Classical Mechanics

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Abstract

"It doesn't matter what we cover. It matters what you discover."
[Attributed to Viktor Weisskopf, theoretical physicist, 1908 – 2002]

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0 Introductory Remarks

Mechanics is the backbone of theoretical physics. Not because we believe (as physicists did in the 19^{th} century) that all phenomena can ultimately be described by mechanical models, but because the *principles of mechanics* – the conservation of energy and momentum, the principle of least action – play a role in all fields of modern physics.

We start the lecture with the exposition of Newton's laws.

1 Newtonian mechanics

1.1 Newton's laws

For a long time, humans were puzzled by the variety of motion they observed: stars seem to move on circles on the sky, while planets follow much more complex trajectories; stones fall to the ground in accelerated motion, while horizontal movements come to a standstill after some time. A clock's pendulum shows a periodic motion. The fundamental problem of classical mechanics (maybe of all physics) is to answer the question: "When is the particle (body) where?" It was the genius of Isaac Newton to condense all these observations into a set of very simple laws of nature and give a very simple universal[1] answer to that question – "Newton's laws":

1. A body remains at rest or in uniform motion unless acted upon by a force.
2. A body acted upon by a force changes its motion in such a manner that the time rate of change of momentum equals the force, and the direction of the change is along the straight line in which that force is acting.
3. If two bodies exert forces on each other, these forces are equal in magnitude and opposite in direction.

There is also a "fourth law" (Newton calls it a "corollary") which states that two forces that act on the same point add up to a resulting force according to the *parallelogram rule*, in other words, forces are described by *vectors*. This law is also called the *superposition principle*.

These laws condensed centuries of experimental observations into theory. We will first discuss the content of these laws in detail and then consider practical applications.

We describe the position in space of a point particle in terms of vectors. In a given coordinate system[2], we can express this vector by three numbers

$$\boldsymbol{r} = (x_1, x_2, x_3) \,. \tag{1.1}$$

The "motion" of the particles through space is then given by its *instantaneous velocity*

$$\boldsymbol{v} = \dot{\boldsymbol{r}} = (\dot{x}_1, \dot{x}_2, \dot{x}_3) \,, \tag{1.2}$$

where

$$v_i = \dot{x}_i \equiv \frac{d}{dt} x_i \,. \tag{1.3}$$

[1] Within classical physics. The theory of Relativity and Quantum Mechanics have refined Newton's laws.

[2] For the moment, we consider only Cartesian or rectangular coordinate systems.

The first law (the *principle of inertia*; in fact due to GALILEO GALILEI) states that the velocity of a *free particle* is constant, i.e. the *acceleration*

$$\boldsymbol{a} \equiv \dot{\boldsymbol{v}} = (\ddot{x}_1, \ddot{x}_2, \ddot{x}_3) = 0\,. \tag{1.4}$$

vanishes. We will discuss in Sec. 1.2 in which coordinate systems these statements actually hold.

The first law does not say what a "force" actually is. The precise definition and its effect on a body was provided by NEWTON in the second law. The *momentum* is defined as the product of mass and velocity,

$$\boldsymbol{p} \equiv m\boldsymbol{v}\,. \tag{1.5}$$

Denoting the *force* by $\boldsymbol{F}$, we can express the second law as

$$\boldsymbol{F} = \frac{d\boldsymbol{p}}{dt} = \frac{d}{dt}(m\boldsymbol{v})\,. \tag{1.6}$$

This equation contains essentially all the dynamical information contained in NEWTON's laws and allows (at least in principle) for the numerical or analytical calculation of the motion of any classical system. Note that this is a vector equation, so it corresponds to three independent equations for each coordinate direction.

NEWTON was aware of only one exact expression for a force, namely, the gravitational force (see Sec. 3). He, however, noticed a general relation among forces that he summarized in his third law. For two isolated bodies, it states that

$$\boldsymbol{F}_1 = -\boldsymbol{F}_2\,, \tag{1.7}$$

or, using Eq. (1.6),

$$\frac{d\boldsymbol{p}_1}{dt} = -\frac{d\boldsymbol{p}_2}{dt}\,. \tag{1.8}$$

We can rearrange that to give

$$\frac{d}{dt}(\boldsymbol{p}_1 + \boldsymbol{p}_2) = 0\,, \tag{1.9}$$

which implies

$$\boldsymbol{p}_1 + \boldsymbol{p}_2 = \text{constant}\,. \tag{1.10}$$

This is a special case of the general law of conservation of momentum. This law is in fact not entirely accurate, due to the finite speed of propagation of (electromagnetic and gravitational) forces. If the particles move with speeds much less than that of light, and if only central forces are involved, conservation of momentum (1.9) holds with excellent accuracy.[3]

[3]The relativistic correction to this law is discussed in Sec. ??. Electromagnetic fields also carry momentum; if they are incorporated consistently (as in classical electrodynamics), momentum conservation continues to hold.

1.2 Inertial frames

The motion of bodies must be measured relative to some reference frame. A reference frame is called an *inertial frame* if NEWTON's laws are valid in that frame, i.e. if a free body moves with constant velocity (that may vanish) along a straight line. Given an inertial frame, all other inertial frame can be obtained by shifting the origin of coordinates, by a *fixed*[4] rotation, or by a frame moving with *constant velocity* relative to the given inertial frame. This is called *Galilean invariance*. Again, these concepts are true to very good approximation if the bodies move with speeds much less than that of light. At *relativistic* speeds, Galilean invariance must be replaced by *Lorentz invariance*.

As an exercise in vector methods, let's prove the statements above. We start with the coordinates in an inertial system $\boldsymbol{r} = (x_1, x_2, x_3)$ and corresponding equations of motion $\boldsymbol{F} = d(m\boldsymbol{v})/dt$. Then we shift the coordinate origin by a constant $\boldsymbol{c} = (c_1, c_2, c_3)$, i.e. $\boldsymbol{r}' = (x_1 - c_1, x_2 - c_2, x_3 - c_3)$. The equations of motion become[5]

$$\boldsymbol{F}' = m\frac{d^2\boldsymbol{r}'}{dt^2} = m\frac{d^2}{dt^2}(\boldsymbol{r} - \boldsymbol{c}) = m\frac{d^2}{dt^2}\boldsymbol{r} = \boldsymbol{F}\,. \tag{1.11}$$

We see that if the force vanishes in the inertial frame, it also vanishes in the shifted frame. Hence, the latter is also an inertial frame. Now it is easy to see why the second frame is allowed to move with constant velocity – the double time derivative will still yield the same value for the force: for $\boldsymbol{r}' = \boldsymbol{r} + \boldsymbol{v}t$, with constant $\boldsymbol{v}$, we have

$$\boldsymbol{F}' = m\frac{d^2\boldsymbol{r}'}{dt^2} = m\frac{d^2}{dt^2}(\boldsymbol{r} + \boldsymbol{v}t) = m\frac{d^2}{dt^2}\boldsymbol{r} = \boldsymbol{F}\,. \tag{1.12}$$

Finally, let's consider a rotation about the 3-axis with angle θ, i.e. $\boldsymbol{r} = (x_1\cos\theta + x_2\sin\theta, x_2\cos\theta - x_1\sin\theta, x_3)$. Obviously, the 3-component of the force does not change, $F_3' = F_3$. For the 1-component of the force in the rotated system we find

$$F_1' = m\frac{d^2x_1'}{dt^2} = m\frac{d^2}{dt^2}(x_1\cos\theta + x_2\sin\theta) = m\cos\theta\frac{d^2x_1}{dt^2} + m\sin\theta\frac{d^2x_2}{dt^2} = \cos\theta F_1 + \sin\theta F_2\,, \tag{1.13}$$

since we assumed the rotation angle to be constant. Similarly, we find $F_2' = \cos\theta F_2 - \sin\theta F_1$. We see that the force is also just rotated, and hence vanishes in the second system if it vanishes in the first.

1.3 Equations of motion for a single body

Assuming for now that the mass of the body does not change with time, NEWTON's equation (1.6) can be written as

$$\boldsymbol{F} = \frac{d}{dt}(m\boldsymbol{v}) = m\frac{d\boldsymbol{v}}{dt} = m\ddot{\boldsymbol{r}}\,. \tag{1.14}$$

[4]Rotating reference frames are discussed in Sec. 8.

[5]For simplicity, we assume here that the mass is constant, $dm/dt = 0$.

The force is generally a function of position, velocity, and time, and may be written as $\boldsymbol{F}(\boldsymbol{r}, \boldsymbol{v}, t)$. If this function is known, we can integrate the second-order differential equation (1.14). The initial values for $\boldsymbol{r}$ and $\boldsymbol{v} = \dot{\boldsymbol{r}}$ fix the integration constants. Thus, the motion of the body is completely determined.

In the following, we will consider linear motion of a mass point (along the x axis), and discuss the three special cases $F = F(t)$, $F = F(x)$, and $F = F(v)$ (with $v = dx/dt$) in turn. If m is constant, the e.o.m. is

$$m\frac{d^2x}{dt^2} = F \,. \tag{1.15}$$

1.3.1 Force as function of time

Integration of Eq. (1.15) with $F = F(t)$ gives

$$\int\limits_{t_0}^{t} \frac{dv}{dt}dt = v - v_0 = \frac{1}{m}\int\limits_{t_0}^{t} F(t)dt \equiv \frac{1}{m}T(t)\,. \tag{1.16}$$

Here the time integral of the force function, $T(t)$, equals the change in momentum during time interval $t - t_0$. Integrating again gives the solution

$$x - x_0 = v_0(t - t_0) + \frac{1}{m}\int\limits_{t_0}^{t} T(t)dt\,. \tag{1.17}$$

1.3.2 Force as function of position

This is the typical case of a *force field*, $F = F(x)$. The integration of the e.o.m. is performed using the conservation of energy. We multiply Eq. (1.15) by dx/dt on both sides and obtain

$$m\frac{dx}{dt}\frac{d^2x}{dt^2} = F(x)\frac{dx}{dt}\,. \tag{1.18}$$

The left side becomes

$$m\frac{dx}{dt}\frac{d^2x}{dt^2} = \frac{d}{dt}\left\{\frac{m}{2}\left(\frac{dx}{dt}\right)^2\right\}. \tag{1.19}$$

On the right side we use the definition of *work*, $dW \equiv F dx$. Moreover, we define the *kinetic energy* T,

$$T = E_{\text{kin}} = \frac{m}{2}v^2\,, \tag{1.20}$$

and the *potential energy* V by

$$dV = -dW = -F dx\,, \tag{1.21}$$

so

$$V = -\int F(x)dx\,. \tag{1.22}$$

(The potential energy is defined only up to a constant.) Eq. (1.18) now becomes

$$T + V = E = \text{const.}\,. \tag{1.23}$$

In the one-dimensional case, this allows for the full solution of the problem. We write Eq. (1.23) in the form

$$\left(\frac{dx}{dt}\right)^2 = \frac{2}{m}\big[E - V(x)\big]\,, \tag{1.24}$$

and obtain

$$dt = \pm\sqrt{\frac{m}{2(E-V)}}dx\,. \tag{1.25}$$

Integration gives

$$t - t_0 = \pm\frac{2}{m}\int\limits_{x_0}^{x}\sqrt{\frac{dx}{(E-V)}}\,, \tag{1.26}$$

so we know t as a function of x. Inverting this relation gives the solution $x(t)$.

1.3.3 Force as function of velocity

The e.o.m. is now

$$m\frac{dv}{dt} = F(v)\,. \tag{1.27}$$

Writing this as

$$dt = m\frac{dv}{F(v)} \tag{1.28}$$

and integrating gives

$$t = t_0 + m\int\limits_{v_0}^{v}\frac{dv}{F(v)} \equiv f(v)\,, \tag{1.29}$$

and inverting gives $v = f^{-1}(t)$. Then

$$\frac{dx}{dt} = f^{-1}(t)\,, \tag{1.30}$$

and

$$x - x_0 = \int\limits_{t_0}^{t} f^{-1}(t)dt\,. \tag{1.31}$$

Example 1.1: Sliding block on inclined plane (no friction). *The angle of the inclined plane is $\theta = 30°$, the mass of the block is $m = 100g$. What is the block's acceleration?*

There are two forces acting on the block (see Fig. 1). The gravitational force $\boldsymbol{F}_g$ is pulling the block downwards and the "normal" force $\boldsymbol{N}$ is preventing the block from entering the plane.

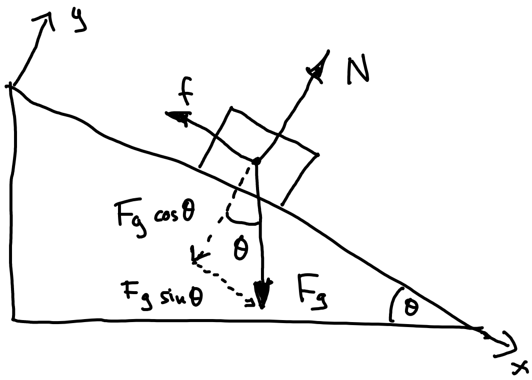

Figure 1: *Sliding block (without and with friction).*

The block will stay on the plane and will only move down the plane which we take to be the x direction. The total force $\boldsymbol{F}$ acting on the block is constant; Eq. (1.14) gives

$$\boldsymbol{F} = \boldsymbol{F}_g + \boldsymbol{N} = m\ddot{\boldsymbol{r}}\,. \tag{1.32}$$

Because there is no "sideways" force, the force vector has essentially two components (directions). We choose them to be the x and y directions (see Fig. 1). Effectively, Eq. (1.32) comprises two equations, one for each component:

$$x \text{ direction:} \quad F_g \sin\theta = m\ddot{x}\,, \tag{1.33}$$

$$y \text{ direction:} \quad 0 = -F_g \cos\theta + N = m\ddot{y}\,. \tag{1.34}$$

The first equality in the second line holds because the normal force does not leads to an acceleration of the block (it does not push the block "off the plane"). It follows that $\ddot{y} = 0$, or $y(t) = y_0 + \dot{y}_0 t$, with y_0, $\dot{y}_0$ two arbitrary constants. As the block does not move up or down orthogonally to the plane, we need to choose the initial conditions $y_0 = 0$, $\dot{y}_0 = 0$, such that $y(t) \equiv 0$ for all times. Using $F_g = mg$, the first equation gives

$$\ddot{x} = \frac{F_g}{m} \sin\theta = g \sin\theta\,. \tag{1.35}$$

The acceleration of the block is constant. Numerically, $\ddot{x} = g\sin(30°) = 4.9\,m/s^2$.

We can easily integrate Eq. (1.35) to get $x(t)$ and $v(t)$ since the right side is just a constant. The solutions can then be solved to give the velocity after the block traveled a distance x_0. There is a neat trick to obtain this last result more directly: we multiply both sides of Eq. (1.35) by $2\dot{x}$ (this is a simple example of an integrating factor*), and obtain*

$$2\dot{x}\ddot{x} = 2\dot{x} g \sin\theta \tag{1.36}$$

$$\Leftrightarrow \quad \frac{d}{dt}(\dot{x}^2) = 2g\sin\theta \frac{dx}{dt} \tag{1.37}$$

$$\Leftrightarrow \quad \int_0^{v_0^2} d(\dot{x}^2) = 2g\sin\theta \int_0^{x_0^2} dx\,. \tag{1.38}$$

Here, we have assumed the initial conditions $x(t = 0) = \dot{x}(t = 0) = 0$, and denoted the final position and velocity by $x(t = t_0) = x_0$, $\dot{x}(t = t_0) = v_0$. We find

$$v_0^2 = 2g \sin\theta x_0\,, \tag{1.39}$$

or

$$v_0 = \sqrt{2g \sin\theta x_0}\,. \tag{1.40}$$

Example 1.2: Sliding block on inclined plane (static friction). *Now we assume that the coefficient of static friction between the block and the plane is $\mu_s = 0.4$. At what angle will the block start sliding if it is initially at rest?*

The size of the static frictional force $\boldsymbol{f}_s$ is proportional to the normal force. Its maximum value is

$$f_{max} = \mu_s N\,. \tag{1.41}$$

The actual frictional force will have some value $f_s \leq f_{max}$ that exactly compensates the force that drives the block downwards, i.e. the block stays at rest. However, for increasing inclination θ, the downward force will at some point exceed f_{max} and the block will start sliding. We call this the friction angle *θ_f. It is defined by*

$$f_{max} = \mu_s N = \mu_s F_g \cos\theta_f\,. \tag{1.42}$$

In general, the equation of motion for the x component is

$$m\ddot{x} = F_g \sin\theta - f_s = F_g \sin\theta - \mu_s F_g \cos\theta\,, \tag{1.43}$$

or

$$\ddot{x} = g(\sin\theta - \mu_s \cos\theta)\,. \tag{1.44}$$

When the block is just about to start sliding, we have $\ddot{x} = 0$, so

$$\sin\theta_f - \mu_s \cos\theta_f = 0 \quad \Rightarrow \quad \tan\theta_f = \mu_s\,. \tag{1.45}$$

Numerically, $\theta_f = \arctan(0.4) = 21.8°$.

Example 1.3: Sliding block on inclined plane (kinetic friction). *After the block begins to slide, the coefficient of kinetic friction becomes $\mu_k = 0.3$. Find the acceleration for the angle $\theta_f = 21.8°$.*

The force of kinetic friction is

$$f_k = \mu_k N = \mu_k F_g \cos\theta_f\,, \tag{1.46}$$

and the acceleration of the block is

$$m\ddot{x} = F_g \sin\theta_f - f_k = mg(\sin\theta_f - \mu_k \cos\theta_f)\,. \tag{1.47}$$

Hence,

$$\ddot{x} = g(\sin\theta_f - \mu_k \cos\theta_f) = 0.09g\,. \tag{1.48}$$

Next, we want to discuss a different type of retarding forces $\boldsymbol{f}_r$ that occur when a body moves through a gas or fluid. Experience shows that this type of friction acts in the negative direction of the velocity and is proportional to a power of v, i.e.

$$\boldsymbol{f}_r = -kv^n \frac{\boldsymbol{v}}{v}\,. \tag{1.49}$$

(Note that we do not include a factor m on the right side.) Empirically, the power is roughly $n = 1$ if the speed is not too small, but at the same time much smaller the the speed of sound ($v \lesssim 24m/s$). For higher velocities, but still below the speed of sound ($v \lesssim 330m/s$), the power is approximately $n = 2$.

Example 1.4: Horizontal motion. *Find the motion of a particle moving horizontally in a medium where the retarding force is proportional to the velocity.*

Eq. (1.14) *gives*

$$m\ddot{x} = m\frac{dv}{dt} = -kv\,. \tag{1.50}$$

This can be integrated:

$$\int \frac{dv}{v} = -\frac{k}{m}\int dt\,, \tag{1.51}$$

or

$$\log v = -\frac{k}{m}t + c_1\,, \tag{1.52}$$

with c_1 an integration constant that can be determined from the initial conditions. For instance, if we denote the velocity at time $t = 0$ by v_0, we see

$$\log v_0 = c_1\,. \tag{1.53}$$

Thus, the solution becomes

$$v = v_0 e^{-kt/m}\,. \tag{1.54}$$

To obtain the displacement at time t, we integrate once more:

$$x = \int v\,dt = v_0 \int dt e^{-kt/m} = -\frac{v_0 m}{k}e^{-kt/m} + c_2\,. \tag{1.55}$$

If the particle starts at $x = 0$ at time $t = 0$, then $c_2 = k/(v_0 m)$, and

$$x = \frac{v_0 m}{k}\left(1 - e^{-kt/m}\right). \tag{1.56}$$

To obtain the velocity as a function of position, we note that

$$\frac{dv}{dx} = \frac{dv}{dt}\frac{dt}{dx} = \frac{dv}{dt}\frac{1}{v}\,, \tag{1.57}$$

and so

$$v\frac{dv}{dx} = \frac{dv}{dt} = -\frac{k}{m}v\,, \tag{1.58}$$

or

$$\frac{dv}{dx} = -\frac{k}{m} \,. \tag{1.59}$$

It follows (using the same initial conditions as above) that

$$v = v_0 - \frac{k}{m}x \,. \tag{1.60}$$

The velocity decreases linearly with distance.

Example 1.5: Terminal velocity. *Find the displacement and velocity of a spherical particle in free fall in an constant gravitational field, with a retarding force proportional to the velocity.*

To be specific, let's consider a small water droplet in air. STOKES' *law tells us that the retarding force for a spherical body is*

$$f_S = 6\pi\eta r v \,, \tag{1.61}$$

where η is the dynamic viscosity of the fluid or gas ($\eta_a = 1.8 \times 10^{-5}$ kg/(ms) for air at temperature 20° C), and r is the radius of the sphere. We assume the droplet starts falling with initial velocity v_0 at height h. The equation of motion (positive z direction downwards) is

$$m\frac{dv}{dt} = mg - 6\pi\eta_a r v \,. \tag{1.62}$$

The mass of the droplet is $m = (4/3)\pi r^3 \rho_w$, with $\rho_w = 1000\,$kg/m^3 the density of water. Rearranging, this yields

$$\frac{dv}{dt} = g - \left(\frac{9\eta_a}{2r^2\rho_w}\right)v \quad \Rightarrow \quad \frac{dv}{kv - g} = -dt \,, \tag{1.63}$$

with $k \equiv 9\eta_a/(2r^2\rho_w)$. Integration gives

$$\frac{1}{k}\log(kv - g) = -t + c \,, \tag{1.64}$$

or

$$kv - g = e^{-kt + c'} \,, \tag{1.65}$$

with $c' = kc$. Using the initial condition $v(t=0) = v_0 = (1/k)e^{c'} + g/k$ yields

$$v = \frac{g}{k} + \left(v_0 - \frac{g}{k}\right)e^{-kt} \,. \tag{1.66}$$

*After some time, the velocity approaches the constant value $v_t = g/k$ (*terminal velocity*). In fact, this result can be directly read off Eq. (1.62). For a droplet of radius $r = 5 \times 10^{-6}$ m in the earth's gravitational field, we find $v_t = 3 \times 10^{-3}$ m/s – the droplet practically floats in the air.*

Integrating once more and evaluating the integration constant with the initial condition $z(t = 0) = 0$ gives

$$z = \frac{g}{k}t + \frac{1}{k}\left(v_0 - \frac{g}{k}\right)\left(1 - e^{-kt}\right) \,. \tag{1.67}$$

In general, the constant k depends on the viscosity of air and the shape of the falling object. For a raindrop with diameter $r = 1$ mm, we find $v_t = 120$ m/s. The terminal velocity of a sky diver in face-down free fall is about 54 m/s. A peregrin falcon can reach 100 m/s. In these cases, it would be more appropriate to use a retarding force proportional to the square of the velocity.

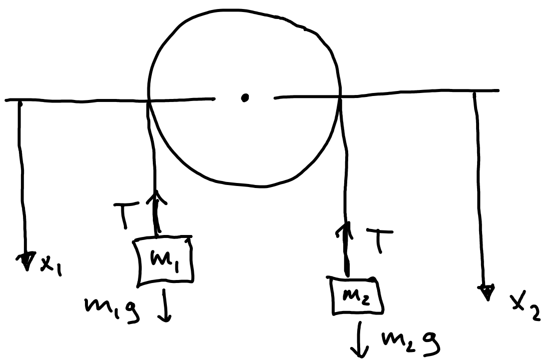

Figure 2: *Atwood's machine.*

Finally, we discuss two classic examples of particle dynamics.

Example 1.6: Atwoods machine. *Atwood's machine is depicted in Fig. 2. Find the acceleration of the masses and the tension of the string.*

We assume that the mass of the string as well as any friction can be neglected. The tension T must be the same throughout the string. The equations of motion for each of the masses are

$$m_1\ddot{x}_1 = m_1 g - T\,, \tag{1.68}$$

$$m_2\ddot{x}_2 = m_2 g - T\,. \tag{1.69}$$

If the string is not elastic, we have $\ddot{x}_2 = -\ddot{x}_1$, so we combine Eqs. 1.68 by eliminating T:

$$m_1\ddot{x}_1 = m_1 g - (m_2 g - m_2\ddot{x}_2) = m_1 g - (m_2 g + m_2\ddot{x}_1)\,. \tag{1.70}$$

We solve this equation for $\ddot{x}_1$:

$$\ddot{x}_1 = \frac{g(m_1 - m_2)}{m_1 + m_2} \tag{1.71}$$

The tension is

$$T = m_1(g - \ddot{x}_1) = \frac{2m_1 m_2 g}{m_1 + m_2}\,. \tag{1.72}$$

Example 1.7: Charged particle in magnetic field. *For instance, consider a cosmic ray particle entering the earth's magnetic field B_0 (near the surface of the earth, we can assume B_0 to be uniform).*

We choose our coordinates such that the y axis is parallel to B_0. Denoting the particle's charge by q, its mass by m, its velocity by $\boldsymbol{v}$, and its acceleration by $\boldsymbol{a}$, we have

$$\boldsymbol{v} = \dot{x}_1\boldsymbol{e}_1 + \dot{x}_2\boldsymbol{e}_2 + \dot{x}_3\boldsymbol{e}_3\,, \tag{1.73}$$

$$\boldsymbol{a} = \ddot{x}_1\boldsymbol{e}_1 + \ddot{x}_2\boldsymbol{e}_2 + \ddot{x}_3\boldsymbol{e}_3\,, \tag{1.74}$$

$$\boldsymbol{B} = B_0\boldsymbol{e}_2\,, \tag{1.75}$$

and the Lorentz force is $\boldsymbol{F} = q\boldsymbol{v}\times\boldsymbol{B}$. The equations of motion are

$$m\boldsymbol{a} = q\boldsymbol{v}\times\boldsymbol{B}\,, \tag{1.76}$$

or in components

$$m\ddot{x}_1 = -qB_0\dot{x}_3\,, \tag{1.77}$$
$$m\ddot{x}_2 = 0\,, \tag{1.78}$$
$$m\ddot{x}_3 = qB_0\dot{x}_1\,. \tag{1.79}$$

The second of these equations is easily integrated, giving

$$x_2(t) = \dot{x}_2^0 t + x_2^0\,, \tag{1.80}$$

where $\dot{x}_2^0$ and x_2^0 are integration constants that may be determined by the initial conditions. To integrate the remaining two equations, we define $\alpha \equiv qB_0/m$, such that

$$\ddot{x}_1 = -\alpha\dot{x}_3\,, \tag{1.81}$$
$$\ddot{x}_3 = \alpha\dot{x}_1\,. \tag{1.82}$$

To decouple these equations, we differentiate once more and substitute,

$$\dddot{x}_1 = -\alpha\ddot{x}_3 = -\alpha^2\dot{x}_1\,, \tag{1.83}$$
$$\dddot{x}_3 = \alpha\ddot{x}_1 = -\alpha^2\dot{x}_3\,. \tag{1.84}$$

These are differential equations of "harmonic oscillator type" for $\dot{x}_1$ and $\dot{x}_3$. The solution for $\dot{x}_1$ is

$$\dot{x}_1 = k_1\cos(\alpha t) + k_2\sin(\alpha t)\,, \tag{1.85}$$

and hence

$$x_1 = A\cos(\alpha t) + B\sin(\alpha t) + x_1^0\,. \tag{1.86}$$

The solution for $\dot{x}_3$ is

$$x_3 = A'\cos(\alpha t) + B'\sin(\alpha t) + x_3^0\,. \tag{1.87}$$

$A, A', B, B', x_1^0, x_3^0$ *are integration constants. The complete solution is*

$$x_1 - x_1^0 = A\cos(\alpha t) + B\sin(\alpha t)\,, \tag{1.88}$$
$$x_2 - x_2^0 = \dot{x}_2^0 t\,, \tag{1.89}$$
$$x_3 - x_3^0 = A'\cos(\alpha t) + B'\sin(\alpha t)\,. \tag{1.90}$$

Inserting this into the first of Eq. (1.81) *gives*

$$-\alpha^2 A\cos(\alpha t) - \alpha^2 B\sin(\alpha t) = \alpha^2 A'\sin(\alpha t) - \alpha^2 B'\cos(\alpha t)\,. \tag{1.91}$$

Evaluating this at $t = 0$ and $t = \pi/(2\alpha)$ gives $A = B'$ and $B = -A'$. This results in

$$x_1 - x_1^0 = A\cos(\alpha t) + B\sin(\alpha t)\,, \tag{1.92}$$
$$x_2 - x_2^0 = \dot{x}_2^0 t\,, \tag{1.93}$$
$$x_3 - x_3^0 = -B\cos(\alpha t) + A\sin(\alpha t)\,. \tag{1.94}$$

If $\dot{x}_3(t=0) = \dot{x}_3^0$ and $\dot{x}_1(t=0) = 0$, Eq. (1.92) yields $\dot{x}_3^0 = \alpha A$ and $B = 0$. This gives the final form of the solution,

$$x_1 - x_1^0 = \left(\frac{\dot{x}_3^0 m}{qB_0}\right)\cos\left(\frac{qB_0 t}{m}\right), \tag{1.95}$$

$$x_2 - x_2^0 = \dot{x}_2^0 t\,, \tag{1.96}$$

$$x_3 - x_3^0 = \left(\frac{\dot{x}_3^0 m}{qB_0}\right)\sin\left(\frac{qB_0 t}{m}\right). \tag{1.97}$$

The particle drifts along the direction of the B field, its trajectory being a circular helix of radius $\dot{x}_3^0 m/(qB_0)$. Given a B field in a laboratory, this can be used to measure the charge-to-mass ratio of elementary particles (e.g. electrons).

1.4 Planar kinematics*

In two-dimensional cartesian coordinates, we have the velocity

$$\boldsymbol{v} = (v_x, v_y) = (\dot{x}, \dot{y})\,, \tag{1.98}$$

with absolute value $|\boldsymbol{v}| = \sqrt{\dot{x}^2 + \dot{y}^2} = v$, as well as the acceleration

$$\dot{\boldsymbol{v}} = (\dot{v}_x, \dot{v}_y) = (\ddot{x}, \ddot{y})\,. \tag{1.99}$$

with absolute value $|\dot{\boldsymbol{v}}| = \sqrt{\ddot{x}^2 + \ddot{y}^2} = a$. Alternatively, we can decompose velocity and acceleration in terms of the components along (index s) and orthogonal to (index n) the trajectory of the mass point. We have

$$\boldsymbol{v}_s = \boldsymbol{v}\,, \qquad v_s = \pm v\,, \qquad \boldsymbol{v}_n = \mathbf{0}\,, \qquad v_n = 0\,. \tag{1.100}$$

This becomes more significant if we decompose $\dot{\boldsymbol{v}}$ in terms of $\dot{\boldsymbol{v}}_s$ and $\dot{\boldsymbol{v}}_n$. If we denote by α the angle between the x direction and the tangent to the trajectory, then the *tangential acceleration* is

$$\dot{v}_s = \dot{v}_x \cos\alpha + \dot{v}_y \sin\alpha\,, \tag{1.101}$$

and the *normal (or centripetal) acceleration* is

$$\dot{v}_n = -\dot{v}_x \sin\alpha + \dot{v}_y \cos\alpha\,. \tag{1.102}$$

We have

$$\cos\alpha = \frac{dx}{ds} = \frac{\dot{x}}{\dot{s}} = \frac{v_x}{v}\,, \qquad \sin\alpha = \frac{dy}{ds} = \frac{\dot{y}}{\dot{s}} = \frac{v_y}{v}\,, \tag{1.103}$$

and so

$$\frac{dv_s}{dt} = \frac{1}{v}\big(v_x\dot{v}_x + v_y\dot{v}_y\big) = \frac{1}{2v}\frac{d}{dt}\big(v_x^2 + v_y^2\big) = \frac{1}{2v}\frac{dv^2}{dt} = \frac{dv}{dt} = \dot{v}\,; \tag{1.104}$$

the tangential acceleration is the change in speed, the change in direction is irrelevant. On the other hand,

$$\frac{dv_n}{dt} = \frac{1}{v}\big(v_x\dot{v}_y - v_y\dot{v}_x\big) = \frac{1}{v}\big(\dot{x}\ddot{y} - \dot{y}\ddot{x}\big) = v^2\frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\big(\dot{x}^2 + \dot{y}^2\big)^{3/2}} = \frac{v^2}{\rho}\,, \tag{1.105}$$

where $1/\rho$ is the curvature of the trajectory. The normal acceleration does not depend on the change in speed, but only on the speed itself and the change in direction (the form of the trajectory). If $\dot{v} = 0$, the acceleration is orthogonal to the velocity and therefore orthogonal to the trajectory.

To clarify the concept of the curvature of the trajectory, we consider the velocities $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ of the mass at two neighboring points, with distance Δs. Let the angle between the two velocities be $\Delta\epsilon$. This is also the angle between the two position vectors $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$, so

$$\Delta s = \rho \Delta\epsilon \,. \tag{1.106}$$

On the other hand, we can write $\boldsymbol{v}_2 = \boldsymbol{v}_1 + \Delta\boldsymbol{v}$, and decompose $\Delta\boldsymbol{v}$ into a tangential component $\Delta\boldsymbol{v}_s$ and a normal component $\Delta\boldsymbol{v}_n$. Then

$$\dot{v}_s = \frac{|\Delta\boldsymbol{v}_s|}{\Delta t} = \frac{v_2 - v_1}{\Delta t} = \frac{\Delta v}{\Delta t} = \dot{v} \,, \tag{1.107}$$

$$\dot{v}_n = \frac{|\Delta\boldsymbol{v}_n|}{\Delta t} = \frac{v\Delta\epsilon}{\Delta t} = \frac{\Delta\epsilon}{\Delta s} v^2 = \frac{v^2}{\rho} \,. \tag{1.108}$$

1.5 Conservation laws*

Here, we briefly discuss some conservation laws that are implied by Newton's laws. The proper framework for dealing with conservation laws is the Lagrangian formalism, and we pick up the topic again in Sec. 5.

The law of *conservation of momentum* has been discussed above for the case of two particles. It will be generalized to an arbitrary number of particles in Sec. 7. For a single free particle, Eq. (1.6) tells us that $\dot{\boldsymbol{p}} = 0$, so we have

I The momentum $\boldsymbol{p}$ of a particle is conserved when the total force on it is zero.

A related concept is the *angular momentum* of a particle, defined as

$$\boldsymbol{L} \equiv \boldsymbol{r} \times \boldsymbol{p} \,. \tag{1.109}$$

Note that the actual value of $\boldsymbol{L}$ depends on the location of the origin of the coordinate system. We define the corresponding *torque* as

$$\boldsymbol{N} \equiv \boldsymbol{r} \times \boldsymbol{F} \,. \tag{1.110}$$

Because $\boldsymbol{F} = \dot{\boldsymbol{p}}$, we have $\boldsymbol{N} = \boldsymbol{r} \times \dot{\boldsymbol{p}}$. The time derivative of the angular momentum is

$$\dot{\boldsymbol{L}} = \frac{d}{dt}(\boldsymbol{r} \times \boldsymbol{p}) = \dot{\boldsymbol{r}} \times \boldsymbol{p} + \boldsymbol{r} \times \dot{\boldsymbol{p}} = \boldsymbol{r} \times \dot{\boldsymbol{p}} = \boldsymbol{N} \,, \tag{1.111}$$

since $\dot{\boldsymbol{r}} \times \boldsymbol{p} = \dot{\boldsymbol{r}} \times m\boldsymbol{v} = m(\dot{\boldsymbol{r}} \times \dot{\boldsymbol{r}}) = 0$. If no torque acts on the particle ($\boldsymbol{N} = 0$), then $\dot{\boldsymbol{L}} = 0$ and $\boldsymbol{L}$ is constant. Hence

II The angular momentum $\boldsymbol{L}$ of a particle subject to no torque is conserved.

Often, a coordinate system can be chosen such that the torque is zero, resulting in a simple solution to a given problem.

The *work* done by a force on a particle on a path from position 1 to position 2 is defined as (see App. B for the definition of line integrals)

$$W_{12} \equiv \int_{\boldsymbol{r}_1}^{\boldsymbol{r}_2} \boldsymbol{F} \cdot d\boldsymbol{r} \,. \tag{1.112}$$

If $\boldsymbol{F}$ is the total force acting on the particle, we can write the integrand as

$$\boldsymbol{F} \cdot d\boldsymbol{r} = m\frac{d\boldsymbol{v}}{dt} \cdot \frac{d\boldsymbol{r}}{dt} dt = m\frac{d\boldsymbol{v}}{dt} \cdot \boldsymbol{v} dt = \frac{m}{2}\frac{d}{dt}(\boldsymbol{v} \cdot \boldsymbol{v})dt = d\left(\frac{1}{2}mv^2\right) . \tag{1.113}$$

We can then easily solve the integral to get

$$W_{12} = \left(\frac{1}{2}mv^2\right)\Bigg|_{\boldsymbol{r}_1}^{\boldsymbol{r}_2} = \frac{1}{2}m\left(v_2^2 - v_1^2\right) = T_2 - T_1 \,. \tag{1.114}$$

This is just the change in *kinetic energy* $T \equiv \frac{1}{2}mv^2$.

If the work required to move a particle from $\boldsymbol{r}_1$ to $\boldsymbol{r}_2$ does not depend on the path between the two locations, but only on the original and final positions, we can fix e.g. position $\boldsymbol{r}_1$, and W_{12} is then just a function of the final position $\boldsymbol{r}_2$. This function, U, is called the *potential energy* of the particle. The work done on the particle is then simply the difference in potential energies at the two positions:

$$\int_{\boldsymbol{r}_1}^{\boldsymbol{r}_2} \boldsymbol{F} \cdot d\boldsymbol{r} \equiv U_1 - U_2 \,. \tag{1.115}$$

For instance, if a body of mass m is raised through a height h in a constant gravitational field with gravitational acceleration g, the amount of work mgh has been done on the body.

The work is independent of the path if the force can be written as the *gradient* of a scalar function[6] (that turns out to be just U):

$$\boldsymbol{F} = -\mathbf{grad}\, U = -\boldsymbol{\nabla} U \,, \tag{1.120}$$

[6] The gradient is defined as follows. Assume we are given a scalar function of several variables (say, three): $\phi(\boldsymbol{r}) = \phi(x_1, x_2, x_3)$. For instance, this could be the temperature distribution in a room. Given the value (temperature) at one point, $\phi(\boldsymbol{r})$, what is the change in value (temperature) $d\phi$ if we move to an infinitesimally nearby point, $\phi(\boldsymbol{r} + d\boldsymbol{r}) = \phi(x_1 + dx_1, x_2 + dx_2, x_3 + dx_3)$? Using the chain rule, we obtain e.g. in x_1 direction

$$\phi(x_1 + dx_1, x_2, x_3) = \phi(x_1, x_2, x_3) + \frac{\partial\phi(x_1, x_2, x_3)}{\partial x_1} dx_1 \,, \tag{1.116}$$

and hence, adding the contribution in the three directions,

$$\phi(x_1 + dx_1, x_2 + dx_2, x_3 + dx_3) = \phi(x_1, x_2, x_3) + \sum_i \frac{\partial\phi(x_1, x_2, x_3)}{\partial x_i} dx_i \,. \tag{1.117}$$

The last term can be written as the scalar product of two vectors,

$$\phi(\boldsymbol{r} + d\boldsymbol{r}) = \phi(\boldsymbol{r}) + (\boldsymbol{\nabla}\phi(\boldsymbol{r})) \cdot d\boldsymbol{r} \,, \tag{1.118}$$

because then

$$\int_{\boldsymbol{r}_1}^{\boldsymbol{r}_2} \boldsymbol{F} \cdot d\boldsymbol{r} = -\int_{\boldsymbol{r}_1}^{\boldsymbol{r}_2} (\boldsymbol{\nabla} U) \cdot d\boldsymbol{r} = -\int_{\boldsymbol{r}_1}^{\boldsymbol{r}_2} dU = U_1 - U_2 \,. \tag{1.121}$$

Here, the potential is a function of position and time, $U = U(\boldsymbol{r}, t)$. (Velocity-dependent potentials arise in electrodynamics and will not be considered here.) It is now easy to see that the potential energy is only defined up to a constant (both U and $U +$ constant give the same force via Eq. (1.120)); only differences in potential energy are physically meaningful. Similarly, since Newton's laws do not change if expressed in terms of a reference frame in uniform relative motion, no absolut kinetic energy can be ascribed to a body.

The *total energy* of a particle is defined as the sum of kinetic and potential energies:

$$E \equiv T + U \,. \tag{1.122}$$

We want to show that E is conserved if the potential energy is time independent. The total time derivative of the energy is

$$\frac{dE}{dt} = \frac{dT}{dt} + \frac{dU}{dt} \,. \tag{1.123}$$

To proceed, we recall Eq. (1.113):

$$\boldsymbol{F} \cdot d\boldsymbol{r} = d\left(\frac{1}{2} m v^2\right) = dT \,. \tag{1.124}$$

It follows

$$\frac{dT}{dt} = \boldsymbol{F} \cdot \frac{d\boldsymbol{r}}{dt} = \boldsymbol{F} \cdot \dot{\boldsymbol{r}} \,. \tag{1.125}$$

Moreover,

$$\frac{dU}{dt} = \sum \frac{\partial U}{\partial x_i} \dot{x}_i + \frac{\partial U}{\partial t} = (\boldsymbol{\nabla} U) \cdot \dot{\boldsymbol{r}} + 0 \,. \tag{1.126}$$

Hence, we find

$$\frac{dE}{dt} = \boldsymbol{F} \cdot \dot{\boldsymbol{r}} + (\boldsymbol{\nabla} U) \cdot \dot{\boldsymbol{r}} = (\boldsymbol{F} + \boldsymbol{\nabla} U) \cdot \dot{\boldsymbol{r}} = 0 \,, \tag{1.127}$$

since, by definition, $\boldsymbol{F} = -\boldsymbol{\nabla} U$. In summary, we have the law of conservation of total energy:

III The total energy E of a particle in a conservative force field is constant in time.

---

where $(d\boldsymbol{r})_i = dx_i$ and we have defined the gradient $(\boldsymbol{\nabla}\phi(\boldsymbol{r}))_i = \partial\phi(x_1, x_2, x_3)/\partial x_i$. So the change in the function ϕ is given by

$$d\phi(\boldsymbol{r}) \equiv \phi(\boldsymbol{r} + \boldsymbol{dr}) - \phi(\boldsymbol{r}) = (\boldsymbol{\nabla}\phi(\boldsymbol{r})) \cdot d\boldsymbol{r} \,. \tag{1.119}$$

Figure 3: *Cable Car. One end of the cable was attached to a winch that was fixed on a pair of heavy sand stones buried in the ground, using a metal cage. The other end was attached to a birch tree to the right (not shown in the photograph).*

1.6 Potential energy and form of motion*

If the potential is given, one can gain a qualitative understanding of the motion by plotting $U(x)$. Because $T \geq 0$, we have $E \geq U(x)$. Depending on the form of the potential and the value of E, one can have bounded periodic motions, unbounded motions, and stable or unstable equilibrium points.

If the particle remains close to the equilibrium point x_0, it is useful to perform a Taylor expansion about x_0. (We choose our coordinate system such that $x_0 = 0$.) This gives

$$U(x) = U(0) + x\left(\frac{dU}{dx}\right)\bigg|_{x=0} + \frac{x^2}{2!}\left(\frac{d^2U}{dx^2}\right)\bigg|_{x=0} + \dots . \tag{1.128}$$

Here, $U(0)$ is just a constant that we can set to zero by adjusting the total energy (recall that only energy differences are physical). At an equilibrium point, the first deriative of U vanishes,

$$\frac{dU}{dx}\bigg|_{x=0} = 0 \quad (\text{equilibrium})\,, \tag{1.129}$$

so the first nonzero term is the second derivative. We can neglect all higher terms if x is "sufficiently small" (you can always include more terms if you want to increase the accuracy). In this approximation, we have

$$U(x) = \frac{x^2}{2}\left(\frac{d^2U}{dx^2}\right)\bigg|_{x=0} . \tag{1.130}$$

The equilibrium is unstable for $(d^2U/dx^2)|_{x=0} < 0$ and stable for $(d^2U/dx^2)|_{x=0} > 0$. The latter case is exactly the potential for a *harmonic oscillator*, to be discussed in detail in Sec. 2.

Example 1.8: Cable car. *Consider the construction of a cable car (a successful example dating back to the teenage years of the instructor is shown in Fig. 3). What force does the winch holding*

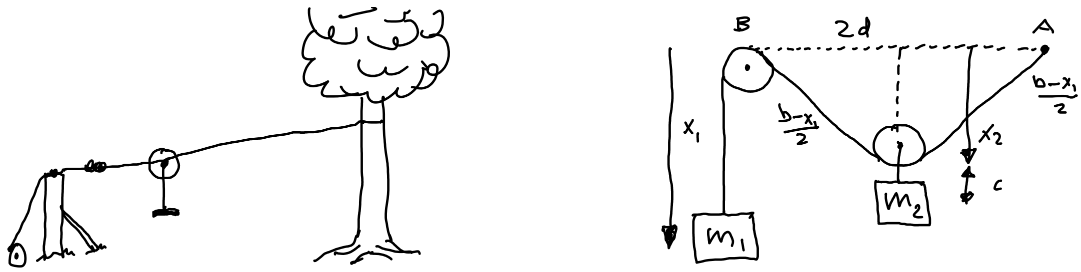

Figure 4: *Schematic sketch of the cable car (left panel), and approximation by a system of pulleys, masses, and strings (right panel).*

the cable need to sustain if a person rides the cable car? We approximate the cable car by a system of pulleys, masses, and strings.

A light string of total length b is attached at the "tree" A. It passes over the pulley at B (the "pole"), located at a distance $2d$ from A, and is attached to the mass m_1 (the "winch"). Another pulley (the "gondola") with mass m_2 (the "passenger") attached passes over the string between A and B, pulling the string down. We want to calculate the distance x_1 when the system is in equilibrium. (We consider the pulleys to be massless. Since we also neglect friction, this quite a bad approximation to the actual cable car.)

We will use the "energy method" to solve the problem. In equilibrium, the kinetic energy is zero, and we can determine the equilibrium position from the condition (1.129). We choose the potential energy such that $U = 0$ along the line AB; then

$$U = -m_1 g x_1 - m_2 g(x_2 + c)\,. \tag{1.131}$$

(The distance c is constant.) Pythagoras tells us that

$$x_2 = \sqrt{(b - x_1)^2/4 - d^2} \tag{1.132}$$

(see Fig. 4, right panel), so

$$U = -m_1 g x_1 - m_2 g\sqrt{(b - x_1)^2/4 - d^2} - m_2 g c\,. \tag{1.133}$$

The derivative of U w.r.t. x_1 at the equilibrium value $(x_1)_0 \equiv x_0$ must vanish:

$$\left.\frac{dU(x_1)}{dx_1}\right|_{x_1 = x_0} = -m_1 g + \frac{m_2 g(b - x_0)}{4\sqrt{(b - x_0)^2/4 - d^2}} = 0\,. \tag{1.134}$$

We solve this condition for x_0:

$$4m_1\sqrt{(b - x_0)^2/4 - d^2} = m_2(b - x_0) \tag{1.135}$$

$$\Rightarrow \quad (b - x_0)^2(4m_1^2 - m_2^2) = 16 m_1^2 d^2 \tag{1.136}$$

$$\Rightarrow \quad x_0 = b - \frac{4m_1 d}{\sqrt{4m_1^2 - m_2^2}}\,. \tag{1.137}$$

We see that for a valid solution, we need $4m_1 > m_2$. If m_2 becomes larger, it will pull m_1 up to the pulley, because for the solution to be stable, the second derivative of the potential

$$\frac{d^2U(x_1)}{dx_1^2} = \frac{-m_2 g}{4\sqrt{(b-x_1)^2/4-d^2}} + \frac{m_2 g(b-x_1)^2}{16\sqrt{(b-x_1)^2/4-d^2}^3} \tag{1.138}$$

must be positive at $x_1 = x_0$:[7]

$$\left.\frac{d^2U(x_1)}{dx_1^2}\right|_{x_1=x_0} = \frac{g(4m_1^2-m_2^2)^{3/2}}{4m_2^2 d}\,. \tag{1.140}$$

2 Oscillations*

We will first consider the one-dimensional problem. As discussed in Sec. 1.3.2, any small motion about a stable equilibrium point can be described by a quadratic potential. To obtain the corresponding force, we could just take the gradient of Eq. (1.130). Alternatively, we can expand the force $F(x)$ about the equilibrium point (chosen to be $x_0 = 0$) into a Taylor series:

$$F(x) = F(0) + x\left.\left(\frac{dF}{dx}\right)\right|_{x=0} + \frac{x^2}{2!}\left.\left(\frac{d^2F}{dx^2}\right)\right|_{x=0} + \ldots\,. \tag{2.1}$$

Here, $F(0)$ must vanish (otherwise $x_0 = 0$ would not be an equilibrium point). For small displacements, we can drop the quadratic and higher terms. We obtain

$$F(x) = -kx\,, \tag{2.2}$$

where we defined $k \equiv -(dF/dx)|_{x=0}$ (for a stable equilibrium point, the force must be directed towards the equilibrium position). Eq. (2.2) is called HOOKE's law.

2.1 Simple harmonic oscillator

Substituting HOOKE's law into NEWTON's equation of motion yields

$$-kx = m\ddot{x}\,. \tag{2.3}$$

[7] The gory details are

$$\begin{aligned}\left.\frac{d^2U(x_1)}{dx_1^2}\right|_{x_1=x_0} &= \frac{-m_2 g}{4\sqrt{\frac{4m_1^2d^2}{4m_1^2-m_2^2}-d^2}} + \frac{m_2 g\frac{16m_1^2d^2}{4m_1^2-m_2^2}}{16\sqrt{\frac{4m_1^2d^2}{4m_1^2-m_2^2}-d^2}^3}\\ &= \frac{-m_2 g}{4\sqrt{\frac{m_2^2d^2}{4m_1^2-m_2^2}}} + \frac{m_2 g\frac{m_1^2d^2}{4m_1^2-m_2^2}}{\sqrt{\frac{m_2^2d^2}{4m_1^2-m_2^2}}^3} = \frac{-m_2 g\frac{m_2^2d^2}{4m_1^2-m_2^2}}{4\sqrt{\frac{m_2^2d^2}{4m_1^2-m_2^2}}^3} + \frac{m_2 g\frac{4m_1^2d^2}{4m_1^2-m_2^2}}{4\sqrt{\frac{m_2^2d^2}{4m_1^2-m_2^2}}^3}\\ &= \frac{m_2 g d^2\frac{4m_1^2-m_2^2}{4m_1^2-m_2^2}}{4\sqrt{\frac{m_2^2d^2}{4m_1^2-m_2^2}}^3} = \frac{m_2 g d^2}{4m_2^3d^3\sqrt{\frac{1}{4m_1^2-m_2^2}}^3} = \frac{(4m_1^2-m_2^2)^{3/2}}{4m_2^2 d}\,.\end{aligned} \tag{1.139}$$

One usually defines

$$\omega_0^2 \equiv \frac{k}{m}\,, \tag{2.4}$$

such that Eq. (2.3) becomes

$$\ddot{x} + \omega_0^2 x = 0\,. \tag{2.5}$$

This is the equation of motion for the (one-dimensional) *simple harmonic oscillator*. The general solution to Eq. (2.5) is

$$x(t) = A\,\sin(\omega_0 t - \delta)\,, \tag{2.6}$$

with the *amplitude* A and *phase* δ as free parameters (determined by the initial conditions to the differential equation).

The amplitude A is related to the total energy of the oscillator in a simple way as we will now show. Using Eq. (2.6), we see that the kinetic energy is

$$T = \frac{1}{2}m\dot{x}^2 = \frac{1}{2}m\omega_0^2 A^2\,\cos^2(\omega_0 t - \delta) = \frac{1}{2}kA^2\,\cos^2(\omega_0 t - \delta)\,. \tag{2.7}$$

We can calculate the potential energy by integrating the work done on the particle over a distance x. For the force (2.2) we have

$$dW = -F dx = kx\,dx\,. \tag{2.8}$$

This gives the potential energy

$$U = \frac{1}{2}kx^2 = \frac{1}{2}kA^2\,\sin^2(\omega_0 t - \delta)\,, \tag{2.9}$$

where we inserted the solution (2.6). Adding Eqs. (2.7) and (2.10), we obtain the total energy

$$E = T + U = \frac{1}{2}kA^2 = \frac{1}{2}m\omega_0^2 A^2\,. \tag{2.10}$$

The total energy is proportional to the square of the amplitude and independent of the time.

We define the period τ_0 of the motion as the time interval between complete repetitions of the particle's motion. This happens when the argument of the sine function in Eq. (2.6) increases by 2π, i.e. $\omega_0\tau_0 = 2\pi$, or

$$\tau_0 = 2\pi\sqrt{\frac{m}{k}}\,. \tag{2.11}$$

We see that ω_0 is the *angular frequency* of the motion, related to the *frequency* ν_0 by

$$\omega_0 = 2\pi\nu_0 = \sqrt{\frac{m}{k}}\,, \tag{2.12}$$

$$\nu_0 = \frac{1}{\tau_0} = \frac{1}{2\pi}\sqrt{\frac{m}{k}}\,. \tag{2.13}$$

The period of the simple harmonic oscillator is independent of the amplitude.

2.2 Harmonic oscillation in two dimensions*

Now we consider motion in two dimensions. For simplicity, we assume that the restoring force is isotropic, i.e. proportional to the distance of the particle from a force center located at the origin:

$$\boldsymbol{F} = -k\boldsymbol{r}\,. \tag{2.14}$$

In coordinates, we find

$$F_x = -kr\,\cos\theta = -kx\,, \tag{2.15}$$
$$F_y = -kr\,\sin\theta = -ky\,. \tag{2.16}$$

The equations of motion are

$$\ddot{x} + \omega_0^2 x = 0\,, \tag{2.17}$$
$$\ddot{y} + \omega_0^2 y = 0\,, \tag{2.18}$$

where $\omega_0^2 = k/m$, with solutions

$$x(t) = A\cos(\omega_0 t - \alpha)\,, \tag{2.19}$$
$$y(t) = B\cos(\omega_0 t - \beta)\,. \tag{2.20}$$

The particle moves in simple harmonic oscillation in each of the two directions, with the same frequency but possibly different amplitudes and phases. To obtain the path of the particles, we combine the two equations (2.31) by eliminating the time t. We start with

$$\begin{aligned} y(t) &= B\cos[\omega_0 t - \alpha + (\alpha - \beta)] \\ &= B\cos(\omega_0 t - \alpha)\cos(\alpha - \beta) - B\sin(\omega_0 t - \alpha)\sin(\alpha - \beta)\,. \end{aligned} \tag{2.21}$$

Defining $\delta = \alpha - \beta$, this can be written as

$$y = \frac{B}{A}x\cos\delta - B\sqrt{1 - \frac{x^2}{A^2}}\sin\delta\,, \tag{2.22}$$

or

$$Ay - Bx\cos\delta = -B\sqrt{A^2 - x^2}\sin\delta\,. \tag{2.23}$$

Squaring this equation gives

$$A^2y^2 - 2ABxy\cos\delta + B^2x^2\cos^2\delta = B^2(A^2 - x^2)\sin^2\delta\,. \tag{2.24}$$

This can be simplified to

$$A^2y^2 - 2ABxy\cos\delta + B^2x^2 = B^2A^2\sin^2\delta\,. \tag{2.25}$$

This is the parametric equation for an ellipse. Let's now assume for simplicity $A = B = 1$. Then, for $\delta = \pm\pi/2$ this is the equation of a circle,

$$y^2 + x^2 = 1\,, \tag{2.26}$$

while for $\delta = 0$

$$(y-x)^2 = 0 \quad \Rightarrow \quad x = y\,, \tag{2.27}$$

and for $\delta = \pm\pi$

$$(y+x)^2 = 0 \quad \Rightarrow \quad x = -y\,. \tag{2.28}$$

In a more general force field, the two frequencies need not be the same, and the trajectories of the solutions,

$$x(t) = A\cos(\omega_x t - \alpha)\,, \tag{2.29}$$
$$y(t) = B\cos(\omega_y t - \beta)\,, \tag{2.30}$$

are described by *Lissajous curves*. They are closed if ω_x/ω_y is a rational fraction.

2.3 Phase diagrams*

The motion of a simple harmonic oscillator is completely determined if the initial conditions $x(t_0)$ and $\dot{x}(t_0)$ are given, i.e. we can calculate $x(t)$ and $\dot{x}(t)$ for arbitrary times. The space with coordinates $x, \dot{x}$ is called *phase space*.[8] For a general oscillator with n degrees of freedom, this is a $2n$-dimensional space. As time passes, the point $(x, \dot{x})$ will trace out a path in phase space. Different initial conditions correspond to different paths. No two (non-identical) paths in phase space can ever cross: If we regard the crossing point as initial condition, this would mean that there are two different solutions to the equations of motion with the same initial conditions. This is impossible, since the solutions to the second-order differential equation are unique.

The phase space can be defined for general dynamical systems, not only harmonic oscillators. The last statement remains true in the general case.

For the simple harmonic oscillator, we have

$$x(t) = A\sin(\omega_0 t - \phi)\,, \tag{2.31}$$
$$\dot{x}(t) = A\omega_0\cos(\omega_0 t - \phi)\,. \tag{2.32}$$

It follows

$$\frac{x^2}{A^2} + \frac{\dot{x}^2}{A^2\omega_0^2} = 1\,. \tag{2.33}$$

This is a family of ellipses. Using $E = kA^2/2$ and $\omega_0^2 = k/m$, we can rewrite this equation as

$$\frac{x^2}{2E/k} + \frac{\dot{x}^2}{2E/m} = 1\,. \tag{2.34}$$

The size of the ellipse corresponds to the total energy of the oscillator.

[8] Frequently, the momentum is used as a variable instead of $\dot{x}$.

2.4 Damped oscillations

Next, we study oscillation in the presence of a retarding (or *damping*) force. We will treat only the one-dimensional case and assume that the damping force is proportional to the velocity. The equations of motion are then

$$m\ddot{x} + b\dot{x} + kx = 0\,, \tag{2.35}$$

with $b > 0$. It will be convenient to write this in the form

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0\,, \tag{2.36}$$

with the *damping parameter* $\beta = b/(2m)$, and $\omega_0^2 = k/m$ as before. The general solution for $\omega_0^2 \neq \beta^2$ is[9]

$$x(t) = e^{-\beta t}\left[A_1 \exp\left(\sqrt{\beta^2 - \omega_0^2}t\right) + A_2 \exp\left(-\sqrt{\beta^2 - \omega_0^2}t\right)\right]. \tag{2.38}$$

We discuss the three cases

$$\omega_0^2 > \beta^2 \;: \text{Underdamping}\,, \tag{2.39}$$
$$\omega_0^2 = \beta^2 \;: \text{Critical damping}\,, \tag{2.40}$$
$$\omega_0^2 < \beta^2 \;: \text{Overdamping} \tag{2.41}$$

in turn.

Underdamped motion

For *underdamped* motion, $\omega_0^2 > \beta^2$, the arguments of the exponentials in the solution (2.38) are imaginary. We define $\omega_1^2 \equiv \omega_0^2 - \beta^2$, such that

$$x(t) = e^{-\beta t}\left[A_1 e^{i\omega_1 t} + A_2 e^{-i\omega_1 t}\right]. \tag{2.42}$$

This can also be written as

$$x(t) = Ae^{-\beta t}\cos(\omega_1 t - \delta)\,. \tag{2.43}$$

We see that the maximum amplitude decreases exponentially with time. The quantity ω_1 is not strictly speaking and angular frequency, as the motion is not strictly periodic. Note that $\omega_1 < \omega_0$, and the total energy of the oscillator is not conserved (the system is not closed, part of the energy is continuously dissipated as heat). See Fig. 5.

[9] To derive this form, insert the ansatz $x(t) = \exp(rt)$ into Eq. (2.36). This yields the equation

$$r^2 + 2\beta r + \omega_0^2 = 0\,, \tag{2.37}$$

with solutions $r_{1,2} = -\beta \pm \sqrt{\beta^2 - \omega_0^2}$.

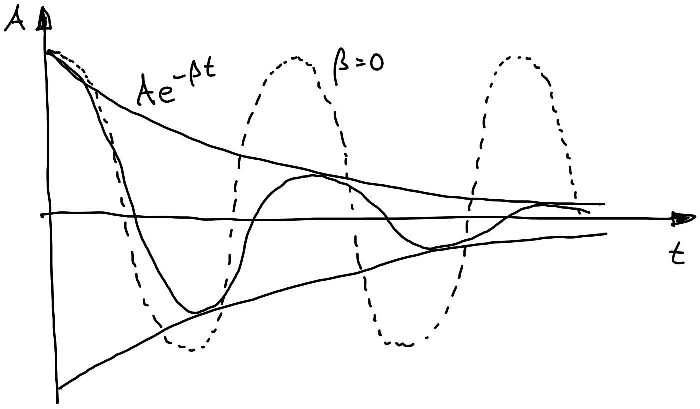

Figure 5: *Underdamped oscillator.*

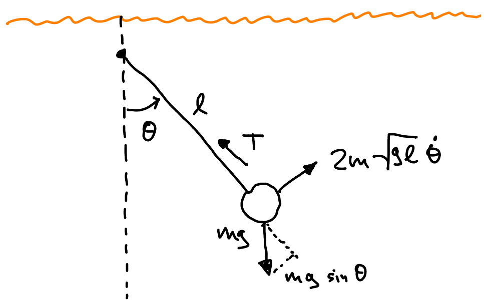

Figure 6: *Pendulum in oil. The bob is moving with decreasing* θ*.*

Critically damped motion

If $\beta^2 \geq \omega_0^2$, the system does not show oscillatory motion. *Critical damping* occurs just at the transition between the two cases, $\beta^2 = \omega_0^2$. The solution is then

$$x(t) = (A + Bt)e^{-\beta t}\,, \tag{2.44}$$

as can be directly verified by substituting into Eq. (2.36). The critically damped oscillator will approach the equilibrium position faster than the under- or overdamped oscillator.

Overdamped motion

The case $\beta^2 > \omega_0^2$ is called *overdamped* motion. The solution in this case is

$$x(t) = e^{-\beta t}\left[A_1 e^{\omega_2 t} + A_2 e^{-\omega_2 t}\right]\,, \tag{2.45}$$

with $\omega_2 = \sqrt{\beta^2 - \omega_0^2}$.

Example 2.1: Pendulum in oil. *Consider a pendulum of length* ℓ *and a spherical bob of mass* m *at its end, moving in oil such that* θ *is decreasing (see Fig. 6). The oil retards the bob's motion* *according to* STOKES*' law* (1.61)*,*

$$f_S = 6\pi\eta r v \,.$$

The bob is initially pulled back such that $\theta(t=0)=\alpha$*, and* $\dot\theta(t=0)=0$*. Assuming* $\eta = m\sqrt{g/\ell}/(3\pi r)$*, find the angular displacement* θ *and velocity* $\dot\theta$ *as a function of time.*

The equation of motion is

$$m\ell\ddot\theta = -mg\sin\theta - 2m\sqrt{g/\ell}(\ell\dot\theta)\,. \tag{2.46}$$

For small angles $\sin\theta \approx \theta$*, and we get*

$$\ddot\theta + 2\sqrt{g/\ell}\dot\theta + \frac{g}{\ell}\theta = 0\,. \tag{2.47}$$

In our previous notation, $\beta = \omega_0 = \sqrt{g/\ell}$*; the motion is critically damped and the solution for* $\theta(t)$ *is given by Eq.* (2.44)*. The constants* A *and* B *can be determined from the initial conditions. We have* $\theta(t=0) = A = \alpha$*, and*

$$\dot\theta(t) = Be^{-\beta t} - \beta(A+Bt)e^{-\beta t}\,, \tag{2.48}$$

and so $\dot\theta(t=0) = B - \beta A = 0$*, hence* $B = \beta\alpha$*. Putting pieces together, we find*

$$\theta(t) = \alpha(1+\sqrt{g/\ell}t)e^{-\sqrt{g/\ell}t}\,, \tag{2.49}$$

$$\dot\theta(t) = -\frac{\alpha g}{\ell}te^{-\sqrt{g/\ell}t}\,. \tag{2.50}$$

2.5 Driving forces

Here, we discuss only the simplest case of a driven oscillator in which the driving force itself exhibits simple harmonic oscillation. We will restrict ourselves to the one-dimensional case. The total force is then

$$F = -kx - b\dot x + F_0\cos(\omega t)\,, \tag{2.51}$$

where we also included a linear damping force. The equations of motion are then

$$\ddot x + 2\beta\dot x + \omega_0^2 x = A\cos(\omega t)\,, \tag{2.52}$$

where $A \equiv F_0/m$, and ω is the angular frequency of the driving force. This is an *inhomogeneous* linear differential equation. Its general solution is given by the general solution of the *homogeneous* system (i.e. with $A=0$), plus a so-called *particular solution.* The solution of the homogeneous equation is given by Eq. (2.38). To find the particular solution, we make the ansatz

$$x_p(t) = D\cos(\omega t - \delta)\,. \tag{2.53}$$

Substituting into Eq. (2.52), we obtain

$$-D\omega^2\cos(\omega t-\delta) - 2\beta D\omega\sin(\omega t-\delta) + D\omega_0^2\cos(\omega t-\delta) = A\cos(\omega t)\,, \tag{2.54}$$

or, using trigonometric identities,

$$\begin{aligned}\{A-D[(\omega_0^2-\omega^2)\cos\delta+2\omega\beta\sin\delta]\}\cos(\omega t)\\ -D[(\omega_0^2-\omega^2)\sin\delta-2\omega\beta\cos\delta]\sin(\omega t)=0\,.\end{aligned} \tag{2.55}$$

The coefficient of $\sin(\omega t)$ must vanish, so

$$\tan\delta=\frac{2\omega\beta}{\omega_0^2-\omega^2}\,, \tag{2.56}$$

or

$$\begin{aligned}\sin\delta&=\frac{2\omega\beta}{\sqrt{(\omega_0^2-\omega^2)^2+4\omega^2\beta^2}}\,,\\ \cos\delta&=\frac{\omega_0^2-\omega^2}{\sqrt{(\omega_0^2-\omega^2)^2+4\omega^2\beta^2}}\,.\end{aligned} \tag{2.57}$$

The coefficient of $\cos(\omega t)$ must also vanish, so

$$D=\frac{A}{(\omega_0^2-\omega^2)\cos\delta+2\omega\beta\sin\delta}=\frac{A}{\sqrt{(\omega_0^2-\omega^2)^2+4\omega^2\beta^2}}\,. \tag{2.58}$$

Hence, the particular solution is

$$x_p(t)=\frac{A}{\sqrt{(\omega_0^2-\omega^2)^2+4\omega^2\beta^2}}\cos(\omega t-\delta)\,, \tag{2.59}$$

with

$$\delta=\arctan\left(\frac{2\omega\beta}{\omega_0^2-\omega^2}\right)\,. \tag{2.60}$$

For nonzero β, there is a delay between the driving force and the motion of the oscillator.

The general solution is

$$x(t)=x_t(t)+x_p(t)\,, \tag{2.61}$$

where the *transient* $x_t(t)$ is given by Eq. (2.38). This part of the solution decays exponentially; in other words, $x(t\gg 1/\beta)=x_p(t)$ (*steady-state solution*).

The amplitude of the forced oscillation is largest for the *resonance frequency* ω_R; setting the derivative $dD/d\omega$ to zero gives

$$\omega_R=\sqrt{\omega_0^2-2\beta^2}\,. \tag{2.62}$$

2.6 Superposition principle and Fourier series*

The one-dimensional oscillations the we discussed above all had equations of motion of the general form

$$\left(\frac{d^2}{dt^2}+a\frac{d}{dt}+b\right)x(t)=F(t)\,. \tag{2.63}$$

The quantity in parentheses is called a *linear differential operator* $\boldsymbol{L}$, i.e. we can write

$$\boldsymbol{L}x(t) = F(t)\,. \tag{2.64}$$

$\boldsymbol{L}$ satisfies the *superposition principle*:

$$\boldsymbol{L}\big(x_1(t) + x_2(t)\big) = \boldsymbol{L}x_1(t) + \boldsymbol{L}x_2(t)\,. \tag{2.65}$$

Hence, given two solutions $x_1(t)$ and $x_2(t)$ for two different force functions $F_1(t)$ and $F_2(t)$, respectively,

$$\boldsymbol{L}x_1(t) = F_1(t)\,, \boldsymbol{L}x_2(t) = F_2(t)\,, \tag{2.66}$$

we can form the linear combination (with constant α_1, α_2) and obtain

$$\boldsymbol{L}\big(\alpha_1 x_1(t) + \alpha_2 x_2(t)\big) = \alpha_1 F_1(t) + \alpha_2 F_2(t)\,, \tag{2.67}$$

or, more generally,

$$\boldsymbol{L}\left(\sum_{n=1}^{N} \alpha_n x_n(t)\right) = \sum_{n=1}^{N} \alpha_n F_n(t)\,. \tag{2.68}$$

This is again of the form (2.64), with

$$x(t) = \sum_{n=1}^{N} \alpha_n x_n(t)\,, \quad F(t) = \sum_{n=1}^{N} \alpha_n F_n(t)\,. \tag{2.69}$$

If each of the individual $F_n(t)$ is of the form $\cos(\omega_n t)$, we know that the solution is given by Eq. (2.59). Therefore, if

$$F_n(t) = \sum_{n=1}^{N} \alpha_n \cos(\omega_n t)\,, \tag{2.70}$$

the steady-state solution is

$$x(t) = \frac{1}{m} \sum_{n} \frac{\alpha_n}{\sqrt{(\omega_0^2 - \omega_n^2)^2 + 4\omega_n^2\beta^2}} \cos(\omega_n t - \delta_n)\,, \tag{2.71}$$

with

$$\delta_n = \arctan\left(\frac{2\omega_n\beta}{\omega_0^2 - \omega_n^2}\right)\,. \tag{2.72}$$

A similar solution can be written down if the force is of the form

$$F_n(t) = \sum_{n=1}^{N} \alpha_n \sin(\omega_n t)\,. \tag{2.73}$$

Now *any* periodic function can be written as a (finite or infinite) *Fourier series*: If $F(t+\tau) = F(t)$ with period $\tau = 2\pi/\omega$, we have

$$F_n(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \big(a_n \cos(n\omega t) + b_n \sin(n\omega t)\big)\,, \tag{2.74}$$

where

$$a_n = \frac{2}{\tau}\int_0^\tau dt' F(t')\cos(n\omega t') = \frac{\omega}{\pi}\int_{-\pi/\omega}^{\pi/\omega} dt' F(t')\cos(n\omega t')\,, \tag{2.75}$$

$$b_n = \frac{2}{\tau}\int_0^\tau dt' F(t')\sin(n\omega t') = \frac{\omega}{\pi}\int_{-\pi/\omega}^{\pi/\omega} dt' F(t')\sin(n\omega t')\,. \tag{2.76}$$

Example 2.2: Sawtooth function. *The sawtooth function*

$$F(t) = A\,\frac{t}{\tau} = \frac{\omega A}{2\pi}t\,, \qquad -\tau/2 < t < \tau/2\,, \tag{2.77}$$

is odd, *i.e.* $F(-t) = -F(t)$, *hence all coefficients* a_n *vanish. The coefficients* b_n *are given by*

$$\begin{aligned} b_n &= \frac{\omega^2 A}{2\pi^2}\int_{-\pi/2}^{-\pi/2} t'\sin(n\omega t')dt' \\ &= \frac{\omega^2 A}{2\pi^2}\left[-\frac{t'\cos(n\omega t')}{n\omega} + \frac{\sin(n\omega t')}{n^2\omega^2}\right]\Bigg|_{-\pi/2}^{-\pi/2} = \frac{A}{n\pi}(-1)^{n+1}\,, \end{aligned} \tag{2.78}$$

or

$$F(t) = \frac{A}{\pi}\left[\sin(\omega t) - \frac{1}{2}\sin(2\omega t) + \frac{1}{3}\sin(3\omega t) \mp \ldots\right]. \tag{2.79}$$

3 Gravitation

3.1 Newton's law of gravitation

Newton's law of universal gravitation states that *each massive particle attracts each other massive particle in the universe with a force that varies directly as the product of the two masses and inversely as the square of the distance between them.* It is easier to write this as a mathematical formula:

$$\boldsymbol{F} = -G\frac{mM}{r^2}\hat{\boldsymbol{r}}\,. \tag{3.1}$$

The unit vector $\hat{\boldsymbol{r}} \equiv \boldsymbol{r}/r$, with $r = \sqrt{|\boldsymbol{r}^2|}$, points from the position of mass M (at the origin) to mass m (at position $\boldsymbol{r}$). Moreover, $G = 6.674\,30(15) \times 10^{-11}\,\mathrm{N\,m^2/s^2}$ [1] is Newton's gravitational constant.

Eq. (3.1) is valid for *point particles.* However, using Newton's "fourth law", we can obtain the gravitational force for extended objects by summing the forces on all individual constituents. For a body with a continuous distribution of matter with mass density ρ, the force on a "test mass" m at position $\boldsymbol{r}$ becomes an integral:

$$\boldsymbol{F}(\boldsymbol{r}) = -Gm\int_V \frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\big(\boldsymbol{r}-\boldsymbol{r}'\big)d^3\boldsymbol{r}'\,. \tag{3.2}$$

The integration is over the volume of the gravitating body, with infinitesimal volume element $d^3\boldsymbol{r}'$ at position $\boldsymbol{r}'$, and $|\boldsymbol{r}-\boldsymbol{r}'|$ is the distance between the volume element and the test mass. If instead of a test mass we consider a second extended body, a second integration is necessary.

The *gravitational field* $\boldsymbol{g}$ generated by a mass distribution is given by

$$\boldsymbol{g}(\boldsymbol{r}) = \frac{\boldsymbol{F}(\boldsymbol{r})}{m} = -G\int_V \frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\big(\boldsymbol{r}-\boldsymbol{r}'\big)d^3\boldsymbol{r}' \,. \tag{3.3}$$

This is a *force per unit mass.* Near the surfaces of the earth, $g \equiv |\boldsymbol{g}|$ is just the usual gravitational acceleration, $g = 9.81\,\text{m/s}^2$. For a point mass M, we have

$$\boldsymbol{g} = -G\frac{M}{r^2}\hat{\boldsymbol{r}} \,. \tag{3.4}$$

3.2 Gravitational potential

The gravitational field can be written as the gradient of a scalar *gravitational potential* Φ,[10]

$$\boldsymbol{g}(\boldsymbol{r}) = -\boldsymbol{\nabla}\Phi(\boldsymbol{r}) \,. \tag{3.5}$$

For a point mass M at the origin, the gravitational potential is given by

$$\Phi(\boldsymbol{r}) = -G\frac{M}{r} \,, \tag{3.6}$$

up to a constant that is conventionally fixed such that $\Phi \to 0$ as $r \to \infty$. For a continuous mass distribution, we have

$$\Phi(\boldsymbol{r}) = -G\int_V \frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}d^3\boldsymbol{r}' \,. \tag{3.7}$$

To get some insight on the physical significance of the gravitational potential, let's calculate the work per unit mass that is required to displace a body in the gravitational field by a small distance $d\boldsymbol{r}$. The work done on the body is

$$dW = -\,\boldsymbol{g}\cdot d\boldsymbol{r} = \boldsymbol{\nabla}\Phi\cdot d\boldsymbol{r} = \sum_i \frac{\partial\Phi}{\partial x_i}dx_i = d\Phi \,, \tag{3.8}$$

the difference in potential at the two points. If we normalize the potential as above, $\Phi(\boldsymbol{r})$ is the work per unit mass that is required to bring a mass from "infinity" to the point $\boldsymbol{r}$. The potential energy U of the mass m in the gravitational field is

$$U = m\Phi \,, \tag{3.9}$$

and the gravitational force is then given by

$$\boldsymbol{F} = -\boldsymbol{\nabla}U \,. \tag{3.10}$$

Time for an important example.

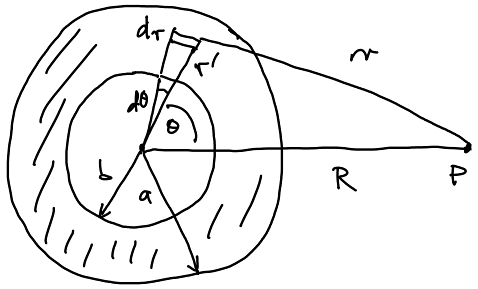

Figure 7: *Gravitational potential of a spherical shell.*

Example 3.1: Gravitational potential of a spherical shell. *We consider a homogeneous spherical shell, centered around the origin, with outer radius a and inner radius b and calculate the potential at a point P which is a distance R from the center of the shell (Fig. 7). Using spherical coordinates, we have*

$$\Phi(\boldsymbol{R}) = -G\int \frac{\rho(\boldsymbol{r}')}{|\boldsymbol{R}-\boldsymbol{r}'|}d^3\boldsymbol{r}' = -2\pi\rho G\int\limits_b^a r'^2 dr' \int\limits_0^\pi \frac{\sin\theta d\theta}{r}\,, \tag{3.11}$$

where we used the rotational symmetry about the axis RP, and $\boldsymbol{r} \equiv \boldsymbol{R} - \boldsymbol{r}'$. We have

$$r^2 = \boldsymbol{r}^2 = (\boldsymbol{R}-\boldsymbol{r}')^2 = R^2 + r'^2 - 2r'R\cos\theta\,. \tag{3.12}$$

For fixed R and r', we may regard r as a function of θ and take the derivative,

$$2rdr = 2r'R\sin\theta d\theta\,, \tag{3.13}$$

or

$$\frac{\sin\theta d\theta}{r} = \frac{dr}{r'R}\,. \tag{3.14}$$

Using this, the potential becomes

$$\Phi(\boldsymbol{R}) = -\frac{2\pi\rho G}{R}\int\limits_b^a r'dr' \int\limits_{r_{min}}^{r_{max}} dr\,. \tag{3.15}$$

The values of $r_{min} = r(\theta = 0)$ and $r_{max} = r(\theta=\pi)$ depend on the location of the point P. For P outside the shell, we find

$$\Phi(\boldsymbol{R}) = -\frac{2\pi\rho G}{R}\int\limits_b^a r'dr' \int\limits_{R-r'}^{R+r'} dr = -\frac{4\pi\rho G}{R}\int\limits_b^a r'^2 dr = -\frac{4\pi\rho G}{3R}(a^3-b^3)\,. \tag{3.16}$$

[10]The reason is that the rotation of the gravitational field vanishes, $\boldsymbol{\nabla}\times\boldsymbol{g} = 0$.

Since the mass of the shell is

$$M = \frac{4\pi\rho}{3}(a^3 - b^3)\,, \tag{3.17}$$

the potential is just

$$\Phi(\boldsymbol{R}) = -\frac{GM}{R}\,. \tag{3.18}$$

The potential outside the shell (or, for $b = 0$, outside the sphere) is the same as if the whole mass were concentrated at the origin.

If P is located inside the shell, we have

$$\Phi(\boldsymbol{R}) = -\frac{2\pi\rho G}{R}\int\limits_b^a r'dr' \int\limits_{r'-R}^{r'+R} dr = -4\pi\rho G\int\limits_b^a r'dr = -2\pi\rho G(a^2 - b^2)\,. \tag{3.19}$$

In this case, the potential is constant, and the gravitational force inside the shell vanishes.

The solution for the case that P is located within the shell is a sum of the results obtained above, where we replace the upper limit of integration by R in the solution for R outside the shell, and the lower limit of integration by R for R inside the shell:

$$\Phi(\boldsymbol{R}) = -\frac{4\pi\rho G}{3R}(R^3 - b^3) - 2\pi\rho G(a^2 - R^2) = -4\pi\rho G\left(\frac{a^2}{2} - \frac{b^3}{3R} - \frac{R^2}{6}\right). \tag{3.20}$$

3.3 Poisson's equation*

It is sometimes useful to carry over some formalism that should be familiar from electrostatics. Consider an arbitrary surface S with a point mass M inside. We want to find the gravitational flux of the mass M through the surface S:

$$\Phi_M = \int\limits_S \boldsymbol{n}\cdot\boldsymbol{g}da\,, \tag{3.21}$$

where $\boldsymbol{n}$ is the unit vector normal to the infinitesimal surface da, and $\boldsymbol{g}$ is the gravitational field generated by the mass m. Substituting Eq. (3.4), we find for the scalar product

$$\boldsymbol{n}\cdot\boldsymbol{g} = -GM\frac{\cos\theta}{r^2}\,, \tag{3.22}$$

where θ is the angle between $\boldsymbol{n}$ and $\boldsymbol{g}$, and the flux becomes

$$\Phi_M = -GM\int\limits_S \frac{\cos\theta}{r^2}da\,. \tag{3.23}$$

The integral over S gives[11]

$$\Phi_M = \int\limits_S \boldsymbol{n}\cdot\boldsymbol{g}da = -4\pi GM\,. \tag{3.24}$$

[11] The proof can be taken from electrostatics. Here is a sketch: Let's first assume a spherical surface with radius r, centered around the point mass. In this case the normal vector $\boldsymbol{n}$ just points in the direction of $\boldsymbol{g}$. The

For several point masses, we can just sum over the contributions, while for a continuous distribution with mass density ρ, we have

$$\Phi_M = \int_S \boldsymbol{n} \cdot \boldsymbol{g} da = -4\pi G \int_V \rho \, dv \,. \tag{3.25}$$

Next, we apply GAUSS's theorem which states

$$\int_S \boldsymbol{n} \cdot \boldsymbol{g} da = \int_V \boldsymbol{\nabla} \cdot \boldsymbol{g} dv \,. \tag{3.26}$$

Comparing Eq. (3.25) and Eq. (3.26), and noting that the volume enclosing the mass distribution is completely arbitrary, we conclude that

$$\boldsymbol{\nabla} \cdot \boldsymbol{g} = -4\pi G \rho \,. \tag{3.27}$$

Finally, inserting Eq. (3.5), we obtain

$$\boldsymbol{\nabla}^2 \cdot \Phi = 4\pi G \rho \,. \tag{3.28}$$

This is *POISSON's equation.* The homogenous equation $\boldsymbol{\nabla}^2 \cdot \Phi = 0$ is called *LAPLACE's equation.*

3.4 Ocean tides

Here, we want to give a simple model of the tides – the movement of water on the earth's surface due to the gravitational potential of the moon and the sun. We consider the earth as spherical and neglect, for now, its own rotation. Also, we first consider only the gravitation due to the moon. Finally, we assume that the whole earth is covered uniformly by water.

First, we establish a fixed coordinate system in space, to avoid complications due to noninertial reference frames (to be discussed in Sec. 8). See Fig. 8. Consider a small volume element of water, with mass m, on the earth's surface. The main gravitational force on the piece of water is due to the earth's gravitation. Tidal forces occur because of the moon's nonuniform gravitational potential, since the earth is an extended object. The force on the mass element m in the primed coordinate system is

$$m\ddot{\boldsymbol{r}}'_m = -G\frac{M_E m}{r^2}\hat{\boldsymbol{r}} - G\frac{M_M m}{R^2}\hat{\boldsymbol{R}} \,, \tag{3.29}$$

while the force of the moon on the center of mass of the earth is

$$M_E \ddot{\boldsymbol{r}}'_E = -G\frac{M_E M_M}{D^2}\hat{\boldsymbol{D}} \,. \tag{3.30}$$

gravitational field has the same value $-GM/r^2$ on the whole surface. Multiplying by the surface area $4\pi r^2$, we obtain $\Phi_M = -4\pi GM$. For a general surface, we can just enclose the point mass with a small sphere and radially project a general surface element $d\boldsymbol{A}$ onto a corresponding element $d\boldsymbol{a}$ on the sphere. If the element $d\boldsymbol{A}$ has a distance R from the point mass, it is larger than $d\boldsymbol{a}$ by a factor $(R/r)^2$, and by a factor $1/\cos\theta$ if it is tilted by an angle θ with regards to the radial lines. The exactly compensates the factor $(r/R)^2 \cos\theta$ that the flux through $d\boldsymbol{A}$ is smaller, thus showing that the total flux is the same.

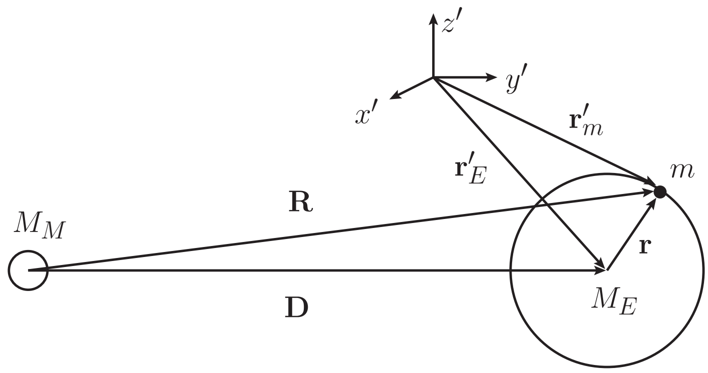

Figure 8: *Inertial coordinate system for describing the ocean tides caused by the moon's gravity.*

The acceleration of the mass point, as measured from the center of the earth, is then given by

$$\ddot{\boldsymbol{r}} = \ddot{\boldsymbol{r}}'_m - \ddot{\boldsymbol{r}}'_E = -G\frac{M_E}{r^2}\hat{\boldsymbol{r}} - GM_M\left(\frac{\hat{\boldsymbol{R}}}{R^2} - \frac{\hat{\boldsymbol{D}}}{D^2}\right). \tag{3.31}$$

The tidal force is

$$\boldsymbol{F}_t = -GmM_M\left(\frac{\hat{\boldsymbol{R}}}{R^2} - \frac{\hat{\boldsymbol{D}}}{D^2}\right). \tag{3.32}$$

Using that $R^2 = D^2 + 2rD\cos\theta + r^2$ and hence

$$\begin{aligned} R^3 &= (D^2 + 2rD\cos\theta + r^2)^{3/2} = D^3(1 + 2r/D\cos\theta + r^2/D^2)^{3/2} \\ &= D^3 + 3D^2 r\cos\theta + \ldots\,, \end{aligned} \tag{3.33}$$

we find

$$\begin{aligned} \frac{\hat{\boldsymbol{R}}}{R^2} - \frac{\hat{\boldsymbol{D}}}{D^2} &= \frac{\boldsymbol{R}}{R^3} - \frac{\boldsymbol{D}}{D^3} = \frac{\boldsymbol{D}+\boldsymbol{r}}{D^3 + 3D^2 r\cos\theta} - \frac{\boldsymbol{D}}{D^3} \\ &= \frac{\boldsymbol{D}+\boldsymbol{r}}{D^3}\left(1 - 3\frac{r}{D}\cos\theta\right) - \frac{\boldsymbol{D}}{D^3} \\ &= \frac{r}{D^3}\big(\hat{\boldsymbol{r}} - 3\cos\theta\hat{\boldsymbol{D}}\big)\,. \end{aligned} \tag{3.34}$$

For instance, the tidal force on the earth's surfaces closest to and farthest from the moon is $F_{t,y} = \mp 2GmM_M r/D^3$ (here, $\cos\theta = \mp 1$, while $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{D}}$ point in the same or opposite directions, respectively). At the "poles" (i.e. $\cos\theta = 0$) we have $F_{t,z} = \pm GmM_M r/D^3$. More generally, writing the unit vectors as $\hat{\boldsymbol{D}} = (0,1,0)$ and $\hat{\boldsymbol{r}} = (\sin\theta\sin\phi, \cos\theta, \sin\theta\cos\phi)$, we have

$$F_{t,x} = \frac{r}{D^3}\sin\theta\sin\phi\,, \tag{3.35}$$

$$F_{t,y} = -\frac{2r}{D^3}\cos\theta\,, \tag{3.36}$$

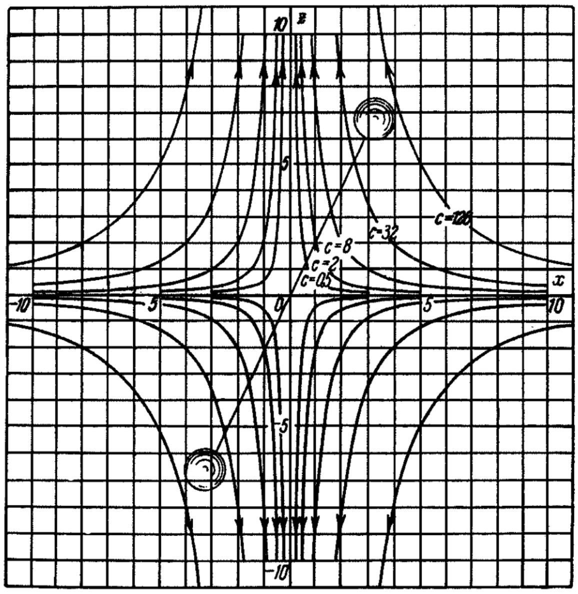

FIGURE 2. Field of perturbation forces.

Figure 9: *Schematic sketch of tidal forces (from V. V. Beletskii, "Motion of an artificial satellite about its center of mass", Moscow 1966).*

$$F_{t,z} = \frac{r}{D^3} \sin\theta \cos\phi \,. \tag{3.37}$$

A cross section of this force field is sketched in Fig. 9. Of course, the Earth turns within this force field, while the moon revolves around the earth, such that a fixed point on earth experiences approximately two high tides per day. (Note also that the torques created by the tidal force tend to align a non-spherical object along the earth-moon axis. This goes a long way to explain why the moon always show the same side towards the earth.)

4 Variational calculus

In this section we discuss the mathematical formalism required for Lagrangian and Hamiltonian mechanics.

4.1 Euler's equation

We want to find a function $y(x)$ that *extremizes* (i.e., *minimizes* or *maximizes*) the *functional*

$$J[y] = \int_{x_1}^{x_2} f\{y(x), y'(x), x\}\, dx \,. \tag{4.1}$$

See Fig. 10. Here, $y'(x) \equiv dy/dx$. For the moment, we regard the integration boundaries as fixed. A function $y(x)$ minimizes (maximizes) J if *any* neighboring function increases

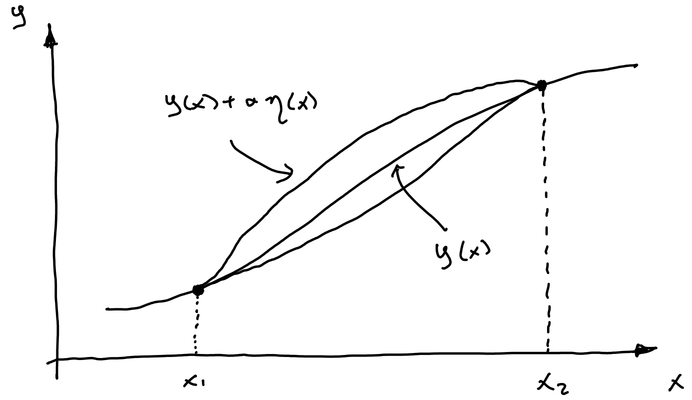

Figure 10: *Variation $\delta y(x)$ of a path $y(x)$*

(decreases) the value of J. We can parameterize the neighboring fuctions by writing $y = y(\alpha, x)$, where α is some parameter such that $y(0, x) = y(x)$; i.e., we write

$$y(\alpha, x) = y(x) + \alpha\eta(x)\,, \tag{4.2}$$

where $\eta(x)$ is an arbitrary smooth function that vanishes at x_1 and x_2. We can then regard J as a function of the parameter α:

$$J(\alpha) = \int_{x_1}^{x_2} f\{y(\alpha, x), y'(\alpha, x), x\}\, dx\,. \tag{4.3}$$

The condition that J be stationary is that J be independent of α to first order, or

$$\left.\frac{dJ}{d\alpha}\right|_{\alpha=0} = 0\,, \tag{4.4}$$

for arbitrary $\eta(x)$. (This is only a necessary, not a sufficient condition.)

Example 4.1: Sinusoidal variation of a straight line. *Consider the function $f = (dy/dx)^2$, and $y(x) = x$. We take $\eta(x) = \sin(x)$ as the variation, and aim to find the stationary value of J between the point 0 and 2π.*

We have

$$y(\alpha, x) = x + \alpha \sin(x)\,, \tag{4.5}$$

and

$$\frac{dy(\alpha, x)}{dx} = 1 + \alpha\cos(x)\,. \tag{4.6}$$

It follows that

$$f\{y, y', x\} = (y')^2 = 1 + 2\alpha\cos(x) + \alpha^2\cos^2(x)\,, \tag{4.7}$$

and[12]

$$J(\alpha) = \int\limits_{0}^{2\pi} \left(1 + 2\alpha\cos(x) + \alpha^2\cos^2(x)\right) dx = 2\pi + 0 + \alpha^2\pi \,. \tag{4.8}$$

J is minimized by the straight line.

We now perform the differentiation in Eq. (4.4) explicitly:

$$\frac{dJ}{d\alpha} = \frac{d}{d\alpha}\int\limits_{x_1}^{x_2} f(y, y', x)\, dx = \int\limits_{x_1}^{x_2} \left(\frac{\partial f}{\partial y}\frac{\partial y}{\partial \alpha} + \frac{\partial f}{\partial y'}\frac{\partial y'}{\partial \alpha}\right) dx \,. \tag{4.9}$$

Using Eq. (4.2) we have

$$\frac{\partial y(\alpha, x)}{\partial \alpha} = \eta(x)\,, \qquad \frac{\partial y'(\alpha, x)}{\partial \alpha} = \frac{\partial \eta(x)}{\partial x}\,, \tag{4.10}$$

and we get

$$\frac{dJ}{d\alpha} = \int\limits_{x_1}^{x_2} \left(\frac{\partial f}{\partial y}\eta(x) + \frac{\partial f}{\partial y'}\frac{\partial \eta(x)}{\partial x}\right) dx \,. \tag{4.11}$$

Next, we integrate the second term by parts:

$$\int\limits_{x_1}^{x_2} \frac{\partial f}{\partial y'}\frac{\partial \eta(x)}{\partial x}\, dx = \frac{\partial f}{\partial y'}\eta(x)\bigg|_{x_1}^{x_2} - \int\limits_{x_1}^{x_2} \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y'}\right)\eta(x)\, dx \,. \tag{4.12}$$

The boundary term vanishes since we assumed $\eta(x_1) = \eta(x_2) = 0$, and we obtain finally

$$\frac{dJ}{d\alpha} = \int\limits_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} - \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y'}\right)\right]\eta(x)\, dx \,. \tag{4.13}$$

Because $\eta(x)$ is an arbitrary function, in order to satisfy the condition (4.4) we need

$$\frac{\partial f}{\partial y} - \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y'}\right) = 0\,, \tag{4.14}$$

where now $\alpha = 0$, i.e. y and y' are the original, unperturbed functions of x. Eq. (9.62) is known as *Euler's equation.*

Example 4.2: Brachistochrone. *Consider a particle moving in a constant force field starting at rest from some point (x_1, y_1) to some other point (x_2, y_2), with $x_2 \neq x_1$ and $y_2 \neq y_1$. See Fig. 11, left panel. Which path allows the point to travel in the least possible time?*

[12] $d[\sin(x)\cos(x) + x]/dx = 2\cos^2(x)$

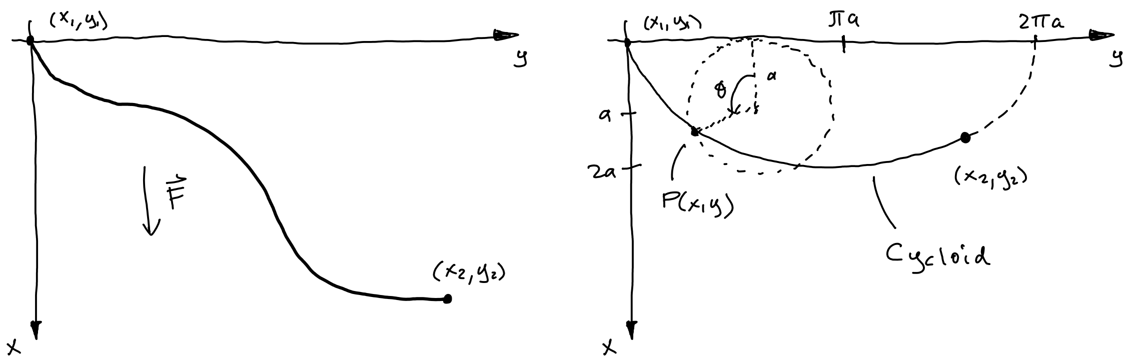

Figure 11: *The solution to the brachistochrone problem (left panel) is a cycloid (right panel).*

Let's choose the coordinate system such that the initial position of the particle is at the origin, and the force is directed downwards along the x direction. We define the potential energy U to be zero at the origin. Then the total energy is $T + U = 0$, or $\frac{1}{2}mv^2 = mgx$. It follows that $v = \sqrt{2gx}$. The time required for the transit is

$$t = \int_{(0,0)}^{(x_2,y_2)} \frac{ds}{v} = \int_{(0,0)}^{(x_2,y_2)} \frac{\sqrt{dx^2 + dy^2}}{\sqrt{2gx}} = \frac{1}{\sqrt{2g}} \int_0^{x_2} \sqrt{\frac{1 + y'^2}{x}}\, dx \,. \tag{4.15}$$

Hence, in our notation

$$f(y, y', x) = \sqrt{\frac{1 + y'^2}{x}} = f(y', x)\,, \tag{4.16}$$

and Euler's equation is just

$$\frac{\partial}{\partial x}\frac{\partial f}{\partial y'} = 0\,, \tag{4.17}$$

or

$$\frac{\partial f}{\partial y'} = \textit{constant} \equiv \frac{1}{\sqrt{2a}}\,. \tag{4.18}$$

We have

$$\frac{\partial f}{\partial y'} = \frac{1}{2}\sqrt{\frac{x}{1 + y'^2}} \cdot \frac{2y'}{x}\,, \tag{4.19}$$

or

$$\frac{y'^2}{x(1 + y'^2)} = \frac{1}{2a}\,. \tag{4.20}$$

Solving for dy and integrating gives

$$y = \int \frac{x dx}{\sqrt{2ax - x^2}}\,. \tag{4.21}$$

Now we change integration variables: $x = a(1 - \cos\theta)$, $dx = a\sin\theta\, d\theta$, and find (using the initial conditions)

$$y = \int a(1 - \cos\theta) d\theta = a(\theta - \sin\theta)\,. \tag{4.22}$$

These are the equations for the cycloid *(see Fig. 11, right panel):*

$$x = a(1 - \cos\theta)\,, \tag{4.23}$$

$$y = a(\theta - \sin\theta)\,. \tag{4.24}$$

If f does not explicitly depend on x, we can find a somewhat simpler version of EULER's equation. We have

$$\frac{df}{dx} = \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} + \frac{\partial f}{\partial x} = y'\frac{\partial f}{\partial y} + y''\frac{\partial f}{\partial y'} + \frac{\partial f}{\partial x}\,. \tag{4.25}$$

Now we substitute the relation

$$\frac{d}{dx}\left(y'\frac{\partial f}{\partial y'}\right) = y''\frac{\partial f}{\partial y'} + y'\frac{d}{dx}\frac{\partial f}{\partial y'}\,, \tag{4.26}$$

into Eq. (4.25) and obtain

$$\frac{d}{dx}\left(y'\frac{\partial f}{\partial y'}\right) = \frac{df}{dx} - \frac{\partial f}{\partial x} + y'\left(\frac{d}{dx}\frac{\partial f}{\partial y'} - \frac{\partial f}{\partial y}\right)\,, \tag{4.27}$$

where the two terms in the parentheses cancel due to EULER's equation (9.62). Therefore, we have

$$\frac{\partial f}{\partial x} - \frac{d}{dx}\left(f - y'\frac{\partial f}{\partial y'}\right) = 0\,. \tag{4.28}$$

If f does not explicitly depend on x, this gives

$$f - y'\frac{\partial f}{\partial y'} = \text{constant}\,. \tag{4.29}$$

Functions with several dependent variables

So far, we have considered the variational problem for a functional of a single function $y(x)$. Frequently, one encounters the case that several such functions $y_i(x)$ are present, i.e.

$$f = f\{y_1(x), y_1'(x), y_2(x), y_2'(x), \ldots; x\} \equiv f\{y_i(x), y_i'(x); x\}\,, \tag{4.30}$$

where $i = 1, \ldots, n$. We can derive the corresponding EULER equations in analogy to before by defining

$$y_i(\alpha, x) = y_i(x) + \alpha\eta_i(x)\,. \tag{4.31}$$

Then, following the same steps as above, we obtain

$$\frac{dJ}{d\alpha} = \int_{x_1}^{x_2}\left[\sum_i \frac{\partial f}{\partial y_i} - \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y_i'}\right)\right]\eta_i(x)\,dx\,. \tag{4.32}$$

Because the individual variations $\eta_i(x)$ are all independent, the condition that $\frac{dJ}{d\alpha}$ vanish at $\alpha = 0$ requires that each terms in the square brackets vanish separately:

$$\frac{\partial f}{\partial y_i} - \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y_i'}\right) = 0\,, \quad i = 1, \ldots, n\,. \tag{4.33}$$

4.2 Euler's equations with constraints

Often, the variables $y_i(x)$ are not all independent, i.e. they satisfy the equations of constraint

$$g\{y_i; x\} = 0\,. \tag{4.34}$$

Let's consider the case with two variables, y_1 and y_2. We write the variations again in the form $y_i(x,\alpha) = y_i(x) + \alpha\eta_i(x)$, for $i = 1, 2$. The variation of J is then

$$\frac{dJ}{d\alpha} = \int_{x_1}^{x_2} \left[\left(\frac{\partial f}{\partial y_1} - \frac{d}{dx}\frac{\partial f}{\partial y_1'} \right) \frac{dy_1}{d\alpha} + \left(\frac{\partial f}{\partial y_2} - \frac{d}{dx}\frac{\partial f}{\partial y_2'} \right) \frac{dy_2}{d\alpha} \right] dx\,. \tag{4.35}$$

Because of the constraint (that must apply also to the varied functions)

$$g\{y_1, y_2; x\} = 0\,, \tag{4.36}$$

the variations $dy_1/d\alpha = \eta_1$ and $dy_2/d\alpha = \eta_2$ are no longer independent, and the two expressions in parentheses do not vanish separately. Differentiating Eq. (4.36) gives

$$\frac{dg}{d\alpha} = \frac{\partial g}{\partial y_1}\frac{dy_1}{d\alpha} + \frac{\partial g}{\partial y_2}\frac{dy_2}{d\alpha} = \frac{\partial g}{\partial y_1}\eta_1 + \frac{\partial g}{\partial y_2}\eta_2 = 0\,, \tag{4.37}$$

or

$$\frac{\partial g}{\partial y_1}\eta_1(x) = -\frac{\partial g}{\partial y_2}\eta_2(x)\,. \tag{4.38}$$

Thus, the variation of J becomes

$$\frac{dJ}{d\alpha} = \int_{x_1}^{x_2} \left[\left(\frac{\partial f}{\partial y_1} - \frac{d}{dx}\frac{\partial f}{\partial y_1'} \right) \eta_1(x) + \left(\frac{\partial f}{\partial y_2} - \frac{d}{dx}\frac{\partial f}{\partial y_2'} \right) \eta_2(x) \right] dx \tag{4.39}$$

$$= \int_{x_1}^{x_2} \left[\left(\frac{\partial f}{\partial y_1} - \frac{d}{dx}\frac{\partial f}{\partial y_1'} \right) - \left(\frac{\partial f}{\partial y_2} - \frac{d}{dx}\frac{\partial f}{\partial y_2'} \right) \frac{\partial g/\partial y_1}{\partial g/\partial y_2} \right] \eta_1(x)\, dx\,. \tag{4.40}$$

From this we read off that

$$\left(\frac{\partial f}{\partial y_1} - \frac{d}{dx}\frac{\partial f}{\partial y_1'} \right) \left(\frac{\partial g}{\partial y_1} \right)^{-1} = \left(\frac{\partial f}{\partial y_2} - \frac{d}{dx}\frac{\partial f}{\partial y_2'} \right) \left(\frac{\partial g}{\partial y_2} \right)^{-1}. \tag{4.41}$$

Since both sides of this equation are a function of x, we can write this as

$$\begin{aligned} \frac{\partial f}{\partial y_1} - \frac{d}{dx}\frac{\partial f}{\partial y_1'} + \lambda(x)\frac{\partial g}{\partial y_1} &= 0\,, \\ \frac{\partial f}{\partial y_2} - \frac{d}{dx}\frac{\partial f}{\partial y_2'} + \lambda(x)\frac{\partial g}{\partial y_2} &= 0\,. \end{aligned} \tag{4.42}$$

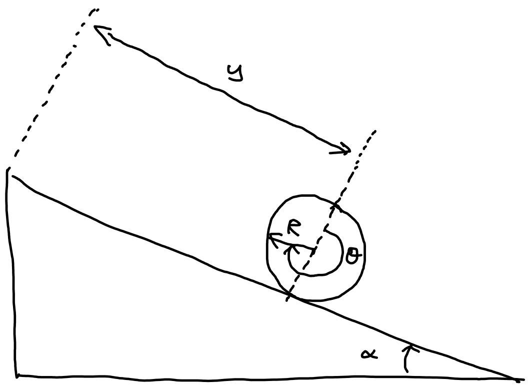

Figure 12: *Disc rolling on an inclined plane without slipping.*

We now have three relations (Eq. (4.36) and Eq. (4.42)) for the three unknown functions $y(x)$, $z(x)$, and $\lambda(x)$. The function $\lambda(x)$ is known as a *Lagrange undetermined multiplier.*

For the general case of several variables and constraints, we have the following set of equations:

$$\frac{\partial f}{\partial y_i} - \frac{d}{dx}\frac{\partial f}{\partial y_i'} + \sum_{j=1}^{m} \lambda_j(x)\frac{\partial g_j}{\partial y_i} = 0\,, \tag{4.43}$$

$$g_j\{y_i; x\} = 0\,. \tag{4.44}$$

If $i = 1, \ldots, n$ and $j = 1, \ldots, m$, these are $n+m$ equations for $n+m$ unknowns. Sometimes, the equations of constraint can be given in differential or integral form. The conditions (4.44) are equivalent to the differential equations

$$\sum_i \frac{\partial g_j}{\partial y_i} dy_i = 0\,. \tag{4.45}$$

Frequently, it is more useful to represent the constraints in differential form, as we will see later.

Example 4.3: Rolling disc on inclined plane. *The relation between the coordinates is*

$$y = R\theta\,, \tag{4.46}$$

where R is the radius of the disc. The equation of constraint is thus

$$g(y, \theta) = y - R\theta = 0\,, \tag{4.47}$$

and we have

$$\frac{\partial g}{\partial y} = 1\,, \quad \frac{\partial g}{\partial \theta} = -R\,. \tag{4.48}$$

4.3 The δ notation

Frequently, another (somewhat less explicit) notation is used in variational calculus. Denoting the variation of the functional J by δJ, the condition of stationarity becomes just $\delta J = 0$. The derivation of EULER's equation looks then somewhat smoother:

$$\delta J = \int_{x_1}^{x_2} \delta f(y, y'; x)\, dx \tag{4.49}$$

$$= \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} \delta y + \frac{\partial f}{\partial y'} \delta y' \right] dx \tag{4.50}$$

$$= \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} \delta y + \frac{\partial f}{\partial y'} \frac{d}{dx} \delta y \right] dx \tag{4.51}$$

$$= \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} \right] \delta y\, dx\,. \tag{4.52}$$

Since the variation δy is arbitrary, EULER's equation follows as before.

5 Lagrangian and Hamiltonian mechanics

NEWTON's laws are, in principle, sufficient to solve any problem in mechanics. However, in practice it can become very difficult to obtain the solutions, if non-rectangular coordinates or complicated constraints are involved. Lagrangian and Hamiltonian mechanics allows us to efficiently deal with such problems.

5.1 HAMILTON's principle

HAMILTON's principle states: *Of all the possible paths along which a dynamical system may move from one point to another within a specified time interval (consistent with any constraints), the actual path followed is that which minimizes the time integral of the difference between the kinetic and potential energies.* Using the variational calculus, we can formulate this principle as follows:

$$\delta \int_{t_1}^{t_2} (T - U)\, dt = 0\,. \tag{5.1}$$

In general, the value of the integral need not be minimal, but only extremal (minimal or maximal).

For instance, for a single particle in a conservative force field, and using rectangual coordinates, we have

$$T = T(\dot{x}_i)\,, \qquad U = U(x_i)\,. \tag{5.2}$$

We define the *Lagrange function* or *Lagrangian* as

$$L \equiv T - U = L(\dot{x}_i, x_i)\,, \tag{5.3}$$

and the condition (5.1) becomes

$$\delta \int\limits_{t_1}^{t_2} L(\dot{x}_i, x_i)\, dt = 0\,. \tag{5.4}$$

We can immediately write down the EULER equations by identifying

$$x \to t\,, \tag{5.5}$$
$$y_i(x) \to x_i(t)\,, \tag{5.6}$$
$$y_i'(x) \to \dot{x}_i(t)\,, \tag{5.7}$$
$$f\{y_i(x), y_i'(x); x\} \to L(\dot{x}_i, x_i)\,. \tag{5.8}$$

This gives the *LAGRANGE equations of motion*

$$\frac{\partial L}{\partial x_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} = 0\,, \quad i = 1, 2, 3\,. \tag{5.9}$$

As an example we consider the simple harmonic oscillator. We have

$$L = T - U = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2\,, \tag{5.10}$$
$$\frac{\partial L}{\partial x} = -kx\,, \tag{5.11}$$
$$\frac{\partial L}{\partial \dot{x}} = m\dot{x}\,, \tag{5.12}$$
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = m\ddot{x}\,, \tag{5.13}$$

and the LAGRANGE equation is

$$m\ddot{x} + kx = 0\,, \tag{5.14}$$

as before.

Next, we consider the mathematical pendulum. Starting with a rectangular coordinate system (with the origin located at the support of the pendulum), we have

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2\,, \tag{5.15}$$
$$U = mgy\,, \tag{5.16}$$
$$L = T - U = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2 - mgy\,. \tag{5.17}$$

Let's transform to polar coordinates, $x = \ell \sin\theta$, $y = -\ell\cos\theta$. We find $\dot{x} = \ell\dot{\theta}\cos\theta$, $\dot{y} = \ell\dot{\theta}\sin\theta$, and so

$$L = \frac{m}{2}\ell^2\dot{\theta}^2\big(\sin^2\theta + \cos^2\theta\big) + mg\ell\cos\theta = \frac{m}{2}\ell^2\dot{\theta}^2 + mg\ell\cos\theta\,. \tag{5.18}$$

The LAGRANGE equation for the variable θ is

$$\frac{\partial L}{\partial \theta} = -mg\ell \sin\theta\,, \tag{5.19}$$

$$\frac{\partial L}{\partial \dot{\theta}} = m\ell^2\dot{\theta}\,, \tag{5.20}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = m\ell^2\ddot{\theta}\,; \tag{5.21}$$

$$\ddot{\theta} + \frac{g}{\ell}\sin\theta = 0\,. \tag{5.22}$$

The method works even if the coordinates are not rectangular.

5.2 Generalized coordinates

Consider a mechanical system consisting of n point particles. We need n position vectors, i.e. $3n$ numbers, to describe the the positions of all particles at a given time. These can always be taken as the $3n$ Cartesian coordinates. However, these coordinates might not all be independent (i.e. if the particles form a rigid body, or if they are constrained to move on given trajectories). If there are m constraints, then only $s = 3n - m$ coordinates are independent, and we say that the system has $3n - m$ *degrees of freedom*. More generally, these coordinates need not be rectangular or curvilinear. Any s independent coordinates are allowed, as long as they uniquely specify the position of the system. They are called *(proper) generalized coordinates*, denoted by $q_i = q_1, q_2, \ldots\ q_s$. ("Proper" in this context means that the coordinates are not restricted by constraints.) Associated with them are their time derivatives or *generalized velocities*, $\dot{q}_i = \dot{q}_1, \dot{q}_2, \ldots$.

For a system of n particles with Cartesian coordinates $x_{a,i}$, $a = 1, \ldots, n$, $i = 1, 2, 3$, the relations are

$$x_{a,i} = x_{a,i}(q_j, t)\,, \tag{5.23}$$

$$\dot{x}_{a,i} = x_{a,i}(q_j, \dot{q}_j, t)\,. \tag{5.24}$$

Note that these relations may depend on time. The expressions for $\dot{x}_{a,i}$ may depend on both q_j and $\dot{q}_j$, while the expressions for $x_{a,i}$ cannot depend on the generalized velocities $\dot{q}_j$.

Example 5.1: Particle on sphere. *Find the generalized coordinates for a particle on the surface of a hemisphere.*

Call the radius R and choose the origin at the center of the sphere. The constraint in Cartesian coordinates is

$$x_1^2 + x_2^2 + x_3^2 - R^2 = 0\,, \quad z \geq 0\,. \tag{5.25}$$

As generalized coordinates we can choose the cosines between the position vector of the particle and the coordinate axes:

$$q_1 = \frac{x_1}{R}\,, \quad q_2 = \frac{x_2}{R}\,, \quad q_3 = \frac{x_3}{R}\,. \tag{5.26}$$

These are not all independent:

$$q_1^2 + q_2^2 + q_3^2 = 1 \quad \Leftrightarrow \quad q_3 = \sqrt{1 - q_1^2 - q_2^2}\,, \tag{5.27}$$

so we may choose q_1 and q_2 as our generalized coordinates.

The space of $s = 3n - m$ independent coordinates is called *configuration space*. The time history of a physical system can be represented by a path in the corresponding configuration space. This "motion" is automatically consistent with any constraints.

5.3 Lagrange's equations in generalized coordinates

We can now slightly reformulate Hamilton's principle as follows:

Of all the possible paths along which a dynamical system may move from one point to another in configuration space within a specified time interval, the actual path followed is that which minimizes the time integral of the Lagrangian function for the system.

The advantage of formulating the equations of motion in terms of a minimum principle is that the minimization condition is independent of the choice of integration variables, i.e., coordinate transformations. We can therefore express the Lagrangian in equivalent ways,

$$L = T(\dot{x}_j) - U(x_j) = T(\dot{x}_j(q_j, \dot{q}_j, t)) - U(x_j(q_j, t))\,. \tag{5.28}$$

We will then generally take the Lagrangian as a function of the generalized (independent) coordinates,

$$L = L(q, \dot{q}, t) \equiv L(q_1, \ldots, q_s; \dot{q}_1, \ldots, \dot{q}_s; t)\,. \tag{5.29}$$

Hamilton's principle becomes

$$\delta \int_{t_1}^{t_2} L(q, \dot{q}, t) = 0\,. \tag{5.30}$$

We can now again translate the results of the previous section,

$$x \to t\,, \tag{5.31}$$

$$y_i(x) \to q_i(t)\,, \tag{5.32}$$

$$y_i'(x) \to \dot{q}_i(t)\,, \tag{5.33}$$

$$f\{y_i(x), y_i'(x); x\} \to L(q_j, \dot{q}_j, t)\,, \tag{5.34}$$

and find the Lagrange equations

$$\frac{\partial L}{\partial q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} = 0\,, \quad j = 1, \ldots, s\,. \tag{5.35}$$

These equations are valid if all forces (apart from forces of constraint) can be derived from a potential[13], and if the $m = 3n - s$ equations of constraint are given in the form

$$f_k(q_j, t) = 0\,, \quad k = 1, \ldots, m\,. \tag{5.36}$$

Such constraints are called *holonomic*. If the constraints do not explicitly depend on time, they are called *scleronomic*, otherwise *rheonomic*.

[13] This is the case, e.g., for conservative forces.

Example 5.2: Particle in a cone. *A particle of mass m is constrained to move on the inside surface of a smooth cone of half angle α. The particle is subject to a constant gravitational force. Determine a set of generalized coordinates and the corresponding constraints. Find* LAGRANGE*'s equations.*

We choose cylindrical coordinates, with the apex of the cone located at the origin. The constraint keeping the mass on the surface is

$$z = r \cot \alpha \,, \tag{5.37}$$

so the system has two degrees of freedom and we need two proper generalized coordinates. We eliminate z. Then

$$v^2 = \dot{r}^2 + r^2\dot{\phi}^2 + \dot{z}^2 = \dot{r}^2\big(1 + \cot^2\alpha\big) + r^2\dot{\phi}^2 = \dot{r}^2 \csc^2\alpha + r^2\dot{\phi}^2 \,, \tag{5.38}$$

while the potential energy can be taken as

$$U = mgz = mgr \cot \alpha \,. \tag{5.39}$$

The Lagrangian is

$$L = \frac{m}{2}\big(\dot{r}^2 \csc^2\alpha + r^2\dot{\phi}^2\big) - mgr \cot\alpha \,. \tag{5.40}$$

L does not explicitly depend on ϕ, so $\partial L/\partial\phi = 0$, and the LAGRANGE *equation for ϕ implies*

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} = 0 \,, \tag{5.41}$$

such that

$$\frac{\partial L}{\partial \dot{\phi}} = mr^2\dot{\phi} = \textit{constant} \,. \tag{5.42}$$

This is the angular momentum about the z-axis. The LAGRANGE *equation for r is*

$$\frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = 0 \,, \tag{5.43}$$

or

$$\ddot{r} - r\dot{\phi}\sin^2\alpha + g \sin\alpha\cos\alpha = 0 \,. \tag{5.44}$$

Example 5.3: Pendulum with rotating support. *The point of support of a pendulum of length b and mass m moves on a circle of radius a rotating with constant angular velocity ω (Fig. 13). Find the equations of motion.*

Choose the center of the circle as the origin. The Cartesian coordinates of the mass are then

$$\begin{aligned} x &= a\cos(\omega t) + b\sin\theta \,, \\ y &= a\sin(\omega t) - b\cos\theta \,. \end{aligned} \tag{5.45}$$

where θ is the angle between the string and the vertical. The velocities are

$$\begin{aligned} \dot{x} &= -a\omega\sin(\omega t) + b\dot{\theta}\cos\theta \,, \\ \dot{y} &= a\omega\cos(\omega t) + b\dot{\theta}\sin\theta \,. \end{aligned} \tag{5.46}$$

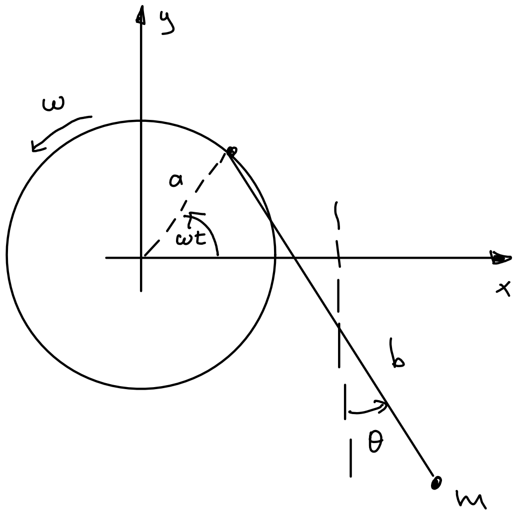

Figure 13: *Pendulum with rotating support.*

Clearly, the single generalized coordinate is θ. The Lagrangian is

$$\begin{aligned} L = T - U &= \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2\right) - mgy \\ &= \frac{m}{2}\left[a^2\omega^2 + b^2\dot{\theta}^2 + 2a\omega b\dot{\theta}\sin(\theta - \omega t)\right] - mg\big(a\sin(\omega t) - b\cos\theta\big)\,. \end{aligned} \tag{5.47}$$

To find the equations of motion, we calculate

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = mb^2\ddot{\theta} + mba\omega(\dot{\theta} - \omega)\cos(\theta - \omega t)\,, \tag{5.48}$$

$$\frac{\partial L}{\partial \theta} = mba\omega\dot{\theta}\cos(\theta - \omega t) - mgb\sin\theta\,, \tag{5.49}$$

and find

$$\ddot{\theta} = \frac{a\omega^2}{b}\cos(\theta - \omega t) - \frac{g}{b}\sin\theta\,. \tag{5.50}$$

5.4 Lagrange's equations with undetermined multipliers

Holonomic constraints, of the form (5.36), can always be solved to give a set of independent generalized coordinates. Any set of constraints that involve the velocities are nonholonomic unless the equations can be integrated to give holonomic constraints. This is the case for constraints of the form

$$\sum_j \frac{\partial f_k}{\partial q_j} dq_j + \frac{\partial f_k}{\partial t} dt = 0\,, \tag{5.51}$$

because this can be written as

$$\frac{df_k}{dt} = 0\,. \tag{5.52}$$

If the constraints are given in differential form, we can alternatively incorporate them into LAGRANGE's equations using the method of undetermined multipliers. For constraints of the form[14]

$$\sum_{j=1}^{s} \frac{\partial f_k}{\partial q_j} dq_j = 0\,, \quad k = 1, \ldots, m\,, \tag{5.53}$$

the LAGRANGE equations are (see Eq. (4.43))

$$\frac{\partial L}{\partial q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} + \sum_k \lambda_k(t) \frac{\partial f_k}{\partial q_j} = 0\,. \tag{5.54}$$

While the advantage of the Lagrangian formalism is that one does not need to specify the forces of constraint, in certain circumstances these forces might be of interest. The method of the undetermined multipliers allows to calculate these forces. It can be shown that the *generalized forces of constraint* are

$$Q_j = \sum_k \lambda_k \frac{\partial f_k}{\partial q_j}\,. \tag{5.55}$$

Example 5.4: Rolling disc. *Find the equations of motion, the forces of constraint, and the angular acceleration of a disc rolling down an inclined plane.*

The kinetic energy is

$$T = \frac{1}{2} M \dot{y}^2 + \frac{1}{2} I \dot{\theta}^2 = \frac{1}{2} M \dot{y}^2 + \frac{1}{4} M R^2 \dot{\theta}^2\,, \tag{5.56}$$

with M and R the mass and radius of the disc, respectively, and $I = \frac{1}{2} M R^2$ its moment of inertia about the central axis (see Sec. 9). The potential energy can be taken as

$$U = Mg(\ell - y) \sin\alpha\,, \tag{5.57}$$

with ℓ the length of the inclined plane and α its angle of inclination. The Lagrangian is

$$L = T - U = \frac{1}{2} M \dot{y}^2 + \frac{1}{4} M R^2 \dot{\theta}^2 - Mg(\ell - y) \sin\alpha\,, \tag{5.58}$$

while the equation of constraint is

$$f(y, \theta) = y - R\theta = 0\,. \tag{5.59}$$

We could use this equation to eliminate either y or θ. Instead, let's use the Lagrangian multiplier, in which case the LAGRANGE equations are

$$\frac{\partial L}{\partial y} - \frac{d}{dt}\frac{\partial L}{\partial \dot{y}} + \lambda \frac{\partial f}{\partial y} = 0\,, \tag{5.60}$$

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} + \lambda \frac{\partial f}{\partial \theta} = 0\,. \tag{5.61}$$

[14] This is equivalent to the form Eq. (5.51), since in the variation of L we hold the time fixed at the endpoints.

Using the Langrangian above, this gives

$$Mg\sin\alpha - M\ddot{y} + \lambda = 0\,, \tag{5.62}$$

$$-\frac{1}{2}MR^2\ddot{\theta} - R\lambda = 0\,. \tag{5.63}$$

Differentiating the equation of constraint gives

$$\ddot{\theta} = \frac{\ddot{y}}{R}\,, \tag{5.64}$$

and inserting this into Eq. (5.63) we find

$$\lambda = -\frac{1}{2}M\ddot{y}\,. \tag{5.65}$$

Using this in Eq. (5.62) gives the equation of motion

$$\ddot{y} = \frac{2g\sin\alpha}{3}\,, \tag{5.66}$$

and, hence, the constraint

$$\lambda = -\frac{Mg\sin\alpha}{3}\,. \tag{5.67}$$

Moreover, we then find

$$\ddot{\theta} = \frac{2g\sin\alpha}{3R}\,. \tag{5.68}$$

The rolling of the disc reduces the acceleration by a factor $2/3$ compared to sliding without friction. The magnitude of force of friction prevent the disc from sliding and forcing it to roll is just $\lambda = -(Mg/3)\sin\alpha$.

The generalized forces of constraint are

$$Q_y = \lambda\frac{\partial f}{\partial y} = \lambda = -\frac{Mg\sin\alpha}{3}\,, \tag{5.69}$$

$$Q_\theta = \lambda\frac{\partial f}{\partial \theta} = -R\lambda = \frac{MgR\sin\alpha}{3}\,. \tag{5.70}$$

Of course, we could have used the relation $\dot{\theta} = \dot{y}/R$ to eliminate $\dot{\theta}$ from the Lagrangian, to obtain

$$L = \frac{3}{4}M\dot{y}^2 - Mg(\ell - y)\sin\alpha\,, \tag{5.71}$$

which immediately leads to Eq. (5.66) via LAGRANGE's equation. However, this does not give us the forces of constraint.

5.5 The method of similarity

We start with proving a theorem. The kinetic energy of a system of n particles of mass m_a is given in rectangular coordinates as

$$T = \frac{1}{2}\sum_{a=1}^{n}\sum_{i=1}^{3} m_a \dot{x}_{a,i}^2 \,. \tag{5.72}$$

Next, we express the system in terms of generalized coordinates and velocities, i.e., we write

$$x_{a,i} = x_{a,i}(q_j, t)\,, \quad j = 1, \ldots, s\,, \tag{5.73}$$

$$\dot{x}_{a,i} = \sum_j \frac{\partial x_{a,i}}{\partial q_j}\dot{q}_j + \frac{\partial x_{a,i}}{\partial t}\,. \tag{5.74}$$

The kinetic energy becomes

$$T = \sum_a \frac{m_a}{2}\left[\sum_{ijk}\frac{\partial x_{a,i}}{\partial q_j}\frac{\partial x_{a,i}}{\partial q_k}\dot{q}_j\dot{q}_k + 2\sum_{ij}\frac{\partial x_{a,i}}{\partial q_j}\frac{\partial x_{a,i}}{\partial t}\dot{q}_j + \sum_i\left(\frac{\partial x_{a,i}}{\partial t}\right)^2\right]. \tag{5.75}$$

If the relations between the coordinates do not explicitly depend on time, this has the form

$$T = \sum_{jk} a_{jk}\dot{q}_j\dot{q}_k\,, \tag{5.76}$$

the kinetic energy is quadratic in the generalized velocities.

Example 5.5: Kepler's third law. *The gravitational potential has spatial dependence $1/r$. What is the relation between the size of the planetary orbits and the times of revolution around the sun?*

The gravitational potential energy scales as

$$U(\alpha r) = \frac{1}{\alpha}U(r)\,. \tag{5.77}$$

If a planet moves along a curve γ, then the rescaled curve $\alpha\gamma$ is also an allowed planetary motion. If we now also rescale time by $t \to \beta t$, we have $\dot{x} \to (\alpha/\beta)\dot{x}$, and the kinetic energy scales as $T \to (\alpha/\beta)^2 T$. The Lagrangian will be invariant up to an irrelevant global factor if $\alpha^3/\beta^2 = 1$, i.e. if the squares of the revolution times are proportional to the cubes of the sizes of the planetary orbits. This is Kepler's third law.

Now we will prove a useful theorem. Taking the derivative w.r.t. $\dot{q}_l$ then gives

$$\frac{\partial T}{\partial \dot{q}_l} = \sum_k a_{lk}\dot{q}_k + \sum_j a_{jl}\dot{q}_j\,, \tag{5.78}$$

and multiplying by $\dot{q}_l$ and summing over l gives

$$\sum_l \dot{q}_l\frac{\partial T}{\partial \dot{q}_l} = \sum_{lk} a_{lk}\dot{q}_l\dot{q}_k + \sum_{jl} a_{jl}\dot{q}_j\dot{q}_l = 2\sum_{jk} a_{jk}\dot{q}_j\dot{q}_k = 2T\,. \tag{5.79}$$

This is a special case of EULER's theorem: For a homogeneous function $f(y_k)$ of degree n, we have[15]

$$\sum_k y_k \frac{\partial f}{\partial y_k} = nf\,. \tag{5.82}$$

5.6 Conservation laws and symmetry

5.6.1 Conservation of energy

Assume that the motion of a system is invariant under time translations, i.e. the Lagrangian does not explicitly depend on time:

$$\frac{\partial L}{\partial t} = 0\,. \tag{5.83}$$

(Note how this is phrased: This does not mean that the system is static, but that, given fixed initial conditions at some time t, the subsequent motion is the same regardless of the value of t.) In this case, the total time derivative of L becomes

$$\frac{dL}{dt} = \sum_j \frac{\partial L}{\partial q_j}\dot{q}_j + \sum_j \frac{\partial L}{\partial \dot{q}_j}\ddot{q}_j\,. \tag{5.84}$$

Using the LAGRANGE equations gives

$$\frac{dL}{dt} = \sum_j \dot{q}_j \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} + \sum_j \frac{\partial L}{\partial \dot{q}_j}\ddot{q}_j = \sum_j \frac{d}{dt}\left(\dot{q}_j \frac{\partial L}{\partial \dot{q}_j}\right), \tag{5.85}$$

or

$$\frac{d}{dt}\left(\sum_j \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} - L\right) = 0\,. \tag{5.86}$$

The quantity in the parentheses is constant in time; it is conventionally denoted by H:

$$\sum_j \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} - L \equiv H = \text{constant}\,. \tag{5.87}$$

Let us now assume that $\partial U/\partial \dot{q}_j = 0$. Then

$$\frac{\partial L}{\partial \dot{q}_j} = \frac{\partial (T-U)}{\partial \dot{q}_j} = \frac{\partial T}{\partial \dot{q}_j}\,. \tag{5.88}$$

[15] This is actually easy to prove. If $f(\lambda y_k) = \lambda^n f(y_k)$, then

$$\frac{\partial}{\partial \lambda} f(\lambda y_k) = n\lambda^{n-1} f(y_k)\,, \tag{5.80}$$

but also

$$\frac{\partial}{\partial \lambda} f(\lambda y_k) = \sum_k \frac{\partial f}{\partial y_k}\frac{\partial(\lambda y_k)}{\partial \lambda} = \sum_k \frac{\partial f}{\partial y_k} y_k\,. \tag{5.81}$$

Eq. (5.82) follows from setting $\lambda = 1$.

In this case, Eq. (5.87) becomes

$$\sum_j \dot{q}_j \frac{\partial T}{\partial \dot{q}_j} - T + U = H\,. \tag{5.89}$$

Using Eq. (5.79), this becomes finally

$$2T - T + U = T + U = H = E = \text{constant}\,. \tag{5.90}$$

The function H, defined in Eq. (5.87), will be discussed in detail in Sec. 5.7. The relation $E = H$ is valid if (a) the kinetic energy is homogeneous of degree 2 in the generalized velocities, and if (b) the potential is independent of the generalized velocities.

5.6.2 Conservation of linear momentum

Now assume that the motion of a system is not affected by a spatial translation. We consider an infinitesimal translation of the each position vector (i.e. the whole system), $\boldsymbol{r}_a \to \boldsymbol{r}_a + \delta\boldsymbol{r}$. Writing $\delta\boldsymbol{r} = \sum_i \delta x_i \boldsymbol{e}_i$, the shift of the Lagrangian $L = L(x_{a,i}, \dot{x}_{a,i})$ is

$$\delta L = \sum_{a,i} \frac{\partial L}{\partial x_{a,i}} \delta x_i = 0 \tag{5.91}$$

(we assume that the displacement is time independent, i.e. $\delta\dot{x}_{a,i} = 0$). Because the three δx_i are independent, it follows that

$$\sum_a \frac{\partial L}{\partial x_{a,i}} = 0\,. \tag{5.92}$$

Usinge Lagrange's equations, this is equivalent to

$$\frac{d}{dt} \sum_a \frac{\partial L}{\partial \dot{x}_{a,i}} = 0\,, \tag{5.93}$$

so

$$\begin{aligned}\sum_a \frac{\partial L}{\partial \dot{x}_{a,i}} &= \sum_a \frac{\partial (T-U)}{\partial \dot{x}_{a,i}} = \sum_a \frac{\partial T}{\partial \dot{x}_{a,i}} = \sum_a \frac{\partial}{\partial \dot{x}_{a,i}} \left(\frac{m_a}{2} \sum_j \dot{x}_{a,j}^2 \right) \\ &= \sum_a m_a \dot{x}_{a,j} = \sum_a p_{a,j} = \text{constant}\,.\end{aligned} \tag{5.94}$$

It follows that the total momentum $\sum_a \boldsymbol{p}_a$ of the system is conserved. (If the system is invariant only under translation along a particular direction, then only the momentum in that direction is conserved.)

5.6.3 Conservation of angular momentum

Now assume that the motion of a system is not affected by a (fixed) rotation. We will again consider an infinitesimal rotation, denoted by $\delta\theta$. Then all position vectors change according to $\boldsymbol{r}_a \to \boldsymbol{r}_a + \delta\boldsymbol{r}_a$, where now

$$\delta\boldsymbol{r}_a = \delta\boldsymbol{\theta} \times \boldsymbol{r}_a\,. \tag{5.95}$$

Now, also the velocity vectors will change, according to

$$\delta\dot{\boldsymbol{r}}_a = \delta\boldsymbol{\theta} \times \dot{\boldsymbol{r}}_a\,. \tag{5.96}$$

The corresponding change in the Lagrangian must vanish:

$$\delta L = \sum_{a,i} \frac{\partial L}{\partial x_{a,i}} \delta x_{a,i} + \sum_{a,i} \frac{\partial L}{\partial \dot{x}_{a,i}} \delta \dot{x}_{a,i} = 0\,. \tag{5.97}$$

Using Eq. (5.94) and the Lagrange equations, we can write this as

$$\delta L = \sum_{a,i} \dot{p}_{a,i} \delta x_{a,i} + \sum_{a,i} p_{a,i} \delta \dot{x}_{a,i} = 0\,, \tag{5.98}$$

or

$$\sum_a \left(\dot{\boldsymbol{p}}_a \cdot \delta\boldsymbol{r}_a + \boldsymbol{p}_a \cdot \delta\dot{\boldsymbol{r}}_a \right) = 0\,. \tag{5.99}$$

Using Eqs. (5.95) and (5.96), this becomes

$$\begin{aligned} &\sum_a \left[\dot{\boldsymbol{p}}_a \cdot (\delta\boldsymbol{\theta} \times \boldsymbol{r}_a) + \boldsymbol{p}_a \cdot (\delta\boldsymbol{\theta} \times \dot{\boldsymbol{r}}_a) \right] \\ = &\sum_a \delta\boldsymbol{\theta} \cdot \left[\boldsymbol{r}_a \times \dot{\boldsymbol{p}}_a + \dot{\boldsymbol{r}}_a \times \boldsymbol{p}_a \right] = \delta\boldsymbol{\theta} \cdot \frac{d}{dt} \sum_a (\boldsymbol{r}_a \times \boldsymbol{p}_a) = 0\,. \end{aligned} \tag{5.100}$$

Because $\delta\boldsymbol{\theta}$ is arbitrary, we have

$$\frac{d}{dt} \sum_a (\boldsymbol{r}_a \times \boldsymbol{p}_a) = 0\,, \tag{5.101}$$

and hence

$$\sum_a (\boldsymbol{r}_a \times \boldsymbol{p}_a) = \sum_a \boldsymbol{L}_a = \text{constant}\,. \tag{5.102}$$

The total angular momentum of the system of particles is conserved. Again, if the system is rotationally invariant only about one axis, then the angular momentum about this symmetry axis is conserved.

These seven constants (or *first integrals*) are the only ones that are *additive.* They are intimately related to the homogeneity of time and the homogeneity and isotropy of space. The discovery of the connection between symmetry and conservation laws goes back to Amalie Emmy Noether (1882 - 1935). See Fig. 14.

Figure 14: *Commemorative plaque for Amalie Emmy Noether on her birth house in Erlangen, Germany. (The instructor happened to be born in the same town.)*

5.7 Canonical equations – Hamiltonian dynamics

In Lagrangian mechanics, the generalized velocities are the time derivatives of the generalized coordinates and thus not strictly independent. In *Hamiltonian dynamics*, one uses *generalized momenta* instead of the generalized velocities. The generalized momenta are on equal footing to the generalized coordinates. This allows for more general coordinate transformations, the so-called canonical transformations. This is the strength of the Hamiltonian formulation.

In analogy to Eq. (5.94) one defines the generalized momentum by

$$p_j \equiv \frac{\partial L}{\partial \dot{q}_j}\,. \tag{5.103}$$

The equation (5.87) defining the Hamiltonian can then be written as

$$H = \sum_j p_j \dot{q}_j - L\,. \tag{5.104}$$

The Lagrangian is (as always) considered as a function of the generalized coordinates and velocities, and possibly time: $L = (q_j, \dot{q}_j, t)$. We can solve Eq. (5.103) for the generalized velocities, $\dot{q}_j = \dot{q}_j(q_k, p_k, t)$. The Hamiltonian is always considered to be a function of the generalized coordinates and momenta (and possibly time): $H = H(q_j, p_j, t)$.

With these conventions, the total differential of the Hamiltonian is

$$dH = \sum_j \left(\frac{\partial H}{\partial q_j} dq_j + \frac{\partial H}{\partial p_j} dp_j \right) + \frac{\partial H}{\partial t} dt\,. \tag{5.105}$$

On the other hand, using Eq. (5.104) we can write

$$dH = \sum_j \left(\dot{q}_j dp_j + p_j d\dot{q}_j - \frac{\partial L}{\partial q_j} dq_j - \frac{\partial L}{\partial \dot{q}_j} d\dot{q}_j \right) - \frac{\partial L}{\partial t} dt \,. \tag{5.106}$$

The second and forth terms in the parentheses cancel, and we find

$$dH = \sum_j \left(\dot{q}_j dp_j - \dot{p}_j dq_j \right) - \frac{\partial L}{\partial t} dt \,. \tag{5.107}$$

Comparing Eqs. (5.105) and (5.111) give *Hamilton's equations of motion*

$$\dot{q}_j = \frac{\partial H}{\partial p_j} \,, \tag{5.108}$$

$$\dot{p}_j = -\frac{\partial H}{\partial q_j} \,, \tag{5.109}$$

as well as

$$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} \,. \tag{5.110}$$

These $2s$ first-order differential equations are equivalent to the s second-order Lagrange equations.

Moreover, using Eqs. (5.108) and (5.109) in the total time derivative of the Hamiltonian gives

$$\frac{dH}{dt} = \sum_j \left(\frac{\partial H}{\partial q_j} \dot{q}_j + \frac{\partial H}{\partial p_j} \dot{p}_j \right) + \frac{\partial H}{\partial t} = \sum_j \left(-\dot{p}_j \dot{q}_j + \dot{q}_j \dot{p}_j \right) + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t} \,. \tag{5.111}$$

The Hamiltonian is constant in time if it does not explicitly depend on time.

Example 5.6: Particle in cylinder. *A particle of mass m is subject to a force $\boldsymbol{F} = -k\boldsymbol{r}$ and is constrained to move on the surface of a cylinder defined by $x^2 + y^2 = R^2$.*

The potential corresponding to the force is given by

$$U = \frac{1}{2} k r^2 = \frac{1}{2} k \left(x^2 + y^2 + z^2 \right) = \frac{1}{2} k \left(R^2 + z^2 \right) . \tag{5.112}$$

The squared velocity is

$$v^2 = \dot{R}^2 + R^2 \dot{\theta}^2 + \dot{z}^2 \,. \tag{5.113}$$

Because $\dot{R} = 0$, the kinetic energy is

$$T = \frac{m}{2} \left(R^2 \dot{\theta}^2 + \dot{z}^2 \right) , \tag{5.114}$$

and we find the Lagrangian

$$L = T - U = \frac{m}{2} \left(R^2 \dot{\theta}^2 + \dot{z}^2 \right) - \frac{1}{2} k \left(R^2 + z^2 \right) . \tag{5.115}$$

The generalized momenta corresponding to θ and z are

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mR^2\dot{\theta}\,, \tag{5.116}$$

$$p_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z}\,. \tag{5.117}$$

The Hamiltonian is equal to the total energy (why?), and we find

$$H = T + U = \frac{p_\theta^2}{2mR^2} + \frac{p_z^2}{2m} + \frac{1}{2}kz^2\,. \tag{5.118}$$

(We have dropped the irrelevant constant $kR^2/2$.) Hamilton's equations are

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = 0\,, \tag{5.119}$$

$$\dot{p}_z = -\frac{\partial H}{\partial z} = -kz\,, \tag{5.120}$$

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mR^2}\,, \tag{5.121}$$

$$\dot{z} = \frac{\partial H}{\partial p_z} = \frac{p_z}{m}\,. \tag{5.122}$$

It follows from Eq. (5.119) that p_θ is conserved, and using Eq. (5.121)

$$p_\theta = mR^2\dot{\theta} = \textit{constant}\,. \tag{5.123}$$

This is the angular momentum about the z axis. Eqs. (5.120) and (5.122) give

$$m\ddot{z} + kz = 0\,. \tag{5.124}$$

This is the equation for a simple harmonic oscillator.

The generalized coordinates q_j and momenta p_j are called *canonically conjugated variables*. We have seen that if the Hamiltonian does not depend on one of the coordinates, the corresponding momentum is conserved. Such a coordinate is called *cyclic*. It follows from

$$\dot{p}_k = \frac{\partial L}{\partial q_k} = -\frac{\partial H}{\partial q_k} \tag{5.125}$$

that a cyclic coordinate will also be absent in the Lagrangian, and Lagrange's equations then imply that a cyclic coordinate leads to a conserved momentum. However, in Lagrangian mechanics, the corresponding generalized velocity is still present, while in Hamiltonina mechanics we can completely eliminate the corresponding two equations, thus reducing the number of first-order equations to $2s - 2$. We can determine the value for the conserved momentum from the initial conditions, say $p_k \equiv \alpha_k$. The equation of motion for the cyclic coordinate q_k is

$$\dot{q}_k = \left.\frac{\partial H}{\partial p_k}\right|_{p_k=\alpha_k} \equiv \omega_k\,, \tag{5.126}$$

with solution

$$q_k = \int dt\, \omega_k\,. \tag{5.127}$$

From the example above it would seem that there is no advantage of Hamilton's over Lagrange's method, since we resubstituted the equations to obtain a second-order equation. However, Hamiltonian mechanics admits a larger class of coordinate transformation (the canonical transformations). It turns out to be always possible to find a coordinate system where *all* coordinates are cyclic (Hamilton-Jacobi method).

5.8 Derivation of Hamilton's equations from a variational principle

Hamilton's equation of motion can be derived from a minimum principle. Using Eq. (5.104), we can write Eq. (5.30) as

$$\delta \int_{t_1}^{t_2} \left(\sum_j p_j \dot{q}_j - H \right) dt = 0\,. \tag{5.128}$$

Performing the variation, we find

$$\int_{t_1}^{t_2} \sum_j \left(p_j \delta\dot{q}_j + \dot{q}_j \delta p_j - \frac{\partial H}{\partial q_j} \delta q_j - \frac{\partial H}{\partial p_j} \delta p_j \right) dt = 0\,. \tag{5.129}$$

We regard the q_j and p_j as independent. Integrating by parts gives

$$\int_{t_1}^{t_2} \sum_j \left\{ \left(\dot{q}_j - \frac{\partial H}{\partial p_j} \right) \delta p_j - \left(\dot{p}_j + \frac{\partial H}{\partial q_j} \right) \delta q_j \right\} dt = 0\,, \tag{5.130}$$

and Hamilton's equations follow.

5.9 Liouville's theorem*

The $2s$-dimensional space consisting of the q_j and p_j is called *phase space*. The motion of a system corresponds to a unique path in phase space. No two paths can ever intersect, due to the uniqueness of the solutions to the equations of motion for given initial conditions.

Frequently (e.g., in statistical mechanics) one considers the density ρ of points in phase space. We will prove that a given volume in phase space is invariant under the phase flow (motion according to the equations of motion).

The number N of "particles" within a phase-space volume dv is given by

$$N = \rho(\boldsymbol{q}, \boldsymbol{p})\, dv = \rho(\boldsymbol{q}, \boldsymbol{p})\, dq_1 dq_2 \ldots dq_s dp_1 dp_2 \ldots dp_s\,. \tag{5.131}$$

Now consider a volume element $dq_k dp_k$.[16] The number of particles entering (or leaving) through the "left" side per unit time is

$$\frac{dN_l}{dt} = \rho(\ldots, q_k, p_k, \ldots)\frac{q_k}{dt}dp_k\,. \tag{5.132}$$

The number of particles leaving (or entering) through the "right" side per unit time is

$$\begin{aligned}\frac{dN_r}{dt} &= \rho(\ldots, q_k + dq_k, p_k, \ldots)(\dot{q}_k + d\dot{q}_k)dp_k \\ &= \left(\rho + \frac{\partial\rho}{\partial q_k}dq_k + \ldots\right)\left(\dot{q}_k + \frac{\partial\dot{q}_k}{\partial q_k}dq_k\right)dp_k \\ &= \rho\dot{q}_k dp_k + \left(\frac{\partial\rho}{\partial q_k}\dot{q}_k + \frac{\partial\dot{q}_k}{\partial q_k}\rho\right)dq_k dp_k + \ldots\,.\end{aligned} \tag{5.133}$$

Hence, the total change in particles (per unit time) in q_k direction is

$$\frac{dN_{q_k}}{dt} = \frac{dN_r}{dt} - \frac{dN_l}{dt} = \left(\frac{\partial\rho}{\partial q_k}\dot{q}_k + \frac{\partial\dot{q}_k}{\partial q_k}\rho\right)dq_k dp_k\,. \tag{5.134}$$

In a similar way, we find the total change in particles (per unit time) in p_k direction as

$$\frac{dN_{p_k}}{dt} = \frac{dN_u}{dt} - \frac{dN_d}{dt}\dot{p}_k = \left(\frac{\partial\rho}{\partial p_k} + \frac{\partial\dot{p}_k}{\partial p_k}\rho\right)dq_k dp_k\,. \tag{5.135}$$

The net change of particles per unit time in the volume is of course the rate of change per unit time of the density in the fixed volume element $dq_k dp_k$, or

$$\frac{\partial\rho}{\partial t}dq_k dp_k = \left(\frac{\partial\rho}{\partial q_k}\dot{q}_k + \frac{\partial\dot{q}_k}{\partial q_k}\rho + \frac{\partial\rho}{\partial p_k}\dot{p}_k + \frac{\partial\dot{p}_k}{\partial p_k}\rho\right)dq_k dp_k\,, \tag{5.136}$$

or, dividing by $dq_k dp_k$ and summing over all k,

$$\frac{\partial\rho}{\partial t} = \sum_k\left(\frac{\partial\rho}{\partial q_k}\dot{q}_k + \frac{\partial\dot{q}_k}{\partial q_k}\rho + \frac{\partial\rho}{\partial p_k}\dot{p}_k + \frac{\partial\dot{p}_k}{\partial p_k}\rho\right)\,. \tag{5.137}$$

Now

$$\frac{\partial\dot{q}_k}{\partial q_k} + \frac{\partial\dot{p}_k}{\partial p_k} = \frac{\partial^2 H}{\partial q_k\partial p_k} - \frac{\partial^2 H}{\partial p_k\partial q_k} = 0\,, \tag{5.138}$$

where we used Hamilton's equations, so we have

$$0 = \frac{\partial\rho}{\partial t} - \sum_k\left(\frac{\partial\rho}{\partial q_k}\dot{q}_k + \frac{\partial\rho}{\partial p_k}\dot{p}_k\right) = \frac{d\rho}{dt}\,. \tag{5.139}$$

This result is known as *Liouville's theorem.*

[16] I.e., with corners (q_k, p_k), $(q_k + dq_k, p_k)$, $(q_k, p_k + dp_k)$, $(q_k + dq_k, p_k + dp_k)$. This is a volume element "fixed in phase space", not a moving element of the phase space "fluid".

6 Central-force motion

In this section, we study systems of two particles affected only by forces directed along the line connecting the two particles ("central forces").

6.1 Reduced mass

We need six coordinates $\boldsymbol{r}_1, \boldsymbol{r}_2$ to fully specify the state of a two-particle system. For central forces, three out of the six coordinates are uninteresting. The potential energy will only depend on $r \equiv |\boldsymbol{r}_2 - \boldsymbol{r}_1|$, and the Lagrangian for the system is

$$L = \frac{m_1}{2}|\dot{\boldsymbol{r}}_1|^2 + \frac{m_2}{2}|\dot{\boldsymbol{r}}_2|^2 - U(r)\,. \tag{6.1}$$

As a preparation, let us figure out how the total momentum of the two particles changes if we go to a different inertial system moving with relative velocity $\boldsymbol{V}$. The relation between the velocities is $\boldsymbol{v}_i = \boldsymbol{v}'_i + \boldsymbol{V}$, $i = 1, 2$. Here, the prime denotes the moving system. Therefore, we have the relation

$$\sum_i m_i \boldsymbol{v}_i = \sum_i m_i \boldsymbol{v}'_i + \boldsymbol{V} \sum_i m_i\,. \tag{6.2}$$

The total momenta in the two systems are $\boldsymbol{P} = \sum_i m_i \boldsymbol{v}_i$, $\boldsymbol{P}' = \sum_i m_i \boldsymbol{v}'_i$, and we find the relation between the momenta in the two systems

$$\boldsymbol{P} = \boldsymbol{P}' + \boldsymbol{V} \sum_i m_i\,. \tag{6.3}$$

Therefore, we can find a system in which the total momentum vanishes, e.g. $\boldsymbol{P}' = 0$. The corresponding relative velocity is

$$\boldsymbol{V} = \frac{\boldsymbol{P}}{\sum_i m_i} = \frac{m_1 \boldsymbol{v}_1 + m_2 \boldsymbol{v}_2}{m_1 + m_2}\,. \tag{6.4}$$

We can write this as the derivative of

$$\boldsymbol{R} = \frac{m_1 \boldsymbol{r}_1 + m_2 \boldsymbol{r}_2}{m_1 + m_2} \tag{6.5}$$

This is the *center of mass* (CM) of the system.

Instead of $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$, we can describe our system by the location of the CM, $\boldsymbol{R}$, and the relative distance of the particles, $\boldsymbol{r} = \boldsymbol{r}_2 - \boldsymbol{r}_1$. Now we choose the origin of our coordinate system to be at the center of mass; i.e., we have $\boldsymbol{R} = 0$ or

$$m_1 \boldsymbol{r}_1 + m_2 \boldsymbol{r}_2 = 0\,. \tag{6.6}$$

Combining this with $\boldsymbol{r} = \boldsymbol{r}_2 - \boldsymbol{r}_1$ gives

$$\begin{aligned} \boldsymbol{r}_1 &= -\frac{m_2}{m_1 + m_2}\boldsymbol{r}\,, \\ \boldsymbol{r}_2 &= \frac{m_1}{m_1 + m_2}\boldsymbol{r}\,. \end{aligned} \tag{6.7}$$

The Lagrangian becomes

$$L = \frac{\mu}{2}|\dot{\boldsymbol{r}}|^2 - U(r)\,, \tag{6.8}$$

where

$$\mu \equiv \frac{m_1 m_2}{m_1 + m_2} \tag{6.9}$$

is the *reduced mass.* We see that for central forces, the *two-body problem* can be reduced to the motion of a single "particle" of mass μ. If desired, the individual motions of the particles can always be obtained using Eq. (6.7).

6.2 Conservation theorems

The central force field is spherically symmetric, hence the angular momentum is conserved:

$$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p} = \text{constant}\,. \tag{6.10}$$

It follows that the motion of the system is restricted to the plane orthogonal to $\boldsymbol{L}$. The problem is effectively two-dimensional, and it is convenient to choose polar coordinates:

$$L = \frac{\mu}{2}\left(\dot{r}^2 + r^2\dot{\varphi}^2\right) - U(r)\,. \tag{6.11}$$

We see that the coordinate φ is cyclic, so the corresponding conjugate momentum is conserved:

$$\dot{p}_\varphi = \frac{\partial L}{\partial \varphi} = 0 = \frac{d}{dt}\frac{\partial L}{\partial \dot{\varphi}}\,, \tag{6.12}$$

or

$$p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = \mu r^2 \dot{\varphi} \equiv l = \text{constant}\,. \tag{6.13}$$

The constant l is called a *first integral* of the motion; it has allowed us to integrate one of the e.o.m. immediately. It has a simple geometric interpretation. The area swept out by the radius vector in a time interval dt is

$$dA = \frac{1}{2}r^2 d\varphi\,, \tag{6.14}$$

and so

$$\frac{dA}{dt} = \frac{1}{2}r^2\dot{\varphi} = \frac{l}{2\mu} = \text{constant}\,. \tag{6.15}$$

The radius vector sweeps out equal areas in equal times. This result depends only on the central nature of the force, and not on the $1/r^2$ behavior. This result was first obtained by Johannes Kepler from observations of planetary motions.

We had eliminated the uniform motion of the CM by our choice of coordinates, so conservation of linear momentum does not add any new information to the solution of the problem. Energy conservation, however, gives a further non-trivial restriction on the motion:

$$E = \frac{\mu}{2}\left(\dot{r}^2 + r^2\dot{\varphi}^2\right) + U(r) = \frac{\mu}{2}\dot{r}^2 + \frac{1}{2}\frac{l^2}{\mu r^2} + U(r) = \text{constant}\,. \tag{6.16}$$

6.3 Equations of motion, orbits, and effective potential

We have seen in Sec. 1.3.2 how to use the conservation of energy to solve the general one-dimensional mechanical problem. For the present case, Eq. (??) gives

$$\frac{dr}{dt} = \pm\sqrt{\frac{2}{\mu}\left(E - U - \frac{l^2}{2\mu r^2}\right)}\,. \tag{6.17}$$

This equation can be integrated to give $r(t)$.

If we are interested in the trajectory, we need the functional relation between r and φ. Using

$$d\varphi = \frac{d\varphi}{dt}\frac{dt}{dr}dr = \frac{\dot\varphi}{\dot r}dr\,, \tag{6.18}$$

substituting $\dot\varphi = l/(\mu r^2)$ and Eq. (6.17), and integrating, we find

$$\varphi(r) = \pm\int dr\frac{l/r^2}{\sqrt{2\mu\left(E - U - \frac{l^2}{2\mu r^2}\right)}}\,. \tag{6.19}$$

The case $U(r) \propto r^2$ is the harmonic oscillator (see Sec. 2.2). The case $U(r) \propto 1/r$ will be treated in Sec. 6.4.

The radial velocity of the motion is given by Eq. (6.17). The vanishing of the left side, $\dot r$, corresponds to a *turning point* of the motion; the condition is

$$E - U(r) - \frac{l^2}{2\mu r^2} = 0\,. \tag{6.20}$$

If one chooses the angle φ to be zero at a turning point, Eq. (6.19) shows that the motion is symmetric about the turning point: we only have to choose the plus or minus sign according to whether r increases or decreases. (Note that, according to Eq. (6.13), φ increases or decreases monotonically.)

The existence and type of turning points depends on the form of the potential. If this equation has two solutions, r_{min} and r_{max}, the motion happens in that annular region. It may happen that $\dot r = 0$ for all times, corresponding to circular motion. There may be only a single condition, $r \geq r_{\text{min}}$, in which case the particles comes from and escapes to infinity.

The motion might be periodic, in which case the orbits are *closed*; otherwise, the orbits are *open*. Eq. (6.19) allows us to calculate the change in the polar angle when the particle moves from r_{min} to r_{max} and back. Because of the symmetry of the motion about the turning point, that angle is given by

$$\Delta\varphi = 2\int_{r_{\text{min}}}^{r_{\text{max}}} dr\frac{l/r^2}{\sqrt{2\mu\left(E - U - \frac{l^2}{2\mu r^2}\right)}}\,. \tag{6.21}$$

The path is closed if $\Delta\varphi$ is a rational fraction of 2π. This is the case for $U(r) \propto r^n$, with $n = -1$ or $n = 2$.

Eq. (6.16) shows that we can regard the radial part of the motion as an one-dimensional motion with the *effective potential energy*

$$V_{\text{eff}}(r) \equiv U(r) + \frac{l^2}{2\mu r^2}\,, \tag{6.22}$$

such that Eq. (6.17) becomes

$$\frac{dr}{dt} = \pm\sqrt{\frac{2}{\mu}\big(E - V_{\text{eff}}(r)\big)}\,. \tag{6.23}$$

We see that we can ontain the turning points by finding the roots of the equation $E - V_{\text{eff}}(r) = 0$.

6.4 Planetary motion: Kepler's problem

Writing the the potential for a central force with $1/r^2$ dependence in the general form

$$U(r) = -\frac{k}{r}\,, \tag{6.24}$$

the trajectory for a particle moving in that force field is given by

$$\varphi(r) = \int_{r_{\text{min}}}^{r} dr' \frac{l/r'^2}{\sqrt{2\mu\Big(E + \frac{k}{r'} - \frac{l^2}{2\mu r'^2}\Big)}}\,. \tag{6.25}$$

To solve the integral, we change the integration variable to $u = l/r'$. This gives ($du = -l dr'/r'^2$; I will be sloppy with the variable names)

$$\varphi = -\int \frac{du}{\sqrt{2\mu E + 2\mu\frac{ku}{l} - u^2}} = -\int \frac{du}{\sqrt{\frac{k^2\mu^2}{l^2} + 2\mu E - (u - k\mu/l)^2}}\,. \tag{6.26}$$

Now we introduce $v = u - k\mu/l$, so we get

$$\varphi = -\int \frac{dv}{\sqrt{\frac{k^2\mu^2}{l^2} + 2\mu E - v^2}}\,, \tag{6.27}$$

and finally $w = v/\sqrt{k^2\mu^2/l^2 + 2\mu E}$, so[17]

$$\varphi = -\int \frac{dw}{\sqrt{1 - w^2}} = \arccos(w) + \text{constant}\,, \tag{6.28}$$

[17] We have $\cos(\arccos(x)) = x$, so using the chain rule $1 = -\sin(\arccos(x))\arccos(x)'$, and using $\sin^2 = 1 - \cos^2$, we obtain $\arccos(x)' = -1/\sqrt{1 - x^2}$.

or

$$\cos\varphi = \frac{v}{\sqrt{k^2\mu^2/l^2 + 2\mu E}} = \frac{u - \frac{k\mu}{l}}{\sqrt{k^2\mu^2/l^2 + 2\mu E}} = \frac{\frac{l}{r} - \frac{k\mu}{l}}{\sqrt{k^2\mu^2/l^2 + 2\mu E}}\,. \tag{6.29}$$

This can be simplified to give the final answer

$$\cos\varphi = \frac{\frac{l^2}{k\mu}\frac{1}{r} - 1}{\sqrt{1 + \frac{2El^2}{k^2\mu}}}\,. \tag{6.30}$$

Now we define

$$p \equiv \frac{l^2}{k\mu}\,, \quad e \equiv \sqrt{1 + \frac{2El^2}{k^2\mu}}\,, \tag{6.31}$$

and we can write the solution as

$$\frac{p}{r} = 1 + e\cos\varphi\,. \tag{6.32}$$

This is the equation for a *conic section*. We have chosen the constant such that $\varphi = 0$ corresponds to the minimal value of r. The parameter e is called the eccentricity of the orbit. In Cartesian coordinates, the equation is

$$(1 - e^2)x^2 + 2epx + y^2 = p^2\,. \tag{6.33}$$

For $x = 0$ we obtain $y = \pm p$. The minimum and maximum values of x for $y = 0$ are

$$x_{\text{min,max}} = \frac{p(e \pm 1)}{e^2 - 1} = \begin{cases} -\frac{p}{1-e} \\ \frac{p}{1+e} \end{cases} \tag{6.34}$$

The case $e = 0$ is a circle of radius p, as can also be seen from Eq. (6.33). For the special case $e = 1$, x_{min} diverges. Eq. (6.33) shows that this is a parabola, $x = (p^2 - y^2)/(2p)$. The case $0 < e < 1$ corresponds to an ellipse (see Fig. 15), with length of the *semimajor axis*

$$a = \frac{x_{\text{max}} - x_{\text{min}}}{2} = \frac{p}{1 - e^2} = \frac{k}{2|E|}\,. \tag{6.35}$$

(Note that $E < 0$ for the elliptic orbit.) x_{max} is also called the *pericenter* (or *perihelion* for the solar system), x_{min} is called the *apocenter* (or *aphelion* for the solar system). The distance between the *focal points* of the ellipse and the center of the ellipse is

$$a - x_{\text{max}} = \frac{p}{1 - e^2} - \frac{p}{1 + e} = \frac{pe}{1 - e^2} = ae\,. \tag{6.36}$$

The length of the *semiminor axis* is obtained by setting $x = -ae$ in Eq. (6.33) and solving for y:

$$b = \sqrt{p^2 - (1 - e^2)\frac{p^2e^2}{(1 - e^2)^2} + 2ep\frac{pe}{1 - e^2}} = \frac{p}{\sqrt{1 - e^2}} = \sqrt{pa} = \frac{l}{\sqrt{2\mu|E|}}\,. \tag{6.37}$$

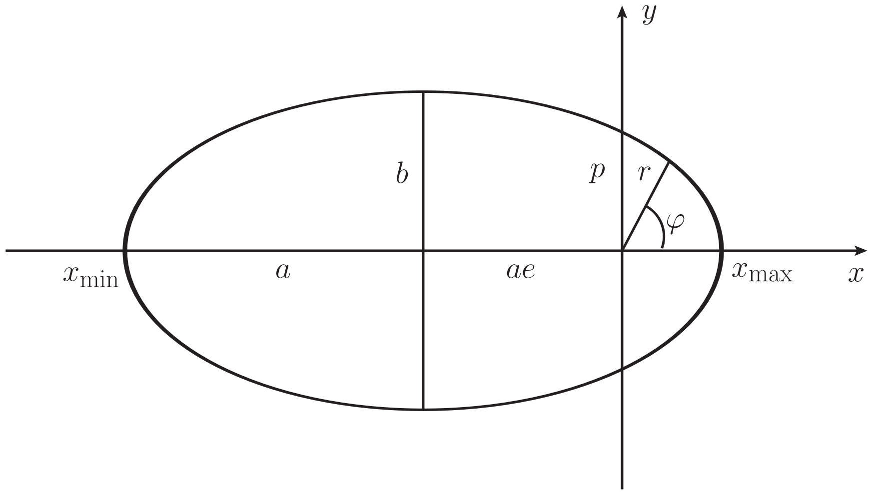

Figure 15: *Ellipse.*

Going back to polar coordinates, we can now write the minimum and maximum distance of the orbit from the focal point as $r_{\rm min} = a(1-e)$ and $r_{\rm max} = a(1+e)$.

The discussion of the hyperbola is left as an exercise for the students. In summary, the different eccentricities correspond to:

$e > 1$	$E > 0$	Hyperbola,
$e = 1$	$E = 0$	Parabola,
$0 < e < 1$	$V_{\rm min} < E < 0$	Ellipse,
$e = 0$	$E = V_{\rm min}$	Circle.

Here, $V_{\rm min}$ is the minimum of the effective potential (6.22); for $U(r) = k/r$ we have $r = l^2/(\mu k)$ and thus

$$V_{\rm min} = -\frac{k\mu}{2l^2}\,. \tag{6.38}$$

So much for the geometry of the orbit. To obtain the period for the elliptic motion, we use the expression for the areal velocity, Eq. (6.15), in the form

$$dt = \frac{2\mu}{l}dA\,, \tag{6.39}$$

or

$$\tau = \frac{2\mu}{l}A\,, \tag{6.40}$$

with $A = ab\pi$ the area of the ellipse. Thus, we have

$$\tau = \frac{2\mu}{l}\frac{k}{2|E|}\frac{l}{\sqrt{2\mu E}}\pi = \pi k\sqrt{\frac{\mu}{2|E|^3}}\,. \tag{6.41}$$

Alternatively, we can write this as

$$\tau^2 = \frac{4\mu^2}{l^2}a^2b^2\pi^2 = \frac{4\pi^2\mu}{k}a^3\,, \tag{6.42}$$

where we used $p = b^2/a = l^2/(k\mu)$.

For the case of Newton's law,

$$F(r) = -G\frac{mM}{r^2}\,, \tag{6.43}$$

we read off $k = GmM$. With the definition of the reduced mass, we can rewrite Eq. (6.42) as

$$\tau^2 = \frac{4\pi^2 a^3}{G(m+M)} \approx \frac{4\pi^2 a^3}{GM} \quad \text{for} \quad m \ll M\,. \tag{6.44}$$

The above discussion can be summarized in Kepler's laws:

1. The planets move around the sun in elliptical orbits, with the sun in one of the focal points.
2. The radius vector pointing from the sun to the planet sweeps out equal areas in equal times.
3. The squares of the planets periods are proportional to the cubes of the planets semi-major axes.

6.5 Time dependence of planetary motion*

Finally, we want to discuss the time dependence of the motion of the planets around the sun. Our starting point is Eq. (6.17), which we can write as (recall that $E < 0$ for closed orbits)

$$\begin{aligned} t &= \int \frac{dr}{\sqrt{\frac{2}{\mu}(E-U) - \frac{l^2}{\mu^2 r^2}}} = \sqrt{\frac{\mu}{2|E|}}\int \frac{r dr}{\sqrt{-r^2 + \frac{k}{|E|}r - \frac{l^2}{2\mu|E|}}} \\ &= \sqrt{\frac{\mu a}{k}}\int \frac{r dr}{\sqrt{-r^2 + 2ar - b^2}} = \sqrt{\frac{\mu a}{k}}\int \frac{r dr}{\sqrt{a^2e^2 - (r-a)^2}}\,. \end{aligned} \tag{6.45}$$

where we used $U = -k/r$, $a = k/(2|E|)$, $b^2 = l^2/(2\mu|E|)$, $a = p/(1-e^2)$, $b = p/\sqrt{1-e^2}$. We substitute $r = a(1 - e\cos\xi)$ (the parameter ξ is called the *eccentric anomaly* for historical reasons) and find

$$t = \sqrt{\frac{\mu a^3}{k}}\int (1 - e\cos\xi)d\xi = \sqrt{\frac{\mu a^3}{k}}(\xi - e\sin\xi) + \tau^*\,, \tag{6.46}$$

where τ^* is the *time of perihelion passage*. Setting the clock such that $\tau^* = 0$, the parameterization of the trajectory is

$$r = a(1 - e\cos\xi)\,, \quad t = \sqrt{\frac{\mu a^3}{k}}(\xi - e\sin\xi)\,. \tag{6.47}$$

From this result, we can also obtain the Cartesian coordinates $x = r\cos\varphi$, $y = r\sin\varphi$, of the trajectory in dependence on ξ. Using Eq (6.32), we can write

$$e\,x = p - r = a(1-e^2) - a(1 - e\cos\xi) = a\,e\,(\cos\xi - e)\,, \tag{6.48}$$

and $y = \sqrt{r^2 - x^2}$, so

$$x = a(\cos\xi - e)\,, \quad y = a\sqrt{1-e^2}\sin\xi = b\sin\xi\,. \tag{6.49}$$

The magnitude of the radial velocity is given (after some algebra) by Eq. (6.17):

$$v_r = \dot{r} = \sqrt{\frac{2|E|}{\mu}}\frac{e}{\sqrt{1-e^2}}\sin\varphi = \sqrt{\frac{k}{\mu p}}\,e\sin\varphi\,, \tag{6.50}$$

while the magnitude of the transversal velocity is given by

$$v_t = r\dot{\varphi} = r\frac{d\varphi}{dr}\frac{dr}{dt}\,. \tag{6.51}$$

Using Eq. (6.19) (and even more algebra) we find

$$v_t = \sqrt{\frac{k}{\mu p}}(1 + e\cos\varphi)\,. \tag{6.52}$$

The total velocity is

$$v = \sqrt{v_r^2 + v_t^2} = \sqrt{\frac{k}{\mu p}}\sqrt{1 + e^2 + 2e\cos\varphi}\,. \tag{6.53}$$

It is maximal at *perihelion* ($\varphi = 0$) and minimal at *aphelion* ($\varphi = \pi$).

6.6 The Laplace-Runge-Lenz vector*

We show that the vector

$$\boldsymbol{l} = \boldsymbol{v} \times \boldsymbol{L} - \frac{k\boldsymbol{r}}{r} \tag{6.54}$$

is constant in time:

$$\frac{d}{dt}\boldsymbol{l} = \dot{\boldsymbol{v}} \times \boldsymbol{L} - \frac{k\boldsymbol{v}}{r} + \frac{k\boldsymbol{r}(\boldsymbol{v}\cdot\boldsymbol{r})}{r^3}\,. \tag{6.55}$$

We insert $\boldsymbol{L} = \mu\boldsymbol{r} \times \boldsymbol{v}$ and obtain

$$\frac{d}{dt}\boldsymbol{l} = \mu\boldsymbol{r}(\dot{\boldsymbol{v}}\cdot\boldsymbol{v}) - \mu\boldsymbol{v}(\dot{\boldsymbol{v}}\cdot\boldsymbol{r}) - \frac{k\boldsymbol{v}}{r} + \frac{k\boldsymbol{r}(\boldsymbol{v}\cdot\boldsymbol{r})}{r^3}\,. \tag{6.56}$$

Inserting the e.o.m. $\mu\dot{\boldsymbol{v}} = -k\boldsymbol{r}/r^3$ shows that this expression vanishes.

The lenght of $\boldsymbol{l}$ is most easily evaluated at perihelion, where $v = v_\pi = \sqrt{k/\mu p}(1+e)$ and $r = r_\pi = p/(1+e)$. Moreover, by definition $|\boldsymbol{L}| = \sqrt{kp\mu}$, so $|\boldsymbol{l}| = \sqrt{v^2|\boldsymbol{L}|^2 + k^2 - 2v|\boldsymbol{L}|k} = \sqrt{k^2(1+e)^2 + k^2 - 2k^2(1+e)} = ke$.

6.7 Perihelion shifts*

The elliptic orbits for a single planet in a Kepler potential $U(r) = -k/r$ are closed and the locations of the perihelions are fixed. The influence of other planets, the oblateness of the sun, and the effects of General Relativity can be taken into account by small corrections δU of the Kepler potential; these corrections result in a constant shift of the perihelion. We will calculate this shift approximately for

$$U(r) = -\frac{k}{r} + \delta U\,, \tag{6.57}$$

with **(a)** $\delta U = \beta/r^2$, and **(b)** $\delta U = \gamma/r^3$. We have

$$\Delta\varphi = \int\limits_{r_{\min}}^{r_{\max}} dr \frac{2l}{r^2\sqrt{2\mu\Big(E - U(r) - \frac{l^2}{2\mu r^2}\Big)}} = -2\frac{d}{dl}\int\limits_{r_{\min}}^{r_{\max}} dr \sqrt{2\mu\Big(E - U(r) - \frac{l^2}{2\mu r^2}\Big)}\,. \tag{6.58}$$

We expand the integrand:

$$\begin{aligned}\Delta\varphi &= -2\frac{d}{dl}\int\limits_{r_{\min}}^{r_{\max}} dr \sqrt{2\mu\Big(E + \frac{k}{r} - \delta U - \frac{l^2}{2\mu r^2}\Big)} \\ &= -2\frac{d}{dl}\int\limits_{r_{\min}}^{r_{\max}} dr \sqrt{2\mu\Big(E + \frac{k}{r} - \frac{l^2}{2\mu r^2}\Big)}\sqrt{1 - \frac{2\mu\delta U}{2\mu\Big(E + \frac{k}{r} - \frac{l^2}{2\mu r^2}\Big)}} \\ &= -2\frac{d}{dl}\int\limits_{r_{\min}}^{r_{\max}} dr \sqrt{2\mu\Big(E + \frac{k}{r} - \frac{l^2}{2\mu r^2}\Big)}\left(1 - \frac{\mu\delta U}{2\mu\Big(E + \frac{k}{r} - \frac{l^2}{2\mu r^2}\Big)}\right)\,.\end{aligned} \tag{6.59}$$

The first term gives 2π for the unperturbed Kepler orbit, so the second term induces the shift

$$\begin{aligned}\delta\varphi &= 2\frac{d}{dl}\int\limits_{r_{\min}}^{r_{\max}} dr \sqrt{2\mu\Big(E + \frac{k}{r} - \frac{l^2}{2\mu r^2}\Big)}\left(\frac{\mu\delta U}{2\mu\Big(E + \frac{k}{r} - \frac{l^2}{2\mu r^2}\Big)}\right) \\ &= 2\frac{d}{dl}\int\limits_{r_{\min}}^{r_{\max}} dr \frac{\mu\delta U}{\sqrt{2\mu\Big(E + \frac{k}{r} - \frac{l^2}{2\mu r^2}\Big)}}\,.\end{aligned} \tag{6.60}$$

Now we use

$$d\varphi(r) = dr\frac{l}{r^2\sqrt{2\mu\Big(E - U(r) - \frac{l^2}{2\mu r^2}\Big)}} \tag{6.61}$$

and obtain

$$\delta\varphi(r)2\frac{d}{dl}\int\limits_0^{\pi}\frac{r^2\mu\delta U}{l}d\varphi\,. \tag{6.62}$$

For case (a) we obtain

$$\delta\varphi(r) = 2\frac{d}{dl}\int_0^\pi \frac{\mu\beta}{l}d\varphi = -\frac{2\pi\mu\beta}{l^2}\,, \tag{6.63}$$

and for case (b) (recall $l^2 = pk\mu$)

$$\begin{aligned}\delta\varphi(r) &= 2\frac{d}{dl}\int_0^\pi \frac{\mu\gamma}{rl}d\varphi = 2\frac{d}{dl}\int_0^\pi d\varphi\frac{\mu\gamma}{l}\frac{1+e\cos\varphi}{p} \\ &= 2\frac{d}{dl}\left(\frac{\mu\gamma}{l}\frac{\pi}{p}\right) = 2\frac{d}{dl}\left(\frac{\pi k\mu^2\gamma}{l^3}\right) = -\frac{6\pi k\mu^2\gamma}{l^4}\,.\end{aligned} \tag{6.64}$$

Using $p = a(1-e^2)$, the shifts for cases (a) and (b) are

$$\delta\varphi = -\frac{2\pi\beta}{a(1-e^2)k}\,, \qquad \delta\varphi = -\frac{6\pi\gamma}{a^2(1-e^2)^2k}\,. \tag{6.65}$$

7 Systems of particles

We will first generalize some of the concepts introduced in Sec. 6.1 to the case of n particles. We will then discuss collisions of two particles.

7.1 Linear momentum and center of mass

We repeat and generalize the analysis of Sec. 6.1. Consider a system on n particles. The relation of momenta in two different inertial systems, related by the relative velocity $\boldsymbol{V}$, is

$$\sum_a m_a\boldsymbol{v}_a = \sum_a m_a\boldsymbol{v}'_a + \boldsymbol{V}\sum_a m_a\,, \tag{7.1}$$

where now $a = 1, \ldots, n$. The total momenta in the two systems are $\boldsymbol{P} = \sum_a m_a\boldsymbol{v}_a$, $\boldsymbol{P}' = \sum_a m_a\boldsymbol{v}'_a$, and again we find the relation between the momenta in the two systems

$$\boldsymbol{P} = \boldsymbol{P}' + \boldsymbol{V}\sum_a m_a\,. \tag{7.2}$$

Therefore, we can find a system in which the total momentum vanishes, e.g. $\boldsymbol{P}' = 0$. The corresponding relative velocity is

$$\boldsymbol{V} = \frac{\boldsymbol{P}}{\sum_a m_a} = \dot{\boldsymbol{R}}\,, \tag{7.3}$$

where

$$\boldsymbol{R} = \frac{\sum_a m_a\boldsymbol{r}_a}{\sum_a m_a} \tag{7.4}$$

is the location of the *center of mass* (CM) of the system. If there are no external forces acting on the system, the CM is either at rest (CM system) or moves with constant velocity. For a continuous mass distribution, the CM is given by

$$\boldsymbol{R} = \frac{1}{M}\int \boldsymbol{r} dm\,, \tag{7.5}$$

with M the total mass of the system ($M = \int dm$).

Example 7.1: CM of a hemisphere. *Find the CM of a hemisphere with constant density.*

Let a be the radius of the sphere, $\rho = M/(2\pi a^3/3)$, $\boldsymbol{R} = (0,0,Z)$ by a suitable choice of the coordinate system. Then

$$Z = \frac{1}{M}\int\limits_0^a z\,dm\,, \tag{7.6}$$

and we choose $dm = \rho dV = \rho\pi(a^2 - z^2)dz$, so

$$Z = \frac{1}{M}\int\limits_0^a dz\rho\pi(za^2 - z^3) = \frac{\pi\rho a^4}{4M} = \frac{3a}{8}\,. \tag{7.7}$$

In some situations it may be useful to separate a subsystem, which is then acted upon by external forces. The force $\boldsymbol{f}_a$ on particle a that is part of the subsystem can be split into a sum of contributions $\boldsymbol{f}_{ab}$ arising from all other particles b in the subsystem, and a net resultant external force $\boldsymbol{F}_a$:

$$\boldsymbol{f}_a = \sum_{b\neq a}\boldsymbol{f}_{ab} + \boldsymbol{F}_a\,. \tag{7.8}$$

Newton's second law for particle a is

$$\frac{d}{dt}\boldsymbol{p}_a = \sum_{b\neq a}\boldsymbol{f}_{ab} + \boldsymbol{F}_a\,. \tag{7.9}$$

Summing over all particles a in the subsystem (and assuming constant masses), we find

$$\frac{d}{dt}\boldsymbol{P} = \sum_a\sum_{b\neq a}\boldsymbol{f}_{ab} + \sum_a \boldsymbol{F}_a\,, \tag{7.10}$$

where $\boldsymbol{P} = \sum_a \boldsymbol{p}_a$ is the total momentum of the subsystem. If we use the CM system and assume for the moment that all external forces vanish, $\boldsymbol{F}_a = 0$, we know the the right side of the equation must vanish, and thus

$$\sum_a\sum_{b\neq a}\boldsymbol{f}_{ab} = 0\,. \tag{7.11}$$

(This result follows also directly from Newton's third law.) Therefore, in the presence of external forces, the total momentum changes according to

$$\frac{d}{dt}\boldsymbol{P} = \sum_a \boldsymbol{F}_a\,. \tag{7.12}$$

Using the definition of the CM, we can write this also as

$$M\ddot{\boldsymbol{R}} = \boldsymbol{F}\,, \tag{7.13}$$

where $M = \sum_a m_a$ is the total mass of the subsystem, and $\boldsymbol{F} = \sum_a \boldsymbol{F}_a$ is the total external force. *The center of mass of a (sub)system moves as if it were a single particle of mass M, acted on by the total external force $\boldsymbol{F}$.* (Note that, in the presence of external forces, the CM system is no longer an inertial system.)

7.2 Angular momentum

Let us calculate how the angular momentum of a system of particles transforms under a change of the coordinate system. Writing

$$\boldsymbol{r}_a = \boldsymbol{R} + \boldsymbol{r}'_a\,, \tag{7.14}$$

we obtain

$$\boldsymbol{L} = \sum_a (\boldsymbol{r}_a \times \boldsymbol{p}_a) = \sum_a (\boldsymbol{r}_a \times m_a \dot{\boldsymbol{r}}_a) \tag{7.15}$$

$$= \sum_a m_a \big[(\boldsymbol{r}'_a + \boldsymbol{R}) \times (\dot{\boldsymbol{r}}'_a + \dot{\boldsymbol{R}}) \big]\,. \tag{7.16}$$

Because in the CM system $\sum_a m_a \boldsymbol{r}'_a = 0$ and $\sum_a m_a \dot{\boldsymbol{r}}'_a = 0$, we have

$$\boldsymbol{L} = \sum_a m_a \big(\boldsymbol{r}'_a \times \dot{\boldsymbol{r}}'_a + \boldsymbol{R} \times \dot{\boldsymbol{R}} \big) = \sum_a \boldsymbol{r}'_a \times \boldsymbol{p}'_a + \boldsymbol{R} \times \boldsymbol{P}\,. \tag{7.17}$$

The total angular momentum does not depend of the choice of the origin of the coordinate system only if the whole system is at rest. In general, *the total angular momentum about an origin is the sum of the angular momentum of the center of mass about that origin and the angular momentum of the system about the position of the center of mass.*

If all internal forces between particles are central (along the line connecting the two particles), then no interal torque arises and the only change in total angular momentum is due to the external torque present. To see this, we write the time derivative of the angular momentum of particle a

$$\dot{\boldsymbol{L}}_a = \boldsymbol{r}_a \times \dot{\boldsymbol{p}}_a\,. \tag{7.18}$$

Expressing the change in momentum in terms of internal and external forces, this gives

$$\dot{\boldsymbol{L}}_a = \boldsymbol{r}_a \times \Big(\sum_{b \neq a} \boldsymbol{f}_{ab} + \boldsymbol{F}_a \Big)\,. \tag{7.19}$$

Summing over a gives the total angular momentum,

$$\dot{\boldsymbol{L}} = \sum_a \dot{\boldsymbol{L}}_a = \sum_a \boldsymbol{r}_a \times \boldsymbol{F}_a + \sum_{a;b\neq a} \boldsymbol{r}_a \times \boldsymbol{f}_{ab}\,. \tag{7.20}$$

The last term can be written

$$\sum_{a;b\neq a} \boldsymbol{r}_a \times \boldsymbol{f}_{ab} = \sum_{a<b} \left[\boldsymbol{r}_a \times \boldsymbol{f}_{ab} + \boldsymbol{r}_b \times \boldsymbol{f}_{ba}\right] = \sum_{a<b} \left[(\boldsymbol{r}_a - \boldsymbol{r}_b) \times \boldsymbol{f}_{ab}\right], \tag{7.21}$$

where we used NEWTON's third law in the last step. If the forces are central, this expression vanishes, and we have

$$\dot{\boldsymbol{L}} = \sum_a \boldsymbol{r}_a \times \boldsymbol{F}_a = \sum_a \boldsymbol{N}_a \equiv \boldsymbol{N}, \tag{7.22}$$

where $\boldsymbol{N}_a$ is the external torque on particle a, and $\boldsymbol{N}$ is the total external torque. *If the resultant external torque about a given axis vanishes, then the total angular momentum about that axis is conserved.*

7.3 Energy

How does the total energy transform if we change into another inertial system that moves with relative velocity $\boldsymbol{V}$? As before, we write $\boldsymbol{v}_a = \boldsymbol{v}'_a + \boldsymbol{V}$, so

$$\begin{aligned} E &= \frac{1}{2}\sum_a m_a \boldsymbol{v}_a^2 + U = \frac{1}{2}\sum_a m_a(\boldsymbol{v}'_a + \boldsymbol{V})^2 + U \\ &= \frac{M}{2}\boldsymbol{V}^2 + \boldsymbol{V}\cdot\sum_a m_a \boldsymbol{v}'_a + \sum_a \frac{m_a \boldsymbol{v}'^2_a}{2} + U, \end{aligned} \tag{7.23}$$

or

$$E = E' + \boldsymbol{V}\cdot\boldsymbol{P}' + \frac{M}{2}\boldsymbol{V}^2. \tag{7.24}$$

If the "primed" system is the CM system, then $\boldsymbol{P}' = 0$, and $E' = E_{\text{int}}$, the internal energy. *The total energy of the system is equal to the sum of kinetic energy of a particle of mass M moving with the velocity of the CM and the internal energy of the individual particles moving relative to the CM.*

We can calculate the work done on the system as it evolves in time as

$$W_{12} = \sum_a \int_1^2 \boldsymbol{F}_a^{\text{tot}} \cdot d\boldsymbol{r}_a, \tag{7.25}$$

where $\boldsymbol{F}_a^{\text{tot}}$ is the total force (internal and external) acting on particle a. As we did before, we can write

$$\boldsymbol{F}_a^{\text{tot}}\cdot d\boldsymbol{r}_a = \frac{d\boldsymbol{p}_a}{dt}\cdot d\boldsymbol{r}_a = m_a\frac{d\boldsymbol{v}_a}{dt}\cdot\frac{d\boldsymbol{r}_a}{dt}dt = m_a\frac{d\boldsymbol{v}_a}{dt}\cdot\boldsymbol{v}_a dt = \frac{m_a}{2}\frac{d}{dt}\big(\boldsymbol{v}_a^2\big)dt = \frac{m_a}{2}d\big(\boldsymbol{v}_a^2\big), \tag{7.26}$$

and so

$$W_{12} = \sum_a \int_1^2 \boldsymbol{F}_a^{\text{tot}}\cdot d\boldsymbol{r}_a = T_2 - T_1. \tag{7.27}$$

On the other hand, explicitly splitting the forces into internal and external as above, we can write the work as

$$W_{12} = \sum_a \int_1^2 \boldsymbol{F}_a^{\text{tot}} \cdot d\boldsymbol{r}_a = \sum_a \int_1^2 \Big(\boldsymbol{F}_a + \sum_{b\neq a} \boldsymbol{f}_{ab}\Big) \cdot d\boldsymbol{r}_a \,. \tag{7.28}$$

As before, $\boldsymbol{f}_{ab}$ is the force originating from particle b and acting on particle a. We will assume that this force does only on the relative distance between the particles and acts along the line connecting the two particles, i.e., we have (using Newton's third law) $\boldsymbol{f}_{ab} = -\boldsymbol{f}_{ba} = f_{ab}\hat{\boldsymbol{r}}_{ab}$, with $f_{ab} = f_{ba}$ and $\hat{\boldsymbol{r}}_{ab} = (\boldsymbol{r}_b - \boldsymbol{r}_a)/|\boldsymbol{r}_b - \boldsymbol{r}_a|$ the unit vector from $\boldsymbol{r}_a$ to $\boldsymbol{r}_b$. Then we can derive this force from a potential. Consider, first, the force between a and b only, $U_{ab} = U_{ab}(|\boldsymbol{r}_b - \boldsymbol{r}_a|)$. Then we have

$$\boldsymbol{f}_{ab} = -\frac{\partial U_{ab}(|\boldsymbol{r}_b - \boldsymbol{r}_a|)}{\partial \boldsymbol{r}_a} = -\frac{\partial U_{ab}(|\boldsymbol{r}_b - \boldsymbol{r}_a|)}{\partial |\boldsymbol{r}_b - \boldsymbol{r}_a|}\frac{\partial |\boldsymbol{r}_b - \boldsymbol{r}_a|}{\partial \boldsymbol{r}_a} = -\frac{\partial U_{ab}(r)}{\partial r}\hat{\boldsymbol{r}}_{ab} = f_{ab}(r)\hat{\boldsymbol{r}}_{ab} \,, \tag{7.29}$$

where in the second-to-last step we denoted $r \equiv |\boldsymbol{r}_b - \boldsymbol{r}_a|$, and used

$$\frac{\partial |\boldsymbol{r}_b - \boldsymbol{r}_a|}{\partial \boldsymbol{r}_a} = \frac{\partial \sqrt{(\boldsymbol{r}_b - \boldsymbol{r}_a)^2}}{\partial \boldsymbol{r}_a} = -\frac{\boldsymbol{r}_b - \boldsymbol{r}_a}{\sqrt{(\boldsymbol{r}_b - \boldsymbol{r}_a)^2}} = -\hat{\boldsymbol{r}}_{ab} \,. \tag{7.30}$$

We see that we can derive this force from a potential of the form

$$U_{ab}(\boldsymbol{r}_a) = \int_{\boldsymbol{r}_b}^{\boldsymbol{r}_a} f_{ab}(r) dr \,. \tag{7.31}$$

The total internal potential is then just $U_{\text{int}} = \sum_{a<b} U_{ab}$. If also the external forces can be derived from a potential, $\boldsymbol{F}_a = -\partial U_a/\partial \boldsymbol{r}_a$, then the total potential energy is $U = U_a + \sum_{a<b} U_{ab}$. Then

$$W_{12} = \sum_a \int_1^2 \boldsymbol{F}_a^{\text{tot}} \cdot d\boldsymbol{r}_a = -\int_1^2 \sum_a \frac{\partial}{\partial \boldsymbol{r}_a}\Big(U_a + \sum_{a<b} U_{ab}\Big) d\boldsymbol{r}_a = -\int_1^2 dU = U_1 - U_2 \,. \tag{7.32}$$

It follows that

$$T_2 - T_1 = U_1 - U_2 \,, \tag{7.33}$$

or

$$T_1 + U_1 = T_2 + U_2 \,. \tag{7.34}$$

The total energy of the system is conserved.

Example 7.2: Exploding satellite. *A satellite of mass M explodes while in flight into three fragments. One mass ($m_1 = M/2$) travels in the original direction of the satellite, mass $m_2 = M/6$ travels in the opposite direction, and mass $m_3 = M/3$ comes to rest. The energy E released* *in the explosion is equal to five times the satellites kinetic energy at explosion. What are the velocities?*

Let the velocity of the satellite be $\boldsymbol{v}$. The three fragments have the following masses and veocities:

$$m_1 = \frac{M}{2}\,, \quad \boldsymbol{v}_1 = k_1\boldsymbol{v}\,; \tag{7.35}$$

$$m_2 = \frac{M}{6}\,, \quad \boldsymbol{v}_2 = -k_2\boldsymbol{v}\,; \tag{7.36}$$

$$m_3 = \frac{M}{3}\,, \quad \boldsymbol{v}_3 = 0\,. \tag{7.37}$$

Conservation of momentum and energy give

$$Mv = \frac{M}{2}k_1 v - \frac{M}{6}k_2 v\,, \tag{7.38}$$

$$E + \frac{M}{2}v^2 = \frac{1}{2}\frac{M}{2}v^2 + \frac{1}{2}\frac{M}{6}v^2\,. \tag{7.39}$$

We have $k_2 = 3k_1 - 6$, and so

$$5\left(\frac{M}{2}v^2\right) + \frac{M}{2}v^2 = \frac{Mv^2}{4}k_1^2 + \frac{Mv^2}{12}(3k_1 - 6)^2\,, \tag{7.40}$$

or $k_1(k_1 - 3) = 0$. We must have $k_1 > 0$, so

$$\boldsymbol{v}_1 = 3\boldsymbol{v}\,, \quad \boldsymbol{v}_2 = -3\boldsymbol{v}\,, \quad \boldsymbol{v}_3 = 0\,. \tag{7.41}$$

7.4 Decay of particles

Frequently, conservation of momentum and energy allows to draw important conclusions about the properties of various mechanical processes, independently of the precise nature of the underlying interactions. Prime examples are decays and collisions of particles.

We start with discussing the decay of a particle into two particles. We will use two different inertial frames to describe the process: the “center-of-mass” (CM) frame and the “laboratory” (LAB) frame. We will generally denote quantities in the CM frame with primes and in the LAB frame without primes.

The discussion is simplest in the CM frame, where the decaying particle is at rest. Momentum conservation implies that the momenta of the two outgoing particles add to zero: $\boldsymbol{p}_1' = -\boldsymbol{p}_2'$. We denote the absolute value by $p' = |\boldsymbol{p}_1'| = |\boldsymbol{p}_2'|$. We can determine p' using energy conservation. Denote the masses of the two final particles by m_1, m_2, their internal energies by E_{1i}, E_{2i}, and the internal energy of the decaying particle by E_i. Energy conservation gives

$$E_i = E_{1i} + \frac{m_1 v_1'^2}{2} + E_{2i} + \frac{m_2 v_2'^2}{2} = E_{1i} + \frac{p'^2}{2m_1} + E_{2i} + \frac{p'^2}{2m_2}\,. \tag{7.42}$$

Let

$$\Delta E = E_i - E_{1i} - E_{2i} > 0 \tag{7.43}$$

be the energy released in the decay. It will be converted into the kinetic energy of the final-state particles:

$$\Delta E = \frac{p'^2}{2}\left(\frac{1}{m_1} + \frac{1}{m_2}\right) = \frac{p'^2}{2\mu}\,, \tag{7.44}$$

with the reduced mass $\mu = m_1 m_2/(m_1 + m_2)$. Their speeds are given by $v_1' = p'/m_1$, $v_2' = p'/m_2$.

We now describe the process in the LAB frame, in which the decaying particle is moving with velocity $\boldsymbol{V}$. We consider particle 1 and denote its velocity in the CM frame by $\boldsymbol{v}_1'$ and in the LAB frame by $\boldsymbol{v}_1$. We have $\boldsymbol{v}_1 = \boldsymbol{v}_1' + \boldsymbol{V}$. It follows

$$v_1'^2 = v_1^2 + V^2 - 2v_1 V\cos\theta\,, \tag{7.45}$$

where $v_1' = |\boldsymbol{v}_1'|$, $v_1 = |\boldsymbol{v}_1|$, $V = |\boldsymbol{V}|$, and θ is the angle between the $\boldsymbol{v}_1$ and $\boldsymbol{V}$ (in the LAB frame). This relation allows to calculate the velocity $\boldsymbol{v}_1$ in the LAB frame (if θ is known). It can be illustrated in a diagram as shown in Fig. 16. Because $\boldsymbol{v}_1 = \boldsymbol{v}_1' + \boldsymbol{V}$, we can obtain $\boldsymbol{v}_1$

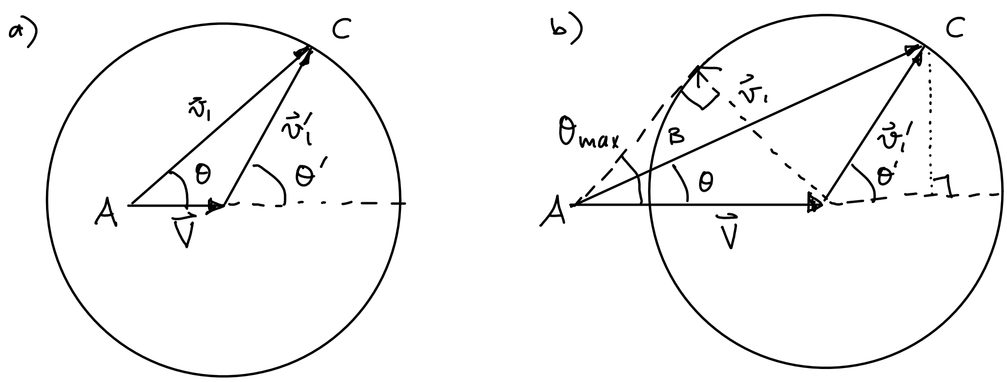

Figure 16: *Disintegration of a particle into two particles. Left panel: $v_1' > V$; right panel: $v_1' < V$.*

by drawing a circle of radius v_1' around the tip of $\boldsymbol{V}$. The velocity $\boldsymbol{v}_1$ will point from point A to any point on the circle (or, rather, the sphere whose intersection with the drawing plane is shown). We distinguish two cases: $v_1' > V$ and $v_1' < V$. In the first case, any angle θ is allowed. In the second case, the particle can be emitted in the forward direction with an angle θ that does not exceed

$$\sin\theta_{\text{max}} = \frac{v_1'}{V}\,. \tag{7.46}$$

For the relation between the emission angles θ and θ' in the LAB and CM frames, we read off

$$\tan\theta = \frac{v_1'\sin\theta'}{v_1'\cos\theta' + V}\,. \tag{7.47}$$

We can solve this expression for $\cos\theta'$:

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{v_1'\sqrt{1-\cos^2\theta'}}{v_1'\cos\theta' + V}\,, \tag{7.48}$$

or

$$\sin^2\theta(v_1'\cos\theta' + V)^2 = v_1'^2(1-\cos^2\theta')\cos^2\theta\,, \tag{7.49}$$

or

$$\cos^2\theta' + 2\frac{V}{v_1'}\sin^2\theta\cos\theta' + \frac{V^2}{v_1'^2}\sin^2\theta - \cos^2\theta = 0\,, \tag{7.50}$$

so

$$\cos\theta' = -\frac{V}{v_1'}\sin^2\theta \pm \cos\theta\sqrt{1-\frac{V^2}{v_1'^2}\sin^2\theta}\,. \tag{7.51}$$

As can be seen from Fig. 16, the relation between θ and θ' is unique for $v_1' > V$, and we have to choose the positive sign such that $\theta' = 0$ for $\theta = 0$. For $v_1' < V$, however, the relation is not unique; for given θ, there are two possible endpoints for $\boldsymbol{v}_1$ on the circle (points B and C in Fig. 16, right panel), corresponding to two different values for θ'.

7.5 Elastic collision of two particles

An elastic collision involves no change in the internal energies of the scattering particles. Accordingly, the internal energy may be neglected when applying the law of conservation of energy.

The scattering process is most easily described in the CM frame. We denote the initial and final velocities of the two particles in the CM frame by u_1', u_2' and v_1', v_2', respectively, and the corresponding velocities in the LAB frame by u_1, u_2 and v_1, v_2. The relation of the initial velocities in the CM and LAB frames is

$$\boldsymbol{u}_1' = \frac{m_2\boldsymbol{u}}{m_1+m_2}\,,\quad \boldsymbol{u}_2' = -\frac{m_1\boldsymbol{u}}{m_1+m_2}\,, \tag{7.52}$$

where $\boldsymbol{u} = \boldsymbol{u}_1 - \boldsymbol{u}_2$ (see Eq. (6.7)).

Momentum conservation implies that the momenta of the two particles remain equal and opposite after the collision, while energy conservation implies that their magnitudes also do not change.[18] The result of the collision is simply a rotation of the velocity vectors. If we

[18] This can be seen as follows: The condition of energy conservation can be written as

$$\frac{m_1^2(u_1')^2}{2m_1} + \frac{m_2^2(u_2')^2}{2m_2} = \frac{m_1^2(v_1')^2}{2m_1} + \frac{m_2^2(v_2')^2}{2m_2}\,. \tag{7.53}$$

In the CM frame we have $m_1u_1' = m_2u_2'$ and $m_1v_1' = m_2v_2'$, so

$$m_1^2(u_1')^2\left(\frac{1}{m_1}+\frac{1}{m_2}\right) = m_1^2(v_1')^2\left(\frac{1}{m_1}+\frac{1}{m_2}\right), \tag{7.54}$$

or $u_1' = v_1'$, and hence also $u_2' = v_2'$.

denote by $\boldsymbol{n}$ a unit vector in the direction of $\boldsymbol{v}_1'$ (velocity of particle 1 after the collision), the final velocities are given by

$$\boldsymbol{v}_1' = \frac{m_2}{m_1+m_2} u\boldsymbol{n}\,, \quad \boldsymbol{v}_2' = -\frac{m_1}{m_1+m_2} u\boldsymbol{n}\,, \tag{7.55}$$

with $u = |\boldsymbol{u}|$. To obtain the final velocities in the LAB frame, we have to add the velocity $\boldsymbol{V}$ of the center of mass:

$$\boldsymbol{v}_1 = \frac{m_2}{m_1+m_2} u\boldsymbol{n} + \frac{m_1\boldsymbol{u}_1 + m_2\boldsymbol{u}_2}{m_1+m_2}\,, \quad \boldsymbol{v}_2 = -\frac{m_1}{m_1+m_2} u\boldsymbol{n} + \frac{m_1\boldsymbol{u}_1 + m_2\boldsymbol{u}_2}{m_1+m_2}\,. \tag{7.56}$$

No further information can be obtained from energy and momentum conservation. The direction of $\boldsymbol{n}$ depends on the relative position of the particles before the collision, as well as on the nature of the interaction.

We can again interpret these results geometrically; this will allow us to obtain some useful relations. To this end, we use momenta rather than velocities. We denote the initial momenta in the LAB frame by $\boldsymbol{q}_1, \boldsymbol{q}_2$ and the final momenta by $\boldsymbol{p}_1, \boldsymbol{p}_2$. Momenta in the CM frame will be denoted by an additional prime. Multiplying the relations (7.56) by m_1 and m_2, respectively, gives

$$\boldsymbol{p}_1 = \mu u\boldsymbol{n} + \frac{m_1}{m_1+m_2}(\boldsymbol{q}_1 + \boldsymbol{q}_2)\,, \quad \boldsymbol{p}_2 = -\mu u\boldsymbol{n} + \frac{m_2}{m_1+m_2}(\boldsymbol{q}_1 + \boldsymbol{q}_2)\,. \tag{7.57}$$

(As usual, $\mu = m_1 m_2/(m_1+m_2)$ is the reduced mass.) The relation between the momenta $\boldsymbol{q}_i$ and $\boldsymbol{p}_i$ is shown in Fig. 17. There, we have the following relations:

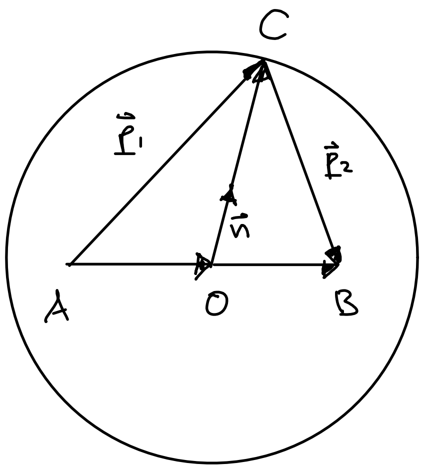

Figure 17: *Elastic collision of two particles.*

$$\overrightarrow{OC} = \mu u\boldsymbol{n}\,, \quad \overrightarrow{AO} = \frac{m_1}{m_1+m_2}(\boldsymbol{q}_1 + \boldsymbol{q}_2)\,, \quad \overrightarrow{OB} = \frac{m_2}{m_1+m_2}(\boldsymbol{q}_1 + \boldsymbol{q}_2)\,, \tag{7.58}$$

(and therefore $\overrightarrow{AB} = \boldsymbol{q}_1 + \boldsymbol{q}_2$). The final momenta $\boldsymbol{p}_1, \boldsymbol{p}_2$ are then given by the vectors $\overrightarrow{AC}$ and $\overrightarrow{CB}$, according to Eq. (7.57). Given $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$, the points A and B are fixed; the point C, on the other hand, can lie anywhere on the circle.

We will now study the case that particle 2 is initially at rest. In this case, the length $OB = \frac{m_2}{m_1+m_2}\boldsymbol{q}_1 = \mu u$ equals the radius of the circle; i.e., the point B is located on the circle. The vector $\overrightarrow{AB}$ represents $\boldsymbol{q}_1$, the momentum before the collision. The point A is located inside the circle for $m_1 < m_2$ and outside the circle for $m_1 > m_2$. See Fig. 18. The angles of

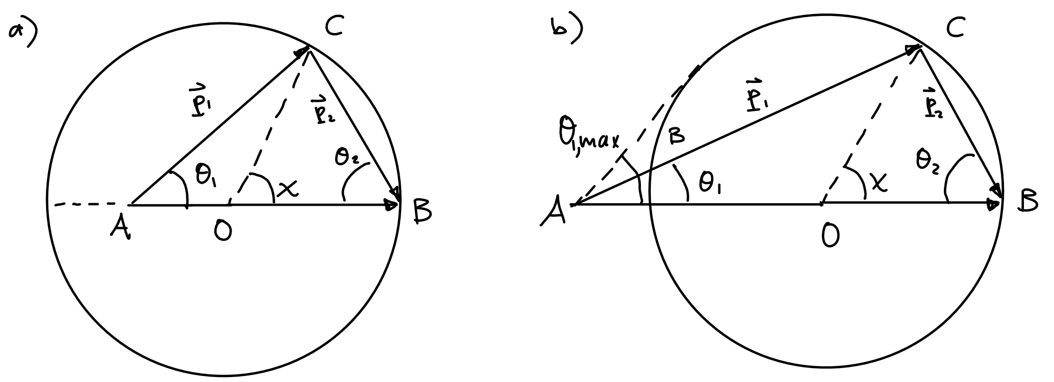

Figure 18: *Elastic collision of two particles, with particle 2 initially at rest. Left panel: $m_1 < m_2$; right panel: $m_1 > m_2$.*

deflection w.r.t. $\boldsymbol{q}_1$ are denoted by θ_1, θ_2; the angle χ, giving the direction of $\boldsymbol{n}$, is the angle of deflection of the first particle in the CM system. From Fig. 18 we read off that

$$\tan\theta_1 = \frac{m_2 \sin\chi}{m_1 + m_2\cos\chi}\,, \quad \theta_2 = \frac{\pi - \chi}{2} \tag{7.59}$$

(note that $AO = m_1 q_1/(m_1+m_2)$ and $OC = OB = m_2 q_1/(m_1+m_2)$). Using $(m_1 + m_2)^2|\boldsymbol{p}_1|^2 = (m_1 m_2 u\boldsymbol{n} + m_1^2\boldsymbol{u})^2$ and $(m_1+m_2)^2|\boldsymbol{p}_2|^2 = (m_1 m_2 u\boldsymbol{n} - m_1 m_2\boldsymbol{u})^2 = 2m_1^2 m_2^2 u^2(1-\cos\chi)$ (from Eq. (7.57) with $\boldsymbol{q}_2 = 0$, so $\boldsymbol{u} = \boldsymbol{u}_1$), we obtain the absolute values of the final velocities as

$$v_1 = \frac{\sqrt{m_1^2 + m_2^2 + 2m_1 m_2\cos\chi}}{m_1+m_2}u\,, \quad v_2 = \frac{2m_1 u}{m_1+m_2}\sin\frac{\chi}{2}\,. \tag{7.60}$$

The angle between the velocities after the collision is given by the sum $\theta_1 + \theta_2$. Fig. 18 shows that $\theta_1 + \theta_2 > \pi/2$ for $m_1 < m_2$, and $\theta_1 + \theta_2 < \pi/2$ for $m_1 > m_2$.

The case that both particle move along a straight line after the collision (in the LAB frame) corresponds to $\chi = \pi$ (central collision). In this case, C is located either to the left of A (Fig. 18, left panel; $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ point in opposite directions), or to the right of A (Fig. 18, right panel; $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ point in the same direction). The final velocities are then

$$\boldsymbol{v}_1 = \frac{m_1 - m_2}{m_1+m_2}\boldsymbol{u}\,, \quad \boldsymbol{v}_2 = \frac{2m_1}{m_1+m_2}\boldsymbol{u}\,. \tag{7.61}$$

The absolute value of $\boldsymbol{v}_2$ is maximal in this case. It follows that the maximal energy that can be transferred to the particle initially at rest is

$$E_{2,\text{out,max}} = \frac{m_2 v_{2,\text{max}}^2}{2} = \frac{4m_1 m_2}{(m_1+m_2)^2} E_{1,\text{in}} \,, \tag{7.62}$$

where $E_{1,\text{in}} = m_1 u_1^2/2$ is the energy of the initially moving particle.

Example 7.3: Scattering of dark matter on atomic nuclei. *Let's assume we have a dark matter particle of mass $m_\chi = 100\, m_p$ (where $m_p = 1.67 \times 10^{-27}$ kg $= 938$ MeV is the proton mass), scattering off an atomic xenon nucleus (mass roughly $m_A = 130\, m_p$). The escape velocity of our galaxy is about 500 km/s, or in units of speed of light 500 km/s$/(299792$ km/s$) \sim 0.002$. This is the maximal speed of dark matter in the galactic halo – dark matter moves with nonrelativistic speeds.*

Let's calculate the maximum transferred energy in the lab frame, scattering on a nucleus at rest. We have $|\boldsymbol{q}_1| = 0.002 m_\chi = 0.2$ GeV, $|\boldsymbol{q}_2| = 0$. The maximal energy transferred to the nucleus is then

$$E_{A,\mathit{out,max}} = \frac{4m_A m_\chi}{(m_A+m_\chi)^2} E_{\chi,\mathit{in}} \,. \tag{7.63}$$

Using $E_{\chi,in} = |\boldsymbol{q}_1|^2/(2m_\chi)$, this gives $E_{A,out,max} \sim 2$ keV. That is the energy of a typical X-ray photon.

For $m_1 < m_2$, the velocity of the first particle after the collision can have any direction. For $m_1 < m_2$, the maximal scattering angle is given by

$$\sin\theta_{1,\text{max}} = \frac{OC}{AO} = \frac{m_2}{m_1} \,. \tag{7.64}$$

The collision is particularly simple if the masses of the participating particles are equal. In this case, both points A and B are located on the circle (see Fig. 19). We have

$$\theta_1 = \frac{\chi}{2} \,, \quad \theta_2 = \frac{\pi-\chi}{2} \,, \tag{7.65}$$

and so

$$v_1 = u\cos\frac{\chi}{2} \,, \quad v_2 = u\sin\frac{\chi}{2} \,. \tag{7.66}$$

Note that $\theta_1 + \theta_2 = \pi/2$. For a central collision ($\chi = \pi$), we have $v_1 = 0$ and $v_2 = u$; i.e. the first (moving) particle stops and the second particle (initially at rest) moves with the velocity of the first particle.

7.6 Scattering cross sections

So far, all information obtained about the elastic collision of two particles was *kinematic* – no information on the value of the scattering angle χ was available. This information is provided by the *dynamics* of the collision, i.e. by specifying the forces between the colliding particles. Here, we will only consider the case of central forces. As usual, we will consider the

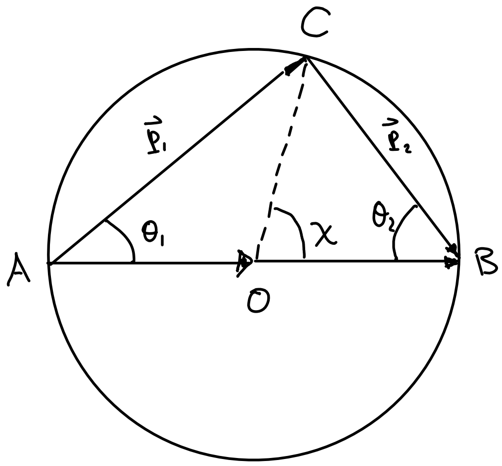

***Figure 19:** Elastic collision of two particles with equal masses.*

equivalent problem of the motion of a single particle (with reduced mass $\mu = m_1 m_2/(m_1 + m_2)$) in a fixed force field (as we did in Sec. 6).

We have seen in Sec. 6 that the trajectory of particle in a central force field is symmetric about the line that connects the force center with the point of closest approach (see Fig. 20). Therefore, the two asymptotes subtend the same angle, ϕ_0, with that line. The scattering

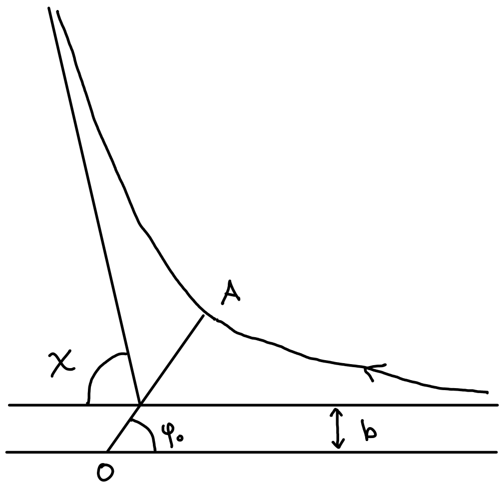

***Figure 20:** Scattering of a particle in a central force field.*

angle is then given by $\chi = |\pi - 2\phi_0|$. Using Eq. (6.19), we calculate the angle ϕ_0 as

$$\phi_0 = \int_{r_{\min}}^{\infty} dr \frac{l/r^2}{\sqrt{2\mu\left[E - U(r)\right] - \frac{l^2}{r^2}}} \tag{7.67}$$

(recall that r_{min} is a root of the radical in the denominator in Eq. (7.69)).

It will turn out to be useful instead of the constants E and l to use the velocity "at infinity" v_∞ and the *impact parameter* b. The impact parameter b is the distance to the force center at which the particle would pass if the force were absent. With the usual convention that the potential energy vanishes at infinity, the relations between these parameters are

$$E = \frac{\mu v_\infty^2}{2}\,, \quad l = \mu b v_\infty\,, \tag{7.68}$$

and we have

$$\phi_0 = \int_{r_{\text{min}}}^{\infty} dr \frac{b/r^2}{\sqrt{1 - \frac{b^2}{r^2} - \frac{2U}{\mu v_\infty^2}}}\,. \tag{7.69}$$

Using this result, we can calculate the scattering angle χ.

In practice, one usually considers the scattering of a *beam* of equal particles that move towards the scattering center with the same velocity v_∞. Different particles in the beam will have different impact parameters and will therefore be scattered with different angles χ. We denote by dR the *scattering rate* – the number of particles dN that will be scattered by an angle between χ and $\chi + d\chi$ within a unit time intervall T, i.e. $dR = dN/T$ (we used the fact that the scattering is symmetric under rotations about the beam axis). The quantity dR is inconvenient as it depends not only on the interaction, but also on the experimental setup (namely, the density of particles in the beam). Therefore, one defines the *(differential) cross section*

$$d\sigma = \frac{dR}{\Phi}\,, \tag{7.70}$$

where the *flux* Φ is the number of particles that passes per unit time interval T through the cross section of the beam. $d\sigma$ has units of an area and is completely determined by the interaction (force).

If the relation between χ and b is unique, then only those particles with impact parameter in an intervall between $b(\chi)$ and $b(\chi) + db(\chi)$ will be deflected by an angle between χ and $\chi + d\chi$. The number of these particles (per unit time interval) is equal to the product of Φ and the area of the annular region between the circles with radius b and $b + db$: $dR = 2\pi b\, db\, \Phi$, and the cross section becomes

$$d\sigma = 2\pi b\, db\,. \tag{7.71}$$

To obtain the dependence of the cross section on the scattering angle, we write this as

$$d\sigma = 2\pi b(\chi)\left|\frac{db(\chi)}{d\chi}\right| d\chi\,. \tag{7.72}$$

Frequently, one relates the differential scross section to the solid angle element $d\Omega$ instead of the planar angle element $d\chi$. The solid angle between two cones with opening angles χ and $\chi + d\chi$ is $d\Omega = 2\pi \sin\chi d\chi$, so

$$d\sigma = \frac{b(\chi)}{\sin\chi}\left|\frac{db(\chi)}{d\chi}\right| d\Omega\,. \tag{7.73}$$

If one is interested in the actual scattering of one particle on another, one can use the relations (7.59) to obtain the cross section in terms of the scattering angles in the LAB frame.

Example 7.4: Scattering of hard spheres. *We consider the scattering of two hard spheres with radii R_1 and R_2, and masses m_1 and m_2, respectively. We start with the scattering in*

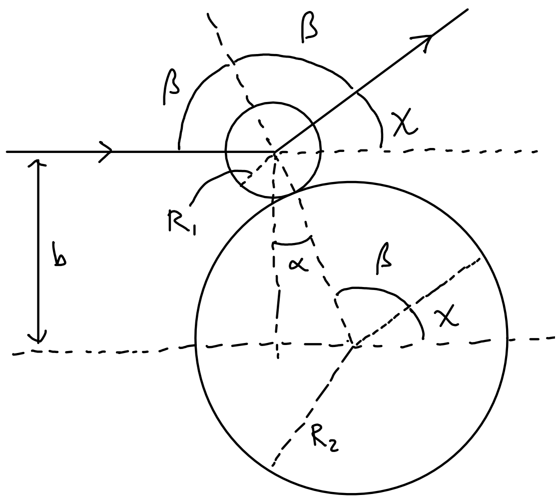

Figure 21: *Scattering of hard spheres.*

the CM system. In this case, we have to consider the scattering of a sphere of reduced mass $\mu = m_1 m_2/(m_1 + m_2)$ on a (fixed) sphere (see Fig. 21).

The impact parameter is given by

$$b = (R_1 + R_2)\cos\alpha\,. \tag{7.74}$$

To obtain $b(\chi)$, we observe that $2\pi = \pi/2 + 3\beta + \chi + \alpha$. We know already that $\beta = (\pi - \chi)/2$, so $\alpha = 3\pi/2 - 3\beta - \chi = \chi/2$, so

$$b(\chi) = (R_1 + R_2)\cos\frac{\chi}{2}\,, \tag{7.75}$$

and

$$db(\chi) = -\frac{R_1 + R_2}{2}\sin\frac{\chi}{2}d\chi\,. \tag{7.76}$$

Inserting into Eq. (7.72) gives

$$d\sigma = \pi(R_1 + R_2)^2\cos\frac{\chi}{2}\sin\frac{\chi}{2}d\chi = \frac{\pi}{2}(R_1 + R_2)^2\sin\chi d\chi\,. \tag{7.77}$$

Expressing this result in terms of the solid angle element (see Eq. (7.73)) gives

$$d\sigma = \frac{1}{4}(R_1 + R_2)^2 d\Omega\,. \tag{7.78}$$

The total cross section is obtained by integrating this expression over the whole sphere. We haved $\int d\Omega = \int_0^\pi 2\pi \sin\chi d\chi = 4\pi$*, so the total cross section is*

$$\sigma = \pi(R_1 + R_2)^2 \,, \tag{7.79}$$

as expected.

To obtain the differential cross section in the LAB frame where on of the spheres is initially at rest, we need to use the relations (7.59) to relate the scattering angles θ_1 *and* θ_2 *to the scattering angle* χ *in the CM frame. The differential cross section is*

$$d\sigma = \pi(R_1 + R_2)^2 \frac{\sin\chi d\chi}{2} \,. \tag{7.80}$$

Solving

$$\tan\theta_1 = \frac{m_2 \sin\chi}{m_1 + m_2 \cos\chi} \tag{7.81}$$

for $\cos\chi$ *gives*

$$\cos\chi = -\frac{m_1}{m_2}\sin^2\theta_1 \pm \cos\theta_1 \sqrt{1 - \frac{m_1^2}{m_2^2}\sin^2\theta_1} \,. \tag{7.82}$$

(see Sec. 7.4). For $m_2 > m_1$ *the solution is unique and we need to choose the positive sign. Using* $\sin\chi d\chi = -d\cos\chi$ *and*

$$\begin{aligned} d\cos\chi &= \sin\theta_1 d\theta_1 \Bigg(-2\frac{m_1}{m_2}\cos\theta_1 - \sqrt{1 - \frac{m_1^2}{m_2^2}\sin^2\theta_1} + \frac{-\frac{m_1^2}{m_2^2}\cos^2\theta_1}{\sqrt{1 - \frac{m_1^2}{m_2^2}\sin^2\theta_1}} \Bigg) \\ &= -\sin\theta_1 d\theta_1 \left(2\frac{m_1}{m_2}\cos\theta_1 + \frac{1 + \frac{m_1^2}{m_2^2}\cos 2\theta_1}{\sqrt{1 - \frac{m_1^2}{m_2^2}\sin^2\theta_1}} \right) , \end{aligned} \tag{7.83}$$

we find

$$d\sigma = \pi(R_1 + R_2)^2 \frac{\sin\theta_1 d\theta_1}{2} \left(2\frac{m_1}{m_2}\cos\theta_1 + \frac{1 + \frac{m_1^2}{m_2^2}\cos 2\theta_1}{\sqrt{1 - \frac{m_1^2}{m_2^2}\sin^2\theta_1}} \right) . \tag{7.84}$$

For $m_1 > m_2$ *we need to take the difference between the two solutions (as one of the LAB scattering angles increases and the other decreases with* χ*) and find*

$$d\sigma = \pi(R_1 + R_2)^2 \sin\theta_1 d\theta_1 \frac{1 + \frac{m_1^2}{m_2^2}\cos 2\theta_1}{\sqrt{1 - \frac{m_1^2}{m_2^2}\sin^2\theta_1}} \,. \tag{7.85}$$

7.7 The Rutherford cross section*

One of the most important applications of the general expressions for the cross section is the scattering of charged particles is a Coulomb field, i.e. $U(r) = k/r$. We insert the potential into Eq. (7.69) and obtain

$$\phi_0 = \int_{r_{\min}}^{\infty} dr \frac{b/r^2}{\sqrt{1 - \frac{b^2}{r^2} - \frac{2k}{r\mu v_\infty^2}}} \,. \tag{7.86}$$

Here,

$$r_{\min} = \frac{k}{\mu v_\infty^2} + \sqrt{\frac{k^2}{\mu^2 v_\infty^4} + b^2} \,. \tag{7.87}$$

Integration gives ($u = 1/r$, $du = -dr/r^2$)

$$\phi_0 = -b \int_{u_{\max}}^{0} du \frac{1}{\sqrt{1 - u^2 b^2 - \frac{2k}{\mu v_\infty^2} u}} = - \int_{u_{\max}}^{0} du \frac{1}{\sqrt{\frac{1}{b^2} - \left(u + \frac{k}{b^2 \mu v_\infty^2}\right)^2 + \frac{k^2}{b^4 \mu^2 v_\infty^4}}} \,. \tag{7.88}$$

where $u_{\max} = 1/r_{\min}$. Shifting the integration variable $y = u + \frac{k}{b^2 \mu v_\infty^2}$ gives

$$\phi_0 = - \int_{y_{\max}}^{\frac{k}{b^2 \mu v_\infty^2}} dy \frac{1}{\sqrt{\frac{1}{b^2} + \frac{k^2}{b^4 \mu^2 v_\infty^4} - y^2}} \tag{7.89}$$

where $y_{\max} = u_{\max} + \frac{k}{b^2 \mu v_\infty^2}$. Next, we rescale $w = y/\sqrt{1/b^2 + k^2/(b^4 \mu^2 v_\infty^4)}$. This gives

$$\phi_0 = - \int_{1}^{w_\infty} dw \frac{1}{1 - w^2} = \arccos(w)\Big|_{1}^{w_\infty} \,, \tag{7.90}$$

where $w_{\max} = y_{\max}/\sqrt{1/b^2 + k^2/(b^4 \mu^2 v_\infty^4)} = 1$ and $w_\infty = k/\sqrt{v_\infty^4 \mu^2 b^2 + k^2}$. Therefore

$$\phi_0 = \arccos \frac{\frac{k}{\mu b v_\infty^2}}{\sqrt{1 + \frac{k^2}{b^2 \mu^2 v_\infty^4}}} \,. \tag{7.91}$$

This implies

$$b^2 = \frac{k^2}{\mu^2 v_\infty^4} \tan^2 \phi_0 \tag{7.92}$$

and with $\chi = |\pi - 2\phi_0|$

$$b^2 = \frac{k^2}{\mu^2 v_\infty^4} \cot^2 \frac{\chi}{2} \,. \tag{7.93}$$

Differientiating gives

$$\frac{db}{d\chi} = - \frac{k}{2\mu v_\infty^2} \frac{1}{\sin^2 \frac{\chi}{2}} \,. \tag{7.94}$$

Inserting into Eq. (7.73) and using $\sin \chi = 2 \sin(\chi/2) \cos(\chi/2)$

$$d\sigma = \frac{k^2}{4\mu^2 v_\infty^4} \frac{1}{\sin^4 \frac{\chi}{2}} d\Omega \,. \tag{7.95}$$

8 Motion in a noninertial reference frame

In certain circumstances, it is convenient to use non-inertial coordinate frames to describe the motion of a system. Examples are motion on the earth (taking into account the earths own motion within, e.g., the solar system) and the motion of rigid bodies.

8.1 Angular velocity

Accelerated coordinate systems are typically associated with extended, rigid bodies. It is therefore useful to distinguish between translations and rotations. (For instance, at a given time, the translational movement could be associated with the CM, and the remaining motion can be described as a rotation about an axis through the CM (why?).) How do we describe this motion? Let us focus on the rotational motion. We denote the position vector of a moving particle by $\boldsymbol{r}$ and its instantaneous velocity by $\boldsymbol{v}$. Denote the angle between the position vector and the axis of rotation by α (see Fig. 22). If within the time interval dt the polar angle

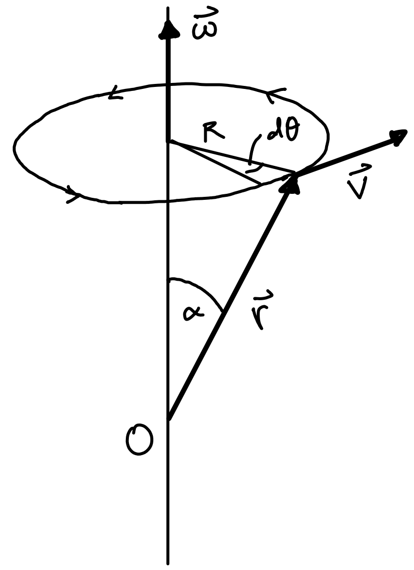

Figure 22: *Particle rotating about an axis.*

changes by an amount $d\theta$, the instantaneous speed of the particle is $v = |\boldsymbol{v}| = Rd\theta/dt = |\boldsymbol{r}|\omega \sin\alpha$, where $\omega = \dot{\theta}$. Because $\boldsymbol{v}$ is orthogonal to both $\boldsymbol{r}$ and the axis of rotation, we can write

$$\boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{r} \,. \tag{8.1}$$

This defines the *angular velocity* $\boldsymbol{\omega}$.

If we denote the velocity of the translational motion by $\boldsymbol{V}$ and include that motion for our particle, we have, more generally,

$$\boldsymbol{v} = \boldsymbol{V} + \boldsymbol{\omega} \times \boldsymbol{r} \,. \tag{8.2}$$

What happens if we choose another origin $\tilde{O}$ of our coordinate system at distance $\boldsymbol{a}$ from O? We denote the translational velocity in this system by $\tilde{\boldsymbol{V}}$ and the angular velocity by $\tilde{\boldsymbol{\omega}}$. The relation between the two position vectors is $\boldsymbol{r} = \tilde{\boldsymbol{r}} + \boldsymbol{a}$, and we obtain

$$\boldsymbol{v} = \boldsymbol{V} + \boldsymbol{\omega} \times \boldsymbol{a} + \boldsymbol{\omega} \times \tilde{\boldsymbol{r}} \equiv \tilde{\boldsymbol{V}} + \tilde{\boldsymbol{\omega}} \times \tilde{\boldsymbol{r}} \,. \tag{8.3}$$

We see that

$$\tilde{\boldsymbol{V}} = \boldsymbol{V} + \boldsymbol{\omega} \times \boldsymbol{a} \,, \quad \tilde{\boldsymbol{\omega}} = \boldsymbol{\omega} \,. \tag{8.4}$$

The angular velocity does not depend on the choice of origin of the coordinate system.

8.2 Accelerating coordinate systems

First, we consider a coordinate system that moves w.r.t. an inertial frame with a time-dependent translational velocity $\boldsymbol{V}(t)$, i.e. there is no rotation of the coordinates axes; the fixed and moving coordinate axes stay parallel (or at fixed angles) to each other, see Fig. 23 (left panel). We consider the motion of a single particle in either coordinate system. The velocity in the inertial frame, $\boldsymbol{v}_0$, and the velocity in the accelerated frame, $\boldsymbol{v}'$, are related by

$$\boldsymbol{v}_0 = \boldsymbol{v}' + \boldsymbol{V}(t) \,. \tag{8.5}$$

How does the Lagrangian change if we change from the inertial to the accelerated frame? The Lagrangian in the inertial frame can be chosen as

$$L_0 = \frac{m\boldsymbol{v}_0^2}{2} - U \,. \tag{8.6}$$

Inserting the relation (8.5), we find

$$L' = \frac{m\boldsymbol{v}'^2}{2} + m\boldsymbol{v}' \cdot \boldsymbol{V} + \frac{m}{2}\boldsymbol{V}^2 - U \,. \tag{8.7}$$

To simplify this expression, we point out that adding to the Lagrangian the total time derivative of an arbitrary function of time and coordinates does not change the equations of motion.[19] Since $\boldsymbol{V}^2(t)$ is an arbitrary function of time, it can be written as the time derivative of

[19] This can be seen as follows: Let us write

$$L'(q, \dot{q}, t) = L(q, \dot{q}, t) + \frac{d}{dt} f(q, t) \,. \tag{8.8}$$

Then

$$S' = \int_{t_1}^{t_2} dt\, L'(q, \dot{q}, t) = \int_{t_1}^{t_2} dt\, L(q, \dot{q}, t) + \int_{t_1}^{t_2} dt \frac{d}{dt} f(q, t) = S + f(q(t_2), t_2) - f(q(t_1), t_1) \,. \tag{8.9}$$

Since $q(t_1)$ and $q(t_2)$ are held fixed in the variation of the action, the equations of motions resulting from S and S' are the same.

another arbitrary function and thus can be dropped from the Lagrangian. Similarly, we can write

$$\boldsymbol{v}'\cdot\boldsymbol{V} = \boldsymbol{V}\cdot\frac{d}{dt}\boldsymbol{r}' = \frac{d}{dt}\left(\boldsymbol{V}\cdot\boldsymbol{r}'\right) - \boldsymbol{r}'\cdot\frac{d}{dt}\boldsymbol{V}\,, \tag{8.10}$$

and drop the total derivative. The resulting Lagrangian is

$$L' = \frac{m\boldsymbol{v}'^2}{2} - m\boldsymbol{r}'\cdot\dot{\boldsymbol{V}} - U\,, \tag{8.11}$$

and the resulting equations of motion are

$$m\frac{d\boldsymbol{v}'}{dt} = -\frac{\partial U}{\partial \boldsymbol{r}'} - m\dot{\boldsymbol{V}}\,. \tag{8.12}$$

Writing $\boldsymbol{V} = \dot{\boldsymbol{R}}$, we can interpret the last term as an additional force $\boldsymbol{f}_a = -m\ddot{\boldsymbol{R}}$ that appears in the accelerated frame. We write the final version of the equations of motion, expressed in the accelerating coordinate system, as

$$m\ddot{\boldsymbol{r}}' = -\frac{\partial U}{\partial \boldsymbol{r}'} - m\ddot{\boldsymbol{R}}\,. \tag{8.13}$$

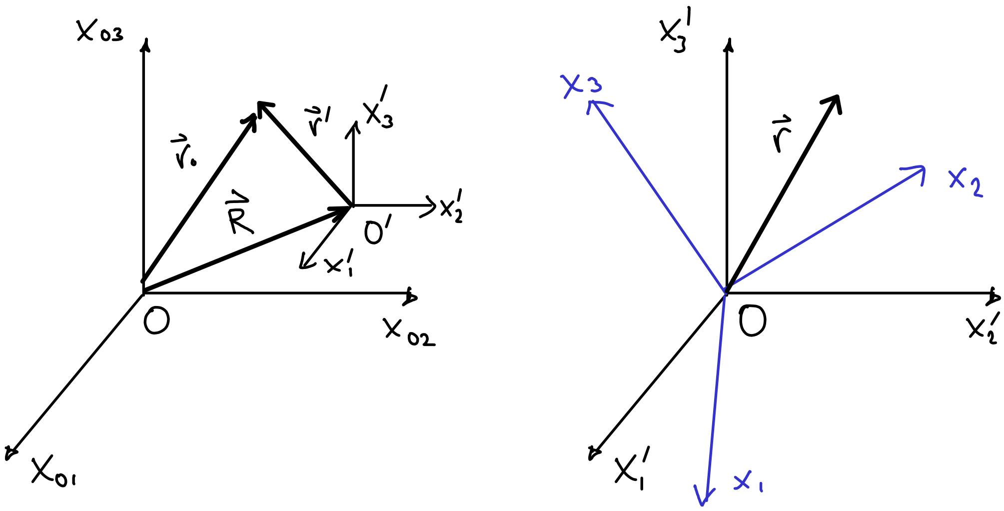

***Figure 23:** Fixed, accelerating, and rotating coordinate systems.*

Now, we consider a (possibly time-dependent) rotation of the coordinate system, see Fig. 23. We neglect the translational acceleration for the moment and further assume that the origins of the two coordinate systems coincide (i.e. $\boldsymbol{R} = \boldsymbol{0}$, and hence $\boldsymbol{r}' = \boldsymbol{r}$, see Fig. 23, right panel). The relation between velocities in the non-rotating and rotating coordinate systems is given by[20]

$$\left(\dot{\boldsymbol{r}}\right)_{\text{fixed}} = \left(\dot{\boldsymbol{r}}\right)_{\text{rotating}} + \boldsymbol{\omega}\times\boldsymbol{r}\,. \tag{8.14}$$

[20] There is somewhat of a notational dilemma here. As the origins of the two coordinate systems coincide, the position of the particle is described by the same position vector. However, this vector has different coordinate representations in the two different systems. We try to indicate this explicitly whereever it is not clear from context.

Generally, the relation between the time rate of change in the non-rotating and rotating frames of any vector $\boldsymbol{q}$ is given by

$$\left(\frac{d\boldsymbol{q}}{dt}\right)_{\text{fixed}} = \left(\frac{d\boldsymbol{q}}{dt}\right)_{\text{rotating}} + \boldsymbol{\omega} \times \boldsymbol{q}\,. \tag{8.15}$$

We could proceed by inserting Eq. (8.14) into the Lagrangian (8.11), and then derive the equations of motion (see App. D). However, in this case it is actually easier to consider NEWTON's equations of motion directly. NEWTON's equation in the inertial frame is simply given by

$$m\big(\ddot{\boldsymbol{r}}\big)_{\text{fixed}} = \boldsymbol{F}\,. \tag{8.16}$$

Here, $\boldsymbol{F} \equiv -\boldsymbol{\nabla} U$ corresponds to the *actual "physical"* forces. Using the general relation (8.15) to take the time derivative of the relation (8.14) gives the relation between the accelerations in the fixed and rotating frames:

$$\begin{aligned}\big(\ddot{\boldsymbol{r}}\big)_{\text{fixed}} &\equiv \left(\frac{d}{dt}\right)_{\text{fixed}}\left(\frac{d\boldsymbol{r}}{dt}\right)_{\text{fixed}} = \left(\frac{d}{dt}\right)_{\text{fixed}}\left[\left(\frac{d\boldsymbol{r}}{dt}\right)_{\text{rotating}} + \boldsymbol{\omega}\times\boldsymbol{r}\right] \\ &= \big(\ddot{\boldsymbol{r}}\big)_{\text{rotating}} + \boldsymbol{\omega}\times\big(\dot{\boldsymbol{r}}\big)_{\text{rotating}} + \big(\dot{\boldsymbol{\omega}}\big)_{\text{rotating}}\times\boldsymbol{r} + \boldsymbol{\omega}\times\left[\big(\dot{\boldsymbol{r}}\big)_{\text{rotating}} + \boldsymbol{\omega}\times\boldsymbol{r}\right].\end{aligned} \tag{8.17}$$

NEWTON's equation (8.16) becomes (*all* time derivatives are evaluated in the *rotating* system, so I drop the corresponding subscripts)

$$m\ddot{\boldsymbol{r}} = \boldsymbol{F} - m\dot{\boldsymbol{\omega}}\times\boldsymbol{r} - m\boldsymbol{\omega}\times\big(\boldsymbol{\omega}\times\boldsymbol{r}\big) - 2m\boldsymbol{\omega}\times\dot{\boldsymbol{r}}\,. \tag{8.18}$$

The last two terms on the right side are called the *centrifugal force* and the *CORIOLIS force*, respectively.[21]

Until now, we assumed that the origins of the two coordinate systems coincide. We obtain the general case by simply adding the translational motion of the CM, according to Eq. (8.13), and get

$$m\big(\ddot{\boldsymbol{r}}\big)_{\text{rotating}} = \boldsymbol{F} - m\big(\ddot{\boldsymbol{R}}\big)_{\text{fixed}} - m\dot{\boldsymbol{\omega}}\times\boldsymbol{r} - m\boldsymbol{\omega}\times\big(\boldsymbol{\omega}\times\boldsymbol{r}\big) - 2m\boldsymbol{\omega}\times\big(\dot{\boldsymbol{r}}\big)_{\text{rotating}}\,. \tag{8.19}$$

Note that the time derivatives in $\ddot{\boldsymbol{R}}$ are still evaluated in the fixed frame, as indicated by the explicit notation.

8.3 Motion relative to the earth

We here assume the earth to be spherical and isotropic. We neglect any motion of the earth related to the motion of the solar system as a whole, or any small disturbances (precession of the earth's axis etc.). We locate the "fixed" (non-rotating) coordinate system at the center

[21] Gaspard-Gustave de Coriolis (1792 - 1843) discovered this effect during his study of water wheels. Only at the end of the nineteenth century it became known under his name.

of the earth, with the z axis pointing towards the North pole. The force on any particle of mass m near the surface of the earth in this fixed system is then

$$\boldsymbol{F} = m\boldsymbol{g}_0 + \boldsymbol{f}\,, \tag{8.20}$$

where

$$\boldsymbol{g}_0 = -G\frac{M_e}{R_e^2}\boldsymbol{e}_R \tag{8.21}$$

is the earth's gravitational field (M_e its mass and R_e its radius), while $\boldsymbol{f}$ denotes any other forces acting on the particle. $\boldsymbol{e}_R$ is an unit vector in the radial direction.

The earth's period of rotation w.r.t. the stellar background is $\tau_e = 23^{\mathrm{h}}\,56^{\mathrm{m}}\,4^{\mathrm{s}} = 86164\,\mathrm{s}$ (sidereal day), so $|\omega| = 2\pi/\tau_e = 7.29 \times 10^{-5}/\mathrm{s}$. The force $\boldsymbol{F}$ acting on the particle, as expressed in a rotating and accelerating system that is attached to the earth's surface is given by Eq. (8.19). To express $(\ddot{\boldsymbol{R}})_{\text{fixed}}$ in terms of time derivatives in the moving frame, note that $\boldsymbol{R}$ is here the vector pointing from the center of the earth towards the position of the rotating frame on the earth's surface, and its only motion is a constant rotation (as measured in the fixed frame) with the same angular velocity $\boldsymbol{\omega}$, i.e. $(\dot{\boldsymbol{R}})_{\text{rotating}} = 0$ and hence $(\dot{\boldsymbol{R}})_{\text{fixed}} = \boldsymbol{\omega} \times \boldsymbol{R}$. Using Eq. (8.15), we find

$$\left(\ddot{\boldsymbol{R}}\right)_{\text{fixed}} = \boldsymbol{\omega} \times \left(\boldsymbol{\omega} \times \boldsymbol{R}\right), \tag{8.22}$$

and the equations of motion become

$$m\ddot{\boldsymbol{r}} = \boldsymbol{f} + m\boldsymbol{g}_0 - m\boldsymbol{\omega} \times \left[\boldsymbol{\omega} \times (\boldsymbol{r} + \boldsymbol{R})\right] - 2m\boldsymbol{\omega} \times \dot{\boldsymbol{r}}\,, \tag{8.23}$$

where all time derivatives are evaluated in the rotating frame. The second and third terms on the right side are the effective gravitational force on the earth's surface; we define the *effective gravitational acceleration* as

$$\boldsymbol{g} = \boldsymbol{g}_0 - \boldsymbol{\omega} \times \left[\boldsymbol{\omega} \times (\boldsymbol{r} + \boldsymbol{R})\right]. \tag{8.24}$$

Near the surface of the earth we can neglect $\boldsymbol{r} \ll \boldsymbol{R}$, so

$$\boldsymbol{g} = \boldsymbol{g}_0 - \boldsymbol{\omega} \times \left(\boldsymbol{\omega} \times \boldsymbol{R}\right). \tag{8.25}$$

This is a correction of order $R\omega^2/g = 0.35\%$. Away from the poles and the equator, the gravitational acceleration is not strictly directed toward the center of the earth. In total, we now have

$$\boldsymbol{F} = \boldsymbol{f} + m\boldsymbol{g} - 2m\boldsymbol{\omega} \times \dot{\boldsymbol{r}}\,. \tag{8.26}$$

For instance, a body moving on the Northern hemisphere is deflected by the Coriolis force towards its right, and towards the left on the Southern hemisphere.

Example 8.1: SURF. *A careless physicist drops a stone into the mine shaft at Sanford Underground Research Facility (SURF; 1478 m depth). How far is the stone deflected from vertical fall?*

We need to solve the equation

$$\ddot{\boldsymbol{r}} = \boldsymbol{g} - 2\boldsymbol{\omega} \times \dot{\boldsymbol{r}}\,. \tag{8.27}$$

We choose the z axis of the rotating frame in direction $-\boldsymbol{g}$, and the x and y axes South and East, respectively. SURF is located at Lead, SD ($44°21'3''$ Northern latitude), so the polar angle is $\theta = 0.77 \approx \pi/4$. We have

$$\begin{aligned} \omega_x &= -\omega\cos\theta\,,\\ \omega_y &= 0\,,\\ \omega_z &= \omega\sin\theta\,. \end{aligned}$$

The equation of motion in z direction is

$$\ddot{z} = -g - 2(\omega_x\dot{y} - \omega_y\dot{x}) = -g + 2\dot{y}\omega\cos\theta\,. \tag{8.28}$$

We can clearly neglect the second term, and obtain $z(t) = -gt^2/2$ if the stone starts falling from rest. The time for falling a distance $-h$ is then $t = \sqrt{2h/g}$.

The remaining equations of motion are

$$\ddot{x} = -2(\omega_y\dot{z} - \omega_z\dot{y}) = 2\dot{y}\omega\sin\theta\,. \tag{8.29}$$

and[22]

$$\ddot{y} = -2(\omega_z\dot{x} - \omega_x\dot{z}) = -2\dot{z}\omega\cos\theta - 2\dot{x}\omega\sin\theta\,. \tag{8.32}$$

The gravitational acceleration acts in direction of z, so we will neglect $\dot{x}$ and $\dot{y}$. This leaves

$$\ddot{y} = -2\dot{z}\omega\cos\theta = -2(-gt)\omega\cos\theta\,, \tag{8.33}$$

so $y = 2g\omega\cos\theta(t^3/6)$, or

$$y = \frac{\omega\cos\theta}{3}\sqrt{\frac{8h^3}{g}} = 14\,cm\,. \tag{8.34}$$

Example 8.2: Foucault's pendulum. *We use coordinate systems similar to those in the previous example. The origin of the rotating system is located at the equilibrium point of the pendulum. The equations of motion are*

$$\ddot{\boldsymbol{r}} = \boldsymbol{g} + \frac{\boldsymbol{T}}{m} - 2\boldsymbol{\omega}\times\dot{\boldsymbol{r}}\,, \tag{8.35}$$

[22] Note that we can write the equations of motion in the following form:

$$\begin{pmatrix}\ddot{x}\\ \ddot{y}\\ \ddot{z}+g\end{pmatrix} = \begin{pmatrix} 0 & 2\omega\sin\theta & 0\\ -2\omega\sin\theta & 0 & -2\omega\cos\theta\\ 0 & 2\omega\cos\theta & 0\end{pmatrix}\begin{pmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\end{pmatrix} \tag{8.30}$$

with an *antisymmetric* coefficient matrix. This implies energy conservation: if we take the scalar product of Eq. (8.30) with the vector $m(\dot{x},\dot{y},\dot{z})$, the right side vanishes due to the antisymmetry and we obtain

$$m\dot{x}\ddot{x} + m\dot{y}\ddot{y} + m\dot{z}\ddot{z} + mg\dot{z} = \frac{m}{2}\frac{d}{dt}\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right) + \frac{d}{dt}(mgz) = 0\,, \tag{8.31}$$

or $T + V =$ constant. The reason is that the Coriolis force acts in a direction orthogonal to the velocity and, thus, no work is done on the particle.

where $\boldsymbol{T}$ is the tension in the string of length ℓ.[23] *We consider only small angles, such that motion in the z direction can be neglected and the pendulum bob moves only in the x-y plane. We then have $T \approx T_z = mg$, and*

$$T_x = -T\frac{x}{\ell}\,, \quad T_y = -T\frac{y}{\ell}\,. \tag{8.36}$$

Reutilizing our results from above, we then find the equations of motion

$$\ddot{x} = -\frac{g}{\ell}x + 2\dot{y}\omega\sin\theta\,, \tag{8.37}$$

$$\ddot{y} = -\frac{g}{\ell}y - 2\dot{x}\omega\sin\theta\,. \tag{8.38}$$

We can solve this coupled system *of differential equations by adding i times the second equation to the first:*

$$\ddot{x} + i\ddot{y} + \frac{g}{\ell}(x+iy) = 2\omega\sin\theta(\dot{y} - i\dot{x}) = -2i\omega\sin\theta(\dot{x} + i\dot{y})\,, \tag{8.39}$$

Writing $s = x + iy$, this is

$$\ddot{s} + 2i\omega_z\dot{s} + \alpha^2 s = 0\,, \tag{8.40}$$

where we defined $\alpha^2 = g/\ell$. This is the equation for a damped harmonic oscillator, with solution (see Eq. (2.38), with $\beta \to i\omega_z$, $\omega_0^2 \to \alpha^2$)

$$s(t) = e^{-i\omega_z t}\left[A\exp\left(i\sqrt{\omega_z^2 + \alpha^2}t\right) + B\exp\left(-i\sqrt{\omega_z^2 + \alpha^2}t\right)\right]. \tag{8.41}$$

Since (for reasonable length of the pendulum) $\alpha^2 \gg \omega_z^2$, we can approximate this as

$$s(t) = e^{-i\omega_z t}\left[Ae^{i\alpha t} + Be^{-i\alpha t}\right]. \tag{8.42}$$

We denote the (hypothetical) solution for $\omega_z = 0$ by

$$s'(t) = x'(t) + iy'(t) = Ae^{i\alpha t} + Be^{-i\alpha t}\,, \tag{8.43}$$

or $s(t) = e^{-i\omega_z t}s'(t)$. This gives

$$\begin{aligned} x + iy &= \left[\cos(\omega_z t) - i\sin(\omega_z t)\right](x' + iy') \\ &= x'\cos(\omega_z t) + y'\sin(\omega_z t) + i(-x'\sin(\omega_z t) + y'\cos(\omega_z t))\,, \end{aligned} \tag{8.44}$$

or, finally,

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \cos(\omega_z t) & \sin(\omega_z t) \\ -\sin(\omega_z t) & \cos(\omega_z t) \end{pmatrix}\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix}. \tag{8.45}$$

We see that the plane of oscillation of the pendulum rotates with angular frequency $\omega_z = \omega\sin\theta$.

[23] In some sense, FOUCAULT's pendulum is analogous to the falling particle of the previous problem, only that the "leading" motion in $-z$ direction is prohibited by the mass being fixed to a string. Alternatively to the method followed here, we could implement the constraint $x^2 + y^2 + z^2 = \ell^2$ using LAGRANGE multipliers. Solving for $z = \ell\sqrt{1 - x^2/\ell^2 - y^2/\ell^2}$ then allows for a more systematic expansion in small parameters.

9 Dynamics of rigid bodies

9.1 The inertia tensor

We want to find an expression for the kinetic energy of a rigid body that takes into account both the translational and rotational motion. If the rigid body consists of n discrete massive particles, its kinetic energy is given by[24]

$$T = \sum_{a=1}^{n} \frac{m_a \boldsymbol{v}_a^2}{2} \,. \tag{9.1}$$

We now use the relation (8.2) to express the $\boldsymbol{v}_a$ in terms of a coordinate system that is fixed w.r.t. the rigid body (i.e., which might have translational and rotational motion w.r.t. to the fixed coordinate system of Eq. (9.1)) and find

$$T = \sum_{a=1}^{n} \frac{m_a}{2} \big(\boldsymbol{V} + \boldsymbol{\omega} \times \boldsymbol{r}_a\big)^2 = \sum_{a=1}^{n} \frac{m_a}{2} \boldsymbol{V}^2 + \sum_{a=1}^{n} m_a \boldsymbol{V} \cdot \big(\boldsymbol{\omega} \times \boldsymbol{r}_a\big) + \sum_{a=1}^{n} \frac{m_a}{2} \big(\boldsymbol{\omega} \times \boldsymbol{r}_a\big)^2 \,. \tag{9.2}$$

The first time on the right side is just $M\boldsymbol{V}^2/2$, where $M = \sum_a m_a$ is the total mass of the rigid body. We can write the second term on the right side as

$$\sum_{a=1}^{n} m_a \boldsymbol{V} \cdot \big(\boldsymbol{\omega} \times \boldsymbol{r}_a\big) = \sum_{a=1}^{n} m_a \boldsymbol{r}_a \cdot \big(\boldsymbol{V} \times \boldsymbol{\omega}\big) = M\boldsymbol{R} \cdot \big(\boldsymbol{V} \times \boldsymbol{\omega}\big) \,, \tag{9.3}$$

where $\boldsymbol{R}$ is the position of the CM. Hence, this term vanishes if we choose the origin of the moving system in the CM. To rewrite the last term in Eq. (9.2), we use the identity

$$\big(\boldsymbol{A} \times \boldsymbol{B}\big) \cdot \big(\boldsymbol{A} \times \boldsymbol{B}\big) = A^2 B^2 - \big(\boldsymbol{A} \cdot \boldsymbol{B}\big)^2 \,, \tag{9.4}$$

and find

$$\begin{aligned} \sum_{a=1}^{n} \frac{m_a}{2} \big(\boldsymbol{\omega} \times \boldsymbol{r}_a\big)^2 &= \sum_{a=1}^{n} \frac{m_a}{2} \left[\omega^2 r_a^2 - \big(\boldsymbol{\omega} \cdot \boldsymbol{r}_a\big)^2\right] \\ &= \sum_{a=1}^{n} \frac{m_a}{2} \left[\Big(\sum_i \omega_i^2\Big)\Big(\sum_k x_{a,k}^2\Big) - \Big(\sum_i \omega_i x_{a,i}\Big)\Big(\sum_j \omega_j x_{a,j}\Big)\right] \\ &= \sum_{a=1}^{n} \frac{m_a}{2} \sum_{ij} \left[\omega_i \omega_j \delta_{ij} \sum_k x_{a,k}^2 - \omega_i \omega_j x_{a,i} x_{a,j}\right] \\ &= \sum_{ij} \omega_i \omega_j \sum_{a=1}^{n} \frac{m_a}{2} \left[\delta_{ij} \sum_k x_{a,k}^2 - x_{a,i} x_{a,j}\right] . \end{aligned} \tag{9.5}$$

Defining the *inertia tensor*

$$I_{ij} \equiv \sum_{a=1}^{n} m_a \left[\delta_{ij} \sum_k x_{a,k}^2 - x_{a,i} x_{a,j}\right] , \tag{9.6}$$

[24] For a "continuous" body, the sum has to be replaced by an integral.

we can write the total kinetic energy as

$$T = \frac{1}{2}M\boldsymbol{V}^2 + \frac{1}{2}\sum_{ij} I_{ij}\omega_i\omega_j \,. \tag{9.7}$$

Note that the inertia tensor is symmetric and, hence, has six independent components. For a continuous mass distribution, it is given by

$$I_{ij} = \int_V \rho(\boldsymbol{r})\Big[\delta_{ij}\sum_k x_k^2 - x_i x_j\Big] \,. \tag{9.8}$$

Example 9.1: Dumbbell. *Consider a dumbbell, with two (point-like) weights of equal mass $m = 1$ kg separated by a massless stick of length 1 m.*

We locate the weights on the z axis, symmetric about the origin, and find (in units of kg m^2):

$$I_{11} = 1(1/2)^2 + 1(-1/2)^2 = \frac{1}{2} = I_{22}\,, \tag{9.9}$$

$$I_{33} = 1[(1/2)^2 - (1/2)^2] + 1[(-1/2)^2 - (-1/2)^2] = 0\,, \tag{9.10}$$

$$I_{12} = I_{13} = I_{23} = 0\,. \tag{9.11}$$

Example 9.2: Cube. *Consider a homogeneous cube of density ρ and side length b. Locate the origin at the CM, and the coordinate axes through the centers of the sides of the cube.*

Because of an obvious symmetry, we need only calculate I_{11} and I_{12}. We find

$$\begin{aligned} I_{11} &= \rho \int_{-b/2}^{b/2} dx \int_{-b/2}^{b/2} dy \int_{-b/2}^{b/2} dz (y^2 + z^2) = \rho b \int_{-b/2}^{b/2} dy \left(by^2 + \frac{b^3}{12}\right) \\ &= \rho b\left(\frac{b^4}{12} + \frac{b^4}{12}\right) = \frac{1}{6}\rho b^5 = \frac{1}{6}Mb^2\,, \end{aligned} \tag{9.12}$$

and

$$I_{12} = -\rho \int_{-b/2}^{b/2} dx \int_{-b/2}^{b/2} dy \int_{-b/2}^{b/2} dz\, xy = 0\,. \tag{9.13}$$

We can write the result in matrix form,

$$I = \begin{pmatrix} \frac{1}{6}\beta & 0 & 0 \\ 0 & \frac{1}{6}\beta & 0 \\ 0 & 0 & \frac{1}{6}\beta \end{pmatrix} \,, \tag{9.14}$$

with $\beta \equiv Mb^2$.

9.2 Steiner's theorem

The result (9.7) is only valid if the origin of the moving coordinate system is located at the CM. However, in some occasions it is simpler to calculate the inertia tensor in a different coordinate system. How does the inertia tensor transform under a shift of coordinates?

Assume we are given (or have calculated) the inertia tensor in a general (non-CM) coordinate system. We denote the position vector by capital letters, $\boldsymbol{R}_a$, with components $X_{a,1}, X_{a,2}, X_{a,3}$. The inertia tensor is then

$$J_{ij} = \sum_{a=1}^{n} m_a \Big[\delta_{ij} \sum_k X_{a,k}^2 - X_{a,i} X_{a,j}\Big] . \tag{9.15}$$

We denote the coordinates of the CM system with small letters, as before. Assume that the CM is located at $\boldsymbol{a}$, such that $\boldsymbol{R} = \boldsymbol{r} + \boldsymbol{a}$. Then

$$\begin{aligned} J_{ij} &= \sum_{a=1}^{n} m_a \Big[\delta_{ij} \sum_k (x_{a,k} + a_k)^2 - (x_{a,i} + a_i)(x_{a,j} + a_j)\Big] \\ &= \sum_{a=1}^{n} m_a \Big[\delta_{ij} \sum_k x_{a,k}^2 - x_{a,i} x_{a,j}\Big] \\ &\quad + \sum_{a=1}^{n} m_a \Big[\delta_{ij} \sum_k (2 x_{a,k} a_k + a_k^2) - (a_i x_{a,j} + a_j x_{a,i} + a_i a_j)\Big] \\ &= I_{ij} + \sum_{a=1}^{n} m_a \Big[\delta_{ij} \sum_k a_k^2 - a_i a_j\Big] + \sum_{a=1}^{n} m_a \Big[2\delta_{ij} \sum_k x_{a,k} a_k - a_i x_{a,j} - a_j x_{a,i}\Big] . \end{aligned} \tag{9.16}$$

However, the last terms on the right side vanish in the CM system. Using $\sum_{a=1}^{n} m_a = M$ and $\sum_k a_k^2 = a^2$, we find our final result

$$I_{ij} = J_{ij} - M\big(\delta_{ij} a^2 - a_i a_j\big) . \tag{9.17}$$

Example 9.3: Cube, again. *Consider again a homogeneous cube of density ρ and side length b, but now locate the origin at one edge of the cube, and the coordinate axes along the sides of the cube.*

Because of an obvious symmetry, we need only calculate J_{11} and J_{12}. We find

$$\begin{aligned} J_{11} &= \rho \int_0^b dX \int_0^b dY \int_0^b dZ (Y^2 + Z^2) = \rho b \int_0^b dY \left(b Y^2 + \frac{b^3}{3}\right) \\ &= \rho b \left(\frac{b^4}{3} + \frac{b^4}{3}\right) = \frac{2}{3} \rho b^5 = \frac{2}{3} M b^2 , \end{aligned} \tag{9.18}$$

and

$$J_{12} = -\rho \int_0^b dX \int_0^b dY \int_0^b dZ\, XY = -\rho b \int_0^b dX \frac{b^2}{2} X = -\rho \frac{b^5}{4} = -\frac{1}{4} M b^2 . \tag{9.19}$$

We can write the result in matrix form,

$$J = \begin{pmatrix} \frac{2}{3}\beta & -\frac{1}{4}\beta & -\frac{1}{4}\beta \\ -\frac{1}{4}\beta & \frac{2}{3}\beta & -\frac{1}{4}\beta \\ -\frac{1}{4}\beta & -\frac{1}{4}\beta & \frac{2}{3}\beta \end{pmatrix}\,, \tag{9.20}$$

with $\beta \equiv Mb^2$.

To obtain the inertia tensor in the CM frame, we need $\boldsymbol{a} = (b/2, b/2, b/2)$, $a^2 = 3b^2/4$, *and calculate*

$$I_{11} = J_{11} - M\left(\frac{3}{4} - \frac{1}{4}\right)b^2 = \frac{2}{3}Mb^2 - \frac{1}{2}Mb^2 = \frac{1}{6}Mb^2\,, \tag{9.21}$$

$$I_{12} = J_{12} - M\left(-\frac{1}{4}\right)b^2 = -\frac{1}{4}Mb^2 + \frac{1}{4}Mb^2 = 0\,. \tag{9.22}$$

The kinetic energy of a rigid body can be expressed solely in terms of the inertia tensor even if the origin is not located at the center of origin, as long as the origin of the system fixed to the rigid body is not moving as seen from the inertial system (e.g. the origins of the two system may be chosen to coincide at all times).

In this case, if we keep the convention that $\boldsymbol{r}_a$ denotes the position vector as measured from the CM, and denote the position of the CM again by $\boldsymbol{a}$, we should set $\boldsymbol{V} = 0$ and replace $\boldsymbol{r}_a \to \boldsymbol{r}_a + \boldsymbol{a}$ in Eq. (9.2). This gives

$$\begin{aligned} T &= \sum_{a=1}^{n} \frac{m_a}{2}\big(\boldsymbol{\omega}\times\boldsymbol{r}_a + \boldsymbol{\omega}\times\boldsymbol{a}\big)^2 \\ &= \sum_{a=1}^{n} \frac{m_a}{2}\big(\boldsymbol{\omega}\times\boldsymbol{r}_a\big)^2 + \sum_{a=1}^{n} m_a\big(\boldsymbol{\omega}\times\boldsymbol{a}\big)\cdot\big(\boldsymbol{\omega}\times\boldsymbol{r}_a\big) + \sum_{a=1}^{n}\frac{m_a}{2}\big(\boldsymbol{\omega}\times\boldsymbol{a}\big)^2\,. \end{aligned} \tag{9.23}$$

As before, the second term in the last line vanishes[25], and we have for the total kinetic energy

$$T = \sum_{a=1}^{n} \frac{m_a}{2}\big(\boldsymbol{\omega}\times\boldsymbol{r}_a\big)^2 + \sum_{a=1}^{n}\frac{m_a}{2}\big(\boldsymbol{\omega}\times\boldsymbol{a}\big)^2 = \frac{1}{2}\sum_{ij} I_{ij}\omega_i\omega_j + \frac{M}{2}\big(\boldsymbol{\omega}\times\boldsymbol{a}\big)^2\,. \tag{9.24}$$

In fact, this must be the same as just using the inertia tensor in the non-CM frame J_{ij} instead of I_{ij} in Eq. (9.7), as we can easily verify, using Steiner's theorem:

$$\begin{aligned} T &= \frac{1}{2}\sum_{ij} J_{ij}\omega_i\omega_j \\ &= \frac{1}{2}\sum_{ij} I_{ij}\omega_i\omega_j + \frac{M}{2}\sum_{ij}\big(\delta_{ij}a^2 - a_i a_j\big)\omega_i\omega_j \\ &= \frac{1}{2}\sum_{ij} I_{ij}\omega_i\omega_j + \frac{M}{2}\big[\omega^2 a^2 - (\boldsymbol{\omega}\cdot\boldsymbol{a})^2\big] \\ &= \frac{1}{2}\sum_{ij} I_{ij}\omega_i\omega_j + \frac{M}{2}(\boldsymbol{\omega}\times\boldsymbol{a})^2\,. \end{aligned} \tag{9.25}$$

[25]Use $(\boldsymbol{A}\times\boldsymbol{B})\cdot(\boldsymbol{A}\times\boldsymbol{C}) = \boldsymbol{A}^2(\boldsymbol{B}\cdot\boldsymbol{C}) - (\boldsymbol{A}\cdot\boldsymbol{C})(\boldsymbol{A}\cdot\boldsymbol{B})$.

Example 9.4: “Fixed” double pendulum. *Consider a planar pendulum of length ℓ with mass m_1 fixed at its end and mass m_2 at half length.*

We take the x_3 direction of the coordinate system fixed to the rigid body along the (positive) angular velocity $\boldsymbol{\omega}$, i.e.,

$$\boldsymbol{\omega} = \omega\, \boldsymbol{e}_3 = \dot{\theta}\, \boldsymbol{e}_3 \,, \tag{9.26}$$

and the x_1 direction along the pendulum. The inertia then tensor has components

$$J_{ij} \equiv \sum_{a=1}^{2} m_a \Big[\delta_{ij} \sum_k x_{a,k}^2 - x_{a,i} x_{a,j}\Big] \,. \tag{9.27}$$

We have $x_{a,2} = x_{a,3} = 0$, $x_{1,1} = b$, $x_{2,1} = b/2$, so

$$J_{22} = J_{33} = m_1 b^2 + m_2 \frac{b^2}{4} \,, \tag{9.28}$$

with all other components zero. According to the discussion above, this gives the kinetic energy

$$T = \frac{1}{2}\dot{\theta}^2 \left(m_1 b^2 + m_2 \frac{b^2}{4} \right) . \tag{9.29}$$

The Lagrangian for the system is then

$$L = \frac{1}{2}\dot{\theta}^2 \left(m_1 b^2 + m_2 \frac{b^2}{4} \right) + m_1 g b \cos\theta + \frac{1}{2} m_2 g b \cos\theta \,, \tag{9.30}$$

and the equations of motion are

$$\ddot{\theta} b^2 \left(m_1 + \frac{m_2}{4} \right) = -g b \sin\theta \left(m_1 + \frac{m_2}{2} \right) . \tag{9.31}$$

The frequency for small oscillations is given by

$$\omega_0^2 = \frac{g}{b} \frac{m_1 + \frac{m_2}{2}}{m_1 + \frac{m_2}{4}} \,. \tag{9.32}$$

9.3 Principal axes of inertia*

By a suitable orientation of the rotating coordinate system (see Fig. 31 for an elementary example), the inertia tensor can be brought in a form where only the diagonal elements are non-zero.

Consider the rotational part of the kinetic energy,

$$T_{\text{rot}} = \frac{1}{2} \sum_{ij} I_{ij} \omega_i \omega_j \,. \tag{9.33}$$

Obviously, T_{rot} does not change if we just choose a different orientation of our coordinate system. In such a rotated system, the components of the angular momentum $\boldsymbol{\omega}'$ will be

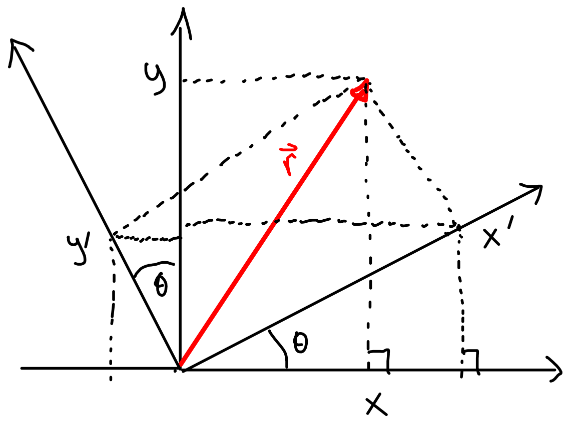

Figure 24: *Rotation of coordinate system. The components of the vector $\boldsymbol{r}$ (in red) can be obtained by orthogonal projection in the primed and unprimed coordinate systems. Their relations are given by trigonometric identities. For instance, projecting the components of $\boldsymbol{r}$ along the x', y' axes onto the x, y axes gives $x = x'\cos\theta - y'\sin\theta$ and $y = x'\sin\theta + y'\cos\theta$.*

linear combinations of the components of the angular momentum $\boldsymbol{\omega}$ in the original frame. For instance, $\omega_1' = R_{11}\omega_1 + R_{12}\omega_2 + R_{13}\omega_3$, or, more generally,

$$\omega_i' = \sum_{j=1}^{3} R_{ij}\omega_j \,. \tag{9.34}$$

In matrix notation we write this as

$$\boldsymbol{\omega}' = \boldsymbol{R}\boldsymbol{\omega} \,. \tag{9.35}$$

Of course, the length (the absolute value) of the angular momentum should not change under such a rotation:

$$\omega^2 = \boldsymbol{\omega}^T\boldsymbol{\omega} \stackrel{!}{=} \left(\boldsymbol{\omega}'\right)^T\boldsymbol{\omega}' = \boldsymbol{\omega}^T\boldsymbol{R}^T\boldsymbol{R}\boldsymbol{\omega} \,, \tag{9.36}$$

so we must have

$$\boldsymbol{R}^T\boldsymbol{R} = \boldsymbol{1} \,. \tag{9.37}$$

Here, the superscript T denotes the transpose of the matrix.

The condition that T_{rot} be invariant under a rotation of the coordinate system is then

$$T_{\text{rot}} = \frac{1}{2}\boldsymbol{\omega}^T\boldsymbol{I}\,\boldsymbol{\omega} = \frac{1}{2}\left(\boldsymbol{\omega}'\right)^T\boldsymbol{I}'\,\boldsymbol{\omega}' = \frac{1}{2}\boldsymbol{\omega}^T\boldsymbol{R}^T\boldsymbol{I}'\,\boldsymbol{R}\boldsymbol{\omega} \,, \tag{9.38}$$

so we must have

$$\boldsymbol{I} = \boldsymbol{R}^T\boldsymbol{I}'\,\boldsymbol{R} \,, \tag{9.39}$$

or, using the orthogonality relation (9.37),

$$\boldsymbol{I}' = \boldsymbol{R}\boldsymbol{I}\boldsymbol{R}^T = \boldsymbol{R}\boldsymbol{I}\boldsymbol{R}^{-1} \,. \tag{9.40}$$

This is the transformation law for the inertia tensor.

We now want to find a coordinate system, or equivalently, a rotation matrix $\boldsymbol{R}$, such that $\boldsymbol{I}' = \boldsymbol{D}$ is a diagonal matrix:

$$\boldsymbol{D} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix} \,. \tag{9.41}$$

Multiplying Eq. (9.40) on the right by $\boldsymbol{R}$ and using Eq. (9.37), we get

$$\boldsymbol{R}\boldsymbol{I} = \boldsymbol{D}\boldsymbol{R} \,, \tag{9.42}$$

or

$$\boldsymbol{R}\boldsymbol{I} - \boldsymbol{D}\boldsymbol{R} = 0 \,. \tag{9.43}$$

We can interpret this equation as a set of three linear systems of equations, one for each row of the matrix $\boldsymbol{R}$. Denoting the rows by $\boldsymbol{v}_i$, the three systems are

$$(\boldsymbol{I} - I_i \mathbf{1})\boldsymbol{v}_i = 0 \,, \quad i = 1, 2, 3 \,, \tag{9.44}$$

one system of equations for each of the $\boldsymbol{v}_i$. Solving the systems will allow us to determine the three rows $\boldsymbol{v}_i$, and hence the matrix $\boldsymbol{R}$. The systems will have a non-trivial solution if and only if the determinant of the coefficient matrix vanishes:

$$\det(\boldsymbol{I} - \lambda \mathbf{1}) = 0 \,. \tag{9.45}$$

The three roots of this cubic equation in λ are the three *principal moments of inertia* I_i, and solving the corresponding systems (9.44) gives the explicit form of the transformation matrix. Since we start with a Cartesian (rectangular) coordinate and perform a pure rotation, we see that the three *principal axes of inertia* are orthogonal to each other.

We did not prove that Eq. (9.45) always has three real solutions. This is a general result of linear algebra. In practice, this is only relevant for a completely asymmetric rigid body. If a rigid body has (at least) one axis of symmetry, we can always choose the x_3 direction along this symmetry axis, and it is straighforward to see that the inertia tensor then takes the form $I_1 = I_2$, and $I_{ij} = 0$ for $i \neq j$.

9.4 Angular momentum

The angular momentum of a rigid body is defined as

$$\boldsymbol{L} = \sum_a (\boldsymbol{r}_a \times \boldsymbol{p}_a) \,. \tag{9.46}$$

Recall that $\boldsymbol{L}$ depends on the choice of origin of the coordinate system; a frequent choice is the CM system. With this choice, the velocity of the rigid body arises from pure rotation, and we can write

$$\boldsymbol{L} = \sum_a m_a(\boldsymbol{r}_a \times \boldsymbol{v}_a) = \sum_a m_a(\boldsymbol{r}_a \times (\boldsymbol{\omega} \times \boldsymbol{r}_a)) \,. \tag{9.47}$$

Now we use the identity

$$\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{A}) = A^2 \boldsymbol{B} - \boldsymbol{A}(\boldsymbol{A} \cdot \boldsymbol{B}) \tag{9.48}$$

to write

$$\boldsymbol{L} = \sum_a m_a \left[r_a^2 \boldsymbol{\omega} - \boldsymbol{r}_a(\boldsymbol{r}_a \cdot \boldsymbol{\omega}) \right] . \tag{9.49}$$

This result can also be expressed in terms of the inertia tensor, as follows:

$$L_i = \sum_a m_a \left[\omega_i \sum_k x_{a,k}^2 - x_{a,i} \sum_j x_{a,j}\omega_j \right] = \sum_j \omega_j \sum_a m_a \left[\delta_{ij} \sum_k x_{a,k}^2 - x_{a,i}x_{a,j} \right] , \tag{9.50}$$

or

$$L_i = \sum_j I_{ij}\omega_j \,. \tag{9.51}$$

In matrix notation,

$$\boldsymbol{L} = \boldsymbol{I}\,\boldsymbol{\omega} \,. \tag{9.52}$$

In general, the angular velocity and the angular momentum need not point in the same direction (if the inertia tensor is not proportional to the unit matrix). They only point in the same direction if the body rotates about one of its principle axes of inertia.

Finally, we derive an alternative expression for the rotational energy. We multiply Eq. (9.51) by $\omega_i/2$ and sum over i; this gives

$$\frac{1}{2}\sum_i L_i\omega_i = \frac{1}{2}\sum_j I_{ij}\omega_i\omega_j = T_{\text{rot}} \,. \tag{9.53}$$

In matrix notation,

$$T_{\text{rot}} = \frac{1}{2}\boldsymbol{\omega}^T \boldsymbol{I}\boldsymbol{\omega} = \frac{1}{2}\boldsymbol{\omega}^T \boldsymbol{L} \,. \tag{9.54}$$

9.5 Euler angles

In the discussion of the motion of rigid bodies, we will not be interested in the motion of the system as a whole (one exception will be the top with a fixed point). In this case it is most useful to locate the origin of both the fixed and the rotating coordinate system into the CM. The relation between the two systems is then a pure rotation. A useful way to parameterize this rotation is using the Euler angles (Fig. 25). We denote the coordinate of the fixed system by X, Y, Z, and the coordinates of the rotating system by x_1, x_2, x_3. The moving x_1x_2 plane intersects the fixed XY plane along the line ON, called the *line of nodes*. The line of nodes

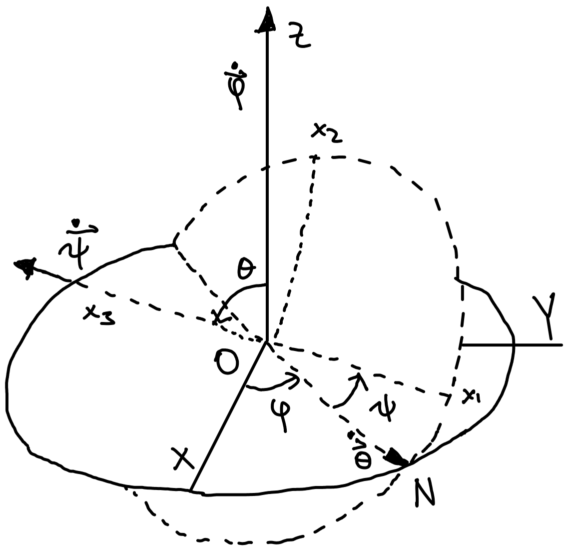

Figure 25: *EULER angles.*

is orthogonal to both the x_3 and Z axes. We choose its direction such that it corresponds to the vector product $\boldsymbol{e}_Z \times \boldsymbol{e}_{x_3}$. To determine the position of the other axes, we choose the angle θ between the Z axis and the x_3 axis, the angle ϕ between the X axis and the line of nodes, and the angle ψ between the line of nodes and the x_1 axis. The angle θ runs from 0 until π, while ϕ and ψ vary between 0 and 2π.

We want to express the angular velocity of the rigid body, $\boldsymbol{\omega}$, w.r.t. to the rotating axes x_1, x_2, x_3. The projections of the angular velocities $\dot{\boldsymbol{\theta}}$, $\dot{\boldsymbol{\phi}}$, $\dot{\boldsymbol{\psi}}$ are

$$\dot{\theta}_1 = \dot{\theta}\cos\psi\,, \quad \dot{\theta}_2 = -\dot{\theta}\sin\psi\,, \quad \dot{\theta}_3 = 0\,; \tag{9.55}$$

$$\dot{\phi}_1 = \dot{\phi}\sin\theta\sin\psi\,, \quad \dot{\phi}_2 = \dot{\phi}\sin\theta\cos\psi\,, \quad \dot{\phi}_3 = \dot{\phi}\cos\theta\,; \tag{9.56}$$

$$\dot{\psi}_1 = 0\,, \quad \dot{\psi}_2 = 0\,, \quad \dot{\psi}_3 = \dot{\psi}\,. \tag{9.57}$$

Therefore, we find for the components of $\boldsymbol{\omega}$:

$$\omega_1 = \dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi\,, \tag{9.58}$$

$$\omega_2 = \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi\,, \tag{9.59}$$

$$\omega_3 = \dot{\phi}\cos\theta + \dot{\psi}\,. \tag{9.60}$$

9.6 EULER's equations for a rigid body

We have at our disposal (at least) two different methods for obtaining the equations of motion for the rigid body: NEWTON's laws, and LAGRANGE's method. Direct application of NEWTON's laws leads to a set of equations called EULER's equations that we will derive here. We will discuss the Lagrangian method in the context of the heavy top in Sec. 9.8.

Starting with our well-known relation (8.15), we can express the time rate of change of the angular momentum, as given in the fixed system, in terms of the angular velocity of the rotating system. If there is no torque applied to the rigid body, the angular momentum is conserved:

$$\frac{d\boldsymbol{L}}{dt} + \boldsymbol{\omega} \times \boldsymbol{L} = 0\,, \tag{9.61}$$

where the time derivative is taken in the rotating system. We now express the angular momentum in terms of the inertia tensor and the angular velocity, using Eq. (9.51). To simplify the expressions, we choose the axes of the rotating coordinate system along the principal axes of inertia, such that $L_i = I_i\omega_i$, and find

$$\begin{aligned} I_1\frac{d\omega_1}{dt} + \omega_2\omega_3(I_3 - I_2) = 0\,,\\ I_2\frac{d\omega_2}{dt} + \omega_3\omega_1(I_1 - I_3) = 0\,,\\ I_3\frac{d\omega_3}{dt} + \omega_1\omega_2(I_2 - I_1) = 0\,. \end{aligned} \tag{9.62}$$

Here, the subscripts denote the components along the rotating x_1, x_2, x_3 axes. The equations (9.62) are called *Euler's equations* for the rigid body. If there is an external torque acting on the body, its components would appear on the right side, instead of the zeros.

9.7 Force-free motion of a symmetric top

As an example,[26] we consider the motion of a free symmetric top. Symmetric here means that two of the moments of inertia coincide: $I_1 = I_2 \neq I_3$. Inserting this into Euler's equations (9.62) gives

$$\begin{aligned} I_1\dot{\omega}_1 + \omega_2\omega_3(I_3 - I_2) = 0\,,\\ I_2\dot{\omega}_2 + \omega_3\omega_1(I_1 - I_3) = 0\,,\\ I_3\dot{\omega}_3 = 0\,. \end{aligned} \tag{9.65}$$

[26] In fact, the motion of the force-free top can be determined from the conservation of angular momentum alone, without the use of Euler's equations. Because the top is symmetric about the x_3 axis, we can choose the x_2 direction orthogonal to the plane spanned by the angular momentum $\boldsymbol{L}$ and the current direction of x_3, such that $L_2 = 0$ and therefore also $\omega_2 = 0$. It follows that $\boldsymbol{L}, \boldsymbol{\omega}$ and the x_3 direction all lie in the same plane. The rotation of the top about its symmetry axis is given by the projection of $\boldsymbol{\omega}$ onto the x_3 axis:

$$\omega_3 = \frac{L_3}{I_3} = \frac{L\cos\theta}{I_3}\,. \tag{9.63}$$

To find the angular velocity of the precession Ω_{Pr}, we need to decompose $\boldsymbol{\omega}$ into a component along the direction of x_3, which does not contribute to the precession, and a component along the direction of $\boldsymbol{L}$, which will give the desired velocity of precession. We have $\omega_1 = \Omega_{\text{Pr}}\,\sin\theta = L_1/I_1 = L/I_1\,\sin\theta$, and so

$$\Omega_{\text{Pr}} = \frac{L}{I_1}\,. \tag{9.64}$$

The last of these equations tells us that ω_3 is constant. Setting $I_1 = I_2$, the first two equations become

$$\dot{\omega}_1 = -\left(\frac{I_3 - I_1}{I_1}\omega_3\right)\omega_2\,, \quad \dot{\omega}_2 = \left(\frac{I_3 - I_1}{I_1}\omega_3\right)\omega_1\,. \tag{9.66}$$

The terms in parentheses are equal and constant, and we define

$$\Omega \equiv \frac{I_3 - I_1}{I_1}\omega_3\,. \tag{9.67}$$

Then

$$\dot{\omega}_1 = -\Omega\omega_2\,, \quad \dot{\omega}_2 = \Omega\omega_1\,. \tag{9.68}$$

We can solve the system by adding i times the second equation to the first:

$$\dot{\omega}_1 + i\dot{\omega}_2 = i\Omega(\omega_1 + i\omega_2)\,, \tag{9.69}$$

or, defining $\eta = \omega_1 + i\omega_2$,

$$\dot{\eta} = i\Omega\eta\,, \tag{9.70}$$

with solution $\eta = Ae^{i\Omega t}$, or

$$\omega_1 = A\cos(\Omega t)\,, \quad \omega_2 = A\sin(\Omega t)\,. \tag{9.71}$$

It follows that the projection of the angular momentum onto the x_1x_2 plane (orthogonal to the symmetry axis of the top), has constant length, $\omega_1^2 + \omega_2^2 = A^2$. Since also ω_3 (and hence the length $|\boldsymbol{\omega}| = \sqrt{A^2 + \omega_3^2}$) is constant, the angular velocity $\boldsymbol{\omega}$ rotates or *precesses* about the symmetry axis x_3 with constant angular velocity Ω. The cone traced out by $\boldsymbol{\omega}$ is called the *body cone* (see Fig. (26)).

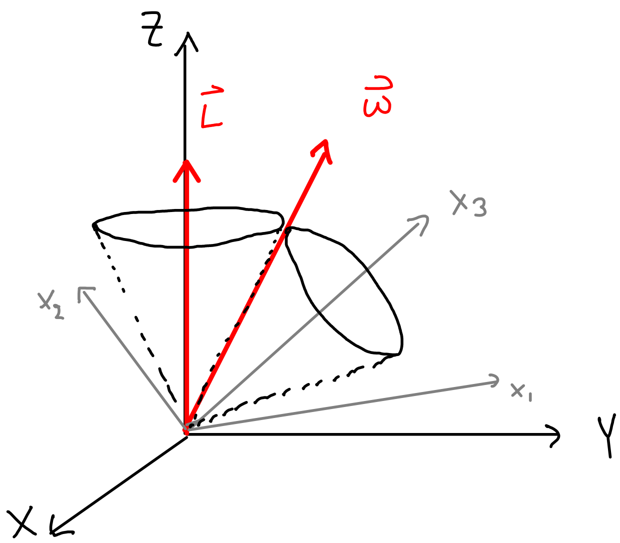

Figure 26: *Space and body cones for the force-free top.*

For force-free motion, the total energy (in the CM system) is equal to the rotational kinetic energy and is conserved:

$$T_{\text{rot}} = \frac{1}{2}\boldsymbol{\omega} \cdot \boldsymbol{L} = \text{constant}\,. \tag{9.72}$$

The angular momentum $\boldsymbol{L}$ is conserved, so the projection of $\boldsymbol{\omega}$ onto $\boldsymbol{L}$ must be constant as $\boldsymbol{\omega}$ evolves in time. That means that $\boldsymbol{\omega}$ precesses about $\boldsymbol{L}$.

We can show that $\boldsymbol{\omega}$, $\boldsymbol{L}$, and the figure axis x_3 always lie in a plane, by showing that $\boldsymbol{L} \cdot (\boldsymbol{\omega} \times \boldsymbol{e}_3)$ vanishes:

$$\boldsymbol{L} \cdot (\boldsymbol{\omega} \times \boldsymbol{e}_3) = \sum_i L_i(\boldsymbol{\omega} \times \boldsymbol{e}_3)_i = L_1\omega_2 - L_2\omega_1 = I_1\omega_1\omega_2 - I_2\omega_1\omega_2 = 0 \tag{9.73}$$

since $I_1 = I_2$. If we choose (as is conventional) the Z axis of the fixed system in direction of $\boldsymbol{L}$, then $\boldsymbol{\omega}$ traces out a cone about the Z axis, called the *space cone*, and the figure axis precesses about the Z axis accordingly. During this motion, the body and space cones just "roll on each other".

For the earth, $\omega_3 \approx \omega$, and we have $I_3 \gtrsim I_1$, so the rate of precession $\Omega = (I_3/I_1 - 1)\omega_3$ is small. Numerically, $(I_3/I_1 - 1) \approx 1/300$. Because $2\pi/\omega$ is one day, we obtain a period of precession $2\pi/\Omega$ of about 300 days or 10 months. The observed precession has a period of about 14 months, which is interpreted as a deviation of the earth from being a homogeneous rigid body (the earth has an elasticity comparable to that of steel).

To interpret the motion in the rotating frame, we express the angular velocity in terms of the Euler angles. We choose the Z axis of the fixed coordinate system in direction of the conserved angular momentum $\boldsymbol{L}$. The components of $\boldsymbol{L}$ are related to the angular velocity via Eq. (9.51), so we have

$$\begin{aligned} L_1 &= I_1\omega_1 = I_1 A\cos(\Omega t)\,, \\ L_2 &= I_1\omega_2 = I_1 A\sin(\Omega t)\,, \\ L_3 &= I_3\omega_3\,. \end{aligned} \tag{9.74}$$

Projecting $\boldsymbol{L}$ onto the moving axes, we find

$$\begin{aligned} L_1 &= L\sin\theta\sin\psi\,, \\ L_2 &= L\sin\theta\cos\psi\,, \\ L_3 &= L\cos\theta\,, \end{aligned} \tag{9.75}$$

where $L = |\boldsymbol{L}| = \sqrt{I_1^2A^2 + I_3^2\omega_3^2}$. Comparing the expressions for L_3 we see that $\omega_3 = L_3/I_3 = (L\cos\theta)/I_3$, which implies that θ is constant. Let us use the symmetry about the x_3 axis and choose the x_2 direction orthogonal to the plane containing the angular momentum $\boldsymbol{L}$ and the symmetry axis of the top, i.e., let us choose $\psi = \pi/2$. Then

$$L_1 = L\sin\theta\,, \quad L_2 = 0\,, \quad L_3 = L\cos\theta\,, \tag{9.76}$$

and we see that this corresponds to $t = 0$. The angular velocity in the rotating frame becomes

$$\omega_1 = \dot\phi\sin\theta\,, \quad \omega_2 = 0\,, \quad \omega_3 = \dot\phi\cos\theta + \dot\psi\,. \tag{9.77}$$

Using $L_1 = I_1\omega_1$, we find the angular velocity of precession

$$\Omega_{\text{Pr}} = \dot{\phi} = \frac{L}{I_1}\,, \tag{9.78}$$

and the angular velocity of the rotation of the top about its axis (as viewed in the moving frame)

$$\dot{\psi} = \omega_3 - \dot{\phi}\cos\theta = \left(\frac{1}{I_3} - \frac{1}{I_1}\right)L\,\cos\theta\,. \tag{9.79}$$

9.8 Motion of a symmetric top with one point fixed

We will solve this problem using the Lagrangian method. It is useful to locate the origin of both the fixed and the rotating coordinate systems at the fixed point of the top, such that there is no translational motion of the system. As we will see, the motion is composed of the three periodic motions rotation, precession, and nutation.

As before, we assume $I_1 = I_2 \neq I_3$. Since the origin of the coordinate system is not located at the CM, we use Steiner's theorem (9.17) (with $a_1 = a_2 = 0$, $a_3 = l$, where l is the distance from the CM to the lowest point of the top) to obtain the kinetic energy

$$T = \frac{1}{2}(I_1 + Ml^2)(\omega_1^2 + \omega_2^2) + \frac{1}{2}I_3\omega_3^2\,. \tag{9.80}$$

where M is the mass of the top. Denoting $I_1' \equiv I_1 + Ml^2$, expressing the angular velocities in terms of Euler angles,

$$\omega_1^2 + \omega_2^2 = \dot{\phi}^2\sin^2\theta + \dot{\theta}^2\,, \quad \omega_3^2 = (\dot{\phi}\cos\theta + \dot{\psi})^2\,, \tag{9.81}$$

and subtracting the potential energy, we obtain the Lagrangian

$$L = \frac{1}{2}I_1'(\dot{\phi}^2\sin^2\theta + \dot{\theta}^2) + \frac{1}{2}I_3(\dot{\phi}\cos\theta + \dot{\psi})^2 - Mgl\cos\theta\,. \tag{9.82}$$

We see that the angles ϕ and ψ are cyclic, so we immediately find the two first integrals

$$p_\phi = \frac{\partial L}{\partial\phi} = I_1'\dot{\phi}\sin^2\theta + I_3(\dot{\phi}\cos\theta + \dot{\psi})\cos\theta = L_Z\,, \tag{9.83}$$

$$p_\psi = \frac{\partial L}{\partial\psi} = I_3(\dot{\phi}\cos\theta + \dot{\psi}) = \omega_3 I_3 = L_3\,. \tag{9.84}$$

We can solve this to find

$$\dot{\phi} = \frac{L_Z - L_3\cos\theta}{I_1'\sin^2\theta}\,, \tag{9.85}$$

$$\dot{\psi} = \frac{L_3}{I_3} - \cos\theta\frac{L_Z - L_3\cos\theta}{I_1'\sin^2\theta}\,. \tag{9.86}$$

Furthermore, the total energy

$$E = \frac{1}{2} I_1'(\dot{\phi}^2 \sin^2\theta + \dot{\theta}^2) + \frac{1}{2} I_3(\dot{\phi}\cos\theta + \dot{\psi})^2 + Mgl\cos\theta \tag{9.87}$$

is conserved. Using the relations above, we can write this as

$$E = \frac{(L_Z - L_3\cos\theta)^2}{2I_1'\sin^2\theta} + \frac{I_1'}{2}\dot{\theta}^2 + \frac{L_3^2}{2I_3} + Mgl\cos\theta\,. \tag{9.88}$$

The combination $E' = E - L_3^2/(2I_3)$ is equally conserved. We can write

$$E' = \frac{I_1'}{2}\dot{\theta}^2 + U_{\text{eff}}(\theta)\,, \tag{9.89}$$

with the *effective potential*

$$U_{\text{eff}}(\theta) = \frac{(L_Z - L_3\cos\theta)^2}{2I_1'\sin^2\theta} + Mgl\cos\theta\,. \tag{9.90}$$

Eq. (9.89) yields

$$t = \int \frac{d\theta}{\sqrt{\frac{2}{I_1'}\big(E' - U_{\text{eff}}(\theta)\big)}}\,. \tag{9.91}$$

Together with Eqs. (9.85) and (9.86) this solves the problem (in principle).

Instead, we discuss the motion qualitatively. We introduce the notation

$$\cos\theta = u\,,\qquad \frac{L_Z}{I_1'} = a\,,\qquad \frac{L_3}{I_1'} = b\,,\qquad \frac{2E'}{I_1'} = \alpha\,,\qquad \frac{2Mgl}{I_1'} = \beta\,. \tag{9.92}$$

We have $\dot{u} = -\dot{\theta}\sin\theta$, or $\dot{u}^2 = \dot{\theta}^2\sin^2\theta = \dot{\theta}^2(1-u^2)$. Then Eq. (9.89) becomes

$$\alpha = \dot{\theta}^2 + \frac{(a-bu)^2}{1-u^2} + \beta u\,, \tag{9.93}$$

or, multiplying by $1-u^2$,

$$\dot{u}^2 = (\alpha - \beta u)(1-u^2) - (a-bu)^2 \equiv f(u)\,. \tag{9.94}$$

Formally, f is a polynomial of degree 3 in u, with $f(+\infty) = +\infty$ and $f(\pm 1) = -(a \mp b)^2 < 0$. Of course, any actual motion of the top corresponds to values for a, b, α, β such that $f(u) \geq 0$ for some range $-1 < u_1 \leq u \leq u_2 < 1$. It follows that $f(u)$ has two real roots for $u_1 \leq u \leq u_2$. The inclination θ of the axis changes periodically between two values θ_1 and θ_2; this periodic motion is called *nutation*.

The angular velocity $\dot{\phi}$ can be expressed as

$$\dot{\phi} = \frac{a-bu}{1-u^2}\,. \tag{9.95}$$

If the root of this equation, $u' = b/a$, lies outside the interval $[u_1, u_2]$, then $(a-bu)/(1-u^2) > 0$ or $(a-bu)/(1-u^2) < 0$ for all allowed values of u, and ϕ varies monotonically. The axis of the top traces out a sinusoidal curve (Fig 27, left panel). If u' lies inside the interval $[u_1, u_2]$, then $\dot{\phi}$ is in opposite directions for $\theta = \theta_1$ and $\theta = \theta_2$, and the axis traces out a looping curve (Fig 27, middle panel). If $u' = u_2$, then the axis traces out a curve with cusps (Fig 27, right panel). The last (exceptional) case is the one usually encountered when releasing a top at inclination θ_2 without initial velocity $\dot{\phi}$. The top first falls, and then rises again. The

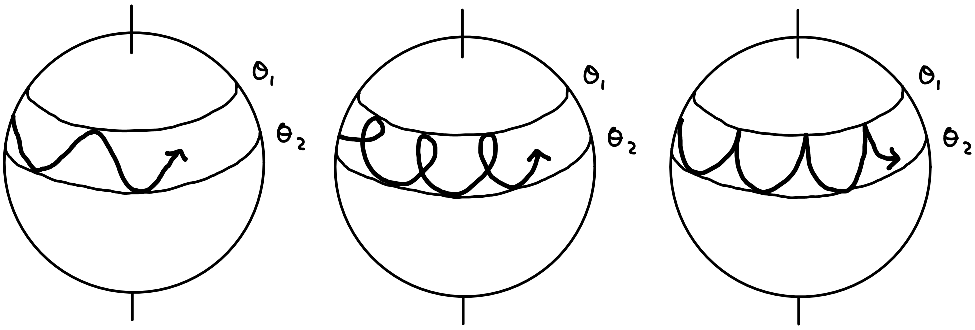

Figure 27: *Path of the top's axis on the unit sphere.*

azimuthal motion of the top is called *precession*. Together with the rotation of the top about its own axis, nutation and precession determine the complete motion of the top.

9.8.1 Motion without nutation

We will show that for specific initial conditions, motion with constant $\theta_0 \neq 0$ is possible (the "trivial" case $\theta_0 \equiv 0$ is discussed in the next subsection). The condition for the motion without nutation follows from Eq. (9.94):

$$0 = (\alpha - \beta u)(1 - u^2) - (a - bu)^2\,, \tag{9.96}$$

or

$$(\alpha - \beta u) = \frac{(a - bu)^2}{1 - u^2} = (a - bu)\dot{\phi}\,, \tag{9.97}$$

where we used Eq. (9.95) in the second step. In addition, we need to require that the zero is of multiplicity two or, equivalently, a local maximum of $f(u)$. This gives the condition

$$0 = -\beta(1 - u^2) - 2(\alpha - \beta u)u + 2b(a - bu)\,, \tag{9.98}$$

or

$$\frac{\beta}{2} = \frac{b(a - bu)}{1 - u^2} - \frac{(\alpha - \beta u)u}{1 - u^2}\,. \tag{9.99}$$

Using Eq. (9.97) and Eq. (9.95), this gives a quadratic equation for $\dot{\phi}$,

$$\frac{\beta}{2} = b\dot{\phi} - \cos\theta_0 \dot{\phi}^2\,, \tag{9.100}$$

where θ_0 is the constant polar angle of the motion. Reinserting the constants b and β, and using Eq. (9.60), this becomes

$$Mg\ell = I_3\dot{\psi}_0\dot{\phi} + (I_3 - I_1')\dot{\phi}^2 \cos\theta_0 \,. \tag{9.101}$$

For the special case $\theta = \pi/2$, this collapses to a linear equation, and we have

$$\dot{\phi} = \frac{\beta}{2b} = \frac{Mg\ell}{I_3\dot{\psi}} \,. \tag{9.102}$$

If $\theta \neq \pi/2$, the solution of Eq. (9.101) still determines the values of $\dot{\phi}$ for which the motion without nutation is possible. The two solutions of Eq. (9.101) are

$$\dot{\phi}_{1,2} = \frac{I_3\dot{\psi} \pm \sqrt{I_3^2\dot{\psi}_0^2 - 4Mg\ell(I_1' - I_3)\cos\theta_0}}{2(I_1' - I_3)\cos\theta_0} \,. \tag{9.103}$$

They are real as long as

$$I_3^2\dot{\psi}_0^2 \geq 4Mg\ell(I_1' - I_3)\cos\theta_0 \,. \tag{9.104}$$

There is just one solution in the case of equality, and two solutions otherwise. Note that for $\theta_0 > \pi/2$ and $I_1' > I_3$, as well as for $\theta_0 < \pi/2$ and $I_1' < I_3$, this condition is satisfied for any value of $\dot{\psi}$, while for the other cases, we must have

$$\dot{\psi}_0 > \frac{2}{I_3}\sqrt{Mg\ell(I_1' - I_3)\cos\theta_0} \,. \tag{9.105}$$

Finally, we note that for very large $\dot{\psi}$, the two (fast and slow) solutions of Eq. (9.103) read

$$\dot{\phi}_1 \approx \frac{I_3\dot{\psi}}{(I_1' - I_3)\cos\theta_0} \,, \qquad \dot{\phi}_2 \approx \frac{Mg\ell}{I_3\dot{\psi}} \,. \tag{9.106}$$

9.8.2 The sleeping top

We now consider the special solution in which the axis of the top is always vertical, i.e., $\theta = 0$, and the angular velocity is constant. In this case, $L_Z = L_3 = I_3\omega_3$. Under what condition will this motion be stable?

We expand the effective potential for small angles,

$$U_{\text{eff}}(\theta) = \frac{(L_Z - L_3\cos\theta)^2}{2I_1'\sin^2\theta} + Mgl\cos\theta = C + A\theta^2 + \dots \,, \tag{9.107}$$

with

$$A = \frac{\omega_3^2 I_3^2}{8I_1'} - \frac{Mgl}{2} \,. \tag{9.108}$$

The equilibrium position $\theta = 0$ is stable if $A > 0$, and not stable otherwise. The condition for stability is therefore

$$\omega_3^2 > \frac{4MglI_1'}{I_3^2} \,. \tag{9.109}$$

The top wakes up when friction reduces the angular velocity to below this limit.

9.9 Stability of rigid-body rotations

Finally, we want to discuss some aspects of the free motion of a rigid body with a general inertia tensor. This problem can be solved in general using EULER's equations, but the solution is "difficult". However, we can obtain some qualitative information about the motion from the conservation laws.

For definiteness, we assume $I_3 > I_2 > I_1$. Energy and momentum conservation imply

$$I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2 = 2E\,, \tag{9.110}$$

$$I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2 = L^2\,. \tag{9.111}$$

We can also write this as

$$\frac{L_1^2}{I_1} + \frac{L_1^2}{I_2} + \frac{L_1^2}{I_3} = 2E\,, \tag{9.112}$$

$$L_1^2 + L_2^2 + L_3^2 = L^2\,. \tag{9.113}$$

These two equations define an ellipsoid with semimajor axes

$$\sqrt{2EI_1}\,, \quad \sqrt{2EI_2}\,, \quad \sqrt{2EI_3}\,, \tag{9.114}$$

and a sphere with radius L in the "three-dimensional angular momentum vector space". The "motion" of $\boldsymbol{L}$ is restricted to lie on the intersection of the ellipsoid and the sphere. See Fig. 28. Starting from $L^2 = 2EI_1$ and increasing L, the allowed curves are closed curves around the L_1 poles, then ellipses crossing the L_2 poles, then closed curves around the L_3 poles. For $L^2 \gtrsim 2EI_1$ or $L^2 \lesssim 2EI_3$, the angular momentum vector performs a periodic motion. Note that the closed curves around the L_1 and L_3 poles stay close to these poles: this motion is "stable". However, the curves through the L_2 poles, on the other hand, are unstable.

9.10 Theory of Billiard*

Here, we give a brief introduction to the theory of billiard. We first consider a horizontal hit in the symmetry plane. First, we calculate at what height the cue needs to hit the billiard ball such that pure rolling motion follows. Let's call the transferred linear momentum p. The transferred angular momentum w.r.t. the center of mass is then ph, with h the distance above the CM. The linear momentum of the billiard ball after the hit is $Mv = p$. The angular momentum is $I_{\text{sphere}}\omega = ph$, with $I_{\text{sphere}} = 2MR^2/5$ the moment of inertial of a sphere. The corresponding velocity of the touching point is $u = R\omega$ and is in opposite direction if $h > 0$. It follows, for $u = v$,

$$p = Mv = \frac{2}{5h}MR^2\frac{v}{R}\,, \tag{9.115}$$

or

$$1 = \frac{2}{5}\frac{R}{h}\,, \tag{9.116}$$

so

$$h = \frac{2}{5}R\,. \tag{9.117}$$

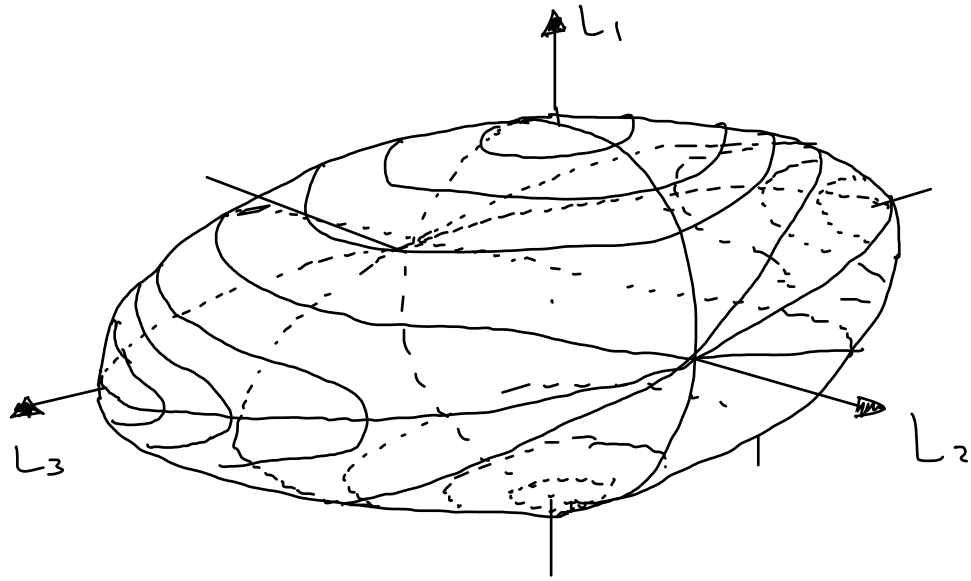

Figure 28: *Ellipsoid of inertia.*

This corresponds to a height $7R/5$ above the table.

Next, for a general height of the contact point, how long does it take until pure rolling motion occurs? Denote by v the CM velocity and by u the radial velocity of the lowest point of the billiard ball (counted in the opposite direction as v). The velocity of the contact point relative to the table is $u - v$ and is opposite in direction to v for $h > 2R/5$. Let μ be the friction coefficient (independent of velocity). For high hits ($h > 2R/5$), the friction opposes the motion of the center of mass; for low hits ($h < 2R/5$), the friction accelerates the motion of the center of mass. Therefore, we have

$$\frac{dv}{dt} = \pm\mu g\,. \tag{9.118}$$

The rotation of the ball is also reduced by the friction; the friction force f exerts a torque such that $I_{\text{sphere}} d\omega/dt = R\mu M g$, or, with $R\omega = u$ (and taking the signs into account),

$$\frac{du}{dt} = \mp\frac{5}{2}\mu g\,. \tag{9.119}$$

Again, the upper sign is for $h > 2R/5$, the lower sign for $h < 2R/5$. The speed of the (center of mass of the) billiard ball right after the hit is $v_0 = p/M$, and hence $v(t) = v_0 \pm \mu g t$. The speed of the lowest point of the ball right after the hit is $u_0 = 5ph/2RM = 5hv_0/2R$, and $u(t) = u_0 \mp 5\mu g t/2$. Pure rolling occurs when $u(\tau) = v(\tau)$, or

$$\frac{5h}{2R}v_0 \mp \frac{5}{2}\mu g\tau = v_0 \pm \mu g\tau \qquad \Leftrightarrow \qquad \mp\frac{7}{2}\mu g\tau = \frac{2R-5h}{2R}v_0\,, \tag{9.120}$$

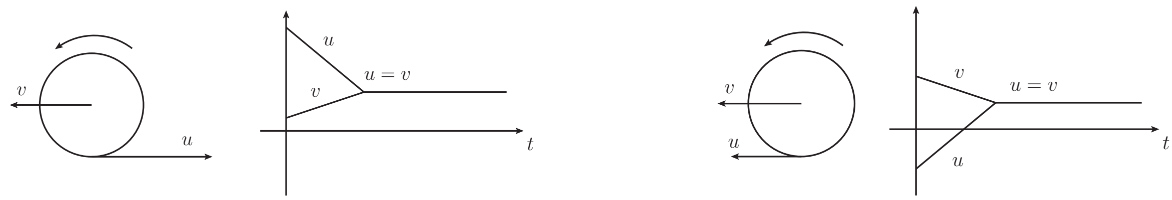

Figure 29: *Motion of billiard balls with friction.*

so

$$\tau = \pm\frac{5h - 2R}{7R}\frac{v_0}{\mu g} \,. \tag{9.121}$$

After that time, the motion is pure rolling. The difference in the center-of-mass velocity is given by Eq. (9.118) as $\Delta v = \pm\mu g\tau$, so the final velocity is

$$v_0 + \Delta v = \frac{5}{7}\frac{R+h}{R}v_0 \,, \tag{9.122}$$

proportional to the height above the table. See Fig. 29.

Next, we would like to discuss what happens if the first billiard ball hits a second billiard ball at rest in an elastic, central collision. The first ball will transfer its linear momentum (and hence its instantaneous velocity), as well as its translational kinetic energy to the second ball. Since the contact point between the two balls is at height R above the table, the first ball will not transfer angular momentum. The second ball will move according to the discussion above, for $h = 0$, so it will reach a final velocity

$$v_{2,\text{final}} = \frac{5}{7}v(\tau_0) \tag{9.123}$$

(unless, of course, it hits a third ball first). Here, $v(\tau_0)$ is the speed of the first ball when it hits the second ball. Depending on whether u is positive or negative at the time of collision, the first ball will follow or recoil from the second ball after the collision.

Nachläufer (“follow shot”). Assume the motion of the first ball is such that, at the time of collision τ_0, the velocity of the balls lowest point is positive (i.e. opposite the direction of v). Note that this might occur either during pure rolling motion, during the accelerating phase with a hit $h > 0$, or after some time during the decelerating phase after a hit with $h < 0$. After the collision, the first ball will be momentarily at rest and then accelerate. Its $u(t)$ will decrease, according to Eq. (9.119). Pure rolling will occur at time τ_1, given by

$$u(\tau_0) - \frac{5}{2}\mu g\tau_1 = \mu g\tau_1 \,, \tag{9.124}$$

or

$$\tau_1 = \frac{2}{7}\frac{u(\tau_0)}{\mu g} \,, \tag{9.125}$$

with $u(\tau_0)$ given explicitly by

$$u(\tau_0) = \frac{5h}{2R} v_0 - \frac{5}{2} \mu g \tau_0 \tag{9.126}$$

for $\tau_0 < \tau$ (i.e. before pure rolling occurs), and

$$u(\tau_0) = v(\tau) = \frac{5}{7} \frac{R+h}{R} v_0 \tag{9.127}$$

for $\tau_0 > \tau$. It follows that

$$v_{1,\text{final}} = v(\tau_1) = \mu g \tau_1 = \frac{2}{7} u(\tau_0)\,. \tag{9.128}$$

Rückzieher (“draw shot”). Assume the ball is hit very low and hits another billiard ball at a time $\tau_0 < \tau$, such that $u(\tau_0) < 0$ is still negative. Again, the first ball transfers its linear momentum, i.e. its velocity $v(\tau_0)$ to the second ball. The second ball will move as discussed above. The first ball will accelerate from rest, but now in the opposite (backward) direction. Its $u(t)$ will increase, according to Eq. (9.119), until it becomes positive and equal to the center-of-mass velocity. Pure rolling will occur at time τ_1, given now by

$$u(\tau_0) + \frac{5}{2} \mu g \tau_1 = -\mu g \tau_1\,, \tag{9.129}$$

or

$$\tau_1 = \frac{2}{7} \frac{|u(\tau_0)|}{\mu g}\,, \tag{9.130}$$

with $u(\tau_0)$ given explicitly by

$$u(\tau_0) = \frac{5h}{2R} v_0 + \frac{5}{2} \mu g \tau_0\,. \tag{9.131}$$

It follows that

$$v_{1,\text{final}} = \frac{2}{7} u(\tau_0) = \frac{5h}{7R} v_0 + \frac{5}{7} \mu g \tau_0 \tag{9.132}$$

(now in the opposite direction).

Parabolic motion of the billiard ball. Now assume the hit is not horizontally, but instead with an angle between the cue and the horizontal plane. The ball will be hit on its upper hemisphere. Choose the x direction along the horizontal component, and the z direction along the vertical. Then the components of $\boldsymbol{p}$ are $(p_x, 0, p_z)$, and the transmitted angular momentum w.r.t. to the CM (origin of the coordinate system) has components

$$L_x = y p_z\,, \qquad L_y = z p_x - x p_z\,, \qquad L_z = -y p_x\,. \tag{9.133}$$

where (x, y, z) is the point where the cue hits the ball. This will induce the following components of angular momentum

$$\omega_x = \frac{5}{2} \frac{L_x}{MR^2}\,, \qquad \omega_y = \frac{5}{2} \frac{L_y}{MR^2} \tag{9.134}$$

(ω_z is irrelevant for the motion). The corresponding components of the CM velocity v and the velocity of the contact point u are

$$v_x = \frac{p_x}{M}\,, \qquad v_y = 0\,, \tag{9.135}$$

and

$$u_x = -R\omega_y\,, \qquad u_y = R\omega_x\,. \tag{9.136}$$

The sliding motion of the contact point with the table has components

$$v_x - u_x = \rho\cos\alpha\,, \qquad v_y - u_y = \rho\sin\alpha\,. \tag{9.137}$$

This induces a frictional force $\boldsymbol{R}$ of magnitude μMg whose direction subtends an angle $\pi + \alpha$ with the x axis, i.e.

$$R_x = -\mu g M\cos\alpha\,, \qquad R_y = -\mu g M\sin\alpha\,. \tag{9.138}$$

For $t > 0$ it modifies the motion according to

$$M\dot{v}_x = R_x\,, \qquad M\dot{v}_y = R_y\,, \tag{9.139}$$

$$I_{\text{sphere}}\dot{\omega}_x = RR_y\,, \qquad I_{\text{sphere}}\dot{\omega}_y = -RR_x\,. \tag{9.140}$$

It follows

$$\dot{v}_x = \mu g\cos\alpha\,, \qquad \dot{v}_y = \mu g\sin\alpha\,, \tag{9.141}$$

and

$$\dot{u}_x = -R\dot{\omega}_y = \frac{5}{2}\mu g\cos\alpha\,, \qquad \dot{u}_y = R\dot{\omega}_x = \frac{5}{2}\mu g\sin\alpha\,. \tag{9.142}$$

Therefore,

$$\dot{v}_x - \dot{u}_x = \frac{d}{dt}(\rho\cos\alpha) = \dot{\rho}\cos\alpha + \rho\dot{\alpha}\sin\alpha = \frac{7}{2}\mu g\cos\alpha\,, \tag{9.143}$$

$$\dot{v}_y - \dot{u}_y = \frac{d}{dt}(\rho\sin\alpha) = \dot{\rho}\sin\alpha - \rho\dot{\alpha}\cos\alpha = \frac{7}{2}\mu g\sin\alpha\,. \tag{9.144}$$

From these equation we find

$$\dot{\alpha} = 0\,, \qquad \dot{\rho} = \frac{7}{2}\mu g \tag{9.145}$$

(recall that in our coordinates, the initial CM velocity is negative). The frictional force is constant in both direction and magnitude, so the billiard ball will follow a parabolic trajectory until purely rolling motion occurs. This will happen after time τ, determined by

$$0 = \rho(\tau) = \rho_0 + \frac{7}{2}\mu g\tau\,, \tag{9.146}$$

or $\tau = -2\rho_0/7\mu g$ (again, recall that $\rho_0 < 0$).

10 Small oscillations

10.1 Two coupled harmonic oscillators

We start with an example: two (one-dimensional) harmonic oscillators with masses $m_1 = m_2 = m$ are connected with a spring of spring constant κ_{12}, and connected to two walls by two springs of spring constant κ. We denote the displacements of m_1 and m_2 from their equilibrium positions by x_1 and x_2, respectively. The kinetic energy of the system is

$$T = \frac{m}{2}\left(\dot{x}_1^2 + \dot{x}_2^2\right), \tag{10.1}$$

and the potential energy[27]

$$U = \frac{1}{2}\left[\kappa(x_1^2 + x_2^2) + \kappa_{12}(x_1 - x_2)^2\right]. \tag{10.3}$$

Lagrange's method then gives the two equations of motion

$$m\ddot{x}_1 + (\kappa + \kappa_{12})x_1 - \kappa_{12}x_2 = 0, \tag{10.4}$$
$$m\ddot{x}_2 + (\kappa + \kappa_{12})x_2 - \kappa_{12}x_1 = 0. \tag{10.5}$$

The ansatz $x_i(t) = A_i \cos(\omega t + \delta)$ leads to the system of equations

$$\begin{aligned}(\kappa + \kappa_{12} - m\omega^2)A_1 - \kappa_{12}A_2 &= 0,\\ -\kappa_{12}A_1 + (\kappa + \kappa_{12} - m\omega^2)A_2 &= 0.\end{aligned} \tag{10.6}$$

Non-trivial solutions for A_1, A_2 exist only if the determinant of the coefficient matrix vanishes:

$$\det\begin{pmatrix}\kappa + \kappa_{12} - m\omega^2 & -\kappa_{12}\\ -\kappa_{12} & \kappa + \kappa_{12} - m\omega^2\end{pmatrix} = 0. \tag{10.7}$$

This gives the condition

$$(\kappa + \kappa_{12} - m\omega^2)^2 - \kappa_{12}^2 = 0. \tag{10.8}$$

It follows that

$$\kappa + \kappa_{12} - m\omega^2 = \pm\kappa_{12}, \tag{10.9}$$

with solutions

$$\omega_1 = \sqrt{\frac{\kappa}{m}}, \quad \omega_2 = \sqrt{\frac{\kappa + 2\kappa_{12}}{m}}. \tag{10.10}$$

Inserting ω_1 into Eq. (10.6) gives $A_1 = A_2 \equiv C_1$, while inserting ω_2 gives $A_1 = -A_2 \equiv C_2$. We can write the two solutions as

$$\begin{pmatrix}x_{11}(t)\\ x_{12}(t)\end{pmatrix} = \begin{pmatrix}1\\ 1\end{pmatrix} C_1\cos(\omega_1 t - \delta_1), \quad \begin{pmatrix}x_{21}(t)\\ x_{22}(t)\end{pmatrix} = \begin{pmatrix}1\\ -1\end{pmatrix} C_2\cos(\omega_2 t - \delta_2), \tag{10.11}$$

[27] The corresponding forces are (as we will see)

$$F_1 = -\kappa x_1 - \kappa_{12}(x_1 - x_2), \quad F_2 = -\kappa x_2 - \kappa_{12}(x_2 - x_1). \tag{10.2}$$

and the general solution is given by

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} C_1 \cos(\omega_1 t - \delta_1) + \begin{pmatrix} 1 \\ -1 \end{pmatrix} C_2 \cos(\omega_2 t - \delta_2) \,. \tag{10.12}$$

It depends on four integration constants, C_1, C_2, δ_1, δ_2 that need to be determined from the initial conditions.

For instance, if $x_1(0) = x_2(0)$ and $\dot{x}_1(0) = \dot{x}_2(0)$, we find only the oscillation with frequency ω_1, while for $x_1(0) = -x_2(0)$ and $\dot{x}_1(0) = -\dot{x}_2(0)$, only the oscillation with frequency ω_2 is generated.

A slightly more interesting case is $x_1(0) = A$, $x_2(0) = \dot{x}_1(0) = \dot{x}_2(0) = 0$. The conditions lead to

$$A = C_1 \cos(\delta_1) + C_2 \cos(\delta_2) \,, \tag{10.13}$$
$$0 = C_1 \cos(\delta_1) - C_2 \cos(\delta_2) \,, \tag{10.14}$$
$$0 = C_1 \omega_1 \sin(\delta_1) + C_2 \omega_2 \sin(\delta_2) \,, \tag{10.15}$$
$$0 = C_1 \omega_1 \sin(\delta_1) - C_2 \omega_2 \sin(\delta_2) \,. \tag{10.16}$$

It follows, in particular, $A = 2C_1 \cos(\delta_1) = 2C_2 \cos(\delta_2)$, and $C_1 \sin(\delta_1) = C_2 \sin(\delta_2) = 0$. On the other hand, we can use trigonometric identities to write

$$x_1(t) = C_1 \big[\cos(\omega_1 t) \cos(\delta_1) + \sin(\omega_1 t) \sin(\delta_1) \big] + C_2 \big[\cos(\omega_2 t) \cos(\delta_2) + \sin(\omega_2 t) \sin(\delta_2) \big] \,, \tag{10.17}$$
$$x_2(t) = C_1 \big[\cos(\omega_1 t) \cos(\delta_1) + \sin(\omega_1 t) \sin(\delta_1) \big] - C_2 \big[\cos(\omega_2 t) \cos(\delta_2) + \sin(\omega_2 t) \sin(\delta_2) \big] \,, \tag{10.18}$$

which become

$$x_1(t) = \frac{A}{2} \big[\cos(\omega_1 t) + \cos(\omega_2 t) \big] \,, \tag{10.19}$$
$$x_2(t) = \frac{A}{2} \big[\cos(\omega_1 t) - \cos(\omega_2 t) \big] \,. \tag{10.20}$$

Using again trigonometric identities, we can write this as[28]

$$x_1(t) = A \cos\left(\frac{\omega_2 - \omega_1}{2} t \right) \cos\left(\frac{\omega_2 + \omega_1}{2} t \right) \,, \tag{10.21}$$
$$x_2(t) = A \sin\left(\frac{\omega_2 - \omega_1}{2} t \right) \sin\left(\frac{\omega_2 + \omega_1}{2} t \right) \,. \tag{10.22}$$

If $\kappa_{12} \ll \kappa$, the difference $\omega_2 - \omega_1$ is small. This is the phenomenon of beats.

[28] We have $\cos\big(\frac{\omega_2 \mp \omega_1}{2} t\big) = \cos\big(\frac{\omega_2}{2} t\big) \cos\big(\frac{\omega_1}{2} t\big) \pm \sin\big(\frac{\omega_2}{2} t\big) \sin\big(\frac{\omega_1}{2} t\big)$, as well as $\cos\big(\frac{\omega}{2} t\big)^2 = (1 + \cos(\omega t))/2$, and $\sin\big(\frac{\omega}{2} t\big)^2 = (1 - \cos(\omega t))/2$.

10.2 General theory of small oscillations

Let us now consider a general system of N coupled oscillators. As in the one-dimensional case, such a system arises naturally as an approximation to *any* conservative system near an equilibrium point.

We assume that any existing constraints are time independent; as we have seen in Sec. 5.5, the kinetic energy is then of the form

$$T = \sum_{ij} a_{ij}(q_k)\dot{q}_i\dot{q}_j \,. \tag{10.23}$$

Near equilibrium, all q_k and $\dot{q}_k$ will be small. We can therefore Taylor-expand the coefficients about the equilibrium positions q_{0k}, and keep only the leading (constant) pieces; we will denote them by $m_{ij} \equiv 2a_{ij}(q_{0k})$. The kinetic energy is then

$$T = \frac{1}{2}\sum_{ij} m_{ij}\dot{q}_i\dot{q}_j \,. \tag{10.24}$$

Similarly, we will expand the potential energy about the equilibrium position $q_{0k} = 0$:

$$U(q_k) = U(q_{0k}) + \sum_i \left.\frac{\partial U}{\partial q_i}\right|_{q_k=0} q_i + \frac{1}{2}\sum_{ij} \left.\frac{\partial^2 U}{\partial q_i \partial q_j}\right|_{q_k=0} q_i q_j + \dots \,. \tag{10.25}$$

The first derivatives vanish in an equilibrium position, and we can drop the constant term, such that the potential energy near the equilibrium becomes, to first approximation,

$$U = \frac{1}{2}\sum_{ij} U_{ij} q_i q_j \,, \tag{10.26}$$

where we defined

$$U_{ij} \equiv \sum_{ij} \left.\frac{\partial^2 U}{\partial q_i \partial q_j}\right|_{q_k=q_{0k}} \,. \tag{10.27}$$

The Lagrangian

$$L = \frac{1}{2}\sum_{ij} m_{ij}\dot{q}_i\dot{q}_j - \frac{1}{2}\sum_{ij} U_{ij} q_i q_j \tag{10.28}$$

then yields the equations of motion

$$\sum_j \left[m_{ij}\ddot{q}_j + U_{ij} q_j\right] = 0\,, \quad i = 1, \dots, N \,. \tag{10.29}$$

We could now proceed in analogy to above. The ansatz $q_k(t) = A_k \cos(\omega t + \delta_k)$ would yield a homogeneous system of N linear equations that we would need to solve. The condition for a non-trivial solution leads to an equation that allows, in principle, to solve for the possible eigenfrequencies.

It is, however, simpler to first perform a coordinate transformation in order to bring the equations of motion in diagonal form. They will then correspond to N *decoupled* equations for N simple harmonic oscillators, for which we already know the solutions.

From our example above, we know that the general solution for the coordinate q_j is given by a linear combination of the solutions

$$q_{jr}(t) = A_{jr} \cos(\omega_r t - \delta_r) \,. \tag{10.30}$$

Inserting this into the equations of motion Eq. (10.29) gives

$$\sum_j U_{ij} A_{jr} = \omega_r^2 \sum_j m_{ij} A_{jr} \,. \tag{10.31}$$

Choosing a different solution $q_{js}(t)$ gives similarly

$$\sum_j U_{ij} A_{js} = \omega_s^2 \sum_j m_{ij} A_{js} \,. \tag{10.32}$$

(For now, we assume that all eigenfrequencies are different.) We multiply the first equation by A_{is}, the second by A_{ir}, sum over the index i, and take the difference of the two equations. Since $m_{ij} = m_{ji}$ and $U_{ij} = U_{ji}$, we obtain

$$0 = (\omega_r^2 - \omega_s^2) \sum_{ij} m_{ij} A_{ir} A_{js} \,. \tag{10.33}$$

It follows that

$$\sum_{ij} m_{ij} A_{ir} A_{js} = 0 \quad \text{for} \quad r \neq s \,. \tag{10.34}$$

For $r = s$ we consider the kinetic energy of the rth oscillation

$$\frac{1}{2} \sum_{ij} m_{ij} \dot{q}_{ir} \dot{q}_{jr} = \frac{1}{2} \omega_r^2 \sin^2(\omega_r + \delta_r) \sum_j m_{ij} A_{ir} A_{jr} > 0 \,, \tag{10.35}$$

and so

$$\sum_{ij} m_{ij} A_{ir} A_{jr} > 0 \,. \tag{10.36}$$

By possibly rescaling the coordinates, we can always achieve

$$\sum_{ij} m_{ij} A_{ir} A_{jr} = 1 \,. \tag{10.37}$$

Together, Eqs. (10.34) and (10.37) just say that

$$\sum_{ij} m_{ij} A_{ir} A_{js} = \delta_{rs} \,, \tag{10.38}$$

or, defining the matrix $\boldsymbol{M}$ with matrix elements m_{ij},

$$\boldsymbol{A}^T\boldsymbol{M}\boldsymbol{A} = \mathbf{1}\,. \tag{10.39}$$

Writing

$$\boldsymbol{\Omega}^2 = \begin{pmatrix} \omega_1^2 & 0 & 0 & 0 \\ 0 & \omega_2^2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \omega_N^2 \end{pmatrix}\,, \tag{10.40}$$

the equations of motion Eq. (10.31) can then be written as

$$\boldsymbol{U}\boldsymbol{A} = \boldsymbol{M}\boldsymbol{A}\boldsymbol{\Omega}^2\,. \tag{10.41}$$

Multiplying on the left by $\boldsymbol{A}^T$ and using Eq. (10.39) gives[29]

$$\boldsymbol{A}^T\boldsymbol{U}\boldsymbol{A} = \boldsymbol{\Omega}^2\,. \tag{10.42}$$

We can introduce *normal coordinates* Q_i via

$$q_i = \sum_r A_{ir}Q_r\,. \tag{10.43}$$

Using these coordinates yields the equations of motion in decoupled form: the Lagrangian is

$$L(Q_i,\dot{Q}_i) = \frac{1}{2}\sum_{ij}\sum_{rs}\left[m_{ij}A_{ir}A_{js}\dot{Q}_r\dot{Q}_s - U_{ij}A_{ir}A_{js}Q_rQ_s\right] = \frac{1}{2}\sum_r\left[\dot{Q}_r^2 - \omega_r^2Q_r^2\right]\,, \tag{10.44}$$

and the equations

$$\ddot{Q}_r^2 + \omega_r^2 Q_r = 0\,, \quad r = 1,\dots,N\,, \tag{10.45}$$

follow.

Example 10.1: Two coupled oscillators, revisited. *Let us re-examine the example in Sec. 10.1. We have the kinetic energy*

$$T = \frac{1}{2}\begin{pmatrix}\dot{q}_1 & \dot{q}_2\end{pmatrix}\begin{pmatrix} m & 0 \\ 0 & m\end{pmatrix}\begin{pmatrix}\dot{q}_1 \\ \dot{q}_2\end{pmatrix}\,, \tag{10.46}$$

[29] Mathematically, the ω_i^2 are the eigenvalues of the "potential energy matrix" $\boldsymbol{U}$ with respect to the "kinetic energy matrix" $\boldsymbol{M}$, i.e. the quadratic form defined by U is diagonalized by a basis that is orthogonal with respect to the inner product defined by the kinetic energy T. In more detail, if we write the kinetic energy as $T = \frac{1}{2}(\boldsymbol{q},\boldsymbol{M}\boldsymbol{q})$ and the potential energy as $U = \frac{1}{2}(\boldsymbol{q},\boldsymbol{U}\boldsymbol{q})$, where $(\cdot,\cdot)$ denotes the usual scalar product, then general theorems of linear algebra tell us that we can first find a coordinate transformation $\boldsymbol{q} = \boldsymbol{A}\boldsymbol{Q}$, with $\boldsymbol{A} \in \mathrm{GL}(N)$ such that $\boldsymbol{A}^T\boldsymbol{M}\boldsymbol{A} = \mathbf{1}$ (since $\boldsymbol{M}$ is positive definite). Then $T = \frac{1}{2}(\boldsymbol{Q},\boldsymbol{Q})$ and $U = \frac{1}{2}(\boldsymbol{Q},\boldsymbol{A}^T\boldsymbol{U}\boldsymbol{A}\boldsymbol{Q})$. Since $\boldsymbol{A}^T\boldsymbol{U}\boldsymbol{A}$ is symmetric, it can be diagonalized with an orthogonal transformation $\boldsymbol{S} \in O(N)$ that leaves T invariant; i.e. $\boldsymbol{S}^T\boldsymbol{A}^T\boldsymbol{U}\boldsymbol{A}\boldsymbol{S} = \boldsymbol{D}$ is diagonal, and $\boldsymbol{S}^T\boldsymbol{S} = \mathbf{1}$.

Now the characteristic equation for the diagonalization of $\boldsymbol{A}^T\boldsymbol{U}\boldsymbol{A}$ is $\det\big(\boldsymbol{A}^T\boldsymbol{U}\boldsymbol{A} - \lambda\mathbf{1}\big) = 0$, which in view of the above can be written as $\det\big(\boldsymbol{A}^T\boldsymbol{U}\boldsymbol{A} - \lambda\boldsymbol{A}^T\boldsymbol{M}\boldsymbol{A}\big) = \det\big[\boldsymbol{A}^T(\boldsymbol{U} - \lambda\boldsymbol{M})\boldsymbol{A}\big] = 0$. Using the multiplication theorem for determinants, this is equivalent to $\det(\boldsymbol{U} - \lambda\boldsymbol{M}) = 0$ – the same condition we found above, but without the necessity to insert the ansatz for solving the equations of motion.

and the potential energy

$$U = \frac{1}{2} \begin{pmatrix} q_1 & q_2 \end{pmatrix} \begin{pmatrix} \kappa + \kappa_{12} & -\kappa_{12} \\ -\kappa_{12} & \kappa + \kappa_{12} \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} . \tag{10.47}$$

According to the general theory, there will be two independent solutions (see Eq. (10.30))

$$q_i(t) = \sum_j A_{ij} \cos(\omega_j t - \delta_j) . \tag{10.48}$$

Inserting this into the equation of motion leads to the system of equations (10.31):

$$\begin{pmatrix} \kappa + \kappa_{12} - \omega^2 m & -\kappa_{12} \\ -\kappa_{12} & \kappa + \kappa_{12} - \omega^2 m \end{pmatrix} \begin{pmatrix} A_{1j} \\ A_{2j} \end{pmatrix} = 0 . \tag{10.49}$$

This system will have non-trivial solutions if the determinant of the coefficient matrix vanishes,

$$0 = \begin{vmatrix} \kappa + \kappa_{12} - \omega^2 m & -\kappa_{12} \\ -\kappa_{12} & \kappa + \kappa_{12} - \omega^2 m \end{vmatrix} = (\kappa + \kappa_{12} - \omega m)^2 - \kappa_{12}^2 , \tag{10.50}$$

with solutions $\omega_1^2 = \kappa/m$, $\omega_2^2 = (\kappa + 2\kappa_{12})/m$. *These two solutions give two systems of equations, one for* ω_1^2

$$\begin{pmatrix} \kappa_{12} & -\kappa_{12} \\ -\kappa_{12} & \kappa_{12} \end{pmatrix} \begin{pmatrix} A_{11} \\ A_{21} \end{pmatrix} = 0 , \tag{10.51}$$

with solution $A_{j1} = (a, a)$, *and one for* ω_2^2

$$\begin{pmatrix} -\kappa_{12} & -\kappa_{12} \\ -\kappa_{12} & -\kappa_{12} \end{pmatrix} \begin{pmatrix} A_{12} \\ A_{22} \end{pmatrix} = 0 , \tag{10.52}$$

with solution $A_{j2} = (b, -b)$. *The coefficient matrix therefore looks like*

$$\boldsymbol{A} = \begin{pmatrix} a & b \\ a & -b \end{pmatrix} . \tag{10.53}$$

We choose values for a *and* b *such that Eq. (10.38) is satisfied. The conditions here read*

$$m \boldsymbol{A}^T \boldsymbol{A} = \mathbb{1} , \tag{10.54}$$

or

$$2ma^2 = 1 , \qquad 2mb^2 = 0 . \tag{10.55}$$

It follows that we can choose the coefficients as $a = b = 1/\sqrt{2m}$. *One can easily check that*

$$\begin{pmatrix} \frac{1}{\sqrt{2m}} & \frac{1}{\sqrt{2m}} \\ \frac{1}{\sqrt{2m}} & -\frac{1}{\sqrt{2m}} \end{pmatrix} \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2m}} & \frac{1}{\sqrt{2m}} \\ \frac{1}{\sqrt{2m}} & -\frac{1}{\sqrt{2m}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , \tag{10.56}$$

and

$$\begin{pmatrix} \frac{1}{\sqrt{2m}} & \frac{1}{\sqrt{2m}} \\ \frac{1}{\sqrt{2m}} & -\frac{1}{\sqrt{2m}} \end{pmatrix} \begin{pmatrix} \kappa + \kappa_{12} & -\kappa_{12} \\ -\kappa_{12} & \kappa + \kappa_{12} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2m}} & \frac{1}{\sqrt{2m}} \\ \frac{1}{\sqrt{2m}} & -\frac{1}{\sqrt{2m}} \end{pmatrix} = \frac{1}{m} \begin{pmatrix} \kappa & 0 \\ 0 & \kappa + 2\kappa_{12} \end{pmatrix} . \tag{10.57}$$

Hence, we have the two decoupled equations of motion

$$\ddot{Q}_1 + \frac{\kappa}{m} Q_1 = 0\,, \qquad \ddot{Q}_2 + \frac{\kappa + 2\kappa_{12}}{m} Q_2 = 0\,, \tag{10.58}$$

where $q_1 = (Q_1 + Q_2)/\sqrt{2m}$*,* $q_2 = (Q_1 - Q_2)/\sqrt{2m}$ *(see Eq. (10.43)). This is the same result that we obtained previously.*

Example 10.2: Coupled pendulum with two different masses. *Let's calculate a less trivial example with two pendula (length* ℓ*, bob masses* m *and* $2m$*) that are coupled with a spring with spring constant* k*. As dynamical parameters we will take the two angles measured as deviations from the vertical.*

The kinetic energy is here

$$T = \frac{1}{2} \begin{pmatrix} \dot{q}_1 & \dot{q}_2 \end{pmatrix} \begin{pmatrix} m\ell^2 & 0 \\ 0 & 2m\ell^2 \end{pmatrix} \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \end{pmatrix} . \tag{10.59}$$

The potential energy can be chosen as $U = mg\ell(1-\cos\theta_1+1-\cos\theta_2)+\frac{1}{2}\kappa(\ell\sin\theta_1-\ell\sin\theta_2)^2$*, which for small angles (*$\cos\theta \approx 1-\theta^2/2, \sin\theta \approx \theta$*) gives*

$$U = \frac{1}{2} \begin{pmatrix} q_1 & q_2 \end{pmatrix} \begin{pmatrix} mg\ell + \kappa\ell^2 & -\kappa\ell^2 \\ -\kappa\ell^2 & mg\ell + \kappa\ell^2 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} . \tag{10.60}$$

Again there will be two independent solutions,

$$q_i(t) = \sum_j A_{ij} \cos(\omega_j t - \delta_j)\,, \tag{10.61}$$

whose coefficients satisfy the system of equations

$$\begin{pmatrix} mg/\ell + \kappa - \omega^2 m & -\kappa \\ -\kappa & mg/\ell + \kappa - 2\omega^2 m \end{pmatrix} \begin{pmatrix} A_{1j} \\ A_{2j} \end{pmatrix} = 0\,. \tag{10.62}$$

The condition for the existence of non-trivial solutions is

$$(\alpha + \beta - \omega^2)(\alpha + \beta - 2\omega^2) - \alpha^2 = 0\,, \tag{10.63}$$

where we defined $\alpha \equiv \kappa/m$ *and* $\beta \equiv \ell/g$*.*

$$2\omega^4 - 3\omega^2(\alpha+\beta) + (\alpha+\beta)^2 - \alpha^2 = 0\,, \tag{10.64}$$

with solutions

$$\omega_{1,2}^2 = \frac{3}{4}(\alpha+\beta) \pm \frac{1}{4}\sqrt{(\alpha+\beta)^2 + 8\alpha^2}\,. \tag{10.65}$$

To simplify the following calculation, let's take $\alpha = \beta = 1$*, such that*

$$\omega_{1,2}^2 = \frac{3 \pm \sqrt{3}}{2}\,. \tag{10.66}$$

Solving the corresponding two linear systems of equations gives the coefficient matrix

$$\boldsymbol{A} = \begin{pmatrix} -1-\sqrt{3} & -1+\sqrt{3} \\ 1 & 1 \end{pmatrix} . \tag{10.67}$$

A straightforward calculation shows that this diagonalizes both the kinetic and potential energy matrices.

10.3 Continuous systems – the wave equation

Consider a linear arrangement of oscillators that are coupled to their neighbors. If we increase the number of these oscillators per unit length, and decrease their masses such that the mass per unit length remains constant, we obtain the limiting case of a vibrating string.

10.3.1 Three coupled oscillators

We consider three equal masses m that are connected to each other and to two walls via four equal springs with spring constant k. Let ℓ_0 be the distance between the masses in equilibrium.

We first consider longitudinal motion only. Denoting by q_j the displacement of mass i from the equilibrium position, the Lagrangian is

$$L = \frac{m}{2}\sum_{i=1}^{3}\dot{q}_i^2 - \frac{k}{2}\big(q_1^2 + (q_2-q_1)^2 + (q_3-q_2)^2 + q_3^2\big)\,. \tag{10.68}$$

We can write this in a somewhat more symmetric way by including the displacements of the two walls (which are, of course, identically zero), calling them $q_0 \equiv 0$ and $q_4 \equiv 0$. Then the Lagrangian becomes

$$L = \frac{m}{2}\sum_{i=1}^{4}\dot{q}_i^2 - \frac{k}{2}\sum_{i=1}^{4}(q_i - q_{i-1})^2\,, \tag{10.69}$$

and we find the equations of motion

$$m\ddot{q}_j + k(2q_j - q_{j-1} - q_{j+1}) = 0\,,\quad j = 1,2,3\,, \tag{10.70}$$

and $q_0(t) = q_4(t) = 0$ for all t.

The case of vertical oscillations is only slightly more complicated if we restrict ourselves to planar oscillations. If in the equilibrium position for the masses the springs are also in equilibrium, there will be no restoring force due to small vertical displacements; therefore, we will assume that the springs (in equilibrium as well as for small vertical displacements) all exert the same tension F on all the masses. (If there is no tension in the springs in equlibrium, the vertical oscillations are not harmonic.) Then, denoting by α_j the angle that the jth spring subtends with the horizontal, the restoring force due to spring j is $F\sin\alpha_j \approx F\alpha_j$ for small displacements. The angles are given by $(q_j - q_{j-1})/\ell_0 = \tan\alpha_j \approx \alpha_j$, so the Lagrangian is

$$L = \frac{m}{2}\sum_{i=1}^{4}\dot{q}_i^2 - \frac{F}{2\ell_0}\sum_{i=1}^{4}(q_i - q_{i-1})^2\,, \tag{10.71}$$

where the q_j now denote the vertical displacements, and $q_0(t) = q_4(t) = 0$ for all t. The Lagrangian (and, hence, the equations of motion) are the same as before upon the replacement $k \to F/\ell_0$.

To solve the equations of motion, we make the ansatz $q_j(t) = a_j \cos(\omega t - \delta)$ and obtain the following system of equations:

$$\begin{aligned} (2k - m\omega^2)a_1 - ka_2 &= 0\,, \\ -ka_1 + (2k - m\omega^2)a_2 - ka_3 &= 0\,, \\ -ka_2 + (2k - m\omega^2)a_3 &= 0\,. \end{aligned} \tag{10.72}$$

It has non-trivial solutions if the determinant of the matrix

$$D = \begin{pmatrix} 2k - m\omega^2 & -k & 0 \\ -k & 2k - m\omega^2 & -k \\ 0 & -k & 2k - m\omega^2 \end{pmatrix} \tag{10.73}$$

vanishes; this gives the condition

$$(2k - m\omega^2)^3 - 2k^2(2k - m\omega^2) = 0\,, \tag{10.74}$$

with solutions $\omega_1^2 = (2 - \sqrt{2})k/m$, $\omega_2^2 = 2k/m$, $\omega_3^2 = (2 + \sqrt{2})k/m$. The corresponding amplitudes can be found by solving the system of equations for each ω_i^2. For ω_2^2:

$$-a_2 = 0\,, \qquad -a_1 - a_3 = 0\,, \qquad -a_2 = 0\,, \tag{10.75}$$

so $a_1 = -a_3$ and $a_2 = 0$; for the other two cases,

$$\pm\sqrt{2}a_1 - a_2 = 0\,, \qquad -a_1 \pm \sqrt{2}a_2 - a_3 = 0\,, \qquad -a_2 \pm \sqrt{2}a_3 = 0\,, \tag{10.76}$$

so $a_1 = a_3 = \pm a_2/\sqrt{2}$. The general solution is then

$$\begin{pmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \end{pmatrix} = c_1 \cos(\omega_1 t - \delta_1) \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 1 \\ \frac{1}{\sqrt{2}} \end{pmatrix} + c_2 \cos(\omega_2 t - \delta_2) \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + c_3 \cos(\omega_3 t - \delta_3) \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -1 \\ \frac{1}{\sqrt{2}} \end{pmatrix}. \tag{10.77}$$

Note that the coefficients a_j follow a sine function: For the first normal mode, we have $a_j = \sin(j\pi/4)$; for the second normal mode, we have $a_j = \sin(2j\pi/4)$; and for the third normal mode, we have $a_j = \sin(3j\pi/4)$.

10.3.2 n coupled oscillators

Now we generalize the example to the case of n oscillators with equal masses, fixed between two walls. With the boundary conditions

$$q_0(t) = 0\,, \qquad q_{n+1}(t) = 0 \tag{10.78}$$

for all t, the equations of motion are just of the same form as before:

$$m\ddot{q}_j + k(2q_j - q_{j-1} - q_{j+1}) = 0\,, \quad j = 1, \ldots, n\,. \tag{10.79}$$

Inspired by our example, we choose the following ansatz for the solution:

$$q_j(t) = C\sin(\alpha j)\cos(\omega t - \delta)\,. \tag{10.80}$$

We can determine α using the boundary conditions Eq. (10.78):

$$0 = \sin(\alpha(n+1)) \quad \Rightarrow \quad \alpha(n+1) = r\pi\,, \tag{10.81}$$

for any $r = 1, 2, \ldots$, so $\alpha = r\pi/(n+1)$. Inserting this into the equations of motion gives the equations for the eigen frequencies

$$\begin{aligned} &- m\omega \sin\left(\frac{rj\pi}{n+1}\right) + 2k\sin\left(\frac{rj\pi}{n+1}\right) - k\sin\left(\frac{r(j-1)\pi}{n+1}\right) - k\sin\left(\frac{r(j+1)\pi}{n+1}\right) \\ =&\left[-m\omega + 2k\left[1 - \cos\left(\frac{-r\pi}{n+1}\right)\right]\right]\sin\left(\frac{rj\pi}{n+1}\right) = 0\,. \end{aligned} \tag{10.82}$$

This gives

$$\omega^2 = \frac{2k}{m}\left[1 - \cos\left(\frac{-r\pi}{n+1}\right)\right] = \frac{4k}{m}\sin^2\left(\frac{r\pi}{2(n+1)}\right)\,, \tag{10.83}$$

so the frequencies of the normal modes are

$$\omega_r = 2\omega_0 \sin\left(\frac{r\pi}{2(n+1)}\right)\,, \quad r = 1, \ldots, n\,, \tag{10.84}$$

with $\omega_0 = \sqrt{k/m}$. (For $n = 3$ this reproduces our earlier results: $\omega_1^2 = 4k/m\sin^2(\pi/8) = 2k/m(1-\cos(\pi/4)) = (2-\sqrt{2})k/m$, $\omega_2^2 = 4k/m\sin^2(\pi/4) = 2k/m$, $\omega_3^2 = 4k/m\sin^2(3\pi/8) = 2k/m(1-\cos(3\pi/4)) = (2+\sqrt{2})k/m$.) The amplitudes are given by

$$a_{jr} = \sin\left(\frac{rj\pi}{n+1}\right)\,, \quad j, r = 1, \ldots, n\,. \tag{10.85}$$

The general solution of the equation of motion is, therefore,

$$q_j = \sum_{r=1}^{n} C_r \sin\left(\frac{rj\pi}{n+1}\right)\cos(\omega_r t + \delta_r)\,. \tag{10.86}$$

The $2n$ constants C_r and δ_r are determined by the initial conditions.

10.3.3 Transition to the continuum

In the continuum limit, $\ell_0 \to 0$, $m \to 0$, $n \to \infty$, such that the length $\ell = (n+1)\ell_0$ of the oscillating chain, the mass density $\rho = m/\ell_0$, and the restoring force $k\ell_0$ remain constant. The equation of motion in the continuum limit is obtained from Eq. (10.79) in the form

$$\frac{m}{k\ell_0^2}\ddot{q}_j = \frac{1}{\ell_0}\left(\frac{q_{j+1} - q_j}{\ell_0} - \frac{q_j - q_{j-1}}{\ell_0}\right)\,. \tag{10.87}$$

We write $x = j\ell_0$ and $q_j(t) = q(x,t)$. Then the limit of the right side is

$$\begin{aligned} &\lim_{\ell_0\to 0} \frac{1}{\ell_0}\left(\frac{q(x+\ell_0,t)-q(x,t)}{\ell_0} - \frac{q(x,t)-q(x-\ell_0,t)}{\ell_0}\right) \\ &= \lim_{\ell_0\to 0} \frac{1}{\ell_0}\left(\left.\frac{\partial q(x,t)}{\partial x}\right|_x - \left.\frac{\partial q(x,t)}{\partial x}\right|_{x-\ell_0}\right) = \frac{\partial^2 q(x,t)}{\partial x^2}\,. \end{aligned} \tag{10.88}$$

The factor $m/k\ell_0^2 \equiv 1/v^2$ remains constant in this limit. We obtain the *wave equation*

$$\frac{1}{v^2}\frac{\partial^2 q(x,t)}{\partial t^2} = \frac{\partial^2 q(x,t)}{\partial x^2}\,. \tag{10.89}$$

10.4 Solutions of the wave equation

We will only consider "standing-wave" (or stationary) solutions of the wave equation. (The general solution can be expressed as a superposition of the stationary solutions.)

10.4.1 The oscillating string

We seek a solution of the form $q(x,t) = g(x)h(t)$. Inserting this into the wave equation (10.89) gives

$$\frac{g(x)}{v^2}\frac{\partial^2 h(t)}{\partial t^2} = h(t)\frac{\partial^2 g(x)}{\partial x^2}\,. \tag{10.90}$$

Denoting time derivatives by dots and derivatives with respect to x by primes, we can write this as

$$\frac{1}{v^2}\frac{\ddot{h}}{h} = \frac{g''}{g}\,. \tag{10.91}$$

Since the left side depends only on t and the right side only on x, both sides must be equal to some constant $-k^2$. It follows that h and g satisfy the equations

$$\ddot{h} = -k^2v^2h\,, \qquad g'' = -k^2 g\,, \tag{10.92}$$

with solutions

$$h(t) = h_1\cos(kvt) + h_2\sin(kvt)\,, \tag{10.93}$$

$$g(x) = g_1\cos(kx) + g_2\sin(kx)\,. \tag{10.94}$$

If both ends of the string (length ℓ) are fixed, then the boundary conditions are

$$g(0) = g(\ell) = 0\,. \tag{10.95}$$

This gives

$$g_1 = 0\,, \qquad k = \frac{r\pi}{\ell} \equiv k_r\,, \tag{10.96}$$

for $r = 1, 2, \ldots$. The allowed frequencies are then

$$\omega_r = k_r v = \frac{r\pi v}{\ell}\,. \tag{10.97}$$

Thus, the partial solutions are *standing waves*

$$q_r(x,t) = \sin(k_r x)\big[h_{1,r}\cos(\omega_r t) + h_{2,r}\sin(\omega_r t)\big]\,, \tag{10.98}$$

while the general solution is given by the superposition

$$q(x,t) = \sum_r^{\infty} \sin(k_r x)\big[h_{1,r}\cos(\omega_r t) + h_{2,r}\sin(\omega_r t)\big]\,. \tag{10.99}$$

We can use the orthogonality relations of the trigonometric functions to express the initial conditions as a Fourier series. Eq. (10.99) tells us that at $t = 0$

$$q(x,0) = \sum_r^{\infty} h_{1,r}\sin(k_r x)\,, \tag{10.100}$$

$$\dot{q}(x,0) = \sum_r^{\infty} \omega_r h_{2,r}\sin(k_r x)\,. \tag{10.101}$$

Multiplying Eq. (10.100) by $\sin(k_s x)$ and integrating over x, we obtain

$$h_{1,r} = \frac{2}{\ell}\int_0^{\ell} q(x,0)\sin(k_r x)dx\,. \tag{10.102}$$

Here, we used

$$\int_0^{\ell} dx \sin\Big(\frac{r\pi x}{\ell}\Big)\sin\Big(\frac{s\pi x}{\ell}\Big) = \frac{\ell}{2}\delta_{rs}\,. \tag{10.103}$$

Similarly, we find

$$h_{2,r} = \frac{2}{\omega_r \ell}\int_0^{\ell} \dot{q}(x,0)\sin(k_r x)dx\,. \tag{10.104}$$

Example 10.3: Guitar string. *The initial condition for a guitar string that is plucked in the center of the string is*

$$q(x,0) = \begin{cases} 2A\frac{x}{\ell} & 0 \le x \le \frac{\ell}{2}\,, \\ 2A\frac{\ell - x}{\ell} & \frac{\ell}{2} \le x \le \ell\,, \end{cases} \tag{10.105}$$

and $\dot{q}(x,0) = 0$. *We find* $h_{2,r} = 0$ *for all* r, *and*

$$h_{1,r} = \frac{4A}{\ell^2}\int_0^{\ell/2} x\sin(k_r x)dx + \frac{4A}{\ell^2}\int_{\ell/2}^{\ell} (\ell - x)\sin(k_r x)dx\,. \tag{10.106}$$

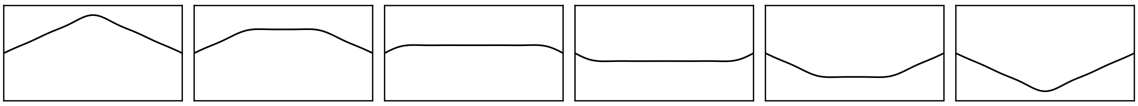

Figure 30: *Plot of the sum of the first ten Fourier modes of the guitar string, at different subsequent times.*

We have

$$\begin{aligned}\int_0^{\ell/2} x\sin(k_r x)dx &= -\frac{1}{k_r}x\cos(k_r x)\Big|_0^{\ell/2} + \frac{1}{k_r}\int_0^{\ell/2}\cos(k_r x)dx \\ &= -\frac{\ell}{2k_r}\cos(r\pi/2) + \frac{1}{k_r^2}\sin(k_r x)\Big|_0^{\ell/2} = -\frac{\ell}{2k_r}\cos(r\pi/2) + \frac{1}{k_r^2}\sin(r\pi/2)\,,\end{aligned} \tag{10.107}$$

as well as

$$\begin{aligned}\int_{\ell/2}^{\ell} (\ell - x)\sin(k_r x)dx &= -\frac{1}{k_r}(\ell - x)\cos(k_r x)\Big|_{\ell/2}^{\ell} - \frac{1}{k_r}\int_{\ell/2}^{\ell}\cos(k_r x)dx \\ &= \frac{\ell}{2k_r}\cos(r\pi/2) - \frac{1}{k_r^2}\sin(k_r x)\Big|_{\ell/2}^{\ell} = \frac{\ell}{2k_r}\cos(r\pi/2) + \frac{1}{k_r^2}\sin(r\pi/2)\,,\end{aligned} \tag{10.108}$$

and so

$$h_{1,r} = \frac{8A}{r^2\pi^2}\sin\Big(\frac{r\pi}{2}\Big)\,. \tag{10.109}$$

Note that only the coefficients with odd r are non-zero. The complete solution is

$$q(x,t) = \frac{8A}{\pi^2}\sum_r^{\infty}\frac{1}{r^2}\sin\Big(\frac{r\pi}{2}\Big)\sin\Big(\frac{r\pi}{\ell}x\Big)\cos\Big(\frac{r\pi v}{\ell}t\Big)\,. \tag{10.110}$$

10.4.2 The oscillating membrane

The wave equation for transversal vibrations of a two-dimensional membrane is

$$\frac{1}{v^2}\frac{\partial^2 q(x,y,t)}{\partial t^2} = \frac{\partial^2 q(x,y,t)}{\partial x^2} + \frac{\partial^2 q(x,y,t)}{\partial y^2}\,. \tag{10.111}$$

Let us consider a rectangular membrane that is fixed at the boundaries, i.e. the boundary conditions are

$$q(0,y,t) = q(a,y,t) = q(x,0,t) = q(x,b,t) = 0\,, \tag{10.112}$$

for all t. We could again try to find a solution of the form $q(x,y,t) = g(x,y)h(t)$. However, for a rectangular membrane we can immediately write down the solution that satisfies the boundary conditions. The partial solutions are

$$q_{rs}(x,y,t) = \sin(k_{x,r}x)\sin(k_{y,s}y)(h_{1,rs}\cos(\omega_{rs}t) + h_{2,rs}\sin(\omega_{rs}t))\,, \tag{10.113}$$

with

$$k_{x,r} = \frac{r\pi}{a}\,, \qquad k_{y,r} = \frac{r\pi}{b}\,. \tag{10.114}$$

Inserting Eq. (10.115) into the wave equation (10.111), we find the frequencies

$$\omega_{rs}^2 = \pi^2 v^2 \left(\frac{r^2}{a^2} + \frac{s^2}{b^2}\right). \tag{10.115}$$

The general solution is

$$q(x,y,t) = \sum_{r,s=1}^{\infty} \sin(k_{x,r}x)\sin(k_{y,s}y)(h_{1,rs}\cos(\omega_{rs}t) + h_{2,rs}\sin(\omega_{rs}t))\,. \tag{10.116}$$

Again, the Fourier coefficients can be calculated using the orthogonality of the trigonometric functions:

$$h_{1,rs} = \frac{4}{ab}\int_0^a\int_0^b q(x,y,0)\sin(k_{x,r}x)\sin(k_{y,s}y)dxdy\,, \tag{10.117}$$

$$h_{2,rs} = \frac{4}{ab\omega_{rs}}\int_0^a\int_0^b \dot{q}(x,y,0)\sin(k_{x,r}x)\sin(k_{y,s}y)dxdy\,. \tag{10.118}$$

As opposed to the vibrating string, here there exist characteristic frequencies that are irrational multiples of the lowest frequency ω_{11}, so the general vibration is not periodic in time. Consider, for instance, a quadratic membrane ($a = b$). Here, $\omega_{11} = \pi v\sqrt{2}/a$, $\omega_{12} = \pi v\sqrt{5}/a$.

For a vibrating membrane it frequently happens that some of the characteristic fequencies coincide. We have $\omega_{r_1 s_1} = \omega_{r_2 s_2}$ for

$$\frac{r_1^2}{a^2} + \frac{s_1^2}{b^2} = \frac{r_2^2}{a^2} + \frac{s_2^2}{b^2}\,, \tag{10.119}$$

or

$$\frac{r_1^2 - r_2^2}{s_1^2 - s_2^2} = \frac{a^2}{b^2}\,. \tag{10.120}$$

Degenerate frequencies can only occur if the ratio a^2/b^2 is rational.

The partial solutions (10.115) have *nodal lines* that are always at rest. These nodal lines are parallel to the edges, $x = 0, a/r, 2a/r, \ldots, a$ and $y = 0, b/s, 2b/s, \ldots, b$. If the corresponding frequency is degenerate, there are even more nodal lines. For instance, consider the modes q_{12} and q_{21} of a quadratic membrane, with frequency $\omega_{12} = \omega_{21} \equiv \omega$. If $h_{1,12} = \lambda h_{1,21}$ and $h_{2,12} = \lambda h_{2,21}$, we have

$$q(x,y,t) = \left[\lambda\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi y}{a}\right) + \sin\left(\frac{2\pi x}{a}\right)\sin\left(\frac{\pi y}{a}\right)\right](h_{1,21}\cos(\omega t) + h_{2,21}\sin(\omega t))\,. \tag{10.121}$$

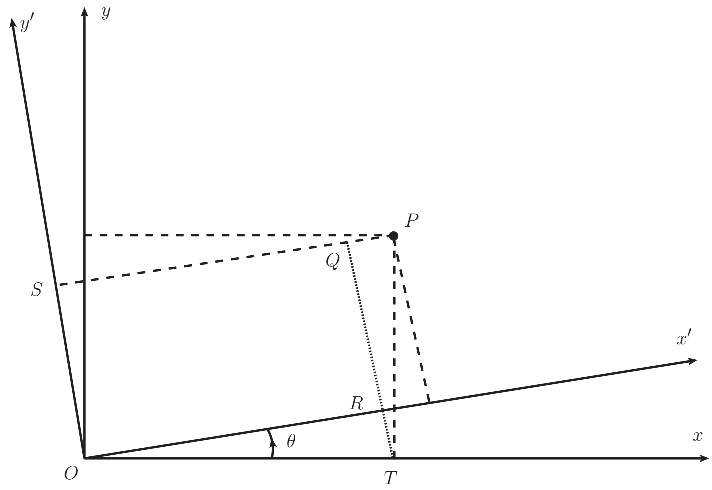

Figure 31: *Rotation of a two-dimensional coordinate system.*

Setting the angle bracket to zero gives the condition

$$\lambda \cos\left(\frac{\pi y}{a}\right) + \cos\left(\frac{\pi x}{a}\right) = 0\,. \tag{10.122}$$

The solutions are diagonal lines for $\lambda = \pm 1$, or otherwise transcendental curves that pass through the center of the membrane.

A Two-dimensional rotations

The relations between the coordinates of a point P in two coordinates systems that are related via a rotation by the angle θ about the origin can be read off Fig. 31. We have $\overline{SQ} = x\cos\theta$, $\overline{QP} = y\sin\theta$, as well as $\overline{QT} = y\cos\theta$, $\overline{RT} = x\sin\theta$, and so

$$x' = x\cos\theta + y\sin\theta\,, \tag{A.1}$$

$$y' = y\cos\theta - x\sin\theta\,. \tag{A.2}$$

B Line integrals

Line integrals are defined as follows. First, consider the integral of a scalar function $f = f(\boldsymbol{r}) = f(x_1, x_2, x_3)$ along a curve Γ from $\boldsymbol{r}_1$ to $\boldsymbol{r}_2$ (Fig. 32, left panel). By the line integral we mean the limit of the sum over the product of small segments of the curve, Δs_i, and the corresponding function values, $f_i = f(s_i)$:

$$\int_\Gamma f\,ds = \lim_{\max(\Delta s_i)\to 0} \sum_i f_i \Delta s_i\,. \tag{B.1}$$

For a vector field $\boldsymbol{A}(\boldsymbol{r})$, we integrate the component of $\boldsymbol{A}$ along the infinitesimal line element represented by the displacement vector $d\boldsymbol{s}$, i.e. $\boldsymbol{A}\cdot d\boldsymbol{s}$, along the curve. If we decompose $\boldsymbol{A}$

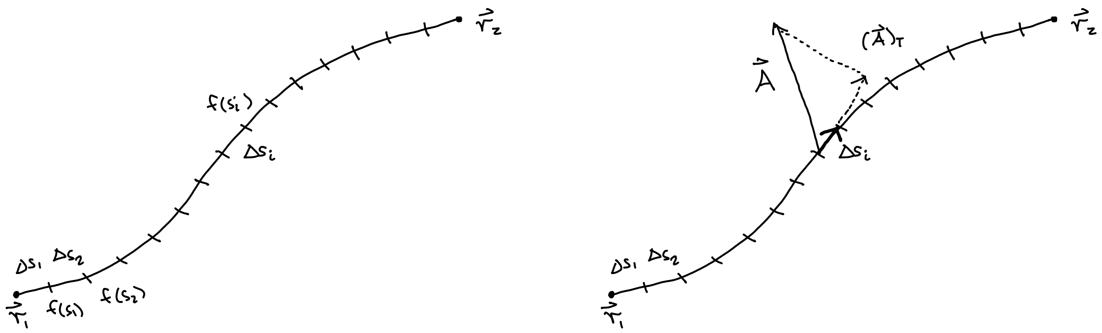

Figure 32: *Integral of a scalar function (left panel) and a vector field (right panel) along a curve.*

and $d\boldsymbol{s}$ into their components, we can write this as

$$\int_\Gamma \boldsymbol{A} \cdot d\boldsymbol{s} = \int_\Gamma \sum_{k=1}^{3} A_k dx_k \,. \tag{B.2}$$

C Cylinder coordinates

Relation to Cartesian coordinates:

$$x_1 = r\cos\phi\,, \quad x_2 = r\sin\phi\,, \quad x_3 = z\,, \tag{C.1}$$

or, vice versa,

$$r = \sqrt{x_1^2 + x_2^2}\,, \quad \phi = \arctan\left(\frac{x_2}{x_1}\right), \quad z = x_3\,. \tag{C.2}$$

We write an infinitesimal line element as

$$d\boldsymbol{s} = dr\,\boldsymbol{e}_r + rd\phi\,\boldsymbol{e}_\phi + dz\,\boldsymbol{e}_z\,, \tag{C.3}$$

and so

$$\dot{\boldsymbol{s}} = \dot{r}\,\boldsymbol{e}_r + r\dot{\phi}\,\boldsymbol{e}_\phi + \dot{z}\,\boldsymbol{e}_z\,, \tag{C.4}$$

and

$$v^2 = \dot{r}^2 + r^2\dot{\phi}^2 + \dot{z}^2 \tag{C.5}$$

D Rotating frame using Lagrangian mechanics

Inserting Eq. (8.14) into the Lagrangian (8.11) gives

$$L = \frac{m\boldsymbol{v}^2}{2} + m\boldsymbol{v}\cdot(\boldsymbol{\omega}\times\boldsymbol{r}) + \frac{m}{2}(\boldsymbol{\omega}\times\boldsymbol{r})^2 - m\boldsymbol{r}\cdot\ddot{\boldsymbol{R}} - U\,. \tag{D.1}$$

To obtain the equations of motion, we calculate

$$\frac{\partial L}{\partial \boldsymbol{r}} = m(\boldsymbol{v}\times\boldsymbol{\omega}) + m((\boldsymbol{\omega}\times\boldsymbol{r})\times\boldsymbol{\omega}) - m\ddot{\boldsymbol{R}} - \frac{\partial U}{\partial \boldsymbol{r}}\,, \tag{D.2}$$

$$\frac{\partial L}{\partial \boldsymbol{v}} = m\boldsymbol{v} + m(\boldsymbol{\omega}\times\boldsymbol{r})\,, \tag{D.3}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \boldsymbol{v}} = m\dot{\boldsymbol{v}} + m(\dot{\boldsymbol{\omega}}\times\boldsymbol{r}) + m(\boldsymbol{\omega}\times\boldsymbol{v})\,. \tag{D.4}$$

Here we used that $(\boldsymbol{\omega}\times\boldsymbol{r})\cdot(\boldsymbol{\omega}\times\boldsymbol{r}) = \boldsymbol{r}\cdot((\boldsymbol{\omega}\times\boldsymbol{r})\times\boldsymbol{\omega})$, so

$$\frac{d}{d\boldsymbol{r}}(\boldsymbol{\omega}\times\boldsymbol{r})^2 = ((\boldsymbol{\omega}\times\boldsymbol{r})\times\boldsymbol{\omega}) + \boldsymbol{r}\cdot\frac{d}{d\boldsymbol{r}}((\boldsymbol{\omega}\times\boldsymbol{r})\times\boldsymbol{\omega}) = 2((\boldsymbol{\omega}\times\boldsymbol{r})\times\boldsymbol{\omega})\,. \tag{D.5}$$

The Lagrangian equations are then

$$m\dot{\boldsymbol{v}} = -m\dot{\boldsymbol{\omega}}\times\boldsymbol{r} - m\,\boldsymbol{\omega}\times\big(\boldsymbol{\omega}\times\boldsymbol{r}\big) - 2m\boldsymbol{\omega}\times\boldsymbol{v} - \frac{\partial U}{\partial \boldsymbol{r}} - m\ddot{\boldsymbol{R}}\,, \tag{D.6}$$

in agreement with Eq. (8.18).

References

[1] Particle Data Group collaboration, P. Zyla et al., *Review of Particle Physics*, *PTEP* **2020** (2020) 083C01.