ This order-eight operator is (non-trivially)
$\dagger$ In Maple, choosing an intertwiner $R$, e.g. $R=D_{x}^{2}$, this amounts to performing $S_{2}=$ rightdivision $\left(\operatorname{LCLM}\left(G_{6}^{5 D f c c}, R\right), R\right)$. Note that the LCLM with $D_{x}$ instead of $D_{x}^{2}$, still yields a symmetric square of order 20 instead of 21 .
$\ddagger$ It is a quite large [28] order-eight linear differential operator of 52 Megabytes.
§ One can, however, check that this operator has no rational solutions.
homomorphic to its adjoint, one intertwiner being of order six, the other one being of order two

$$
\begin{align*}
& \operatorname{adjoint}\left(S_{6}^{6 D f c c}\right) \cdot G_{8}^{6 D f c c}=\operatorname{adjoint}\left(G_{8}^{6 D f c c}\right) \cdot S_{6}^{6 D f c c}  \tag{69}\\
& G_{8}^{6 D f c c} \cdot T_{2}^{6 D f c c}=\operatorname{adjoint}\left(T_{2}^{6 D f c c}\right) \cdot \operatorname{adjoint}\left(G_{8}^{6 D f c c}\right) \tag{70}
\end{align*}
$$

where, again, noticeably, the two intertwiners are, after the same rescaling self-adjoint operators. Let us introduce

$$
\begin{equation*}
a(x)=\frac{x^{6} \cdot p_{5}^{4}}{p_{25}} \cdot \lambda(x), \tag{71}
\end{equation*}
$$

where the polynomial $p_{5}$ reads

$$
\begin{equation*}
p_{5}=56 x^{5}+625 x^{4}-1251 x^{3}-24840 x^{2}-65556 x-38880 \tag{72}
\end{equation*}
$$

where $\lambda(x)$ reads:

$$
\begin{align*}
\lambda(x)=(x-1) & (x-3)(x+24)(2 x+15)(7 x+60)(2 x+3)(4 x+15) \\
& \times(x+9)(x+5)(x+4)(x+15)^{4} \tag{73}
\end{align*}
$$

and where the polynomial $p_{25}$ is a quite large polynomial of degree 25 .
The intertwiners $T_{2}^{6 D f c c}$ and $S_{6}^{6 D f c c}$ are, respectively, of the form $T_{2}^{6 D f c c}=$ $a(x) \cdot Y_{2}^{s}$ and $S_{6}^{6 D f c c}=a(x) \cdot Y_{6}^{s}$, where $Y_{2}^{s}$ and $Y_{6}^{s}$ are two irreducible self-adjoint order-two and order-six operators

$$
\begin{equation*}
Y_{2}^{s}=\frac{1}{W_{2}(x)} \cdot\left(D_{x}^{2}-\frac{d \ln \left(W_{2}(x)\right)}{d x} \cdot D_{x}+\cdots\right) \tag{74}
\end{equation*}
$$

and:

$$
\begin{equation*}
Y_{6}^{s}=\frac{1}{W_{6}(x)^{1 / 3}} \cdot\left(D_{x}^{6}-\frac{d \ln \left(W_{6}(x)\right)}{d x} \cdot D_{x}^{5}+\cdots\right) \tag{75}
\end{equation*}
$$

Their corresponding two Wronskians $W_{2}(x)$ and $W_{6}(x)$ read respectively:

$$
\begin{equation*}
W_{2}(x)=\frac{x^{11} \cdot \lambda(x)^{2} \cdot p_{5}^{3}}{(x+15)^{3} \cdot p_{25}}, \quad \quad W_{6}(x)^{1 / 3}=\frac{(x+15)^{3} \cdot p_{5}}{\lambda(x) \cdot x^{5}} \tag{76}
\end{equation*}
$$

These self-adjoint operators are not globally nilpotent [8], however we see, again, the emergence of some structure: the (Jordan form of the) $p$-curvature of these two self-adjoint operators read

$$
\left[\begin{array}{cc}
0 & 1  \tag{77}\\
S_{1}^{p} & 0
\end{array}\right], \quad\left[\begin{array}{cccccc}
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
R_{1}^{p} & 0 & R_{2}^{p} & 0 & R_{3}^{p} & 0
\end{array}\right]
$$

where $S_{1}$ is a different rational function for each prime $p$, for instance for $p=11$

$$
\begin{equation*}
S_{1}=\frac{4 \cdot(5+x)^{2}}{(x+7) \cdot x \cdot\left(x^{2}+7 x+5\right) \cdot(x+4) \cdot(x+2)}, \tag{78}
\end{equation*}
$$

and similarly for the $R_{n}$ 's of the $6 \times 6$ matrix in (77).

The intertwining relations (69) give, in terms of the self-adjoint operators (74) and (75):

$$
\begin{align*}
& Y_{6}^{s} \cdot a(x) \cdot G_{8}^{6 D f c c}=\operatorname{adjoint}\left(G_{8}^{6 D f c c}\right) \cdot a(x) \cdot Y_{6}^{s}, \\
& G_{8}^{6 D f c c} \cdot a(x) \cdot Y_{2}^{s}=Y_{2}^{s} \cdot a(x) \cdot \operatorname{adjoint}\left(G_{8}^{6 D f c c}\right), \tag{79}
\end{align*}
$$

which yield

$$
\begin{align*}
& \mathcal{K}_{8} \cdot \mathcal{M}_{8}=\mathcal{M}_{8} \cdot \mathcal{K}_{8} \quad \text { and }  \tag{80}\\
& \operatorname{adjoint}\left(\mathcal{M}_{8}\right) \cdot \operatorname{adjoint}\left(\mathcal{K}_{8}\right)=\operatorname{adjoint}\left(\mathcal{K}_{8}\right) \cdot \operatorname{adjoint}\left(\mathcal{M}_{8}\right), \\
& \mathcal{K}_{8}=a(x) \cdot G_{8}^{6 D f c c} \quad \text { and: } \quad \mathcal{M}_{8}=a(x) \cdot Y_{2}^{s} \cdot a(x) \cdot Y_{6}^{s} .
\end{align*}
$$

where:

A commutation relation of linear differential operators, like (80), is a drastic constraint on the operators. As $\mathcal{K}_{8}$ is irreducible, the commutation (80) forces $\mathcal{K}_{8}$ to be of the form $\alpha \cdot \mathcal{M}_{8}+\beta$, where $\alpha$ and $\beta$ are constants. We may thus guess, from the intertwining relations (79), a decomposition of the order-eight operator $G_{8}^{6 D f c c}$, similar to the one we had for $G_{6}^{5 D f c c}$, of the form

$$
\begin{equation*}
G_{8}^{6 D f c c}=Y_{2}^{s} \cdot a(x) \cdot Y_{6}^{s}+\frac{\alpha}{a(x)}, \tag{82}
\end{equation*}
$$

where $Y_{2}^{s}$ and $Y_{6}^{s}$ are two self-adjoint operators of even order (instead of odd order for $\left.G_{6}^{5 D f c c}\right)$. This is, indeed, the case. The operator $G_{8}^{6 D f c c}$ has the noticeable decomposition:

$$
\begin{equation*}
G_{8}^{6 D f c c}=Y_{2}^{s} \cdot a(x) \cdot Y_{6}^{s}+\frac{87480}{a(x)} \tag{83}
\end{equation*}
$$

Again, and similarly to what has been done for $G_{6}^{5 D f c c}$ (see (61)), one can rewrite (83) as:

$$
\begin{equation*}
a(x) \cdot Y_{2}^{s} \cdot a(x) \cdot Y_{6}^{s}=-87480+a(x) \cdot G_{8}^{6 D f c c} \tag{84}
\end{equation*}
$$

which means that the two intertwiners $a(x) \cdot Y_{2}^{s}$ and $a(x) \cdot Y_{6}^{s}$ are inverse of each other modulo the operator $a(x) \cdot G_{8}^{6 D f c c}$. From (84) one sees that a solution of $G_{8}^{6 D f c c}$ is an eigenfunction of $a(x) \cdot Y_{2}^{s} \cdot a(x) \cdot Y_{6}^{s}$ with the eigenvalue -87480 .

The examination of the formal series solutions, at $x=0$, of the self-adjoint ordersix $Y_{6}^{s}$ operator shows a MUM structure. The $Y_{6}^{s}$ operator has one analytic solution at $x=0$, (the other solutions have log terms), which has the following expansion:

$$
\begin{align*}
& \operatorname{Sol}\left(Y_{6}^{s}\right)=1+\frac{197}{11520} x^{2}+\frac{8559443}{1889568000} x^{3}+\frac{381585241573}{154793410560000} x^{4} \\
& \quad+\frac{35207145815207429}{27209779200000000000} x^{5}+\frac{150944307721060740999089}{182807922003148800000000000} x^{6}+\cdots \tag{85}
\end{align*}
$$

This solution-series (85), again, is no母 globally bounded [18, 19]. One deduces immediately, from decomposition (83), an interesting eigenvalue result: the order-eight operator $a(x) \cdot G_{8}^{6 D f c c}$ has the not globally bounded eigenfunction (85), corresponding to the integer eigenvalue 87480.

The self-adjoint order-five irreducible operator $Y_{6}^{s}$ is such that its exterior square is of order 14 instead of the order 15 expected generically (its symmetric square is of order 21 as it should, with no rational solution).

[^4]The symmetric square of $G_{8}^{6 D f c c}$ is of the (generic) order 36. However the exterior square of $G_{8}^{6 D f c c}$ is of order 27 instead of the (generic) order 28.

Remark: The adjoint of $G_{8}^{6 D f c c}$ has the following decomposition, straightforwardly deduced from (83):

$$
\begin{equation*}
\operatorname{adjoint}\left(G_{8}^{6 D f c c}\right)=Y_{6}^{s} \cdot a(x) \cdot Y_{2}^{s}+\frac{87480}{a(x)} \tag{86}
\end{equation*}
$$

So we can expect the Wronskian of $Y_{2}^{s}$ to be a (rational) solution of its exterior square. We have verified that this is indeed the case.

Similarly to the previous order-six operator $G_{6}^{5 D f c c}$, one could try to switch to equivalent operators (see (651), calculate the exterior square of these equivalent operators, and try to find the corresponding rational solution (see (66)). These, at first sight, straightforward calculations are, in fact, too "massive". The way to get the rational solution is, in fact, to switch to differential systems (see (63)).

### 3.9.1. System representation of $G_{8}^{6 D f c c}$

In fact, even after switching to a differential system using the same tools 41 that we used for obtaining (63), we found that the resulting calculation exceeded our computational capacity. These calculations mostly amount to finding a transformation that reduces the system to a system with simple poles. We need a second "trick" to be able to achieve these calculations and get the rational solution of the differential system. An easy way is to rewrite the system in terms of the homogeneous derivative $\theta=x \cdot D_{x}$. Switching to this companion system in $\theta$, one automatically has simple poles for the system.

With all these tricks and tools, we finally found that the linear differential system for the exterior square of the order-eight $G_{8}^{6 D f c c}$ operator is of order 28 and has a rational solution. Note that the intertwiners (69) have been found from this rational solution: seeking straightforwardly for the intertwiners (69) needs too massive calculations!

The rational solution reads

$$
\begin{align*}
& {\left[\begin{array}{ccccc}
c_{1}, & c_{2}, & c_{3}, & \cdots, & c_{28}
\end{array}\right]=}  \tag{87}\\
& =\left[0,0,0,0, \frac{P_{5}}{\delta}, \frac{x^{4} P_{6}}{\delta^{2}}, \frac{x^{8} P_{7}}{\delta^{3}}, 0,0, \frac{P_{10}}{\delta}, \frac{x^{4} P_{11}}{\delta^{2}}, \frac{x^{8} P_{12}}{\delta^{3}}, \frac{x^{12} P_{13}}{\delta^{4}}\right. \text {, } \\
& \frac{P_{14}}{\delta}, \frac{x^{4} P_{15}}{\delta^{2}}, \frac{x^{8} P_{16}}{\delta^{3}}, \frac{x^{12} P_{17}}{\delta^{4}}, \frac{x^{16} P_{18}}{\delta^{5}}, \frac{x^{3} P_{19}}{\delta^{2}}, \frac{x^{7} P_{20}}{\delta^{3}}, \frac{x^{11} P_{21}}{\delta^{4}}, \\
& \left.\frac{x^{15} P_{22}}{\delta^{5}}, \frac{x^{6} P_{23}}{\delta^{3}}, \frac{x^{10} P_{24}}{\delta^{4}}, \frac{x^{14} P_{25}}{\delta^{5}}, \frac{x^{9} P_{26}}{\delta^{4}}, \frac{x^{13} P_{27}}{\delta^{5}}, \frac{x^{12} P_{28}}{\delta^{5}}\right],
\end{align*}
$$

where

$$
\begin{align*}
& \delta=(x-1)(x-3)(x+24)(2 x+15)(7 x+60)(2 x+3) \\
& \times(x+15)(4 x+15)(x+9)(x+5)(x+4) \cdot x^{5} \tag{88}
\end{align*}
$$

$\ddagger$ In the Maple TensorConstruction tools found at 41, the command Theta_companion_system(L) returns two matrices $\frac{1}{p(x)} A_{\theta}$ and $P_{\theta}$ such that, for $Y=\left(y, y^{\prime}, \ldots, y^{(n-1)}\right)^{T}$, we have $Y=P_{\theta} Y_{\theta}$ and $Y_{\theta}^{\prime}=\frac{1}{p(x)} A_{\theta} Y_{\theta}$, where $A_{\theta}$ has no finite poles and $p(x)$ is squarefree, it has only simple roots. This gives the correspondence between the original companion system and the $\theta$-companion system. $\S$ If one is reluctant to switch to companion systems in $\theta$, another way to achieve these calculations is to perform a reduction on the matrix of the corresponding system (moser-reduce in Maple) before calculating the symmetric powers of the system (an operation that preserves the order of the poles).
and where the polynomial $P_{n}$ in (87) are too large to be displayed here. The polynomials $P_{28}, P_{27}, P_{25}, P_{22}, P_{18}$ are of degree 49 , polynomials $P_{26}, P_{24}, P_{21}, P_{17}$ are of degree $38, P_{13}$ is of degree 37 , polynomials $P_{23}, P_{20}, P_{16}, P_{12}, P_{7}$ are of degree 27, polynomials $P_{19}, P_{15}, P_{11}, P_{6}$ are of degree 16 , and $P_{14}, P_{10}, P_{5}$ are of degree 5. Furthermore we have some equalities like $P_{14}=-P_{10}=P_{5}, P_{11}=-2 \cdot P_{15}=$ $-2 / 3 \cdot P_{6}$

Having this rational solution at our disposal, we can, now, find the rational solutions for the exterior square of the equivalent operators:

$$
\begin{equation*}
G_{2}^{(n)} \cdot G_{8}^{6 D f c c}=G_{8}^{(n)} \cdot D_{x}^{n} \tag{89}
\end{equation*}
$$

Recalling (72), the rational solution for $n=2$ reads $p_{5} / \delta$. The differential Galois group of $G_{8}^{6 D f c c}$ is included in (and probably equal to) $S p(8, C)$.

Remark: The same calculations can be performed on all the linear differential operators we have encountered in lattice statistical mechanics [8, $11,12,13,14,15,16$, 17, 19, 20: all the examples we have tested give operators whose irreducible factors are actually equivalent to their adjoint.

### 3.10. Generalization of the decomposition for higher order operators

The remarkable decompositions (60) and (83), encountered with $G_{6}^{5 D f c c}$ and $G_{8}^{6 D f c c}$ can easily be generalized. In fact, one can systematically introduce the even order operators

$$
\begin{equation*}
M_{2 p}^{(n, 2 p-n)}=L_{2 p-n} \cdot a(x) \cdot L_{n}+\frac{\lambda}{a(x)} \tag{90}
\end{equation*}
$$

or, after rescaling $\ddagger$,

$$
\begin{equation*}
\tilde{M}_{2 p}^{(n, 2 p-n)}=a(x) \cdot L_{2 p-n} \cdot a(x) \cdot L_{n}+\lambda \tag{91}
\end{equation*}
$$

where the $L_{m}$ 's are self-adjoint operators of order $m$. They are, naturally, homomorphic to their adjoint, with intertwiners corresponding to these decompositions (90) and (91):

$$
\begin{gathered}
a(x) \cdot L_{n} \cdot M_{2 p}^{(n, 2 p-n)}=\operatorname{adjoint}\left(M_{2 p}^{(n, 2 p-n)}\right) \cdot a(x) \cdot L_{n}, \\
\\
M_{2 p}^{(n, 2 p-n)} \cdot a(x) \cdot L_{2 p-n}=L_{2 p-n} \cdot a(x) \cdot \operatorname{adjoint}\left(M_{2 p}^{(n, 2 p-n)}\right), \\
L_{n} \cdot \tilde{M}_{2 p}^{(n, 2 p-n)}=\operatorname{adjoint}\left(\tilde{M}_{2 p}^{(n, 2 p-n)}\right) \cdot L_{n} \\
\tilde{M}_{2 p}^{(n, 2 p-n)} \cdot a(x) \cdot L_{2 p-n} \cdot a(x)=a(x) \cdot L_{2 p-n} \cdot a(x) \cdot \operatorname{adjoint}\left(\tilde{M}_{2 p}^{(n, 2 p-n)}\right) .
\end{gathered}
$$

Experimentally we have seen (for instance with our two previous lattice Green functions examples of order six and eight, see (57) and (58) for (60), and (69) and (70) for (83)), that this corresponds to two different types of operators: the operators with even $n$, for which the exterior square of an equivalent operator (or of the corresponding differential system) will have a rational solution (yielding a symplectic differential Galois group), and the operators with odd $n$, for which the symmetric square of the corresponding differential system will have a rational solution (yielding an orthogonal differential Galois group).
$\ddagger$ Do note that the rescaled operators (91) are such that the functions annihilated by $L_{n}$ are automatically eigenfunctions of $\tilde{M}_{2 p}^{(n, 2 p-n)}$ with eigenvalue $\lambda$.

## 4. Focus on order-four differential operators: Calabi-Yau conditions

It has been underlined by A.J. Guttmann that these lattice Green functions are (most of the time) solutions of Calabi-Yau ODEs, or higher order Calabi-Yau ODEs [2, 3]. The definition of Calabi-Yau ODEs, and some large lists of Calabi-Yau ODEs, can be found in 4, 5, 43, 44. Calabi-Yau ODEs are defined by several constraints, some are natural like being MUM, others (like some cyclotomic constraints) are essentially introduced, in a classification perspective like [5] to provide hopefully exhaustive lists of Calabi-Yau ODEs, some are related to the concept of "modularity", requiring the integrality of various series like the nome or the Yukawa coupling. Therefore, in the definition of Calabi-Yau ODEs, there is some "mix" between analytic and differential constraints, and constraints of a more arithmetid, or algebraic geometry character. Among all these constraints defining the Calabi-Yau ODEs, the most important one is the so-called "Calabi-Yau condition". Let us consider a (monic) order-four linear differential operator:

$$
\begin{equation*}
\Omega_{4}=D_{x}^{4}+a_{3}(x) \cdot D_{x}^{3}+a_{2}(x) \cdot D_{x}^{2}+a_{1}(x) \cdot D_{x}+a_{0}(x) \tag{92}
\end{equation*}
$$

The exterior square of (92), $E x t^{2}\left(\Omega_{4}\right)$, reads, up to an overall factor:

$$
\begin{equation*}
C_{6}(x) \cdot \operatorname{Ext}^{2}\left(\Omega_{4}\right)=\sum_{n=0}^{6} C_{n}(x) \cdot D_{x}^{n} \tag{93}
\end{equation*}
$$

where the $C_{i}(x)$ 's are polynomial expressions of $a_{3}(x), a_{2}(x), a_{1}(x), a_{0}(x)$ and of their derivatives (up to the third derivative).

The vanishing condition $C_{6}(x)=0$, which reads

$$
\begin{gather*}
8 a_{1}(x)+a_{3}(x)^{3}-4 \cdot a_{3}(x) \cdot a_{2}(x)+6 \cdot a_{3}(x) \cdot \frac{d a_{3}(x)}{d x} \\
-8 \cdot \frac{d a_{2}(x)}{d x}+4 \cdot \frac{d^{2} a_{3}(x)}{d x^{2}}=0, \tag{94}
\end{gather*}
$$

is satisfied if, and only if, the exterior square is of order five, instead of the order six one expects generically. It is called "Calabi-Yau condition" by some authors [6] and is one of the conditions for the ODE to be a Picard-Fuchs equation of a family of Calabi-Yau manifolds (see (11) in [45). This Calabi-Yau condition (94) is actually preserved by pullbacks, but not by operator equivalence. Note that this Calabi-Yau condition (94) is preserved by the adjoint transformation (see Appendix B.3).

Do note that such condition is actually independent of $a_{0}(x)$ in (92). Also note that all the order-four operators $M_{4}$ that can be written as the sum of the symmetric-cube of an order-two operatord, and a function, $M_{4}=\operatorname{Sym}^{3}\left(M_{2}\right)+f(x)$, automatically verify the Calabi-Yau condition (94). This gives a practical way to quickly provide examples of order-four operators satisfying the Calabi-Yau condition (94).

Of course similar Calabi-Yau conditions can be introduced for higher order operators, imposing, for order- $N$ operators, that their exterior squares are of order less than the generic $N \cdot(N-1) / 2$ order. These higher order Calabi-Yau conditions actually correspond to consider self-adjoint operators (see Appendix B.1, see also [22).

[^5]Furthermore similar "Calabi-Yau conditions" can be introduced for symmetric squares instead of exterior squares, imposing, for order- $N$ operators, that their symmetric squares are of order less than the generic $N \cdot(N+1) / 2$ order. For an order-three operator written in a monic form

$$
\begin{equation*}
\Omega_{3}=D_{x}^{3}+a_{2}(x) \cdot D_{x}^{2}+a_{1}(x) \cdot D_{x}+a_{0}(x) \tag{95}
\end{equation*}
$$

the "symmetric Calabi-Yau condition" read $\ddagger$ :

$$
\begin{gather*}
4 a_{2}(x)^{3}-18 a_{1}(x) a_{2}(x)+9 \cdot \frac{d^{2} a_{2}(x)}{d x^{2}}+18 \cdot a_{2}(x) \cdot \frac{d a_{2}(x)}{d x} \\
+54 a_{0}(x)-27 \frac{d a_{1}(x)}{d x}=0 \tag{96}
\end{gather*}
$$

Operators satisfying this "symmetric Calabi-Yau condition" actually correspond to the situation described in (3.4). If their Wronskian $W\left(\Omega_{3}\right)$ is a $N$-th root of a rational function they are conjugated to their adjoint $\left(f(x)=W\left(\Omega_{3}\right)^{2 / 3}\right)$ :

$$
\begin{equation*}
\Omega_{3} \cdot f(x)=f(x) \cdot \operatorname{adjoint}\left(\Omega_{3}\right) \quad \text { with: } \quad a_{2}(x)=-\frac{3}{2} \frac{1}{f(x)} \frac{d f(x)}{d x} \tag{97}
\end{equation*}
$$

In order to disentangle the main focus of this very paper, namely the algebraicdifferential structures, from other structures of more analytical, or arithmetic, character (series integrality [18, [19], MUM property, etc.), we concentrate, in this section, on (mostly order-four) linear differential operators satisfying the Calabi-Yau condition (94), or homomorphic to operators satisfying (94).

### 4.1. Weak and strong Calabi-Yau conditions

If one considers an operator that is homomorphic to an operator with a rational Wronskian, satisfying the Calabi-Yau condition (94), with intertwiners that are of order greater or equal to ont, the exterior square of that operator actually has a rational solution. Unfortunately, in contrast with (94), the condition for an order-four operator to be such that its exterior square has a rational solution, cannot be written directly and explicitly on its coefficients $a_{n}(x)$ (see (92)). We will call "weak CalabiYau condition" this condition that the exterior square of an operator has a rational solution, the Calabi-Yau condition (94) being the "strong" Calabi-Yau condition. Note that the weak Calabi-Yau condition is preserved by the adjoint transformation (see Appendix B.3.)

As far as classifications of Calabi-Yau operators are concerned 4, 5, 43, 44, an operator non-trivially homomorphic to a "Calabi-Yau operator" is certainly as interesting for physics as these "Calabi-Yau operators", and an operator non-trivially homomorphic to an operator verifying the "strong" Calabi-Yau condition (94), or satisfying the "weak Calabi-Yau condition" is certainly as interesting as an operator verifying the "strong" Calabi-Yau condition.

Let us explore the relation between the "weak Calabi-Yau condition" and the "strong Calabi-Yau condition".
$\ddagger$ The "symmetric Calabi-Yau condition" for order-four operators can be found but is drastically larger than (96).
$\dagger$ We must exclude intertwiners of order zero (namely functions): in that case, it is a straightforward calculation to see that the operators are conjugated by a function, both operators satisfying the Calabi-Yau condition (94).
§ With intertwiners of order greater or equal to one.
4.2. A decomposition of operators equivalent to operators satisfying the Calabi-Yau condition

Let us consider an order-four operator $\Omega_{4}$, of Wronskian $w(x)=u(x)^{2}$, which satisfies the Calabi-Yau condition (94). Let us also consider a monic order-four operator $\tilde{\Omega}_{4}$ which is (non-trivially) homomorphic (equivalent in the sense of the equivalence of operators [31]) to the order-four operator $\Omega_{4}$ satisfying the Calabi-Yau condition (94). This amounts to saying that there exist two intertwiners, $U_{3}$ and $L_{3}$, of order less or equal to thre $\ddagger$, such that:

$$
\begin{equation*}
\tilde{\Omega}_{4} \cdot U_{3}=L_{3} \cdot \Omega_{4} \tag{98}
\end{equation*}
$$

It is shown in Appendix A that the order-four operator $\tilde{\Omega}_{4}$ can, in fact, be written in terms of a remarkable decomposition

$$
\begin{equation*}
\tilde{\Omega}_{4}=Z_{2}^{s} \cdot \frac{1}{A_{0}} \cdot A_{2}+A_{0} \tag{99}
\end{equation*}
$$

where $Z_{2}^{s}$ and $A_{2}$ are two self-adjoint operators, $A_{0}$ being a function. Appendix A shows how to get $Z_{2}^{s}, A_{2}$ and $A_{0}$ in such a decomposition: they can simply be obtained as the intertwiners of $\tilde{\Omega}_{4}$ with its adjoints (use (A.10), (A.13), (A.15), (A.16) in Appendix A). Experimentally we have checked that an operator (non trivially) homomorphic to an operator of the form (99) (see (A.2)) can always be decomposed in a form (99): the decomposition (99) is preserved by operator equivalence.

Byproduct: As a byproduct one finds out that the left and right intertwiners of an order-four operator satisfying the weak Calabi-Yau condition are necessarily of order two. Note, however, that this is not true for the intertwiners of an order-four operator satisfying the symmetric weak Calabi-Yau condition which are of odd orders (see (117) in the section 4.5) on the anisotropic simple cubic lattice Green function).

### 4.3. Rational solutions for the exterior square of operators satisfying the weak

 Calabi-Yau conditionWe have the following general result. Any order-four linear differential operator of the form $\dagger$

$$
\begin{equation*}
M_{4}=L_{2} \cdot c_{0}(x) \cdot M_{2}+\frac{\lambda}{c_{0}(x)} \tag{100}
\end{equation*}
$$

where $L_{2}$ and $M_{2}$ are two (general) self-adjoint operators

$$
\begin{align*}
& L_{2}=\alpha_{2}(x) \cdot D_{x}^{2}+\frac{d \alpha_{2}(x)}{d x} \cdot D_{x}+\alpha_{0}(x)  \tag{101}\\
& M_{2}=\beta_{2}(x) \cdot D_{x}^{2}+\frac{d \beta_{2}(x)}{d x} \cdot D_{x}+\beta_{0}(x) \tag{102}
\end{align*}
$$

is such that its exterior square has $1 / \beta_{2}(x)$ as a solution (up to an overall multiplicative constant). This result can be seen to be the consequence of a non trivial identity (B.25) given in Appendix B.4.

Byproduct: Thus the exterior square of $\tilde{\Omega}_{4}$ has a rational solution, which is the inverse of the head polynomial of the second order self-adjoint operator $A_{2}$ in the decomposition (99).
$\ddagger$ Higher order intertwiners can always be reduced to intertwiners with an order less, or equal, to three.
$\dagger$ Note that one can always restrict to $\lambda=1$ rescaling $c_{0}(x)$ into $\lambda^{1 / 2} \cdot c_{0}(x)$.

To sum-up: The operators, non-trivially homomorphic to operators satisfying the (strong) Calabi-Yau condition (94), necessarily satisfy the "weak Calabi-Yau condition": their exterior square have a rational solution. Furthermore this rational solution corresponds to the Wronskian of a self-adjoint order-two operator $L_{2}$ of a remarkable decomposition (100). Decomposition (100) is the most general form of an operator satisfying the "weak Calabi-Yau condition".

Conversely: This naturally raises the reciprocal question. Is any order-four operator satisfying the "weak Calabi-Yau condition" (its exterior square has a rational solution) non trivially homomorphic to an operator satisfying the (strong) CalabiYau condition (94) ? In view of the remarkable decomposition (100), we can also ask the following questions. Is any order-four operator satisfying the "weak CalabiYau condition" necessarily of the form (100), i.e. homomorphic to its adjoint with order-two intertwiners ? Is any order-four operator of the form (100) homomorphic to an operator satisfying the (strong) Calabi-Yau condition (94) ? These questions will be revisited in a forthcoming publication $\ddagger$. The reason why these questions are difficult to answer in general, beyond specific examples, comes from the fact that such a reduction by operator equivalence of operators satisfying the weak Calabi-Yau condition to operators satisfying the strong Calabi-Yau condition, is not unique (an infinite number of equivalent operators can satisfy the Calabi-Yau condition (94), see Appendix B.2).

### 4.4. Calabi-Yau conditions and rational solutions of the exterior square for order-one intertwiners

Let us consider an order-four operator $\Omega_{4}$, satisfying the Calabi-Yau condition (94), and let us introduce $u(x)$ the square root of its Wronskian: $w(x)=u(x)^{2}$. Let us consider the LCLM of $\Omega_{4}$ and of the order-one operator $D_{x}$. This LCLM reads:

$$
\begin{equation*}
L_{1} \cdot \Omega_{4}=M_{4} \cdot D_{x} \quad \text { where: } \quad L_{1}=D_{x}-\frac{1}{a_{0}(x)} \cdot \frac{d a_{0}(x)}{d x} . \tag{103}
\end{equation*}
$$

The order-four operator $M_{4}$ verifies the "weak Calabi-Yau condition". Its exterior square has $u(x)$ as a solution. More precisely, the exterior square of $M_{4}$ is, in fact, the direct sum of an order-one operator and of an order-five operator

$$
\begin{equation*}
\operatorname{Ext}^{2}\left(M_{4}\right)=M_{5} \oplus M_{1} \quad \text { where: } \quad M_{1}=D_{x}-\frac{1}{u(x)} \cdot \frac{d u(x)}{d x} \tag{104}
\end{equation*}
$$

where the order-five operator is homomorphic to the exterior square of $\Omega_{4}$, with two order-two intertwiners $U_{2}$ and $V_{2}$, namely $M_{5} \cdot U_{2}=V_{2} \cdot \operatorname{Ext}^{2}\left(\Omega_{4}\right)$, the order-two intertwiner $U_{2}$ reading

$$
\begin{align*}
U_{2}= & D_{x}^{2}-\frac{1}{u(x)} \cdot \frac{d u(x)}{d x} \cdot D_{x}+r(x) \\
& \quad \text { where: }  \tag{105}\\
& r(x)=a_{2}(x)+\frac{1}{u(x)} \cdot \frac{d^{2} u(x)}{d x^{2}}-2 \cdot\left(\frac{1}{u(x)} \cdot \frac{d u(x)}{d x}\right)^{2} .
\end{align*}
$$

$\ddagger$ If one switches to a representation in terms of differential systems, such a system with Galois group $S p(4, \mathbb{C})$ can always be reduced, via a "gauge-like" transformation 46, 47, to a system with a hamiltonian matrix $M$. Such a system is such that the exterior square system associated with a $6 \times 6$ matrix, has a constant solution, namely $[0,1,0,0,1,0]$ (see 46, 47). Switching back to the operator representation, one actually finds that this operator is homomorphic to its adjoint with order-two intertwiners (themselves homomorphic to their adjoints). Consequently they can always be decomposed into a form (90).

These results can be generalized to more general order-one intertwiners. One can easily deduce the result (here $O^{F}$ denotes the conjugate by $F(x)$ of an operator $O$ : $\left.O^{F}=F(x) \cdot O \cdot F(x)^{-1}\right)$

$$
\begin{equation*}
L_{1}^{F} \cdot \Omega_{4}^{F}=M_{4}^{F} \cdot\left(D_{x}-\frac{1}{F(x)} \cdot \frac{d F(x)}{d x}\right) \tag{106}
\end{equation*}
$$

where, again, $\Omega_{4}^{F}$ satisfies the Calabi-Yau condition (94), and where $M_{4}^{F}$ satisfies the weak Calabi-Yau condition. Along this more general order-one line, see ( (B.24) in Appendix B.3) and (108) below.

Switching to a linear differential system representation of an order-four operator satisfying the Calabi-Yau condition (94), one has the following solution for the differential system :

$$
\begin{equation*}
\left[0, \quad 0,-u(x), \quad u(x), \frac{d u(x)}{d x}, r(x) \cdot u(x)\right] \tag{107}
\end{equation*}
$$

In that heuristic case one remarks that $A_{0}=a_{0}(x)$ in (92), and that $U_{2}$ in (105) is simply related to $A_{2}$ in the decomposition (99): $U_{2}=u(x) \cdot A_{2}$.

If one assumes that the Wronskian of the order-four linear differential operator is a square of a rational function, one, thus, finds a rational solution for the exterior square of the differential system (resp. hyperexponential solution 42 for a Wronskian $N$-th root of a rational function).

If one prefers to stick with differential operators instead of differential systems, one can see the emergence of a rational solution for the exterior square of the orderfour operator, exchanging the order-four operator satisfying the Calabi-Yau condition (94) for an equivalent operator (in the sense of the equivalence of operators [31, see next subsection). This equivalent operator does not satisfy the Calabi-Yau condition (94) since, as we noticed, the Calabi-Yau condition (94) is preserved by pullback but not by operator equivalence.

Remark 1: In this case of an operator, like $M_{4}$ in (103), satisfying the weak Calabi-Yau condition i.e. homomorphic to an operator $\Omega_{4}$ satisfying the Calabi-Yau condition (94) with an order-one intertwiner, one can, for a given $M_{4}$, find $\Omega_{4}$ from (104), (105). In this case (order-one intertwiner) the order-four operator $\Omega_{4}$ is (up to overall factors) unique.

Remark 2: Recalling the general decomposition result (99), one actually finds that an order-one intertwiner situation $L_{1} \cdot \Omega_{4}=M_{4} \cdot\left(D_{x}+q_{0}(x)\right)$, corresponds to the self-adjoint operators $Z_{2}^{s}$ and $A_{2}$ in (99) of the form ( $Z_{2}^{s}$ factors in order-one operatorsti):

$$
\begin{align*}
& Z_{2}^{s}=u(x) \cdot A_{0} \cdot\left(D_{x}+\frac{1}{u(x)} \cdot \frac{d u(x)}{d x}+q_{0}(x)\right) \cdot\left(D_{x}+\frac{1}{A_{0}} \cdot \frac{d A_{0}}{d x}-q_{0}(x)\right) \\
& u(x) \cdot A_{2}=D_{x}^{2}-\frac{1}{u(x)} \cdot \frac{d u(x)}{d x} \cdot D_{x}+r(x) \tag{108}
\end{align*}
$$

where $r(x)$ reduces to (105) when $q_{0}(x)=0$.
Remark 3: These results are specific of order-one intertwiners. One can consider order-two intertwiners introducing the LCLM of $\Omega_{4}$ and of an order-two operator $M_{2}$, yielding the intertwining relation $L_{2} \cdot \Omega_{4}=M_{4} \cdot M_{2}$. Again one still has a decomposition (99) with $Z_{2}^{s}$ and $A_{2}$, two order-two self-adjoint operators, but where $Z_{2}^{s}$ no longer factorizes.
$\dagger \dagger$ Operators $Z_{2}^{s}$ and $A_{2}$ of the form (108) are automatically self-adjoint.

### 4.5. The lattice Green function of the anisotropic simple cubic lattice

At this step it is important to recall the results of Delves and Joyce 48, 49 for the lattice Green function of the anisotropic simple cubic lattice, generalizing the results displayed in section (3.1). The lattice Green function of that anisotropic lattice is solution of an order-four operator (see (14) in 49]), depending on an anisotropy parameter $\alpha$. This order-four operator reads in terms of the homogeneous derivative $\theta=x \cdot D_{x}$ :

$$
\begin{align*}
G_{4}^{a s c}= & 24 \cdot \theta^{3} \cdot(\theta-1)-4 \cdot x \cdot \theta \cdot P_{1}(\theta)+2 \cdot x^{2} \cdot(2 \theta+1) \cdot P_{2}(\theta) \\
& -A \cdot x^{3} \cdot(2 \theta+3)(2 \theta+1) \cdot P_{3}(\theta)  \tag{109}\\
& +5 \cdot(A+4) \cdot A^{3} \cdot x^{4} \cdot(2 \theta+5)(2 \theta+3)(2 \theta+1)(\theta+1),
\end{align*}
$$

where $A=\alpha^{2}-4$ and

$$
\begin{aligned}
P_{1}(\theta)= & 6 \cdot(2 \theta+1)\left(10 \theta^{2}+10 \theta+3\right)+A \cdot\left(28 \theta^{3}+7 \theta^{2}+16 \theta+3\right) \\
P_{2}(\theta)= & 12 \cdot(4 \theta+5)(2 \theta+3)(4 \theta+3)+2 A \cdot\left(172 \theta^{3}+252 \theta^{2}+234 \theta+81\right) \\
& \quad+3 A^{2} \cdot\left(16 \theta^{3}+21 \theta^{2}+18 \theta+6\right) \\
P_{3}(\theta)= & 40 \cdot(4 \theta+3)(4 \theta+1)+12 A \cdot\left(22 \theta^{2}+29 \theta+12\right)+A^{2} \cdot\left(36 \theta^{2}+57 \theta+31\right) .
\end{aligned}
$$

Operator (109) is globally nilpotent [8, 40 for any rational value of the parameter $A$, the Jordan reduction of its $p$-curvature [8, 40] reading

$$
J_{4}=\left[\begin{array}{llll}
0 & 0 & 0 & 0  \tag{110}\\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{array}\right]
$$

Its characteristic polynomial $P(\lambda)$ is $\lambda^{4}$, and its minimal polynomial is $\lambda^{3}$ (mod. any prime $p$ ).

This order-four operator $G_{4}^{a s c}$ is not $M U M$. It has two solutions analytic at $x=0$

$$
\begin{gather*}
1+\frac{1}{2}\left(\alpha^{2}+2\right) \cdot x+\frac{3}{8}\left(\alpha^{4}+8 \alpha^{2}+6\right) \cdot x^{2}+\frac{5}{16}\left(\alpha^{6}+18 \alpha^{4}+54 \alpha^{2}+20\right) \cdot x^{3}+\cdots \\
x+\frac{3}{8}\left(3 \alpha^{2}+11\right) \cdot x^{2}+\frac{5}{48}\left(11 \alpha^{4}+119 \alpha^{2}+146\right) \cdot x^{3}  \tag{111}\\
+\frac{35}{768}\left(25 \alpha^{6}+537 \alpha^{4}+2049 \alpha^{2}+1217\right) \cdot x^{4}+\cdots
\end{gather*}
$$

together with a solution with a $\log$ and a solution with a $\log ^{2}$. The first analytic solution is globally bounded [18, 19 for generic rational values of $\alpha$, or, even, generic rational values of $A$ : for $A=p / q$ the rescaling $x \rightarrow 4 q \cdot x$ changes this series into a series with integer coefficients. The second analytic solution (111) is not globally bounded for generic rational values of $\alpha$, but becomes globally bounded for $\alpha= \pm 1$ : with a rescaling $x \rightarrow 4 x$, the series becomes a series with integer coefficients.

The exterior square of $G_{4}^{a s c}$ (depending on the parameter $\alpha$ ) is of order six with no rational (or hyperexponential [24]) solutions. The symmetric square of $G_{4}^{a s c}$ is of order nine, instead of the order ten one could expect. In other words $G_{4}^{a s c}$ verifies the symmetric Calabi- Yau condition for order-four operators (see (96) above for orderthree symmetric condition).

If, as previously done, we introduce an order-four operator $\tilde{G}_{4}^{\text {asc }}$ equivalent to $G_{4}^{a s c}$

$$
\begin{equation*}
S_{1}^{a s c} \cdot G_{4}^{a s c}=\tilde{G}_{4}^{a s c} \cdot D_{x} \tag{112}
\end{equation*}
$$

the symmetric square of that equivalent order-four operator has a rational solution $r(x)$ :

$$
\begin{equation*}
r(x)=\frac{\left(\alpha^{2}-4\right) \cdot x+3}{x^{2} \cdot\left(1-\alpha^{2} \cdot x\right) \cdot\left(1-(\alpha-2)^{2} \cdot x\right) \cdot\left(1-(\alpha+2)^{2} \cdot x\right)} . \tag{113}
\end{equation*}
$$

The order-four operator (109) can be decomposed in terms of two self-adjoint operators of order one and three, $Y_{1}^{(s)}$ and $Y_{3}^{(s)}$, namely

$$
\begin{align*}
& G_{4}^{a s c}=Y_{1}^{(s)} \cdot \rho(x) \cdot Y_{3}^{(s)}+\frac{8 \cdot\left(\alpha^{2}-1\right)^{2}}{\rho(x)}, \quad \rho(x)=\frac{\left(\left(\alpha^{2}-4\right) \cdot x+3\right)^{4}}{\left(5\left(\alpha^{2}-4\right) \cdot x-3\right)^{3}},  \tag{114}\\
& \text { where: } \quad \begin{aligned}
\rho(x) \cdot Y_{1}^{(s)} & =2 \cdot\left(\left(\alpha^{2}-4\right) \cdot x+3\right)\left(5\left(\alpha^{2}-4\right) \cdot x-3\right) \cdot D_{x} \\
& +\left(\alpha^{2}-4\right)\left(5\left(\alpha^{2}-4\right) \cdot x+69\right)
\end{aligned}
\end{align*}
$$

$Y_{3}^{(s)}$ being slightly more involved. One more time, and similarly to what has been done for $G_{6}^{5 D f c c}$ and $G_{8}^{6 D f c c}$ (see (61) and (84)), one can rewrite (114) as

$$
\begin{equation*}
\rho(x) \cdot Y_{1}^{(s)} \cdot \rho(x) \cdot Y_{3}^{(s)}=-8 \cdot\left(\alpha^{2}-1\right)^{2}+\rho(x) \cdot G_{4}^{a s c} \tag{116}
\end{equation*}
$$

which means that the two intertwiners $\rho(x) \cdot Y_{1}^{(s)}$ and $\rho(x) \cdot Y_{3}^{(s)}$ are inverse of each other modulo the operator $\rho(x) \cdot G_{4}^{\text {asc }}$.

The order-four operator (109) is homomorphic to its adjoint with the intertwining relations:

$$
\begin{align*}
& Y_{3}^{(s)} \cdot \rho(x) \cdot G_{4}^{a s c}=\operatorname{adjoint}\left(G_{4}^{a s c}\right) \cdot \rho(x) \cdot Y_{3}^{(s)} \\
& G_{4}^{a s c} \cdot \rho(x) \cdot Y_{1}^{(s)}=Y_{1}^{(s)} \cdot \rho(x) \cdot \operatorname{adjoint}\left(G_{4}^{a s c}\right) \tag{117}
\end{align*}
$$

If one denotes $W$ the Wronskian of $G_{4}^{a s c}$ one has the relation $r(x)^{20}=W^{8} \cdot \rho(x)^{5}$ $\left(5\left(\alpha^{2}-4\right) \cdot x-3\right)^{7}$.

Recalling the example of the order-six lattice Green operator $G_{6}^{5 D f c c}$, one sees that the fact that it is the symmetric square (and not the exterior square) of that order-four operator which has a rational solution, is related to the odd order of the two intertwiners. This anisotropic example shows that all the differential algebra structures we display in this paper can be generalized, mutatis mutandis, to problems with more than one variable (see also [50]).

## 5. Exceptional differential Galois groups

Recently a set of Calabi-Yau type operators whose differential Galois group is $G_{2}(C)$, the exceptiona迅 subgroup [51] of $S O(7)$, were explicitly given 29, 30. These examples read (see page 18 section 4.3 of [29], $\theta$ denotes the homogeneous derivative $\left.\theta=x \cdot D_{x}\right)$ :

$$
\begin{align*}
E_{1}= & \theta^{7}-128 \cdot x \cdot\left(48 \theta^{4}+96 \theta^{3}+124 \theta^{2}+76 \theta+21\right)(2 \theta+1)^{3} \\
& +4194304 \cdot x^{2} \cdot(\theta+1) \cdot\left(12 \theta^{2}+24 \theta+23\right) \cdot(2 \theta+1)^{2} \cdot(2 \theta+3)^{2} \\
& -34359738368 \cdot x^{3} \cdot(2 \theta+5)^{2} \cdot(2 \theta+1)^{2} \cdot(2 \theta+3)^{3}, \tag{118}
\end{align*}
$$

$\ddagger$ The compact form of $G_{2}$, subgroup of $S O(7)$, can be described as the automorphism group of the octonion algebra.
and:

$$
\begin{align*}
& E_{2}=\theta^{7}-128 \cdot x \cdot\left(8 \theta^{4}+16 \theta^{3}+20 \theta^{2}+12 \theta+3\right)(2 \theta+1)^{3} \\
&+1048576 x^{2} \cdot(2 \theta+1)^{2} \cdot(2 \theta+3)^{2} \cdot(\theta+1)^{3}, \\
& E_{3}= \theta^{7} \\
&-3^{3} \cdot x \cdot\left(81 \theta^{4}+162 \theta^{3}+198 \theta^{2}+117 \theta+28\right)(2 \theta+1)(3 \theta+1)(3 \theta+2) \\
&+3^{12} x^{2} \cdot(3 \theta+5) \cdot(3 \theta+1) \cdot(\theta+1) \cdot(3 \theta+2)^{2} \cdot(3 \theta+4)^{2}, \quad(119)  \tag{119}\\
& E_{4}= \theta^{7} \\
&-2^{7} \cdot x \cdot\left(128 \theta^{4}+256 \theta^{3}+304 \theta^{2}+176 \theta+39\right)(4 \theta+1)(4 \theta+3)(2 \theta+1) \\
&+2^{26} x^{2} \cdot(4 \theta+7)(4 \theta+5)(4 \theta+3)(4 \theta+1)(2 \theta+1)(2 \theta+3)(\theta+1), \\
& E_{5}= \theta^{7} \\
&-2^{7} 3^{3} \cdot x \cdot\left(648 \theta^{4}+1296 \theta^{3}+1476 \theta^{2}+828 \theta+155\right)(6 \theta+5)(6 \theta+1)(2 \theta+1) \\
&+2^{20} 3^{12} x^{2} \cdot(6 \theta+11)(6 \theta+7)(6 \theta+5)(6 \theta+1)(3 \theta+5)(3 \theta+1)(\theta+1) .
\end{align*}
$$

Note that for the five $E_{n}$, their conjugate $x^{-1 / 2} \cdot E_{n} \cdot x^{1 / 2}$ are self-adjoint operators, and of courst $x^{-1} \cdot E_{n}$ are also self-adjoint operators. Also note that the homogeneous derivative $\theta=x \cdot D_{x}$ is just shifted by $1 / 2$ by this conjugation:

$$
\begin{equation*}
x^{-1 / 2} \cdot \theta \cdot x^{1 / 2}=\theta+\frac{1}{2} \tag{120}
\end{equation*}
$$

Therefore one gets easily the expressions of these new self-adjoint operators by changing $\theta$ into $\theta+1 / 2$ in the previous definitions.

The Wronskians $w_{n}$ of these order-seven operators $E_{n}$ read respectively:

$$
\begin{equation*}
w_{1}=\left(\frac{1}{p_{1}^{w}}\right)^{21 / 2} \quad \text { and: } \quad w_{n}=\left(\frac{1}{p_{n}^{w}}\right)^{7} \quad n=2,3,4,5 \tag{121}
\end{equation*}
$$

where the $p_{n}^{w}$ 's are the following polynomials:

$$
\begin{align*}
p_{1}^{w}= & (16384 x-1) \cdot x^{2}, \quad p_{2}^{w}= \\
& p_{4}^{w}=(262144 x-1) \cdot x^{3}, \quad p_{5}^{w}=(80621568 x-1) \cdot x^{3} \tag{122}
\end{align*}
$$

The solution-series, analytic at $x=0$, of these order-seven operators $E_{n}$ are actually series with integer coefficients. These series are displayed in Appendix C These order-seven operators are MUM and are globally nilpotent (see (C.2) in Appendix C]. The corresponding nomes (called "special coordinates" in 29]) defined as $q^{(n)}=\exp \left(y_{1}^{(n)} / y_{0}^{(n)}\right)=x \cdot \exp \left(\tilde{y}_{1}^{(n)} / y_{0}^{(n)}\right)$, as well as the various Yukawa couplings [18, 19] of these order-seven operators, correspond to series with integer coefficients (see Appendix C).

Note that, after performing the following rescalings $x \rightarrow x / 4096, x / 19683$, $x / 262144, x / 80621568$ on the $E_{n}$ 's for $n=2,3,4,5$, these four rescaled $E_{n}$ 's have now the same Wronskian: $(1-x)^{-7} \cdot x^{-21}$. The homogeneous derivative being invariant by these rescalings the previous rescalings just amount to modifying the coefficients in front of the $x^{n}$ 's in the previous definitions, for instance:

$$
\begin{gathered}
E_{2} \longrightarrow \quad \hat{E}_{2}=\quad 2^{5} \cdot \theta^{7}-x \cdot\left(8 \theta^{4}+16 \theta^{3}+20 \theta^{2}+12 \theta+3\right)(2 \theta+1)^{3} \\
+2 x^{2} \cdot(2 \theta+1)^{2} \cdot(2 \theta+3)^{2} \cdot(\theta+1)^{3} .
\end{gathered}
$$

$\dagger$ If an operator $\Omega$ is self-adjoint, the operator $f(x) \cdot \Omega \cdot f(x)$ is also self-adjoint for any function $f(x)$.

After these rescalings these rescaled operators $\hat{E}_{i}$ for $i=2, \cdots 5$, have, now, their singularities in 0,1 and $\infty$. They read:

$$
\begin{equation*}
\hat{E}_{i}=(x-1)^{2} \cdot x^{7} \cdot D_{x}^{7}+7 \cdot(4 x-3)(x-1) \cdot x^{6} \cdot D_{x}^{6}+\cdots \tag{123}
\end{equation*}
$$

while for $\hat{E}_{1}$ one has:

$$
\begin{equation*}
\hat{E}_{1}=(x-1)^{3} \cdot x^{7} \cdot D_{x}^{7}+\frac{21}{2}(3 x-2)(x-1)^{2} \cdot x^{6} \cdot D_{x}^{6}+\cdots \tag{124}
\end{equation*}
$$

Combining these rescalings with the shift of $\theta$ by $1 / 2$ one easily deduces selfadjoint operators, for instance:

$$
\begin{aligned}
E_{2} \longrightarrow \quad(2 \theta+1)^{7} & -16 x \cdot\left(16 \theta^{4}+64 \theta^{3}+112 \theta^{2}+96 \theta+33\right)(\theta+1)^{3} \\
& +16 x^{2} \cdot(\theta+1)^{2} \cdot(\theta+2)^{2} \cdot(2 \theta+3)^{3} .
\end{aligned}
$$

From now on, let us consider the "rescaled" $\hat{E}_{n}$. The exterior squares of these order-seven operators $\hat{E}_{n}$ are of order 14 instead of the order 21 one could expect generically. The exterior cube of these order-seven operators are of order 27 (instead of order 35). The symmetric squares of these order-seven operators are of order 27, instead of the order 28 one could expect generically (see (125)).

Note that any operator such that its symmetric square is of order less than the generic expected order (here 28) is such that its solutions verifies a quadratic relation. For instance, the seven formal solutions of the order-seven operator $\hat{E}_{1}$ verifies the simple quadratic identity:

$$
\begin{equation*}
2 S_{1} S_{7}-2 S_{6} S_{2}+2 S_{5} S_{3}-S_{4}^{2}=0 \tag{125}
\end{equation*}
$$

Let us consider the operators $E_{n}^{(m)}$ non trivially homomorphic to the $\hat{E}_{n}$ 's:

$$
\begin{equation*}
E_{n}^{(m)} \cdot D_{x}^{m}=L_{m} \cdot \hat{E}_{n} \tag{126}
\end{equation*}
$$

where $L_{m}$ is an order- $m$ operator. For $m=1$ the exterior squares of the $E_{n}^{(m)}$,s are of order 21 (as expected generically), but the symmetric squares of the $E_{n}^{(m)}$, s are still of order 27. The exterior cube of the $E_{n}^{(m)}$,s are of order 34 instead of the order 35 expected generically.

For $m=2$ the symmetric squares of the $E_{n}^{(m)}$ 's are still of order 27 , however for $m=3$ the symmetric squares of the $E_{n}^{(m)}$,s is of the order 28 expected generically. The exterior cube of the $E_{n}^{(m)}$,s are of order 35 expected generically.

The exterior squares of the $E_{n}^{(1)}$ 's are actually a direct sum of an order-fourteen operator and an order-seven operator

$$
\begin{equation*}
\operatorname{Ext}^{2}\left(E_{n}^{(1)}\right)=L_{14}^{(n)} \oplus L_{7}^{(n)} \tag{127}
\end{equation*}
$$

where the order-seven operators are actually simply conjugated to the $\hat{E}_{n}$ 's:

$$
L_{7}^{(1)}=\frac{1}{(1-x)^{3 / 2} x^{3}} \cdot \hat{E}_{1} \cdot(1-x)^{3 / 2} x^{3}, \quad L_{7}^{(n)}=\frac{1}{(1-x) x^{3}} \cdot \hat{E}_{n} \cdot(1-x) x^{3}
$$

The symmetric squares of the $E_{n}^{(3)}$ 's are actually a direct sum of an order- 27 operator and an order-one operator

$$
\begin{equation*}
\operatorname{Sym}^{2}\left(E_{n}^{(3)}\right)=L_{27}^{(n)} \oplus L_{1}^{(n)} \tag{128}
\end{equation*}
$$

where the order-one operators $L_{1}^{(n)}$ have the following rational solutions $r_{n}$ :

$$
\begin{equation*}
r_{1}=\frac{1}{(x-1)^{3} \cdot x^{6}}, \quad r_{n}=\frac{1}{(x-1)^{2} \cdot x^{6}} \quad n=2, \cdots, 5 \tag{129}
\end{equation*}
$$

The exterior cubes of the $E_{n}^{(2)}$,s are actually a direct sum of an order-27 operator $M_{27}^{(n)}$, an order-seven operator $M_{7}^{(n)}$ and an order-one $M_{1}^{(n)}$ operator

$$
\begin{equation*}
\operatorname{Ext}^{3}\left(E_{n}^{(2)}\right)=M_{27}^{(n)} \oplus M_{7}^{(n)} \oplus M_{1}^{(n)} \tag{130}
\end{equation*}
$$

where the order-one operators $M_{1}^{(n)}$ have an algebraic solution for $E_{1}^{(2)}$ :

$$
\begin{equation*}
a_{1}=\frac{1}{(x-1)^{9 / 2} \cdot x^{9}} \tag{131}
\end{equation*}
$$

and the following rational solutions for the exterior cube of the other $E_{n}^{(2)}$,s:

$$
\begin{equation*}
\rho_{n}=\frac{1}{(x-1)^{3} \cdot x^{9}}, \quad n=2, \cdots, 5 \tag{132}
\end{equation*}
$$

Remark 1: The emergence of the exceptional group $G_{2}$ corresponds to the appearance of rational (or square root of rationatit) solutions for the symmetric square and exterior cube of these operators. This is reminiscent (see page 320 of Chapter 9 of [7]) of the (non-Fuchsian) order-seven operator $D_{x}^{7}-x \cdot D_{x}-1 / 2$, which has the differential Galois group $G_{2}$, namely the exceptional subgroup [51] of $S O(7)$. If we had only rational (or square root of rational) solutions of the symmetric square of the operators we would have $S O(7)$ differential Galois groups: the appearance of the rational (or square root of rational) solutions for the exterior cube of these operators explains the emergence of this exceptional subgroup of $S O(7)$.

Remark 2: Throughout the paper, we see the systematic emergence of decompositions as direct sums (see for instance (127), (128), (130)) instead of just a factorization, each time we find a rational solution for some symmetric or exterior power. This should not be seen as a surprise. Indeed, the linear differential operator $E_{n}^{(m)}$ is irreducible. This implies that its differential Galois group is reductive, i.e. all its representations are semi-simple (i.e. decompose as a direct sum of irreducible representations): see section 2.2 , specially discussion before lemma 2.3 in 32. In practice, this means that if we perform any construction like $S y m^{m}$, Ext ${ }^{r}$, etc, and if the corresponding operator factors, then it decomposes as an LCLM of irreducible operators (because the differential Galois group acts on the solution space of this operator, so the above applies).

Remark 3: All these results on symmetric squares, exterior squares and exterior cubes of (equivalent) order-seven operators have not been obtained using the Maple's DEtools command "ratsols" and "expsols", the corresponding algorithms being not powerful enough to cope with such examples of too large order. Furthermore, if one switches to differential systems representations, using the package 41 one findst, again the corresponding algorithms $\ddagger$ are not powerful enough to cope with such examples of too large order. One needs to go a step further, switching to $\theta$-systems, a method that yields systematically simple poles.
$\dagger \dagger$ In that case, the group is not connected but its Lie algebra is still $\mathfrak{g}_{2}$, i.e the connected component of the group which contains the identity is $G_{2}$.
$\dagger$ Use the commands with(TensorConstructions); with(IntegrableConnections); then companionsystem $\left({ }^{*}\right)$, exterior-power-system $\left({ }^{*}, \mathrm{~N}\right)$, symmetric-power-system $(*, \mathrm{~N})$, RationalSolutions $([*],[\mathrm{x}])$, HyperexponentialSolutions([*],[x]).
$\ddagger$ We try to promote, in this paper the idea that switching to differential systems is a more intrinsic and powerful method that working on the operators (seen at first sight by physicists, as simpler). With these examples we see that even switching to differential systems is not enough: one needs to switch to $\theta$-systems.

Seeking for rational solutions of differential systems (for regular systems like these), one (roughly) needs to find a transformation that cast them in simple poles. Then one finds the exponents, and then one reduces to polynomial solutions. Switching to $\theta$-systems $\mathbb{H}$, one automatically has simple poles: this bypasses the first reduction step. However, when one considers the companion matrix, one needs to perform a reduction of each singularity: this can, in general, yield a lot of reductions.

Using the TensorConstruction package, the command to be used is Theta-companion-system or full-theta-companion-system. One needs to perform a "reduction at $\infty "$ in order to find polynomial solutions. Doing all these tricks, one finally finds these results almost immediately for the symmetric squares, and for the exterior cubes.

### 5.1. Two-parameter and three-parameter deformations of $\hat{E}_{1}$

Another operator $\Omega(p, q)$ (generalizing $\hat{E}_{1}$ ), depending on two parameters $p$ and $q$, is given in Appendix D (see also the operator $L$ in section 5.2 of 30).

Let us denote $\Omega(p, q)^{(m)}$ the order-seven linear differential operator homomorphic to $\Omega(p, q)$ with a $D_{x}^{m}$ intertwiner (perform the LCLM of $\Omega(p, q)$ and $D_{x}^{m}$ and rightdivide by $D_{x}^{m}$ ).

One has the following direct sum decompositions, for arbitrary values of $p$ and $q$, for the symmetric square and exterior cube of these equivalent operators:

$$
\begin{equation*}
\operatorname{Sym}^{2}\left(\tilde{\Omega}(p, q)^{(3)}\right)=L_{27} \oplus L_{1}, \tag{133}
\end{equation*}
$$

and

$$
\begin{equation*}
\operatorname{Ext}^{3}\left(\tilde{\Omega}(p, q)^{(2)}\right)=M_{27}^{(n)} \oplus M_{7} \oplus M_{1} \tag{134}
\end{equation*}
$$

where the order-one operator $L_{1}$ has the rational solution $1 / x^{6} /(x-1)^{3}$, and where $M_{1}$ has the square root of rational solution $1 / x^{9} /(x-1)^{9 / 2}$. Furthermore one has

$$
\begin{align*}
& \operatorname{Ext}^{2}\left(\tilde{\Omega}(p, q)^{(1)}\right)=L_{14} \oplus L_{7},  \tag{135}\\
& \quad L_{7}=\frac{1}{(x-1)^{3 / 2} \cdot x^{3}} \cdot \Omega(p, q) \cdot(x-1)^{3 / 2} \cdot x^{3}
\end{align*}
$$

We have the same results as (133), (134) and (135) for (the equivalent of) a simple one-parameter deformation of $\hat{E}_{1}$, namely:

$$
\begin{equation*}
\hat{E}_{1}(r)=\hat{E}_{1}+r \cdot x \cdot(2 \theta+1) \tag{136}
\end{equation*}
$$

Note, however, that combining the previous deformation (136) on $\Omega(p, q)$ does yield a three parameters deformation satisfying direct sum decompositions like (133), (134) and (135)

$$
\begin{gather*}
\Omega(p, q, r)=\Omega(p, q)+r \cdot x \cdot(2 \theta+1), \quad \text { where }  \tag{137}\\
\operatorname{Sym}^{2}\left(\tilde{\Omega}(p, q, r)^{(3)}\right)=L_{27} \oplus L_{1},  \tag{138}\\
E x t^{3}\left(\tilde{\Omega}(p, q, r)^{(2)}\right)=M_{27}^{(n)} \oplus M_{7} \oplus M_{1}, \tag{139}
\end{gather*}
$$

where $L_{1}$ and $M_{1}$ have respectively the solutions $1 / x^{6} /(x-1)^{3}$ and $1 / x^{9} /(x-1)^{9 / 2}$.
$\sharp$ An alternative way, if one does not want to switch to $\theta$-systems, would be to perform a "Moserreduction" on the companion matrix before calculating the symmetric or exterior powers (these powers preserving the order of the poles).

Furthermore one has

$$
\begin{align*}
& \operatorname{Ext}^{2}\left(\tilde{\Omega}(p, q, r)^{(1)}\right)=L_{14} \oplus L_{7},  \tag{140}\\
& \quad L_{7}=\frac{1}{(x-1)^{3 / 2} \cdot x^{3}} \cdot \Omega(p, q, r) \cdot(x-1)^{3 / 2} \cdot x^{3}
\end{align*}
$$

Remark: If one wants to get rid of algebraic solution like $1 / x^{9} /(x-1)^{9 / 2}$, just perform a conjugation of $\hat{E}_{1}: \hat{E}_{1} \rightarrow(x-1)^{-1 / 2} \cdot \hat{E}_{1} \cdot(x-1)^{1 / 2}$, the rational solution of $L_{1}$ will become $1 / x^{6} /(x-1)^{4}$ and the algebraic solution of $M_{1}$ will become $1 / x^{9} /(x-1)^{6}$

This three-parameter conjugated operator $(x-1)^{-1 / 2} \cdot \Omega(p, q, r) \cdot(x-1)^{1 / 2}$ probably also has the exceptional group $G_{2}(C)$ as its differential Galois group.

### 5.2. Three-parameter family of order-seven operators with exceptional Galois groups

The order-seven operators $\hat{E}_{i}$ for $i=2 \cdots 5$ can also be seen as special cases of another order-seven operator $\Omega_{a, b, c}$ depending on three parameters (see Appendix D.2, see also operator $P_{1}$ in section 5.1 of (30).

Let us denote again $\tilde{\Omega}_{a, b, c}^{(m)}$ the order-seven linear differential operator homomorphic to $\Omega_{a, b, c}$ with a $D_{x}^{m}$ intertwiner (perform the LCLM of $\Omega_{a, b, c}$ and $D_{x}^{m}$ and rightdivide by $D_{x}^{m}$ ). The operator $\Omega_{a, b, c}$ is generically irreducible. Again this implies direct sum decompositions for any construction Sym $^{m}$, Ext ${ }^{r}$.

The symmetric square of $\tilde{\Omega}_{a, b, c}^{(3)}$ is actually a direct sum of an order- 27 operator and an order-one operator, $L_{1}$, with the rational solution $1 /(x-1)^{2} / x^{6}$

$$
\begin{equation*}
\operatorname{Sym}^{2}\left(\tilde{\Omega}_{a, b, c}^{(3)}\right)=L_{27} \oplus L_{1} \tag{141}
\end{equation*}
$$

and the exterior cube of $\tilde{\Omega}_{a, b, c}^{(2)}$ is actually a direct sum of an order-27 operator $M_{27}^{(n)}$, an order-seven operator $M_{7}^{(n)}$, and an order-one $M_{1}^{(n)}$ operator which has the rational solution $1 /(x-1)^{3} / x^{9}$ :

$$
\begin{equation*}
\operatorname{Ext}^{3}\left(\tilde{\Omega}_{a, b, c}^{(2)}\right)=M_{27}^{(n)} \oplus M_{7}^{(n)} \oplus M_{1}^{(n)} \tag{142}
\end{equation*}
$$

Furthermore one has $\ddagger$

$$
\begin{array}{cc}
\operatorname{Ext}^{2}\left(\tilde{\Omega}_{a, b, c}^{(1)}\right)=L_{14} \oplus L_{7}, & \text { where: }  \tag{143}\\
L_{7}=\frac{1}{(x-1) \cdot x^{3}} \cdot \Omega_{a, b, c} \cdot(x-1) \cdot x^{3}
\end{array}
$$

This three-parameter operator probably also has the exceptional group $G_{2}(C)$ as its differential Galois group.

## 6. Comments and speculations: diagonal of rational functions

Let us recall that the (minimal) linear differential operators for the $\chi^{(n)}$ 's, the $n$ particle contributions of the magnetic susceptibility of the square Ising model, are not irreducible, but factor into many irreducible operators of various orders [11, 12, 13, 14, 52 (two, three, four, ...). For all the factors for which the calculations can
$\dagger$ For $\Omega(p, q, r)$ the group is the exceptional group $G_{2}(C)$ up to a center, as a consequence of the emergence of a square root $1 / x^{9} /(x-1)^{9 / 2}$.
$\dagger$ Note that these results (141), (142), (143), are obtained for arbitrary values of the three parameters $a, b, c$, of $\Omega_{a, b, c}$ independently of the fact that $\Omega_{a, b, c}$ is MUM or not.
be performed we have seen that these irreducible factors are actually homomorphic to their adjoint. Thus, the interesting question is to see whether all the factors of these (minimal) operators for the $\chi^{(n)}$ 's are homomorphic to their adjoint, i.e. have a "special" differential Galois group, possibly as a consequence of the fact that the $\chi^{(n)}$ 's are diagonals of rational functions (see [18, 19 for a definition).

In this paper we underline selected linear differential operators having selected differential structures (special differential Galois groups) characterized in a differential algebra way (homomorphisms to their adjoint, rational, or hyperexponential 42, solutions of their exterior or symmetric powers). The idea is to disentangle these selected geometrical properties from other selected structures of a more arithmetic properties (globally bounded series solutions [18, 19]), both kinds of selected properties occurring simultaneously with the concept of "modularity". It is important to understand the relationship between these two kinds of properties. Operators with selected differential Galois groups do not necessarily correspond to globally bounded solution series [18, 19]. It is thus natural to see whether operators with globally bounded solution series [18, 19] necessarily correspond to selected differential Galois groups. This question being probably too difficult to address, let us ask the following question: if a linear differential operator has solutions that are diagonals of rational function\& does it necessarily correspond to selected differential Galois groups, or, more simply, are such operators homomorphic to their adjoint (possibly in an algebraic extension) ? Note that we have accumulated a quite large number of operators with solutions that are Hadamard products [55, 56] of algebraic functions (and are thus simple examples of diagonals of rational functions [18, 19]). They all have been seen to be homomorphic to their adjoint (sometimes up to algebraic extensions). Let us recall that diagonals of rational functions are (most of the time transcendental) functions that are the simplest extensions of algebraic functions [18, 19] (modulo each prime, they are algebraic functions). It is worth noting that linear differential operators with algebraic solutions are always homomorphic to their adjoint (up to an algebraic extension). It is thus tempting to see whether (the factors of minimal) differential operators with solutions that are diagonal of rational functions are necessarily homomorphic to their adjoint (possibly in an algebraic extension). In order to get some hint on this question, we consider a set of simple but, hopefully, generic enough $\ddagger$, diagonals of rational functions, find the minimal operators that annihilate them, and check wether the factors of these operators could all be homomorphic to their adjoint (up to algebraic extensions).

### 6.1. Diagonal of rational function: a heuristic simple example of an arbitrary number of variables

Let us first consider one of the simplest example of diagonal of rational functions of $N$ variables, namely the diagonal of the rational function

$$
\begin{equation*}
S_{N}=\operatorname{Diag}\left(\frac{1}{1-x_{1}-x_{2} \cdots-x_{N}}\right)=\sum_{k=0}^{\infty} \frac{(k N)!}{(k!)^{N}} \cdot x^{k} \tag{144}
\end{equation*}
$$

§ There are factors of order 12 or 23 , that are too large to see, by brute-force calculations, if these operators are homomorphic to their adjoint, or such that their exterior or symmetric square could have a rational solution.
$\dagger$ Diagonals of rational functions are necessarily solutions of linear differential operators (see Lipshitz [531 54).
$\ddagger$ We try to avoid operators with hypergeometric or Hadamard product solutions.

The series $S_{N}$ are solutions of the order $N-1$ linear differential operators $L_{N-1}$

$$
\begin{align*}
L_{N-1}= & \sum_{k=0}^{N-1} x^{k} \cdot\left(\alpha_{k} \cdot x-\beta_{k}\right) \cdot D_{x}^{k}=x^{N-2} \cdot\left(N^{N} \cdot x-1\right) \cdot D_{x}^{N-1}  \tag{145}\\
& +x^{N-3} \cdot \frac{N-1}{2} \cdot\left((N-1) \cdot N^{N} \cdot x-(N-2)\right) \cdot D_{x}^{N-2}+\cdots
\end{align*}
$$

which, are remarkably, self-adjoint operators. The first operators read:

$$
\begin{aligned}
L_{4}= & (3125 x-1) \cdot x^{3} D_{x}^{4}+2(12500 x-3) \cdot x^{2} D_{x}^{3}+(45000 x-7) \cdot x D_{x}^{2} \\
& +(15000 x-1) \cdot D x+120 \\
L_{5}= & (46656 x-1) \cdot x^{4} D_{x}^{5}+10(58320 x-1) \cdot x^{3} D_{x}^{4}+25(79056 x-1) \cdot x^{2} D_{x}^{3} \\
& +15(126360 x-1) \cdot x D_{x}^{2}+(331920 x-1) \cdot D x+720
\end{aligned}
$$

or, more simply in $\theta=x D_{x}$ :

$$
\begin{aligned}
& x \cdot L_{4}=5 x \cdot(5 \theta+1) \cdot(5 \theta+2) \cdot(5 \theta+3) \cdot(5 \theta+4)-\theta^{4} \\
& x \cdot L_{5}=6 x \cdot(6 \theta+1) \cdot(6 \theta+2) \cdot(6 \theta+3) \cdot(6 \theta+4) \cdot(6 \theta+5)-\theta^{5}, \\
& x \cdot L_{6}=7 x \cdot(7 \theta+1) \cdot(7 \theta+2) \cdot(7 \theta+3) \cdot(7 \theta+4) \cdot(7 \theta+5) \cdot(7 \theta+6)-\theta^{6},
\end{aligned}
$$

and generally

$$
\begin{equation*}
x \cdot L_{N-1}=N \cdot x \cdot(N \theta+1) \cdot(N \theta+2) \cdots(N \theta+N-1)-\theta^{N-1} \tag{146}
\end{equation*}
$$

which makes crystal clear that these operators are hypergeometric operators with ${ }_{N-1} F_{N-2}$ solutions. For instance, for $L_{4}$, we recover the ${ }_{4} F_{3}$ hypergeometric solution occurring in Candelas et al. paper [57]:

$$
\begin{equation*}
{ }_{4} F_{3}\left(\left[\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}\right],[1,1,1], 5^{5} \cdot x\right) \tag{147}
\end{equation*}
$$

For arbitrary values of $N$ we get, for $L_{N-1}$, the ${ }_{N-1} F_{N-2}$ hypergeometric solution:

$$
\begin{equation*}
{ }_{N-1} F_{N-2}\left(\left[\frac{1}{N}, \frac{2}{N}, \cdots, \frac{N-1}{N}\right],[1,1, \cdots, 1], N^{N} \cdot x\right) . \tag{148}
\end{equation*}
$$

It is worth noting, for larger values of $N$, that the $L_{N-1}$ operators are such that, not only the series-solution, associated with (148), is a globally bounded series [18, 19 , (with the $N^{N}$ factor in the argument of the hypergeometric function it is even a series with integer coefficients), but that the series for the nome and all the Yukawa couplings, are all series with integer coefficients, thus corresponding to a modularity of the operators. These results can be seen to be a consequence of [58] which gives the special parameters of generalized hypergeometric equations leading to mirror maps with integral Taylor coefficients at $x=0$. For instance, the nome $q\left(L_{N-1}\right)$ of the first $L_{N-1}$ 's read:

$$
\begin{aligned}
q\left(L_{3}\right)= & x+104 x^{2}+15188 x^{3}+2585184 x^{4}+480222434 x^{5}+\cdots \\
q\left(L_{4}\right)= & x+770 x^{2}+1014275 x^{3}+1703916750 x^{4}+3286569025625 x^{5}+\cdots \\
q\left(L_{5}\right)= & x+6264 x^{2}+87103188 x^{3}+1736438167584 x^{4} \\
& \quad+42329034160944354 x^{5}+\cdots \\
q\left(L_{6}\right)= & x+56196 x^{2}+9646450758 x^{3}+2718983725393656 x^{4} \\
& \quad+1002323538601613453169 x^{5}+\cdots
\end{aligned}
$$

$$
\begin{align*}
q\left(L_{7}\right)= & x+554112 x^{2}+1360868807232 x^{3}+6344223280197623808 x^{4} \\
& +41288465594250793127633184 x^{5}+\cdots \\
q\left(L_{8}\right)= & x+5973264 x^{2}+240205268638728 x^{3}+21222454448347058876544 x^{4} \\
& +2781115919369621686237935319524 x^{5}+\cdots \\
q\left(L_{9}\right)= & x+69998400 x^{2}+52035672968460000 x^{3}+98009214813679052640000000 x^{4} \\
& +289691284689365345860892113743750000 x^{5}+\cdots \tag{149}
\end{align*}
$$

and all the Yukawa series, including the "higher order Yukawa couplingst', $K_{n}$, are globally bounded, and, even, series with integer coefficients. We actually found the following relations between the Yukawa couplings:

$$
\begin{aligned}
& K_{4}\left(L_{4}\right)= K_{3}\left(L_{4}\right)^{2}, \\
& K_{4}\left(L_{5}\right)= K_{3}\left(L_{5}\right)^{3}, \\
& K_{5}\left(L_{6}\right)= K_{4}\left(L_{6}\right)^{2}, \\
& K_{7}\left(L_{5}\right)=K_{3}\left(L_{5}\right)^{5}, \\
& K_{7}\left(L_{7}\right)=\left(\frac{K_{4}\left(L_{7}\right)}{K_{3}\left(L_{7}\right)}\right)^{7}, \quad K_{6}\left(L_{7}\right)=\left(\frac{K_{4}\left(L_{7}\right)}{K_{3}\left(L_{7}\right)}\right)^{5}, \quad \text { and: } \quad K_{5}\left(L_{7}\right)=\left(\frac{K_{4}\left(L_{7}\right)}{K_{3}\left(L_{7}\right)}\right)^{3}, \\
& K_{8}\left(L_{8}\right)=\left(\frac{K_{5}\left(L_{8}\right)}{K_{3}\left(L_{8}\right)}\right)^{4}, \quad K_{7}\left(L_{8}\right)=\left(\frac{K_{5}\left(L_{8}\right)}{K_{3}\left(L_{8}\right)}\right)^{3} \quad \text { and: } \quad K_{6}\left(L_{8}\right)=\left(\frac{K_{5}\left(L_{8}\right)}{K_{3}\left(L_{8}\right)}\right)^{2}, \\
& K_{9}\left(L_{9}\right)=\left(\frac{K_{5}\left(L_{9}\right)}{K_{4}\left(L_{9}\right)}\right)^{9}, \quad K_{8}\left(L_{9}\right)=\left(\frac{K_{5}\left(L_{9}\right)}{K_{4}\left(L_{9}\right)}\right)^{7}, \quad K_{7}\left(L_{9}\right)=\left(\frac{K_{5}\left(L_{9}\right)}{K_{4}\left(L_{9}\right)}\right)^{5} \\
& \text { and: } \quad K_{6}\left(L_{9}\right)=K_{3}\left(L_{9}\right) \cdot\left(\frac{K_{5}\left(L_{9}\right)}{K_{4}\left(L_{9}\right)}\right)^{3}, \\
& K_{10}\left(L_{10}\right)=\left(\frac{K_{6}\left(L_{10}\right)}{K_{4}\left(L_{10}\right)}\right)^{5}, \quad K_{9}\left(L_{10}\right)=\left(\frac{K_{6}\left(L_{10}\right)}{K_{4}\left(L_{10}\right)}\right)^{4}, \quad K_{8}\left(L_{10}\right)=\left(\frac{K_{6}\left(L_{10}\right)}{K_{4}\left(L_{10}\right)}\right)^{3}, \\
& \text { and: } \quad K_{7}\left(L_{10}\right)=K_{3}\left(L_{10}\right) \cdot\left(\frac{K_{6}\left(L_{10}\right)}{K_{4}\left(L_{10}\right)}\right)^{2},
\end{aligned}
$$

where these Yukawa couplings read respectively

$$
\begin{aligned}
K_{3}\left(L_{4}\right)= & 1+575 x+1418125 x^{2}+3798200625 x^{3}+10597067934375 x^{4} \\
& +30287765070550575 x^{5}+\cdots \\
K_{3}\left(L_{5}\right)= & 1+10080 x+357073920 x^{2}+13943124679680 x^{3} \\
& +570470634728386560 x^{4}+23986351416805190461440 x^{5}+\cdots
\end{aligned}
$$

For $L_{6}$ and $L_{7}$ one has two independent Yukawa couplings. For $L_{8}$ and $L_{9}$ one has three independent Yukawa couplings, for $L_{10}$ and $L_{11}$ one has four independent Yukawa couplings, ... These series are displayed in Appendix E,

For $N$ an odd integer the exterior square of the $(N-1)$-order operator (145) is of order $(N-1)(N-2) / 2-1$ instead of $(N-1)(N-2) / 2$. For $N$ an even integer the symmetric square of the $(N-1)$-order operator (145) is of order $N(N-1) / 2-1$ instead of $N(N-1) / 2$. Similarly to (65) or (89), introducing an equivalent operator $\tilde{L}_{N-1}^{n}$, such that $S^{n} \cdot L_{N-1}=\tilde{L}_{N-1}^{n} \cdot D_{x}^{n}$, the exterior or symmetric square of that equivalent operator has, for a well-suited value of $n$ the rational solution $1 / x^{N-2} /\left(N^{N} x-1\right)$.

[^6]6.2. Diagonal of rational function: heuristic simple examples of three variables

Increasing the degree of the rational functions, another example corresponds to the diagonal of

$$
\begin{equation*}
R(x, y, z)=\frac{1-x+y z}{1-3 x z-5 y^{2}} \tag{150}
\end{equation*}
$$

Unfortunately this diagonal is solution of an order-two operator with algebraic solutions, $\left(4-1215 x^{2}\right) \cdot D_{x}^{2}-3645 \cdot x D_{x}-1080$ (which is conjugated to its adjoint).

We have a similar algebraic result with

$$
\begin{equation*}
R(x, y, z)=\frac{1-7 x+2 y z}{1+3 x z-5 y^{3}} \tag{151}
\end{equation*}
$$

Its diagonal is the lacunary series

$$
\begin{align*}
1-540 & x^{3}+510300 x^{6}-541282500 x^{9}+604514137500 x^{12} \\
& -695204544150000 x^{15}+814769502147562500 x^{18}+\cdots \tag{152}
\end{align*}
$$

which is an algebraic function solution of an order-three operator (homomorphic to its adjoint).

Another example corresponds to the diagonal of

$$
\begin{equation*}
R(x, y, z)=\frac{1-x}{1-3 x+z-5 y^{2}} \tag{153}
\end{equation*}
$$

which also corresponds to the series expansion of a hypergeometric function:

$$
\begin{align*}
& \frac{2}{15}+\frac{13}{15} \cdot{ }_{4} F_{3}\left(\left[\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}\right],\left[\frac{1}{2}, \frac{1}{2}, 1\right], 5^{6} \cdot\left(\frac{3}{4} \cdot x\right)^{2}\right) \\
& =1+1170 x^{2}+5528250 x^{4}+33202669500 x^{6}  \tag{154}\\
& \quad+221602408706250 x^{8}+1569831463275075000 x^{10}+\cdots
\end{align*}
$$

The corresponding order-five operator is a direct sum $D_{x} \oplus M_{4}$ where the order-four operator $M_{4}$, which annihilates the ${ }_{4} F_{3}$ in (154), is homomorphic to its adjoint and such that its exterior square has a rational solution namely $1 / x /\left(9 \cdot 5^{6} x^{2}-16\right)$. The order-four operator $M_{4}$ is of the form (100), namely:

$$
\begin{equation*}
M_{4}=L_{2} \cdot c_{0}(x) \cdot M_{2}+\frac{9}{5^{4}} \cdot \frac{1}{c_{0}(x)}, \quad c_{0}(x)=x^{2} \cdot\left(x^{2}-\frac{16}{9 \cdot 5^{6}}\right) \tag{155}
\end{equation*}
$$

where $L_{2}$ and $M_{2}$ are two order-two self-adjoint operatorst.
Noting that the diagonal of

$$
\begin{equation*}
\tilde{R}(x, y, z)=\frac{1}{1-3 x+z-5 y^{2}} \tag{156}
\end{equation*}
$$

is solution of the order-four operator $M_{4}$, and is nothing but

$$
\begin{equation*}
{ }_{4} F_{3}\left(\left[\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}\right],\left[\frac{1}{2}, \frac{1}{2}, 1\right], \frac{9 \cdot 5^{6}}{16} \cdot x^{2}\right) \tag{157}
\end{equation*}
$$

and introducing $\hat{R}(x, y, z)=15 R(x, y, z)-13 \tilde{R}(x, y, z)-2$, one easily deduces, from (157), that the diagonal of

$$
\begin{equation*}
\hat{R}(x, y, z)=\frac{10 y^{2}-9 x-2 z}{1-3 x+z-5 y^{2}} \quad \text { or } \quad \frac{2-15 x}{1-3 x+z-5 y^{2}} \tag{158}
\end{equation*}
$$

is equal to zero.
These cases, reducible to algebraic or hypergeometric situations, are still too simple to be representative of the "generic" situation.
$\ddagger M_{2}$ is the product of two order-one operators the right factor having the polynomial solution $x^{2}\left(9 \cdot 5^{6} x^{2}-16\right)$.

### 6.3. Towards a "generic" diagonal of rational function example

Trying to avoid these too simple case reducible to hypergeometric functions (or Hadamard product of algebraic functions), we have considered the operator annihilating the diagonal of a rational function of three variables, hopefully involved enough, with no symmetry between the three variables, to be seen as a "generic" diagonal of a rational function.

### 6.3.1. Towards a "generic" diagonal of rational function: a first example

The rational function we have considered reads:

$$
\begin{equation*}
R(x, y, z)=\frac{1}{1-3 x-5 y-7 z+x y+2 y z^{2}+3 x^{2} z^{2}} \tag{159}
\end{equation*}
$$

The diagonal of this rational function readst:

$$
\begin{align*}
S_{0}^{(0)} & =\operatorname{Diag}(R(x, y, z))=1+616 x+947175 x^{2}+1812651820 x^{3} \\
& +3833011883965 x^{4}+8582819380142616 x^{5}+19946071353510410136 x^{6} \\
& +47578122531207001944168 x^{7}+115702070514540009854741415 x^{8} \\
& +285583642613093627090885877280 x^{9} \\
& +713269435359072253352128013072035 x^{10}+\cdots \tag{160}
\end{align*}
$$

The minimal order operator that annihilates the diagonal of this rational function (165) is a quite large order-six linear differential operatory. Again, this operator is too large to check that it is homomorphic to its adjoint. We can, however, check that its exterior square is of order 15 . However, switching to the associated differential thetasystem, we have been able to see that it is actually homomorphic to its adjoint: one actually finds the exterior square of the associated differential system has a rational solution (but not its symmetric square). The differential Galois group thus corresponds to a symplectic structure.

In fact this operator is not MUM. It has four solutions, analytic at $x=0$, namely $S_{0}^{(0)}$ given by (160) and

$$
\begin{align*}
S_{0}^{(1)}=x- & \frac{947569825302083891091227422045}{3191686441638931584990008514} x^{4} \\
& -\frac{13038344513942350315758249091274688499}{19626034561464639086279672353532} x^{5}+\cdots, \\
S_{0}^{(2)}=x^{2} & +\frac{60}{7^{3}} x^{4}-\frac{576}{7^{4}} x^{5}+\cdots, \\
S_{0}^{(3)}=x^{3}+ & \frac{30608172563777847511388970395}{14474768442806945963673508} x^{4} \\
& +\frac{6637738302888023001730565011179544651}{1401859611533188506162833739538} x^{5}+\cdots \tag{161}
\end{align*}
$$

$\S$ When the operators annihilating diagonal of rational functions are of order two, one often finds modular forms, the corresponding nome being seen to be a globally bounded series [18, 19]. A set of examples of diagonal of Szego's rational functions can be found in 59.
$\ddagger$ Use the maple command mtaylor( $\mathrm{F},[\mathrm{x}, \mathrm{y}, \mathrm{z}]$, terms), to get the series in three variables, then take the diagonal. Other method, in Mathematica install the risc package Riscergosum 60, and in HolonomicFunctions‘ use the command FindCreativeTelescoping. $\dagger$ We thank Alin Bostan for providing this order six operator from a creative telescopic code (not by guessing).
the last series $S_{0}^{(3)}$ being not globally bounded. The two other solutions have a log (but no $l o g^{2}, l o g^{3}, \ldots$ ):

$$
\begin{equation*}
S_{1}^{(0)}=S_{0}^{(0)} \cdot \ln (x)+T_{0}^{(0)}, \quad S_{1}^{(2)}=S_{0}^{(2)} \cdot \ln (x)+T_{0}^{(2)} \tag{162}
\end{equation*}
$$

the two series $T_{0}^{(0)}$ and $T_{0}^{(2)}$ being analytic at $x=0$, for instance:

$$
\begin{align*}
& T_{0}^{(0)}=\frac{1769904090259426475015551868948047756831494229112489}{6347493572699380825284454014187955842800} x^{4}  \tag{163}\\
& \quad+\frac{21577983707661117706708514436988691858431632715744973527227853}{21340441599994994868198204433731283524323434200} x^{5}+\cdots
\end{align*}
$$

Of course there is an ambiguity in all these solutions, except $S_{0}^{(3)}$ which is well defined: one can add $S_{0}^{(3)}$ to $S_{0}^{(2)}$, etc ... There is an ambiguity on $S_{0}^{(0)}$, and $S_{0}^{(2)}$, but the match of $S_{0}^{(0)}$ diagonal of the rational function and the form $S_{1}^{(2)}=$ $S_{0}^{(2)} \cdot \ln (x)+T_{0}^{(2)}$ fixes the normalization of $S_{0}^{(0)}$, and $S_{0}^{(2)}$. Thus $T_{0}^{(0)}$ is defined by (163), up to the solutions $S_{0}^{(0)}, S_{0}^{(1)}$ and $S_{0}^{(2)}$. Trying to go further the fact that the operator is not MUM, one can try to define two nomes by

$$
\begin{equation*}
q_{1}=\exp \left(\frac{S_{1}^{(0)}}{S_{0}^{(0)}}\right), \quad q_{2}=\exp \left(\frac{S_{1}^{(2)}}{S_{0}^{(2)}}\right) \tag{164}
\end{equation*}
$$

and, using this ambiguity, seek for $T_{0}^{(0)}$ and $T_{0}^{(2)}$ such that the two nomes are globally bounded series. Unfortunately, it is almost impossible to see if one can build nomes, such that their series are globally bounded series [18, 19 .

With this example that is not MUM, we exclude any simple modularity property for the operator, where the series for the nome, Yukawa couplings, etc ... would be globally bounded. Diagonal of rational functions do not necessarily yield modularity.

### 6.3.2. Towards a "generic" diagonal of rational function: a second example

Let us consider another simpler example with the diagonal of another rational function of three variables:

$$
\begin{equation*}
R(x, y, z)=\frac{1}{1+z-x y+x z+y^{2}} \tag{165}
\end{equation*}
$$

The diagonal of this rational function reads:

$$
\begin{gather*}
S_{0}^{(0)}=\operatorname{Diag}(R(x, y, z))=1-2 x+3 x^{2}+40 x^{3}-545 x^{4}+3948 x^{5} \\
-14910 x^{6}-55176 x^{7}+1544895 x^{8}-14999270 x^{9}+82528303 x^{10} \\
-29585712 x^{11}-5093494406 x^{12}+\cdots \tag{166}
\end{gather*}
$$

It is solution of an irreducible order-four operator $L_{4}$ that is not MUM. This operator is non-trivially homomorphic to its adjoint, with order-two intertwiners and is actually of the form (100):

$$
\begin{aligned}
& L_{4}=L_{2} \cdot c_{0}(x) \cdot M_{2}+\frac{\lambda}{c_{0}(x)}, \quad \lambda=\left(\frac{26}{639}\right)^{2}, \quad c_{0}(x)=\frac{\lambda}{9} \cdot \frac{p_{3}(x)^{2} \cdot p_{4}(x)}{p_{6}(x)}, \\
& p_{6}(x)=676 x^{6}+10514 x^{5}-2047 x^{4}+82424 x^{3}-15796 x^{2}+10304 x+448 \\
& p_{4}(x)=729 x^{4}-1568 x^{3}+984 x^{2}+192 x+16, \quad p_{3}(x)=(71 x+14)(x-2) \cdot x,
\end{aligned}
$$

where $L_{2}$ and $M_{2}$ are two self-adjoint operators, their Wronskian reading respectively

$$
\begin{equation*}
\frac{x^{2} \cdot p_{3}(x) \cdot p_{4}(x)^{2}}{p_{6}(x)}, \quad \frac{p_{3}(x)}{x^{2} \cdot p_{4}(x)} \tag{167}
\end{equation*}
$$

The exterior square of $L_{4}$ is an order-six operator with a rational function solution $R(x)$, corresponding to the direct sum decomposition:

$$
\begin{equation*}
E x t^{2}\left(L_{4}\right)=L_{5} \oplus\left(D_{x}-\frac{d \ln (R(x)}{d x}\right), \quad R(x)=\frac{p_{3}(x)}{x^{2} \cdot p_{4}(x)} \tag{168}
\end{equation*}
$$

where $L_{5}$ is an irreducible order-five operator. The operator $L_{4}$ has the following four solutions: the diagonal $S_{0}^{(0)}$ (see (166)), an analytic solution $S_{0}^{(1)}$

$$
\begin{align*}
S_{0}^{(1)} & =x-\frac{35}{2^{2}} x^{2}+\frac{3185}{2^{6}} x^{3}-\frac{35035}{2^{8}} x^{4}-\frac{16207191}{2^{14}} x^{5}+\frac{1217957741}{2^{16}} x^{6} \\
& -\frac{165312417127}{2^{20}} x^{7}+\frac{3091190741925}{2^{22}} x^{8}+\frac{1071079996954825}{2^{30}} x^{9}+\cdots \tag{169}
\end{align*}
$$

and two formal series solution with a $\log$, namely $S_{1}^{(0)}+\ln (x) \cdot S_{0}^{(0)}$ and also $S_{1}^{(1)}+\ln (x) \cdot S_{0}^{(1)}$

$$
\begin{aligned}
S_{1}^{(0)} & =2-\frac{35}{2} x+\frac{5703}{56} x^{2}-\frac{321597}{896} x^{3}-\frac{2659681}{3584} x^{4}+\frac{29836311703}{1146880} x^{5} \\
& -\frac{1156839045933}{4587520} x^{6}+\frac{722563886554257}{513802240} x^{7}-\frac{550307089986855}{411041792} x^{8} \\
& -\frac{22561115451957783769}{315680096256} x^{9}+\cdots
\end{aligned}
$$

and:

$$
\begin{aligned}
& S_{1}^{(1)}=\frac{11}{2} x^{2}-\frac{30163}{576} x^{3}+\frac{792323}{2304} x^{4}-\frac{219473079}{163840} x^{5}-\frac{16188947647}{5898240} x^{6} \\
& \quad+\frac{70828996802681}{660602880} x^{7}-\frac{27874560487367}{25165824} x^{8}+\frac{5457381128456532577}{811748818944} x^{9}+\cdots
\end{aligned}
$$

If one introduces the nome, its series expansion does not seem to be globally bounded [18, 19]:

$$
\begin{gathered}
q\left(L_{4}\right)=x \cdot \exp \left(\frac{S_{1}^{(1)}}{S_{0}^{(1)}}\right)=x+\frac{11}{2} x^{2}+\frac{6269}{576} x^{3}+\frac{43165}{1152} x^{4}+\frac{1040535941}{13271040} x^{5} \\
-\frac{11364935021}{26542080} x^{6}+\frac{851517278314609}{160526499840} x^{7}-\frac{1854100924158503}{64210599936} x^{8} \\
+\frac{790034414470824586787}{14794122225254400} x^{9}+\cdots
\end{gathered}
$$

Do note that the second analytic series is actually globally bounded. With a rescaling $x \rightarrow 2^{4} x$, the series (169) becomes a series with integer coefficients:

$$
\begin{gather*}
16 x-2240 x^{2}+203840 x^{3}-8968960 x^{4}-1037260224 x^{5}+311797181696 x^{6} \\
-42319978784512 x^{7}+3165379319731200 x^{8}+\cdots \tag{170}
\end{gather*}
$$

### 6.3.3. Towards a "generic" diagonal of rational function: a third example

Let us consider another example with the diagonal of another rational function of three variables:

$$
\begin{equation*}
R(x, y, z)=\frac{1-x-y+x y z}{1-x-y-x y-y^{2} z^{3}} \tag{171}
\end{equation*}
$$

The diagonal of this rational function reads:

$$
\begin{gather*}
S_{0}^{(0)}=\operatorname{Diag}(R(x, y, z))=1+x+10 x^{3}+32 x^{4}+966 x^{6}+3192 x^{7} \\
+120340 x^{9}+401720 x^{10}+16712150 x^{12}+56066920 x^{13} \\
+2466298800 x^{15}+8298484992 x^{16}+378403867380 x^{18}  \tag{172}\\
+1275714885984 x^{19}+59651272137600 x^{21}+\cdots
\end{gather*}
$$

It is solution of an order-five operator $L_{5}$ which factors as $L_{5}=L_{4} \cdot D_{x}$, where $L_{4}$ is an irreducible order-four operator that is not MUM. The exterior square of $L_{4}$ is an order-six operator with a rational function solution $R(x)$, corresponding to the direct sum decomposition:

$$
\begin{align*}
\operatorname{Ext}^{2}\left(L_{4}\right)= & L_{5} \oplus\left(D_{x}-\frac{d \ln (R(x)}{d x}\right), \quad R(x)=\frac{p_{10}(x)}{x^{2} \cdot p_{6}(x)^{2}} \\
p_{10}(x)= & 11008 x^{10}+165760 x^{9}-637392 x^{8}+383388 x^{7}+196287 x^{6} \\
& -281004 x^{5}-66582 x^{4}-45360 x^{3}+15660 x^{2}-810 x+162 \\
p_{6}(x)= & 1024 x^{6}-9909 x^{3}+54 . \tag{173}
\end{align*}
$$

This order-four operator $L_{4}$ is non-trivially homomorphic to its adjoint, with ordertwo intertwiners and is actually of the form (100):

$$
\begin{aligned}
& L_{4}=L_{2} \cdot c_{0}(x) \cdot M_{2}+\frac{1305}{29584} \cdot \frac{1}{c_{0}(x)}, \quad c_{0}(x)=\frac{145}{473344} \cdot \frac{p_{10}(x)^{2}}{p_{16}(x)} \\
& \begin{aligned}
p_{16}(x)= & 37120 x^{16}+1255680 x^{15}-48870560 x^{14}+594756560 x^{13}-31084335 x^{12} \\
& +2785358960 x^{11}+4430975954 x^{10}-8858296096 x^{9}-1107376429 x^{8} \\
& -369545240 x^{7}+4215494304 x^{6}-1487095128 x^{5}-466418052 x^{4} \\
& +228523680 x^{3}-21096612 x^{2}+2737800 x-717336
\end{aligned}
\end{aligned}
$$

where $L_{2}$ and $M_{2}$ are two self-adjoint operators, their Wronskian reading respectively

$$
\begin{equation*}
\frac{x^{2} \cdot p_{10}(x) \cdot p_{6}(x)^{2}}{p_{16}(x)}, \quad \frac{p_{10}(x)}{x^{2} \cdot p_{6}(x)^{2}} \tag{174}
\end{equation*}
$$

The operator $L_{4}$ is not MUM: it has three solution analytic at $x=0$, namely the derivative of diagonal (172)

$$
\begin{equation*}
\frac{d S_{0}^{(0)}}{d x}=1+30 x^{2}+128 x^{3}+5796 x^{5}+22344 x^{6}+1083060 x^{8}+\cdots \tag{175}
\end{equation*}
$$

and the two series solutions

$$
\begin{gather*}
x-\frac{595}{1107} x^{2}+\frac{3515617}{1225449} x^{3}+\frac{227188435}{1225449} x^{4}-\frac{2520602}{15129} x^{5}+\frac{8346429274}{11029041} x^{6} \\
+\frac{2633989297550}{77203287} x^{7}-\frac{42751323143}{1225449} x^{8}+\cdots, \\
x^{2}-\frac{595}{1107} x^{3}+\frac{4523}{1107} x^{4}+\frac{37758}{205} x^{5}-\frac{1412590}{9963} x^{6}+\frac{63250564}{69741} x^{7} \\
+\frac{187516948}{5535} x^{8}+\cdots, \tag{176}
\end{gather*}
$$

One also has a formal series solution with a log, namely

$$
\begin{equation*}
S_{1}(x)+\ln (x) \cdot \frac{d S_{0}^{(0)}}{d x} \tag{177}
\end{equation*}
$$

where $S_{1}(x)$ is a series analytic at $x=0$ :

$$
\begin{align*}
& S_{1}(x)=\frac{1}{5 x}+\frac{3083}{2214}+\frac{5222887}{4084830} x+\frac{956031447781}{22609534050} x^{2}+\frac{661652916345161}{2502875419335} x^{3} \\
& +\frac{2061248440531939}{12514377096675} x^{4}+\frac{25090312744949777}{3089969653500} x^{5}+\frac{43029529492015002359}{901035150960600} x^{6} \\
& \quad+\frac{21734062670504361827}{788405757090525} x^{7}+\frac{5314508101399026611659}{3504025587069000} x^{8}+\cdots \tag{178}
\end{align*}
$$

The last series in (176), as well as $S_{1}(x)$ (see (178)), are not globally bounded series [18, 19.

Remark 1: The series (176) is, of course, also a diagonal of a rational function:

$$
\begin{align*}
x \cdot \frac{d S_{0}^{(0)}}{d x} & =\operatorname{Diag}\left(x \cdot \frac{\partial R(x, y, z)}{\partial x}\right)  \tag{179}\\
& =\operatorname{Diag}\left(y \cdot \frac{\partial R(x, y, z)}{\partial y}\right)=\operatorname{Diag}\left(z \cdot \frac{\partial R(x, y, z)}{\partial z}\right)
\end{align*}
$$

Remark 2: With this order-five example, we see that the minimal order operator $L_{5}$, that annihilates the diagonal of a rational function, is not necessarily irreducible. Let us recall the results of [18, 19] where we have shown that the $\tilde{\chi}^{(n)}$ 's of the susceptibility of the square Ising model are actually diagonals of rational functions. The corresponding (globally nilpotent) linear differential operators annihilating the $\tilde{\chi}^{(n)}$ 's are not irreducible, on the contrary they factor into many linear differential operators, of various orders [8, 11, 12, 13, 14, 17] (one, two, three, ..., 12, 23, ...). The interesting property we must focus on, is not that the (minimal order) linear differential operators annihilating the $\tilde{\chi}^{(n)}$ 's are homomorphic to their adjoint, but that all their factors could be homomorphic to their adjoint. It is the differential Galois group of all these factors that we expect to be "special".

To sum up: One may consider the following conjecture: all the irreducible factors of the minimal order linear differential operator annihilating a diagonal of a rational function should be homomorphic to their adjoint (possibly on an algebraic extension).

Remark 3: Let us recall that the series of the hypergeometric function considered in 18, 19

$$
\begin{equation*}
{ }_{3} F_{2}\left(\left[\frac{1}{9}, \frac{4}{9}, \frac{5}{9}\right],\left[\frac{1}{3}, 1\right], 729 x\right)=1+60 x+20475 x^{2}+9373650 x^{3}+\cdots \tag{180}
\end{equation*}
$$

still remains a series with integer coefficients such that one cannot prove that it is the diagonal of a rational function, or discard that option. The minimal order operator annihilating this series is an order-three operator $L_{3}$ which is not homomorphic to its adjoint.

If our conjecture above was correct, this would be a way to show that the series (180) cannot be the diagonal of a rational function.

## 7. Conclusion

Selected differential Galois groups correspond to symmetric square or exterior square, and possibly higher powers (as seen with order-seven operators with exceptional such that $M_{2} \cdot L_{3}=\operatorname{adjoint}\left(L_{3}\right) \cdot \operatorname{adjoint}\left(M_{2}\right)$ has transcendental coefficients.
differential Galois groups of section (5)) of operators, or equivalent operators, having rational solutions (or $N$-th root of rational solutions, i.e. hyperexponential [24, 42] solutions). We have focused, in this paper, on a concept of "Special Geometry" corresponding to operators homomorphic to their adjoint. In a forthcoming publication, more focused on differential systems, and on demonstrations, we will show the equivalence of the homomorphism of an operator with its adjoint (possibly with algebraic extension), and the fact that its symmetric, or exterior, square of the corresponding differential systems have a rational (resp. $N$-th root of rational) solution.

In [22, 29] Bogner has been able, from the very existence of underlying Calabi-Yau varieties, to show that the Calabi-Yau differential operators are actually conjugated to their adjoint (Poincaré pairing). If one does not assume strong hypotheses like this one, it is not simple to disentangle the differential algebra structures (corresponding to selected differential Galois groups) we have addressed in this paper, and more "arithmetic" concepts associated to the notion of modularity, and the integrality, or globally bounded properties [18, [19] of the various series occurring with these differential operators (solution of the operator, the nome, the Yukawa couplings, ...). Recalling section 6, it is clear that the concept of "Special Geometry", which we address in this paper, does not necessarily yield arithmetic properties like the globally bounded [18, 19 character of various series associated with the operators. Conversely, we know that globally bounded series [18, 19] do not necessarily correspond to holonomic functions (see the example of the non-holonomic susceptibility of the Ising model and its series with integer coefficients [52]). Along such "modularity" line, the idea that operators annihilating diagonals of rational functions should always correspond to a modularity property that the corresponding nome and all the Yukawa's [18, 19 are globally bounded series, has been ruled out (see, for instance, the example of subsection 6.3.1). From a mathematics viewpoint, there is still a lot of work to be performed to clarify the relations between these various neighboring concepts around the notion of "modularity". Along this line, it is probably useful to keep in mind all the simple example $\dagger$ of section 6. From a physics viewpoint, one would like to identify, more specifically, what kind of "Special Geometry" we encounter (Calabi-Yau, selected hypergeometric functions up to pull-backs [20, ...).

In this paper, the emergence of selected differential Galois groups has been seen, in a down-to-earth physicist's viewpoint, as differential algebra properties: one calculates various exterior, or symmetric, powers, and looks (up to operator equivalence) for their rational solution (or hyperexponential [24, 42] solution), and one calculates the homomorphisms of an operator with its adjoint. We have shown that quite involved lattice Green operators of order six and eight are non trivially homomorphic to their adjoint, and that this yields the non trivial decompositions (60) and (83), where their intertwiners emerge in a crystal clear way (see also (114) in section 4.5). Such decompositions enable to understand why the lattice Green operator (47) has a differential Galois group included in the orthogonal group $O(6, \mathbb{C})$ instead of the symplectic $S p(6, \mathbb{C})$ differential Galois group, that one might expect for an order-six operator: the intertwiners are of odd orders.

[^7]Decompositions such as (90), (91) can be generalized for linear differential operators of any even order. In fact, one can actually use the decompositions (90), (91) as an ansatz to provide linear differential operators of any even order, that will automatically have selected differential Galois groups.

With these lattice Green operators, we see that the simple generalization of the Calabi-Yau condition (94) for operators of order $N>4$ (namely the condition that their exterior square is of order less than the generic $N \cdot(N-1) / 2$ order $)$, is a too restrictive concept for physics. These lattice Green operators do not satisfy such higher-order generalization of the Calabi-Yau condition (94), but must be seen as higher-order generalization of a "weak Calabi-Yau condition" (see section 4.1) which amounts to saying that their exterior or symmetric squares have rational solutions, and that they are non-trivially homomorphic to their adjoint.

For order-four operators, Calabi-Yau operators are defined, among several other conditions (see Almkvist et al. [5]), essentially by the Calabi-Yau condition (94). It is, however, quite clear that any equivalent operator (in the sense of the equivalence of operator, i.e. homomorphic to the Calabi-Yau operator), is also a selected operator interesting for physics. We have shown, in this paper, that any orderfour operator, non-trivially homomorphic to an irreducible operator satisfying the Calabi-Yau condition (94), has the following properties: it is homomorphic to its adjoint with order-two intertwiners, it has a simple decomposition (99), and it is such that its exterior square necessarily has a rational solution. Conversely, showing that "irreducible order-four operators such that their exterior square have a rational solution, or, even, have a decomposition (99)" are necessarily equivalent to irreducible operators satisfying the Calabi-Yau condition (94) is a difficult question.

To illustrate the differential algebra structures corresponding to higher-order symmetric or exterior powers, we have also analysed some families of order-seven self-adjoint operators with exceptional differential Galois groups, where one sees, very clearly, the emergence of rational solutions for symmetric square and exterior cube of equivalent operators. Finally, since among the Derived From Geometry $n$-fold integrals ("Periods") occurring in physics, we have seen that they are quite often diagonals of rational functions [18, [19], we have also addressed many examples of (minimal order) operators annihilating diagonals of rational functions, and remarked that they have irreducible factors homomorphic to their adjoint.

The $n$-fold integrals we encounter in theoretical physics are solutions of PicardFuchs differential equations, or in a more modern mathematical language [61, 62], variation of Hodge structurest and Gauss-Manin systems [30, 22, 63, 64]. According to mathematicians one should necessarily have for such variation of Hodge structures, a "polarization" $\ddagger$ necessarily yielding to a "duality" which would send differential operators into their adjoint§. In our physical examples one seems to systematically inherit this "duality" on each factor of the minimal order operator, each irreducible factor being homomorphic to its adjoint. Along this line, section 6 strongly suggests to consider the conjecture that (minimal) operators annihilating diagonal of rational functions solutions, always factor into irreducible operators homomorphic to their

[^8]adjoint, may be on algebraic extensions (these factors thus corresponding to "special" differential Galois groups).

This paper tries to promote the idea that, before deciphering the obfuscation of mathematicians on this subject, physicists should, in a down-to-earth way, use all the differential algebra toolstit they have at their disposal, checking systematically if the linear differential operators they work on, have factors which are homomorphic to their adjoint, or are such that, up to operator equivalence, their exterior (resp. symmetric) square have a rational solution. The emergence of this "duality" on all the irreducible factors of a large class of differential operators of physics needs to be understood.

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## Appendix A. A decomposition of operators equivalent to operators satisfying the Calabi-Yau condition

Let us again consider an order-four operator $\Omega_{4}$ which satisfies the Calabi-Yau condition (94), of Wronskian $w(x)=u(x)^{2}$.

Let us consider a monic order-four operator $\tilde{\Omega}_{4}$ which is (non-trivially) equivalent to the order-four operator $\Omega_{4}$ satisfying the Calabi-Yau condition (94). This amounts to saying that there exist two (at most) order-three intertwiners $U_{3}$ and $L_{3}$

$$
\begin{equation*}
U_{3}=b_{3}(x) \cdot D_{x}^{3}+b_{2}(x) \cdot D_{x}^{2}+b_{1}(x) \cdot D_{x}+b_{0}(x), \tag{A.1}
\end{equation*}
$$

such that

$$
\begin{equation*}
\tilde{\Omega}_{4} \cdot U_{3}=L_{3} \cdot \Omega_{4} \tag{A.2}
\end{equation*}
$$

We choose $L_{3}$ such that $\tilde{\Omega}_{4}$ is monic $\left(\tilde{\Omega}_{4}=D_{x}^{4}+\cdots\right)$. Of course, we also have the (adjoint) relation:

$$
\begin{equation*}
\operatorname{adjoint}\left(U_{3}\right) \cdot \operatorname{adjoint}\left(\tilde{\Omega}_{4}\right)=\operatorname{adjoint}\left(\Omega_{4}\right) \cdot \operatorname{adjoint}\left(L_{3}\right) \tag{A.3}
\end{equation*}
$$

Furthermore, it is shown in Appendix B.1 that any operator satisfying the Calabi-Yau condition (94), is homomorphic to its adjoint, up to a conjugation by the square root of its Wronskian:

$$
\begin{equation*}
u(x) \cdot \operatorname{adjoint}\left(\Omega_{4}\right)=\Omega_{4} \cdot u(x) . \tag{A.4}
\end{equation*}
$$

Combining (A.2), (A.3) and (A.4) one straightforwardly deduces:

$$
\begin{equation*}
\tilde{\Omega}_{4} \cdot Y_{6}=\operatorname{adjoint}\left(Y_{6}\right) \cdot \operatorname{adjoint}\left(\tilde{\Omega}_{4}\right) \tag{A.5}
\end{equation*}
$$

where the order-six operator $Y_{6}$ reads:

$$
\begin{equation*}
Y_{6}=U_{3} \cdot u(x) \cdot \operatorname{adjoint}\left(L_{3}\right) . \tag{A.6}
\end{equation*}
$$

$\dagger \dagger$ DEtools in Maple.
$\ddagger$ Given $\Omega_{4}$ and $U_{3}$, in order to build the equivalent operator $\tilde{\Omega}_{4}$, just perform, in Maple, the LCLM of $\Omega_{4}$ and $U_{3}$, and, then, right-divide by $U_{3}$.

Let us introduce the two operators $N_{2}$ and $Z_{2}$ corresponding to the euclidean division of $Y_{6}$ by $\operatorname{adjoint}\left(\tilde{\Omega}_{4}\right)$ :

$$
\begin{equation*}
Y_{6}=N_{2} \cdot \operatorname{adjoint}\left(\tilde{\Omega}_{4}\right)+Z_{2} . \tag{А.7}
\end{equation*}
$$

$N_{2}$ is of course an order-two operator, but, noticeably, $Z_{2}$ is also an order-two operator instead of an order-three operator one could expect generically.

Furthermore, and noticeably, $N_{2}$ is an order-two self-adjoint operator such that:

$$
\begin{align*}
\frac{1}{b_{3}(x)} \cdot N_{2} \cdot \frac{1}{b_{3}(x)}= & u(x) \cdot\left(D_{x}^{2}-\frac{d \ln (1 / u(x))}{d x} \cdot D_{x}\right) \\
-u(x) \cdot & \left(\frac{d b_{2}(x)}{d x}+b_{2}(x)^{2}-2 b_{1}(x)\right) \\
& -u(x)-\frac{d^{2} u(x)}{d x^{2}}+\frac{2}{x} \cdot\left(\frac{d u(x)}{d x}\right)^{2} \tag{A.8}
\end{align*}
$$

A consequence of the self-adjoint character of $N_{2}$ is that one also has the "adjoint" relation of (A.7):

$$
\begin{equation*}
\operatorname{adjoint}\left(Y_{6}\right)=\tilde{\Omega}_{4} \cdot N_{2}+\operatorname{adjoint}\left(Z_{2}\right) \tag{A.9}
\end{equation*}
$$

Combining (A.5), A.7) and (A.9) one deduces the following homomorphisms of $\tilde{\Omega}_{4}$ with its adjoint, with an order-two intertwiner:

$$
\begin{equation*}
\tilde{\Omega}_{4} \cdot Z_{2}=\operatorname{adjoint}\left(Z_{2}\right) \cdot \operatorname{adjoint}\left(\tilde{\Omega}_{4}\right) \tag{A.10}
\end{equation*}
$$

Let us now perform the euclidean division of $\operatorname{adjoint}\left(\tilde{\Omega}_{4}\right)$ by $Z_{2}$ :

$$
\begin{equation*}
\operatorname{adjoint}\left(\tilde{\Omega}_{4}\right)=A_{2} \cdot Z_{2}+A_{0} \tag{A.11}
\end{equation*}
$$

where $A_{2}$ is an order-two operator and, surprisingly, $A_{0}$ is not an order-one operator, but a function (order zero). Of course (and using the fact that the adjoint of two even order operators is the sum of the adjoints) one also has the "adjoint relation" of (A.11), namely

$$
\begin{equation*}
\tilde{\Omega}_{4}=\operatorname{adjoint}\left(Z_{2}\right) \cdot \operatorname{adjoint}\left(A_{2}\right)+A_{0} \tag{A.12}
\end{equation*}
$$

In fact, and noticeably $A_{2}$ is a self-adjoint operator. Combining (A.10), A.11) and (A.12), one immediately deduces that $Z_{2}$ is conjugated to its adjoint, or equivalently, that the following order-two operator $Z_{2}^{s}$ is self-adjoint:

$$
\begin{equation*}
Z_{2}^{s}=A_{0} \cdot Z_{2}=\operatorname{adjoint}\left(Z_{2}\right) \cdot A_{0} \tag{A.13}
\end{equation*}
$$

One finds out that the order-four operator $\tilde{\Omega}_{4}$ can, in fact, be written in terms of a remarkable decomposition with two order-two self-adjoint operators:

$$
\begin{equation*}
\tilde{\Omega}_{4}=\quad Z_{2}^{s} \cdot \frac{1}{A_{0}} \cdot A_{2}+A_{0} \tag{A.14}
\end{equation*}
$$

One then deduces the homomorphisms of $\tilde{\Omega}_{4}$ with its adjoint:

$$
\begin{equation*}
A_{2} \cdot \frac{1}{A_{0}} \cdot \tilde{\Omega}_{4}=\operatorname{adjoint}\left(\tilde{\Omega}_{4}\right) \cdot \frac{1}{A_{0}} \cdot A_{2} . \tag{A.15}
\end{equation*}
$$

to be compared with

$$
\begin{equation*}
\tilde{\Omega}_{4} \cdot \frac{1}{A_{0}} \cdot Z_{2}^{s}=Z_{2}^{s} \cdot \frac{1}{A_{0}} \cdot \operatorname{adjoint}\left(\tilde{\Omega}_{4}\right) \tag{A.16}
\end{equation*}
$$

$\dagger$ One uses the fact that the adjoint of the sum of an order-six and an order-two operator is the sum of these adjoints.

Note that one also has a relation similar to A.5 but on $\Omega_{4}$. If one introduces

$$
\begin{equation*}
Z_{8}=\operatorname{adjoint}\left(L_{3}\right) \cdot \frac{1}{A_{0}} \cdot A_{2} \cdot U_{3} \cdot u(x) \tag{A.17}
\end{equation*}
$$

one has

$$
\begin{equation*}
\operatorname{adjoint}\left(Z_{8}\right) \cdot \Omega_{4} \cdot u(x)=\Omega_{4} \cdot u(x) \cdot Z_{8} \tag{A.18}
\end{equation*}
$$

where $\Omega_{4} \cdot u(x)$ also verifies the Calabi-Yau condition and is actually self-adjoint. Denoting $\Omega_{4}^{(s)}=\Omega_{4} \cdot u(x)$ this gives

$$
\begin{equation*}
\operatorname{adjoint}\left(Z_{8}\right) \cdot \Omega_{4}^{(s)}=\Omega_{4}^{(s)} \cdot Z_{8} \tag{A.19}
\end{equation*}
$$

One discovers that

$$
\begin{equation*}
Z_{8}=X_{4} \cdot \Omega_{4}^{(s)}-1 \tag{A.20}
\end{equation*}
$$

where $X_{4}$ is actually self-adjoint. One straightforwardly deduces

$$
\begin{align*}
& \operatorname{adjoint}\left(X_{4}\right) \cdot \operatorname{adjoint}\left(Z_{8}\right)= \\
& \quad X_{4} \cdot \operatorname{adjoint}\left(Z_{8}\right)=Z_{8} \cdot X_{4} . \tag{A.21}
\end{align*}
$$

If one does not normalize the order-four operators $\tilde{\Omega}_{4}$ and $\Omega_{4}$ to be monic, these results are easily modified, mutatis mutandis:

$$
\begin{aligned}
\Omega_{4} & \rightarrow \alpha(x) \cdot \Omega_{4}, & \tilde{\Omega}_{4} \rightarrow \beta(x) \cdot \tilde{\Omega}_{4}, & L_{3} \rightarrow \beta(x) \cdot L_{3} \cdot \alpha(x)^{-1} \\
u(x) & \rightarrow \alpha(x) \cdot u(x), & Y_{6} \rightarrow Y_{6} \cdot \beta(x), & Z_{2} \rightarrow Z_{2} \cdot \beta(x), \quad \text { (A.22) } \\
S Z_{2} & \rightarrow & \beta(x) \cdot S Z_{2} \cdot \beta(x), \quad A_{0} \rightarrow \beta(x) \cdot A_{0}, & \left(A_{2}, N_{2}, U_{3}\right) \rightarrow\left(A_{2}, N_{2}, U_{3}\right) .
\end{aligned}
$$

These calculations could have been performed, in another way, considering the other homomorphism relation between $\tilde{\Omega}_{4}$ and $\Omega_{4}$, instead of (A.2):

$$
\begin{equation*}
V_{3} \cdot \tilde{\Omega}_{4}=\Omega_{4} \cdot M_{3} \tag{A.23}
\end{equation*}
$$

and its "adjoint" relation:

$$
\begin{equation*}
\operatorname{adjoint}\left(\tilde{\Omega}_{4}\right) \cdot \operatorname{adjoint}\left(V_{3}\right)=\operatorname{adjoint}\left(M_{3}\right) \cdot \operatorname{adjoint}\left(\Omega_{4}\right) \tag{A.24}
\end{equation*}
$$

Combining (A.23), (A.24) and (A.4), one easily deduces:

$$
\begin{equation*}
\operatorname{adjoint}\left(\tilde{\Omega}_{4}\right) \cdot Z_{6}=\operatorname{adjoint}\left(Z_{6}\right) \cdot \tilde{\Omega}_{4} \tag{A.25}
\end{equation*}
$$

where:

$$
\begin{equation*}
Z_{6}=\operatorname{adjoint}\left(V_{3}\right) \cdot \frac{1}{u(x)} \cdot M_{3} . \tag{A.26}
\end{equation*}
$$

The rightdivision of this order-six operator by $\tilde{\Omega}_{4}$ yields:

$$
\begin{equation*}
Z_{6}=P_{2} \cdot \tilde{\Omega}_{4}+A_{2} \cdot \frac{1}{A_{0}} \tag{A.27}
\end{equation*}
$$

where $P_{2}$ is an order-two self-adjoint operator (much more involved than $N_{2} \ldots$ ).

## Appendix B. Calabi-Yau conditions

## Appendix B.1. Calabi-Yau conditions and self-adjoint operators

Since most of the linear differential operators of order four with polynomial coefficients, we have encountered in lattice statistical mechanics [17, 11, 13, 52, 16, 20, enumerative combinatorics, are globally nilpotent [8], and thus their Wronskian are $N$-th root of rational functions, let us write, without any loss of generality, the coefficient $a_{3}(x)$ of operator $\Omega_{4}$ (see (92)), in the log-derivative form:

$$
\begin{equation*}
a_{3}(x)=-\frac{d \ln (w(x))}{d x} \quad \text { with: } \quad w(x)=u(x)^{2} \tag{B.1}
\end{equation*}
$$

Global nilpotence, even being Fuchsian, corresponds to the Wronskian $w(x)=u(x)^{2}$ being $N$-th root of a rational function.

It is straightforward to verify that if $\Omega_{4}$ satisfies the Calabi-Yau condition (94), then:

$$
\begin{equation*}
u(x) \cdot \operatorname{adjoint}\left(\Omega_{4}\right)=\Omega_{4} \cdot u(x) \tag{B.2}
\end{equation*}
$$

In other words, the Calabi-Yau condition (94) necessarily means that the order-four operator (92) is, not only homomorphic to its adjoint, but conjugated to its adjoint. The following conjugate of $\Omega_{4}$ is self-adjoint:

$$
\begin{equation*}
\tilde{\Omega}_{4}=u(x)^{-1 / 2} \cdot \Omega_{4} \cdot u(x)^{1 / 2} \tag{B.3}
\end{equation*}
$$

This can be easily checked on all the ODEs of the large list of Calabi-Yau ODEs displayed in 5.

Conversely, let us impose that an order-four operator (92) is conjugated to its adjoint. Again it is straightforward to see that relation (B.3) yields:

$$
\begin{align*}
a_{1}(x)= & \frac{d a_{2}(x)}{d x}-\frac{a_{2}(x)}{u(x)} \cdot \frac{d u(x)}{d x}+\frac{1}{u(x)} \cdot \frac{d^{3} u(x)}{d x^{3}}  \tag{B.4}\\
& +6 \cdot\left(\left(\frac{1}{u(x)} \cdot \frac{d u(x)}{d x}\right)^{3}-\frac{1}{u(x)^{2}} \cdot \frac{d u(x)}{d x} \cdot \frac{d^{2} u(x)}{d x^{2}}\right)
\end{align*}
$$

which is nothing but the Calabi-Yau condition (94) taking into account (B.1).
In other words, the Calabi-Yau condition (94) is equivalent to say that an orderfour operator is conjugated to its adjoint. This result can also be found in Bogner (see [22]).

Appendix B.1.1. Strong Calabi-Yau conditions versus self-adjoint conditions on operators: higher order operators

An operator of order five is self-adjoint if it is of the form:

$$
\begin{align*}
L_{5}= & a_{5}(x) \cdot D_{x}^{5}+\frac{5}{2} \cdot \frac{d a_{5}(x)}{d x} \cdot D_{x}^{4}+a_{3}(x) \cdot D_{x}^{3} \\
& +\left(\frac{3}{2} \frac{d a_{3}(x)}{d x}-\frac{5}{2} \frac{d^{3} a_{5}(x)}{d x^{3}}\right) \cdot D_{x}^{2}+a_{1}(x) \cdot D_{x}  \tag{B.5}\\
& \quad+\left(\frac{1}{2} \cdot \frac{d a_{1}(x)}{d x}+\frac{1}{2} \cdot \frac{d^{5} a_{5}(x)}{d x^{5}}-\frac{1}{4} \cdot \frac{d^{3} a_{3}(x)}{d x^{3}}\right) .
\end{align*}
$$

Its symmetric square is of order fourteen instead of the order fifteen one could expect generically. In other words this (exactly) self-adjoint operator, or an orderfive operator conjugated of (B.5) by an arbitrary function, satisfies the symmetric Calabi-Yau condition (that its symmetric square is of order fourteen).

An operator of order six is self-adjoint if it is of the form:

$$
\begin{align*}
L_{6}= & a_{6}(x) \cdot D_{x}^{6}+3 \cdot \frac{d a_{6}(x)}{d x} \cdot D_{x}^{5}+a_{4}(x) \cdot D_{x}^{4} \\
& +\left(2 \cdot \frac{d a_{4}(x)}{d x}-5 \cdot \frac{d^{3} a_{6}(x)}{d x^{3}}\right) \cdot D_{x}^{3}+a_{2}(x) \cdot D_{x}^{2}  \tag{B.6}\\
& \quad+\left(\frac{d a_{2}(x)}{d x}-\frac{d^{3} a_{4}(x)}{d x^{3}}+3 \cdot \frac{d^{5} a_{6}(x)}{d x^{5}}\right) \cdot D_{x}+a_{0}(x)
\end{align*}
$$

Its exterior square is of order fourteen instead of the order fifteen one could expect generically. In other words this (exactly) self-adjoint operator, or an order-six operator conjugated of (B.6) by an arbitrary function, satisfies the Calabi-Yau condition (that its exterior square is of order fourteen), generalization to order six of the order-four Calabi-Yau condition (94).

It is straightforward to verify that an operator conjugated of a self-adjoint operator of order $N$ verifies, for any even order $N$, the generalization to order $N$ of the order-four Calabi-Yau condition (94) and for any odd order $N$, the generalization to order $N$ of the order-three symmetric Calabi-Yau condition (96).

Of course the reciprocal, which is true for order-three and four operators (see (97) and ( $\overline{\mathrm{B} .4})$ ), is not true for higher orders. For instance, let us introduce the order-five operator $M_{5}$ non-trivially homomorphic to the self-adjoint operator (B.5):

$$
\begin{align*}
& M_{5} \cdot D_{x}=\frac{1}{a_{5}(x)} \cdot\left(D_{x}-\frac{1}{W(x)} \cdot \frac{d W(x)}{d x}\right) \cdot L_{5}  \tag{B.7}\\
& \text { where: } \quad W(x)=2 \frac{d a_{1}(x)}{d x}-\frac{d^{3} a_{3}(x)}{d x^{3}}+2 \frac{d^{5} a_{5}(x)}{d x^{5}}
\end{align*}
$$

This operator also verifies the order-five symmetric Calabi-Yau condition (96): its symmetric square is also of order fourteen. This result generalizes with $M_{5}$

$$
\begin{equation*}
M_{5} \cdot\left(D_{x}+\rho(x)\right)=\frac{1}{a_{5}(x)} \cdot\left(D_{x}-z(x)\right) \cdot L_{5} \tag{B.8}
\end{equation*}
$$

where $z(x)$ is a quite involved rational expression of $a_{1}(x), a_{3}(x), a_{5}(x), \rho(x)$ and their derivatives.

Similarly the order-six operator $M_{6}$ equivalent of the self-adjoint operator (B.6):

$$
M_{6} \cdot D_{x}=\frac{1}{a_{6}(x)} \cdot\left(D_{x}-\frac{1}{a_{0}(x)} \cdot \frac{d a_{0}(x)}{d x}\right) \cdot L_{6},
$$

verifies the order-six Calabi-Yau condition (94): its exterior square is also of order fourteen.

This result generalizes with $M_{6}$ given by

$$
\begin{equation*}
M_{6} \cdot\left(D_{x}+\rho(x)\right)=\frac{1}{a_{6}(x)} \cdot\left(D_{x}-z(x)\right) \cdot L_{6} \tag{B.9}
\end{equation*}
$$

where $z(x)$ is a quite involved rational expression of $a_{0}(x), a_{2}(x), a_{4}(x), a_{6}(x), \rho(x)$ and their derivatives.

The order-seven operator $M_{7}$ equivalent of the order-seven self-adjoint operator $L_{7}$ :

$$
M_{7} \cdot\left(D_{x}^{2}+\rho_{1}(x) \cdot D_{x}+\rho_{2}(x)\right)=\frac{1}{a_{7}(x)} \cdot\left(D_{x}^{2}+c_{1}(x) \cdot D_{x}+c_{2}(x)\right) \cdot L_{7}
$$

verifies the order-seven Calabi-Yau condition that its symmetric square is of order 27 (instead of the generic order 28).

Similarly the order-eight operator $M_{8}$ equivalent of the order-seven self-adjoint operator $L_{8}$ :

$$
M_{8} \cdot\left(D_{x}^{2}+\rho_{1}(x) \cdot D_{x}+\rho_{2}(x)\right)=\frac{1}{a_{7}(x)} \cdot\left(D_{x}^{2}+c_{1}(x) \cdot D_{x}+c_{2}(x)\right) \cdot L_{8}
$$

verifies the order-seven symmetric Calabi-Yau condition that its symmetric square is of order 27 (instead of the generic order 28). These last results can easily be generalized. For instance for the order-nine and order-ten self-adjoint operators $L_{9}$, $L_{10}$ the corresponding equivalent operators $M_{9}, M_{10}$ obtained from the LCLM of $L_{9}$ or $L_{10}$ with an order-three operator verify respectively the order-nine symmetric Calabi-Yau condition (namely the symmetric square of $M_{9}$ is of order 44 instead of $45)$ and the order-ten Calabi-Yau condition (namely that the symmetric and exterior squares of $M_{9}$ and $M_{10}$ are of order 44 instead of 45 ), and so on $\ldots$

## Appendix B.2. Equivalence of operators satisfying the Calabi-Yau conditions

Let us recall some examples (see equation (O.37) in [19]) of operators satisfying the Calabi-Yau conditions (94). The order-four ( $\mu$-dependent) linear differential operator

$$
\begin{align*}
\mathcal{C}(\mu)= & 16 \cdot \theta^{2} \cdot(\theta-1)^{2}  \tag{B.10}\\
& -x \cdot(2 \theta+1-\mu) \cdot(2 \theta+1+\mu) \cdot(2 \theta-1-\mu) \cdot(2 \theta-1+\mu),
\end{align*}
$$

is such that its exterior square is actually of order five. These operators are irreducible for even integer values of $\mu$, but factor in a product of order-two and two order-one operators for odd integer values of $\mu$. Operator (B.10) has simple hypergeometric solutions for any value of $\mu$ :

$$
\begin{equation*}
x \cdot{ }_{4} F_{3}\left(\left[\frac{\mu+3}{2}, \frac{-\mu+3}{2}, \frac{\mu+1}{2}, \frac{-\mu+1}{2}\right],[1,2,2] ; x\right) . \tag{B.11}
\end{equation*}
$$

These operators ( (B.10) are, for different even integer values of $\mu$, non-trivially homomorphic (and similarly, these operators (B.10) are, non-trivially homomorphic for different odd integer values of $\mu$ ). One of the simplest examples of homomorphism reads:

$$
\begin{array}{ll}
\mathcal{C}(0) \cdot U_{3}=V_{3} \cdot \mathcal{C}(2), & \text { with: } \\
U_{3}=\left(32 \theta^{3}-80 \theta^{2}+72 \theta-27\right) & -4 \cdot x \cdot(2 \theta+1)(2 \theta-1)(2 \theta-3), \\
V_{3}=\left(32 \theta^{3}-80 \theta^{2}+72 \theta-27\right) & -4 \cdot x \cdot(2 \theta+1)(2 \theta-1)^{2}
\end{array}
$$

More generally, for $\mu=2 N$ one has

$$
\begin{equation*}
\mathcal{C}(0) \cdot U_{3}=V_{3} \cdot \mathcal{C}(2 N) \tag{B.12}
\end{equation*}
$$

in the order-three operators $U_{3}$ and $V_{3}$, the degree in $x$ being $N$. We have an infinite number of equivalent operators satisfying the (strong) Calabi-Yau condition. This is a consequence of the following homomorphism between $\mathcal{C}(\mu)$ and $\mathcal{C}(\mu+2)$ valid for any value of $\mu$ ( $\mu$ being not even a rational number):

$$
\begin{equation*}
\mathcal{C}(\mu) \cdot X_{3}=Y_{3} \cdot \mathcal{C}(\mu+2) \tag{B.13}
\end{equation*}
$$

where $X_{3}$ reads

$$
\begin{gather*}
4 x \cdot\left(2\left(3 \mu^{2}+6 \mu-1\right) \cdot \theta+4 \mu^{3}+15 \mu^{2}+14 \mu-1\right)(2 \theta-\mu-1)(2 \theta-\mu-3) \\
-32\left(3 \mu^{2}+6 \mu-1\right) \cdot \theta^{3}+16\left(2 \mu^{3}+15 \mu^{2}+20 \mu-5\right) \cdot \theta^{2} \\
-8\left(\mu^{2}+2 \mu-1\right)(\mu+3)^{2} \cdot \theta+(\mu+3)^{3}\left(\mu^{2}-1\right), \tag{B.14}
\end{gather*}
$$

and $Y_{3}$ reads:

$$
\begin{gather*}
4 x \cdot\left(2\left(3 \mu^{2}+6 \mu-1\right) \cdot \theta+4 \mu^{3}+9 \mu^{2}+2 \mu+1\right)(2 \theta-\mu+1)(2 \theta-\mu-1) \\
-32\left(3 \mu^{2}+6 \mu-1\right) \cdot \theta^{3}+16\left(2 \mu^{3}+15 \mu^{2}+20 \mu-5\right) \cdot \theta^{2} \\
-8\left(\mu^{2}+2 \mu-1\right)(\mu+3)^{2} \cdot \theta+(\mu+3)^{3}\left(\mu^{2}-1\right), \tag{B.15}
\end{gather*}
$$

the two order-three operators being themselves homomorphid
To sum-up: This shows that a solution of the reduction (by equivalence of operators) of operators satisfying the weak Calabi-Yau condition to operators satisfying the (strong) Calabi-Yau condition is not unique since one can find, for some examples, an infinite number of homomorphic irreducible operators satisfying the (strong) Calabi-Yau condition.

## Appendix B.2.1. Decomposition

Let us consider, for instance, the order-four operator obtained by the rightdivision by $\theta$ of the LCLM of $\mathcal{C}(\mu)$ and $\theta$. This operator $\mathcal{M}(\mu)$ satisfies the weak CalabiYau condition: its exterior square has a rational solution $1 /(x-1)$ independent of $\mu$, and homomorphic to the infinite number of operators $\mathcal{C}(\mu+2 N)$ ( $N$ is any integer), satisfying the (strong) Calabi-Yau condition that their exterior square is of order five.

One has a decomposition similar to (100), namely:

$$
\begin{array}{rlr}
\mathcal{M}(\mu)=L_{2} \cdot a(x) \cdot M_{2}+\frac{\left(\mu^{2}-1\right)^{2}}{16 a(x)}, & a(x)=x^{3} \cdot(x-1), \\
L_{2} & =\frac{1}{(x-1)^{4} x^{5}} \cdot(x \cdot(\theta-4)-(\theta-2)) \cdot(x \cdot(\theta-3)-(\theta-2)), \\
M_{2} & =\frac{1}{x^{2}} \cdot\left(x \cdot\left(\theta^{2}-\frac{\mu^{2}+1}{2}\right)-\theta \cdot(\theta-1)\right), \tag{B.16}
\end{array}
$$

where $L_{2}$ and $M_{2}$ are self-adjoint operators.
Let us consider, instead of $\mathcal{C}(\mu)$, the self-adjoint operator $C_{s}(\mu)=1 / x^{2} \cdot \mathcal{C}(\mu)$, which satisfies the Calabi-Yau condition and instead of $\mathcal{M}(\mu)$ the operator $M(\mu)=$ $a(x) \cdot \mathcal{M}(\mu)$ :

$$
\begin{equation*}
M(\mu)=-16 \cdot a(x) \cdot L_{2} \cdot a(x) \cdot M_{2}-\left(\mu^{2}-1\right)^{2} \tag{B.17}
\end{equation*}
$$

One has the following homomorphism between $C_{s}(\mu)$ and the adjoint of $M(\mu)$ :

$$
\begin{equation*}
(\theta+1) \cdot \operatorname{adjoint}(M(\mu))=C_{s}(\mu) \cdot x \cdot(\theta+1) \tag{B.18}
\end{equation*}
$$

One notes that the exterior square of $\operatorname{adjoint}(M(\mu))$, as well as the exterior square of $(\theta+1) \cdot \operatorname{adjoint}(M(\mu))$, have the rational solution $1 / x^{3}$. This is in agreement with the fact that the exterior square of $\operatorname{adjoint}(\mathcal{M}(\mu))$ is nothing but the Wronskian of the self-adjoint operator $L_{2}$, namely, $x^{3} \cdot(x-1)^{2}=a(x)^{2} / x^{3}$. One thus sees that, if the exterior square of $C_{s}(\mu)$ has no rational solution, $C_{s}(\mu) \cdot x \cdot(\theta+1)$ has a rational solution, namely $1 / x^{3}$.

## Appendix B.3. Calabi-Yau conditions preserved by the formal adjoint

Let us consider a (monic) order-four operator with a rational (resp. $N$-th root of rational) Wronskian $W(x)$ :

$$
\Omega_{4}=D_{x}^{4}-\frac{d \ln (W(x))}{d x} \cdot D_{x}^{3}+a_{2}(x) \cdot D_{x}^{2}+a_{1}(x) \cdot D_{x}+a_{0}(x)
$$

$\dagger$ We have the following "tower" of homomorphisms: $X_{3} \cdot X_{2}=Y_{2} \cdot Y_{3}, X_{2} \cdot X_{1}=Y_{1} \cdot Y_{2}, X_{1}$. $(x-1)=(x-1) \cdot Y_{1}$.

One has the following conjugation relation between the exterior square of this operator and the exterior square of its adjoint:

$$
\begin{equation*}
W(x) \cdot \operatorname{Ext}^{2}\left(\operatorname{adjoint}\left(\Omega_{4}\right)\right)=\operatorname{Ext}^{2}\left(\Omega_{4}\right) \cdot W(x) \tag{B.19}
\end{equation*}
$$

From this conjugation relation it is straightforward to deduce that if $\Omega_{4}$ satisfies the Calabi-Yau condition (94), i.e. its exterior square is of order five instead of six, the exterior square of its adjoint will also be of order five: the Calabi-Yau condition (94) is thus preserved by the adjoint transformation. From this conjugation relation (B.19) it is also straightforward to deduce that if the operator satisfies the weak Calabi-Yau condition, i.e. its exterior square has a rational (resp. $N$-th root of rational) solution, this will also be the case for its adjoint: the exterior square of the adjoint of this operator will also have a rational solution, which is nothing but the previous rational (resp. $N$-th root of rational) solution divided by $W(x)$ the Wronskian of the operator. In other words, the weak Calabi-Yau condition is preserved by the adjoint.

Appendix B.3.1. Decomposition of order-four operators
Let us consider an operator $M_{4}$ of the form (100) where $L_{2}$ and $M_{2}$ are two (general) self-adjoint operators (101) and (102). Note that $a(x) \cdot M_{2} \cdot a(x)$ is also a self-adjoint operator, so up to an overall factor $\rho(x)$, one can consider, without any restriction, the form

$$
\begin{equation*}
M_{4}=M_{2} \cdot L_{2}+\lambda, \tag{B.20}
\end{equation*}
$$

where $L_{2}$ and $M_{2}$ are two (general) self-adjoint operators (see (101) and (102)).
The exterior square of an order-four operator of the form (B.20) (up to an overall factor $\rho(x)$ ) has the rational solution $1 / \alpha_{2}(x)$. Switching to the adjoint amounts to permuting the two self-adjoint operators $L_{2}$ and $M_{2}$. The order-four operator $M_{4}$ is not monic. Denoting $M_{4}^{(u)}$ the order-four operator $M_{4}$ in a monic form we have $M_{4}=\rho(x) \cdot M_{4}^{(u)}$ with $\rho(x)=\alpha_{2}(x) \cdot \beta_{2}(x)$. Thus one has the following relations between their adjoints:

$$
\begin{align*}
& \operatorname{adjoint}\left(M_{4}\right)=\text { adjoint }\left(M_{4}^{(u)}\right) \cdot \rho(x), \\
& \begin{aligned}
\operatorname{Ext}^{2}\left(\operatorname{adjoint}\left(M_{4}\right)\right)= & \operatorname{Ext}^{2}\left(\operatorname{adjoint}\left(M_{4}^{(u)}\right) \cdot \rho(x)\right) \\
& =\frac{1}{\rho(x)^{2}} \cdot \operatorname{Ext}^{2}\left(\operatorname{adjoint}\left(M_{4}^{(u)}\right)\right) \cdot \rho(x)^{2} .
\end{aligned}
\end{align*}
$$

One has the previous relation (B.19) between the exterior square of the monic operator, its Wronskian, and the adjoint of the monic operator:

$$
\begin{align*}
& W(x) \cdot \operatorname{Ext}^{2}\left(\operatorname{adjoint}\left(M_{4}^{(u)}\right)\right)=\operatorname{Ext}^{2}\left(M_{4}^{(u)}\right) \cdot W(x), \\
& \qquad W(x)=\frac{\beta_{2}(x)}{\rho(x)^{2} \cdot \alpha_{2}(x)}=\frac{1}{\beta_{2}(x) \cdot \alpha_{2}(x)^{3}}, \tag{B.22}
\end{align*}
$$

which can be rewritten on the exterior square of $M_{4}$ and its adjoint:

$$
\begin{equation*}
\frac{\beta_{2}(x)}{\alpha_{2}(x)} \cdot \operatorname{Ext}^{2}\left(\operatorname{adjoint}\left(M_{4}\right)\right)=\operatorname{Ext}^{2}\left(M_{4}\right) \cdot \frac{\beta_{2}(x)}{\alpha_{2}(x)} . \tag{B.23}
\end{equation*}
$$

Since the exterior square of $M_{4}$ of the form (B.20) has the rational solution $1 / \alpha_{2}(x)$, the last relation (B.23) is compatible with the fact that the exterior square of the adjoint of $M_{4}$ (of the form (B.20)) has the rational solution $1 / \beta_{2}(x)$.
$\dagger$ Since by definition the exterior square of an operator is normalized to be a monic operator.

Remark: Let us consider an operator $C_{4}$ with head coefficient $A_{4}(x)$ and Wronskian $W(x)$, satisfying the (strong) Calabi-Yau condition that its exterior square is of order five. If one has an intertwining relation

$$
M_{4} \cdot \tilde{R}(x) \cdot\left(D_{x}-\frac{d \ln (R(x))}{d x}\right)=\tilde{S}(x) \cdot\left(D_{x}-\frac{d \ln (S(x))}{d x}\right) \cdot C_{4}
$$

one has the simple relations:

$$
\begin{align*}
\tilde{S}(x) & =\frac{\beta_{2}(x) \cdot a(x)^{2} \cdot \alpha_{2}(x) \cdot \tilde{R}(x)}{A_{4}(x)} \\
W(x) & =\frac{A_{4}(x) \cdot R(x)}{\tilde{R}(x)^{4} \cdot \beta_{2}(x) \cdot \alpha_{2}(x)^{3} \cdot a(x)^{2} \cdot S(x)} \tag{B.24}
\end{align*}
$$

Therefore there are some simple relations between $\alpha_{2}(x)$ and $\beta_{2}(x)$ that will correspond to the rational solutions of the exterior square of $M_{4}$, or of the adjoint of $M_{4}$.

Appendix B.4. Rational solutions for the exterior square of operators satisfying the weak Calabi-Yau condition

The fact that an order-four operator satisfying the weak Calabi-Yau condition, namely having a decomposition of the form (100), is such that its exterior square has a rational solution which is the inverse of the head coefficient of the right most order-two selfadjoint operator is, in fact, the consequence of an identity on the difference of two exterior squares.

A non-trivial identity exists between the difference of the following two exterior squares:

$$
\begin{align*}
& \operatorname{Ext}^{2}\left(L_{2} \cdot M_{2}+\lambda\right)-\operatorname{Ext}^{2}\left(L_{2} \cdot M_{2}\right)=  \tag{B.25}\\
& -4 \cdot \frac{\lambda}{\alpha_{2}(x) \beta_{2}(x)} \cdot\left(D_{x}-\frac{1}{2} \cdot \frac{d \ln \left(\rho_{L}(x)\right)}{d x}\right) \cdot\left(D_{x}-\frac{1}{2} \cdot \frac{d \ln \left(\rho_{R}(x)\right)}{d x}\right), \quad \text { where: } \\
& \rho_{L}(x)=\frac{\mathcal{P}^{2}}{\alpha_{2}(x)^{2} \cdot \beta_{2}(x)^{2} \cdot w_{L}(x)^{5} \cdot w_{M}(x)^{5}}, \quad \rho_{R}(x)=\frac{w_{L}(x) \cdot w_{M}(x) \cdot \alpha_{2}(x)}{\beta_{2}(x)}
\end{align*}
$$

where $\mathcal{P}$ is a slightly involved polynomial expression, on exterior square of product of the two (not necessarily self-adjoint) operators

$$
\begin{align*}
& L_{2}=\alpha_{2}(x) \cdot\left(D_{x}^{2}-\frac{d \ln \left(w_{L}(x)\right.}{d x} \cdot D_{x}\right)+\alpha_{0}(x)  \tag{B.26}\\
& M_{2}=\beta_{2}(x) \cdot\left(D_{x}^{2}-\frac{d \ln \left(w_{M}(x)\right.}{d x} \cdot D_{x}\right)+\beta_{0}(x) \tag{B.27}
\end{align*}
$$

For self-adjoint operators like (101) and the decomposition (100), one has $\alpha_{2}(x)=$ $a(x) / w_{L}(x)$ and $\beta_{2}(x)=a(x) / w_{M}(x), \rho_{R}=w_{M}(x)^{2}$. Furthermore it is simple to see that the exterior square of the product $L_{2} \cdot M_{2}$ has the Wronskian of $M_{2}$ as a solution. Therefore identity (B.25) means that the exterior square of $L_{2} \cdot M_{2}+\lambda$ has $w_{M}(x)$ the Wronskian of $M_{2}$ as a solution.
$\ddagger$ Quadratic in $\alpha_{2}(x)$ and $\beta_{2}(x)$, linear in $\alpha_{0}(x)$ and $\beta_{0}(x)$, and polynomial in $w_{L}(x)$ and $w_{M}(x)$ and their derivatives up to third derivative.

## Appendix C. Analysis of the order-seven operators associated with the exceptional Galois group $G_{2}(C)$

## Appendix C.1. Solution-series of the order-seven operators

The solution-series $y_{0}^{(n)}$, analytic at $x=0$, of the order-seven operators $E_{n}$, $n=1,2 \cdots$, given in section (5), are actually series with integer coefficients and read respectively

$$
\begin{aligned}
& y_{0}^{(1)}=1+2688 x+19707264 x^{2}+191647334400 x^{3}+2133255623587200 x^{4} \\
& +25707449648409919488 x^{5}+\cdots \\
& y_{0}^{(2)}=1+384 x+537984 x^{2}+1097318400 x^{3}+2680866518400 x^{4} \\
& +7283382738960384 x^{5}+\cdots \\
& y_{0}^{(3)}=1+1512 x+9885240 x^{2}+95782780800 x^{3}+1117658718099000 x^{4} \\
& +14536396497887776512 x^{5}+\cdots \\
& y_{0}^{(4)}=1+14976 x+1254798720 x^{2}+159551671910400 x^{3} \\
& +24603126146687088000 x^{4}+4241337041632715022974976 x^{5}+\cdots \\
& y_{0}^{(5)}=1+2678400 x+65172299068800 x^{2}+2494516941707677286400 x^{3} \\
& +116986156694543894801624380800 x^{4} \\
& +6160069364202852097613676563979878400 x^{5}+\cdots
\end{aligned}
$$

These order-seven linear differential operators are globally nilpotent [8, 40, the Jordan reduction of their $p$-curvature 40, 8 reading:

$$
J_{7}=\left[\begin{array}{lllllll}
0 & 1 & 0 & 0 & 0 & 0 & 0  \tag{C.2}\\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]
$$

of characteristic polynomial equal to the minimal polynomial $P(\lambda)=\lambda^{7}$.
These operators are MUM, so they have the traditional "triangular log structure" [20], the formal series solutions with a log being of the form $y_{1}^{(n)}=$ $y_{0}^{(n)} \cdot \ln (x)+\tilde{y}_{1}^{(n)}, y_{2}^{(n)}=y_{0}^{(n)} \cdot \ln (x)^{2} / 2+\tilde{y}_{1}^{(n)} \cdot \ln (x)+\tilde{y}_{2}^{(n)}$, etc. The corresponding nomes (called "special coordinates" in [29]) are defined as $q^{(n)}=\exp \left(y_{1}^{(n)} / y_{0}^{(n)}\right)$ $=x \cdot \exp \left(\tilde{y}_{1}^{(n)} / y_{0}^{(n)}\right)$. These nomes correspond to series with integer coefficients, and read respectively:

$$
\begin{aligned}
q^{(1)}=x & +7040 x^{2}+67555904 x^{3}+747082784768 x^{4} \\
& +8968272297124128 x^{5}+\cdots, \\
q^{(2)}=x & +1152 x^{2}+2150976 x^{3}+4983447552 x^{4} \\
& +13054714896672 x^{5}+\cdots,
\end{aligned}
$$

$$
\begin{align*}
q^{(3)}=x & +5562 x^{2}+49552317 x^{3}+547802062578 x^{4} \\
& +6855142017357054 x^{5}+\cdots, \\
q^{(4)}=x & +72576 x^{2}+8462979648 x^{3}+1230038144557056 x^{4} \\
& +203018472128017391904 x^{5}+\cdots, \\
q^{(5)}=x & +20200320 x^{2}+689499895026240 x^{3} \\
& +29916247864887732510720 x^{4}  \tag{C.3}\\
& +1488739080271271648779215102240 x^{5}+\cdots
\end{align*}
$$

Appendix C.1.1. Yukawa couplings
The Yukawa couplings

$$
\begin{equation*}
K(q)=\left(q \cdot \frac{d}{d q}\right)^{2}\left(\frac{y_{2}}{y_{0}}\right) \tag{C.4}
\end{equation*}
$$

of the five order-seven linear differential operators of section (5) are series with integer coefficients. Their expansion read respectively:

$$
\left.\begin{array}{rl}
K^{(1)}=1 & +768 q-2188032 q^{2}+2883403776 q^{3}-1360234636032 q^{4} \\
& \quad-3787008084959232 q^{5}+\cdots \\
K^{(2)}=1 & +256 q+728320 q^{2}+1640611840 q^{3}+3618799525120 q^{4} \\
& +8043817914720256 q^{5}+\cdots
\end{array}\right\} \begin{aligned}
& K^{(3)}=1+1485 q+19708515 q^{2}+206970715890 q^{3} \\
&+2188620549305955 q^{4}+23409935555891063985 q^{5}+\cdots \\
& K^{(4)}=1+29440 q+4438662400 q^{2}+621410936504320 q^{3} \\
&+88605364227964837120 q^{4}+12835248124604913684029440 q^{5}+\cdots \\
& K^{(5)}=1+17342208 q+687629971954944 q^{2} \\
&+30848876097264182771712 q^{3} \\
&+1428770297588004620323742981376 q^{4} \\
&+67528440221394152640448454407310942208 q^{5}+\cdots
\end{aligned}
$$

These Yukawa couplings, in the $x$ variable, read respectively:

$$
\begin{aligned}
& K^{(1)}=1+768 x+3218688 x^{2}+23958847488 x^{3} \\
& +229225505561856 x^{4}+2508123114368335872 x^{5}+\cdots \\
& K^{(2)}=1+256 x+1023232 x^{2}+3869310976 x^{3} \\
& +14664270683392 x^{4}+56048323595665408 x^{5}+\cdots \\
& K^{(3)}=1+1485 x+27968085 x^{2}+499793427495 x^{3} \\
& +9018524688844995 x^{4}+164714785807791646845 x^{5}+\cdots \\
& K^{(4)}=1+29440 x+6575299840 x^{2}+1514841782026240 x^{3} \\
& +358624525635384843520 x^{4} \\
& +86502979031531419474001920 x^{5}+\cdots
\end{aligned}
$$

$$
\begin{aligned}
K^{(5)}= & 1+17342208 x+1037948123061504 x^{2} \\
& +70587017642949191073792 x^{3} \\
& \quad+5045886607522553002548393221376 x^{4} \\
& \quad+370665145887525483931062348265527902208 x^{5}+\cdots
\end{aligned}
$$

Recalling our results in [18, 19, one can, for higher order-operators, define several Yukawa couplings from Wronskian-like determinants of the solutions, instead of a only one (C.4) for order-four operators:

$$
\begin{gather*}
K_{3}=\frac{W_{1}^{3} \cdot W_{3}}{W_{2}^{3}}, \quad K_{4}=\frac{W_{1}^{8} \cdot W_{4}}{W_{2}^{6}}, \quad K_{5}=\frac{W_{1}^{15} \cdot W_{5}}{W_{2}^{10}}, \\
K_{6}=\frac{W_{1}^{24} \cdot W_{6}}{W_{2}^{15}}, \quad K_{7}=\frac{W_{1}^{35} \cdot W_{7}}{W_{2}^{21}} . \tag{C.5}
\end{gather*}
$$

The well-known Yukawa coupling (C.4) is denoted $K_{3}$ in the previous set of "higher orders" Yukawa couplings (C.5) (see Appendix C. 1 in [18]).

One remarks the following non-trivial identities for the five order-seven operators $E_{i}$ :

$$
\begin{gather*}
K_{4}^{(i)}=\left(K_{3}^{(i)}\right)^{2}, \quad K_{5}^{(i)}=\left(K_{3}^{(i)}\right)^{3}, \quad K_{6}^{(i)}=\left(K_{3}^{(i)}\right)^{5}, \\
K_{7}^{(i)}=\left(K_{3}^{(i)}\right)^{7}, \tag{C.6}
\end{gather*} \quad i=1, \cdots, 5 .
$$

Therefore, for the five $E_{i}$ 's, these various invariants just reduce to the unique Yukawa coupling $K_{3}$. These relations correspond respectively to the identities on the Wronskian-like determinants $W_{n}$ :

$$
\begin{aligned}
W_{1}^{2} \cdot W_{4}= & W_{3}^{2}, \quad W_{1}^{6} \cdot W_{5}=W_{2} \cdot W_{3}^{3}, \quad W_{1}^{9} \cdot W_{6}=W_{3}^{5} \\
& W_{1}^{14} \cdot W_{7}=W_{3}^{7}
\end{aligned}
$$

It had been seen (see (C.17) in [18, [19), for order-four operators, that being self-adjoint up to a conjugation which yields that the Yukawa coupling $K_{3}$ is equal to the Yukawa coupling for the adjoint $K_{3}^{*}$, is nothing but relation $K_{4}=K_{3}^{2}$, namely $W_{1}^{2} \cdot W_{4}=W_{3}^{2}$.

These relations are no longer valid for non self-adjoint (up to conjugation) orderseven operators. For order-seven operators taking the adjoint operator amounts to performing the following involutive transformation on the $W_{n}$ 's $\left(W_{0}=1\right)$ :

$$
\begin{equation*}
W_{n} \quad \longleftrightarrow \quad W_{n}^{*}=\frac{W_{7-n}}{W_{7}}, \quad W_{n} \quad \longleftrightarrow \quad W_{7}^{*}=\frac{1}{W_{7}} \tag{C.7}
\end{equation*}
$$

and, consequently, the Yukawa couplings (C.5) for the adjoint operator read:

$$
\begin{array}{rlrl}
K_{3}^{*} & =\frac{W_{6}^{3} \cdot W_{4}}{W_{7} \cdot W_{5}^{3}}=\frac{K_{6}^{3} \cdot K_{4}}{K_{5}^{3} \cdot K_{7}}, & K_{4}^{*}=\frac{W_{6}^{8} \cdot W_{3}}{W_{7}^{3} \cdot W_{5}^{6}}=\frac{K_{6}^{8} \cdot K_{3}}{K_{5}^{6} \cdot K_{7}^{3}} \\
K_{5}^{*} & =\frac{W_{6}^{15} \cdot W_{2}}{W_{7}^{6} \cdot W_{5}^{10}}=\frac{K_{6}^{15}}{K_{5}^{10} \cdot K_{7}^{6}}, & K_{6}^{*}=\frac{W_{6}^{24} \cdot W_{1}}{W_{7}^{10} \cdot W_{5}^{15}}=\frac{K_{6}^{24}}{K_{5}^{15} \cdot K_{7}^{10}}, \\
K_{7}^{*} & =\frac{W_{6}^{35}}{W_{7}^{15} \cdot W_{5}^{21}}=\frac{1}{K_{7}} . & & \tag{C.8}
\end{array}
$$

Note, for the order-seven operator $\hat{E}_{1}^{(1)}$ (obtained from $\hat{E}_{1}$ by taking the LCLM with $D_{x}$ and rightdividing by $D_{x}$ ), the Yukawa couplings (C.5), as well as these adjoint Yukawa couplings (C.8), as well as the $W_{n}$ 's, as well as the $W_{n}$ 's of the adjoint operators, are still globally bounded (see [18, 19]).

Do note that the relations (C.6) are such that $K_{3}^{*}=K_{3}, K_{4}^{*}=K_{4}, K_{5}^{*}=K_{5}$, $K_{6}^{*}=K_{6}$.

Taking into account the transformations of the $W_{n}$ 's by a pullback $p(x)$ one finds (with $v=p^{\prime}$ ):

$$
\begin{align*}
& \left(W_{1}, W_{2}, W_{3}, W_{4}, W_{5}, W_{6}, W_{7}\right) \quad \longrightarrow  \tag{C.9}\\
& \quad\left(W_{1}, v \cdot W_{2}, v^{3} \cdot W_{3}, v^{6} \cdot W_{4}, v^{10} \cdot W_{5}, v^{15} \cdot W_{6}, v^{21} \cdot W_{7}\right)
\end{align*}
$$

together with the obvious homogeneity

$$
\begin{align*}
& \left(W_{1}, W_{2}, W_{3}, W_{4}, W_{5}, W_{6}, W_{7}\right) \longrightarrow  \tag{C.10}\\
& \quad\left(u \cdot W_{1}, u^{2} \cdot W_{2}, u^{3} \cdot W_{3}, u^{4} \cdot W_{4}, u^{5} \cdot W_{5}, u^{6} \cdot W_{6}, u^{7} \cdot W_{7}\right)
\end{align*}
$$

one finds, if one seeks for invariants of the form $W_{1}^{a} \cdot W_{n} / W_{2}^{b} / W_{3}^{c}$ that the only invariants compatible with these symmetries are the (C.5) together with $\downarrow$

$$
\begin{equation*}
\frac{K_{4}}{K_{3}^{2}}, \quad \frac{K_{5}}{K_{3}^{3}}, \quad \frac{K_{6}}{K_{3}^{5}}, \quad \frac{K_{7}}{K_{3}^{7}} \tag{C.11}
\end{equation*}
$$

## Appendix D. Exceptional Galois groups: two and three parameter operators

Appendix D.1. Exceptional Galois groups: two-parameter deformation of $E_{1}$
Let us introduce the following two-parameter deformation of $E_{1}$ (it amounts to changing $F(x) \rightarrow F(x) \cdot(x /(1-x))^{1 / 2}$ in the $P_{2}$ of 30] then changing $\left.x \rightarrow(x-1) / x\right)$ :

$$
\begin{align*}
& \Omega(p, q)=\theta \cdot\left(\theta^{2}-p^{2}\right) \cdot\left(\theta^{2}-q^{2}\right) \cdot\left(\theta^{2}-(p+q)^{2}\right) \\
& -128 \cdot x \cdot\left(\left(48 \theta^{4}+96 \theta^{3}+124 \theta^{2}+76 \theta+21\right)(2 \theta+1)^{3}+16 \Sigma^{2} \cdot(2 \theta+1)^{2}\right. \\
& \left.\quad-8 \Sigma \cdot\left(8 \theta^{2}+8 \theta+5\right) \cdot(2 \theta+1)^{2}-64 p^{3} q^{3}\right) \\
& \quad+4194304 \cdot x^{2} \cdot(\theta+1) \cdot\left(12(\theta+1)^{2}+11-8 \Sigma\right) \cdot(2 \theta+1)^{2} \cdot(2 \theta+3)^{2} \\
& -34359738368 \cdot x^{3} \cdot(2 \theta+5)^{2} \cdot(2 \theta+1)^{2} \cdot(2 \theta+3)^{3}, \tag{D.1}
\end{align*}
$$

where $\Sigma=p^{2}+p q+q^{2}$. In the $p=q=0$ limit the previous two-parameter order-seven operator reduces to $\hat{E}_{1}$.

For arbitrary values of $p$ and $q$ the operator $x^{-1} \cdot \Omega(p, q)$ is actually self-adjoint. Note that $\Omega(p, 0)$ and $\Omega(p+1,0)$ are homomorphic with an order-six intertwiner for any value of $p$. Also note that $\Omega(p, 1 / 2)$ and $\Omega(p+r, 1 / 2)$ are homomorphic with an order-six intertwiner for any value of $p$, for $r=1 / 2,1 / 3,1 / 6,1 / 11$.
$\dagger$ These ratio being equal to 1 in our case.

Appendix D.2. Exceptional Galois groups: families with three parameters
Let us consider the following order-seven operator depending on three parameters $a$, $c, d$, (here $\sigma$ denotes $b^{2}+b c+c^{2}$ ):

$$
\begin{align*}
& \Omega_{a, b, c}= \theta \cdot\left(\theta^{2}-b^{2}\right) \cdot\left(\theta^{2}-c^{2}\right) \cdot\left(\theta^{2}-(b+c)^{2}\right) \\
&- x \cdot(2 \theta+1) \cdot(\theta+a) \cdot(\theta+1-a) \cdot\left(\theta \cdot(\theta+1) \cdot\left(\theta^{2}+\theta+1-\sigma\right)\right. \\
&\left.+2 a \cdot(1-a) \cdot\left(\theta^{2}+\theta+1-\sigma-a \cdot(1-a)\right)\right) \\
&+ x^{2} \cdot(\theta+1) \cdot(\theta+a) \cdot(\theta+1-a) \cdot(\theta+a+1) \cdot(\theta+(1-a)+1) \\
& \quad \times(\theta+2 a)(\theta+2 \cdot(1-a)) . \tag{D.2}
\end{align*}
$$

On this explicit expression one sees obviously that (D.2) is $(b, c)$-symmetric, $\Omega_{a, b, c}=$ $\Omega_{a, c, b}$ and that it is invariant by the $a \leftrightarrow 1-a$ involution, $\Omega_{a, b, c}=\Omega_{1-a, b, c}$. Less obviously one notes that $\Omega_{a, b, c}$ and $\Omega_{a+N, b+M, c+P}$ are homomorphic for any value of the three integers $N, M, P$. This operator can easily be turned into a self-adjoint operator $\Omega_{a, b, c}^{s}=x^{-1 / 2} \cdot \Omega_{a, b, c} \cdot x^{1 / 2}$ (or the self-adjoint operator $x^{-1} \cdot \Omega_{a, b, c}$ ).

The previous order-seven rescaled operators (123), namely $\hat{E}_{i}$ for $i=2 \cdots 5$ can actually be seen as special cases of the rescaled (D.2). For instance $\hat{E}_{2}=$ $\Omega_{1 / 2,0,0}, \hat{E}_{3}=\Omega_{1 / 3,0,0}, \hat{E}_{4}=\Omega_{1 / 4,0,0}, \hat{E}_{5}=\Omega_{1 / 6,0,0}$. Note that $\Omega_{0,0,0}$ factors into seven products of order-one operators, namely $\Omega_{0,0,0}=x^{7} \cdot(x-1)^{2} \cdot \omega_{0,0,0}$, where $\omega_{0,0,0}$ reads:

$$
\begin{align*}
\omega_{0,0,0}= & \left(D_{x}+\frac{8 \cdot x-6}{(x-1) \cdot x}\right)\left(D_{x}+\frac{7 \cdot x-5}{(x-1) \cdot x}\right)\left(D_{x}+\frac{5 \cdot x-4}{(x-1) \cdot x}\right)  \tag{D.3}\\
& \times\left(D_{x}+\frac{4 \cdot x-3}{(x-1) \cdot x}\right)\left(D_{x}+\frac{3 \cdot x-2}{(x-1) \cdot x}\right)\left(D_{x}+\frac{x-1}{(x-1) \cdot x}\right) \cdot D_{x}
\end{align*}
$$

## Appendix E. Yukawa couplings of the operators (146)

Let us recall the operators

$$
\begin{equation*}
x \cdot L_{N-1}=N x \cdot(N \theta+1) \cdot(N \theta+2) \cdots(N \theta+N-1)-\theta^{N-1} \tag{E.1}
\end{equation*}
$$

annihilating the diagonal of rational functions

$$
\begin{equation*}
S_{N}=\operatorname{Diag}\left(\frac{1}{1-x_{1}-x_{2} \cdots-x_{N}}\right)=\sum_{k=0}^{\infty} \frac{(k N)!}{(k!)^{N}} \cdot x^{k} \tag{E.2}
\end{equation*}
$$

and let us also recall, for these higher orders operators, the "higher order" Yukawa couplings (C.5), (see Appendix C. 1 in [18]), one gets the following series expansions.

For $L_{6}$ one has two independent Yukawa couplings:

$$
\begin{aligned}
& K_{3}\left(L_{6}\right)=1+10097920 x+381994497763200 x^{2}+16633254043776570088000 x^{3} \\
& \quad+775506882960998615640344320000 x^{4} \\
& \quad+37663047736228445917647206103076800000 x^{5}+\cdots, \\
& K_{4}\left(L_{6}\right)=1+37273810 x+1993144925004100 x^{2}+110716785445910533561000 x^{3} \\
& \quad+6240527867851744863088075810000 x^{4} \\
& \quad+354307497308094243698303790171562900000 x^{5}+\cdots
\end{aligned}
$$

$\dagger$ See operator $P_{1}$ in 30.

For $L_{7}$ one also has two independent Yukawa couplings:

$$
\begin{aligned}
K_{3}\left(L_{7}\right)= & 1+1998080 x+17805741956352 x^{2}+194576429723517255680 x^{3} \\
& +2352770839522203863766605056 x^{4} \\
& \quad+30251355556001122775209879097376768 x^{5}+\cdots, \\
K_{4}\left(L_{7}\right)= & 1+8684032 x+117020081027584 x^{2}+1699286765410547138560 x^{3} \\
+ & 25562078087040978837930064384 x^{4} \\
& +392649379685173887316823248478666752 x^{5}+\cdots,
\end{aligned}
$$

For $L_{8}$ one has three independent Yukawa couplings:

$$
\begin{aligned}
K_{3}\left(L_{8}\right)= & 1+28165644 x+4505049006911820 x^{2}+956135990658824836437024 x^{3} \\
& +233949266493282926229755622721356 x^{4} \\
& +62439728262268133355948266259742771574160 x^{5}+\cdots, \\
K_{4}\left(L_{8}\right)= & 1+141215076 x+35931446901528372 x^{2}+10427983259188646965239144 x^{3} \\
+ & 3234974169704568107122039167181620 x^{4} \\
& +1045613138106144593968102802858498597502432 x^{5}+\cdots, \\
K_{5}\left(L_{8}\right)= & 1+373047525 x+145724830276964841 x^{2}+56876157804695752594934289 x^{3} \\
+ & 22158779200978185414267897869823861 x^{4} \\
& +8621411505524839637858169288895426578119277 x^{5}+\cdots,
\end{aligned}
$$

For $L_{9}$ one also has three independent Yukawa couplings:

$$
\begin{aligned}
& K_{3}\left(L_{9}\right)= 1+412077600 x+1289641316659740000 x^{2}+5866947627331695510672000000 x^{3} \\
&+32243888417489985271109666517337500000 x^{4} \\
&+198873128042921363581235281228819150221047577600 x^{5}+\cdots, \\
& K_{4}\left(L_{9}\right)= 1+2351650400 x+12183365592555300000 x^{2}+77390125534275385218992000000 x^{3} \\
&+546304879451395256841297089282462500000 x^{4} \\
&+4116981156738304057315870224531007355517746150400 x^{5}+\cdots, \\
& K_{5}\left(L_{9}\right)= 1+7068028000 x+59007489670532260000 x^{2} \\
&+517997397361455559649200000000 x^{3} \\
&+4673650695052899977682502065243662500000 x^{4} \\
&+42920054481728604570276319721869711743549574528000 x^{5}+\cdots,
\end{aligned}
$$

For $L_{10}$ one has four independent Yukawa couplings:

$$
\begin{aligned}
& K_{3}\left(L_{10}\right)=1+6309779256 x+420625737971786884680 x^{2} \\
& \quad+45079261240559432713629269385600 x^{3} \\
& \quad+6138317582117485921071955303191615926205000 x^{4} \\
& \quad+966585253200336527311871686585471862322931324462664256 x^{5}+\cdots
\end{aligned}
$$

```
\(K_{4}\left(L_{10}\right)=1+40564413164 x+4622591813564354886036 x^{2}\)
    \(+702661675261046914875321464472560 x^{3}\)
            \(+124030582023288017842696295577576233194974100 x^{4}\)
            \(+24007059648071495889025807725100100128489899260922131664 x^{5}+\cdots\),
\(K_{5}\left(L_{10}\right)=1+138021017970 x+26412005921656543623886 x^{2}\)
    \(+5668431522859800858070100130586968 x^{3}\)
            \(+1295519605238134595212195097688057025652309166 x^{4}\)
                        \(+308199874468915161097074199781704462799392654807438814820 x^{5}+\cdots\),
\(K_{6}\left(L_{10}\right)=1+317814173215 x+94769164483661457482561 x^{2}\)
    \(+27681891546591722088579495934023171 x^{3}\)
            \(+8013406679969563547817150713512983722786791981 x^{4}\)
            \(+2308882670142336707807831384186329396677206570700741275495 x^{5}+\cdots\)
```

Remark: Note that, we have relations between these Yukawa couplings and the Yukawa couplings of exterior or symmetric powers of the same operators. For instance the exterior square of $L_{4}$ is an irreducible order five operator of nome

$$
\begin{aligned}
q=x+ & 1345 x^{2}+2552775 x^{3}+5602757375 x^{4}+13320846541250 x^{5} \\
& +33314508430778394 x^{6}+86273174430421418330 x^{7} \\
& +229182120170130009397850 x^{8} \\
& +620754459813846824189800125 x^{9}+\cdots
\end{aligned}
$$

and of Yukawa couplings:

$$
\begin{gathered}
K_{3}\left(E x t^{2}\left(L_{4}\right)\right)=1-575 x-1087500 x^{2}-2357466250 x^{3}-5515348543750 x^{4} \\
-13549400590159950 x^{5}-34443162481829737500 x^{6} \\
-89801360565832417275000 x^{7}-238760519646901921788093750 x^{8} \\
-644794600714076957552558593750 x^{9}+\cdots \\
K_{4}\left(E x t^{2}\left(L_{4}\right)\right)=1-1725 x-2270625 x^{2}-3510633125 x^{3}-5943482381250 x^{4} \\
\quad-10616175881261100 x^{5}-19525058497227228750 x^{6} \\
\quad-36235211885062595043750 x^{7}-66371326425035067092906250 x^{8} \\
\\
-116122894233894656970457656250 x^{9}+\cdots
\end{gathered}
$$

$$
\begin{aligned}
K_{5}\left(E x t^{2}\left(L_{4}\right)\right) & =1-2875 x-2131250 x^{2}-1182175000 x^{3}+1120605484375 x^{4} \\
& +7242481554278375 x^{5}+23221270993221987500 x^{6} \\
& +64082729395471190031250 x^{7} \\
& +167312972891732043150312500 x^{8} \\
& +426590930940677272029245312500 x^{9}+\cdots
\end{aligned}
$$

One has the relations

$$
K_{4}\left(E x t^{2}\left(L_{4}\right)\right)=K_{3}\left(E x t^{2}\left(L_{4}\right)\right)^{3}, \quad K_{5}\left(E x t^{2}\left(L_{4}\right)\right)=K_{3}\left(E x t^{2}\left(L_{4}\right)\right)^{5}
$$

and one notes the following relation between the Yukawa coupling of $L_{4}$ with the Yukawa coupling of its exterior square:

$$
\begin{equation*}
K_{3}\left(L_{4}\right)=\frac{1}{K_{3}\left(E x t^{2}\left(L_{4}\right)\right)^{2}} \tag{E.3}
\end{equation*}
$$

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[^0]:    $\dagger$ Sorbonne Universités (previously the UPMC was in Paris Universitas).

[^1]:    § Do not expect a simple match between the dimension of the lattice and this order $n$.

[^2]:    $\dagger$ Which is exactly a rational modular curve already found for the order-three operator $F_{3}$ in [20]. - In 35 Joyce adopted the Heun function notation used by Snow 36], which corresponds to a change of sign in the accessory parameter $q$ in the Heun function $\operatorname{Heun} G(a, q, \alpha, \beta, \gamma, \delta, x)$. Therefore HeunG $(9,3 / 4,1 / 4,3 / 4,1,1 / 2, *)$ is denoted $F(9,-3 / 4,1 / 4,3 / 4,1,1 / 2, *)$ in 35 . Unfortunately this old notation, different from the one used, for instance, in Maple, may contribute to some confusion in the literature.

[^3]:    $\dagger$ The operator $\tilde{G}_{3}^{f c c}$ can be obtained from the rightdivision of the $L C L M\left(G_{3}^{f c c}, D_{x}\right)$ by $D_{x}$, the operator $S_{1}^{f c c}$ being obtained from the rightdivision of the $L C L M\left(G_{3}^{f c c}, D_{x}\right)$ by $G_{3}^{f c c}$. $\S$ A consequence of the fact that, in this case, the differential Galois group is reductive so its representations are semi-simple.

    - Conjugated to its adjoint by the function $(1-12 x)^{1 / 2} \cdot x$

[^4]:    $\dagger$ This is also the case for the self-adjoint order-two $Y_{2}^{s}$ operator. Its solution analytic at $x=0$ is not globally bounded [18 19.

[^5]:    $\dagger$ In order to disentangle these various constraints see 18,19 .

    - If the order-two operator $M_{2}$ is chosen to be globally nilpotent its Wronskian is an $N$-th root of a rational function, and $M_{4}=\operatorname{Sym}^{3}\left(M_{2}\right)+f(x)$ is conjugated to its adjoint up to the cube of this Wronskian.

[^6]:    $\dagger$ See appendic C, and especially C.2, in 1819 for the definition of these "higher order" Yukawa couplings $K_{n}$.

[^7]:    $\ddagger$ Katz’s book [7] provides examples of self-adjoint operators with special differential Galois groups that are not even Fuchsian (see also one of our first (hypergeometric) examples (21).
    $\dagger$ For instance the order-six and order-eight operators $G_{6}^{5 D f c c}$ and $G_{8}^{6 D f c c}$ of sections 3.8 and 3.9 are not MUM.

[^8]:    $\dagger$ Corresponding to the integrands in the $n$-fold integrals, namely one-parameter families of algebraic varieties.
    $\ddagger$ Which is a non-degenerated bilinear map (dual to an intersection mapping, see also the Poincaré duality 65 66).
    § The Picard-Fuchs linear differential operators associated with a family of smooth projective manifolds are homomorphic to their adjoint. This can be seen using the Poincaré duality [68].

