BASIC CONTINUUM MECHANICS

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Preface

The following is a basic course in continuum mechanics, for the fourth, undergraduate year at KTH.

The experience over the years is that, first of all, vector analysis has to be trained. The next difficulty is that of Cartesian tensor notation. For most students it takes quite some effort to learn the Einstein summation convention, to see the deep difference between dummy indices and free indices. It is necessary to master these mathematical methods. Then it is possible to apply them to continuum mechanics. Otherwise, all the physics and mechanics will be hidden in formulas containing a lot of symbols and indices. But once one has mastered this technique, it turns out to be very powerful.

The aim of the course is to integrate into a coherent whole the knowledge the student already has of strength of materials and fluid mechanics. This, then will create a much more stable basis for continued work or study in the field of mechanics of continua, be they solid or fluid.

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Lars Söderholm

Chapter 1

Vectors and second order tensors

1 The Einstein summation convention

You are probably most familiar with writing the components of a vector **a** as

$$(a_1, a_2, a_3).$$

By the way, we shall in this course all the time use an orthonormal basis. - But sometimes it is convenient to think of a vector as a column matrix

$$[\mathbf{a}] = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}.$$

The scalar product of two vectors $\mathbf{a}, \mathbf{b}$ is then written in matrix form

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

So the dot of the scalar product changes the column matrix to the left of the dot into a row matrix.

Dummy indices

Sums of this kind, over a pair of equal indices occur so frequently that one often omits the summation sign and writes

$$a_i b_i = \mathbf{\Sigma}_{i=1}^{3} a_i b_i = a_1 b_1 + a_2 b_2 + a_3 b_3 = \mathbf{a} \cdot \mathbf{b}.$$

So, *an index which occurs twice in an expression is summed over.* Such an index is called *dummy.* The rule is called the *Einstein summation convention.* So, now we write simply

$$\mathbf{a} \cdot \mathbf{b} = a_i b_i. \tag{1.1}$$

It is clear that *we can use any summation index.* So

$$a_i b_i = a_1 b_1 + a_2 b_2 + a_3 b_3 = a_j b_j = \mathbf{a} \cdot \mathbf{b}.$$

Never more than two equal indices

Some times we have to change the names of dummy indices. Let us as an example take a look at the square of the scalar product $\mathbf{a} \cdot \mathbf{b}$. Now we have to be careful.

$$(\mathbf{a} \cdot \mathbf{b})^2 = (a_i b_i)^2.$$

We could feel tempted to write this as $a_i b_i a_i b_i$. But this does not work. Let us take a close look at what is happening.

$$\begin{aligned}
(\mathbf{a} \cdot \mathbf{b})^2 &= (a_1 b_1 + a_2 b_2 + a_3 b_3)(a_1 b_1 + a_2 b_2 + a_3 b_3) \\
&= a_1 b_1 a_1 b_1 + a_1 b_1 a_2 b_2 + a_1 b_1 a_3 b_3 + a_3 b_3 a_3 b_3 \\
&\quad + a_2 b_2 a_1 b_1 + a_2 b_2 a_2 b_2 + a_2 b_2 a_3 b_3 \\
&\quad + a_3 b_3 a_1 b_1 + a_3 b_3 a_2 b_2 + a_3 b_3 a_3 b_3 \\
&= \Sigma_{i=1}^{3} \Sigma_{j=1}^{3} a_i b_i a_j b_j = a_i b_i a_j b_j.
\end{aligned}$$

There are in total nine terms in the expression. *We had to change name of one pair of dummies* here and sum over i and j independently. - *If you find more than two equal indices in an expression, then there is simply something wrong!*

Free indices

The i-component of the vector $\mathbf{a}$ we write a_i. This index is a *free index.* It is not summed over. It is very important to distinguish between free indices and dummy indices.

Distinguishing between dummy indices and free indices

Now we take three vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and take a look at the expression

$$a_i b_j c_i.$$

Here, j occurs once only, so it is a free index. But i occurs twice. i must be a dummy index and is summed over. So this expression has one free index. The order of the factors can be changed arbitrarily: writing out the expression explicitly, we have

$$\begin{aligned} a_i b_j c_i &= a_i c_i b_j \\ &= (a_i c_i) b_j = (\mathbf{a} \cdot \mathbf{c}) b_j. \end{aligned}$$

It is the j-component of the vector

$$(\mathbf{a} \cdot \mathbf{c})\mathbf{b}.$$

In other words,

$$[(\mathbf{a} \cdot \mathbf{c})\mathbf{b}]_j = a_i b_j c_i.$$

If you are not convinced, let us take it in a little more detail. The is are summed over, but not the j

$$\begin{aligned} a_i b_j c_i &= a_1 b_j c_1 + a_2 b_j c_2 + a_3 b_j c_3 \\ &= (a_1 c_1 + a_2 c_2 + a_3 c_3) b_j = (a_i c_i) b_j = (\mathbf{a} \cdot \mathbf{c}) b_j. \end{aligned}$$

Remember that we can change the dummy indices if we like. But we cannot change free indices

$$[(\mathbf{a} \cdot \mathbf{c})\mathbf{b}]_j = a_i b_j c_i = a_k b_j c_k.$$

It is also very important that we are *not* allowed to name the dummy indices j, as j is already used as a free index.

Exercise 1.1 *We define the expressions*

$$\begin{aligned} a_i &= A_{ij} c_j \\ b_i &= B_{ij} d_j \end{aligned}$$

Write $a_i b_i$ in the form

$$a_i b_i = E_{kl} c_k d_l.$$

and find E_{kl}
Answer: $E_{kl} = A_{ik} B_{il}$

Exercise 1.2 *Show that*

$$E_{kl} = A_{ik}B_{il}$$

means that $\mathbf{E} = \mathbf{A}^T\mathbf{B}$.

Exercise 1.3 *Also write down the components of* $\mathbf{F} = \mathbf{A}\mathbf{B}^T$

Answer: $F_{ij} = A_{ik}B_{jk}$.

2 Tensors of second order

Just as a vector can be thought of as column matrix, tensors can be defined as square matrices. As an example

$$[\mathbf{T}] = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}. \tag{1.2}$$

But you have probably also met a more geometrical definition of a vector as an object which goes from one point to another point in space. This definition is a bit more satisfactory, as it does not use a basis.

Now we take an arbitrary vector $\mathbf{a}$. We remember that the vector can be thought of as a column matrix

$$[\mathbf{a}] = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}.$$

We can now form the matrix product $[\mathbf{T}]\,[\mathbf{a}]$.

$$[\mathbf{T}]\,[\mathbf{a}] = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}.$$

Let us calculate it according to the rules of matrix multiplication.

$$[\mathbf{T}]\,[\mathbf{a}] = \begin{bmatrix} T_{11}a_1 + T_{12}a_2 + T_{13}a_3 \\ T_{21}a_1 + T_{22}a_2 + T_{23}a_3 \\ T_{31}a_1 + T_{32}a_2 + T_{33}a_3 \end{bmatrix} = \begin{bmatrix} T_{1j}a_j \\ T_{2j}a_j \\ T_{3j}a_j \end{bmatrix}.$$

The result is a column matrix, or a vector, which we write $\mathbf{Ta}$.

We see that the components of $\mathbf{Ta}$ are given by

$$(\mathbf{Ta})_i = T_{ij}a_j.$$

We can then *think of a tensor as a linear operator*, which takes an arbitrary vector $\mathbf{a}$ into a new vector $\mathbf{Ta}$. The matrix of this linear operator or components of the tensor are given by (1.2). If we think of a tensor as a linear operator taking vectors into vectors, we don't need a set of basis vectors for the definition of a tensor.

Unit tensor

The simplest example of a tensor is perhaps the unit tensor

$$[\mathbf{1}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{1.3}$$

Its components are 1 on the diagonal and 0 off the diagonal

$$(\mathbf{1})_{ij} = \delta_{ij}.$$

Here, the Kronecker delta is given by

$$\delta_{ij} = \left\{ \begin{array}{l} 1 \text{ } \mathit{if} \text{ } i = j \\ 0 \text{ } \mathit{if} \text{ } i \neq j \end{array} \right. .$$

Exercise 2.1 *Let us check in component notation how the unit tensor acts on a vector* $\mathbf{a}$. *We have*

$$\begin{aligned} (\mathbf{1a})_i &= (\mathbf{1})_{ij}a_j = \delta_{ij}a_j \\ &= \delta_{i1}a_1 + \delta_{i2}a_2 + \delta_{i3}a_3. \end{aligned}$$

In this sum only one of the terms is non-zero. It is the one where the index summed over also takes the value i. *The Kronecker delta then has the value* 1 *and the result is*

$$\delta_{i1}a_1 + \delta_{i2}a_2 + \delta_{i3}a_3 = a_i.$$

If this seems strange, try in the first formula to give i *the value* 1. *Then it goes like this*

$$\begin{aligned} (\mathbf{1a})_1 &= (\mathbf{1})_{1j}a_j = \delta_{1j}a_j = \delta_{11}a_1 + \delta_{12}a_2 + \delta_{13}a_3 \\ &= 1a_1 + 0a_2 + 0a_3 = a_1. \end{aligned}$$

Tensor multiplication = matrix multiplication

If we have two tensors $\mathbf{T}, \mathbf{U}$ we can multiply them. If

$$[\mathbf{U}] = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ U_{21} & U_{22} & U_{23} \\ U_{31} & U_{32} & U_{33} \end{bmatrix}, \tag{1.4}$$

the product $\mathbf{TU}$ is given by

$$\begin{aligned} [\mathbf{T}]\,[\mathbf{U}] &= \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ U_{21} & U_{22} & U_{23} \\ U_{31} & U_{32} & U_{33} \end{bmatrix} \\ &= \begin{bmatrix} T_{11}U_{11} + T_{12}U_{21} + T_{13}U_{31} & T_{11}U_{12} + T_{12}U_{22} + T_{13}U_{32} & ... \\ T_{21}U_{11} + T_{22}U_{21} + T_{23}U_{31} & T_{21}U_{12} + T_{22}U_{22} + T_{23}U_{32} & ... \\ ... & ... & ... \end{bmatrix}. \end{aligned}$$

Or

$$[\mathbf{T}]\,[\mathbf{U}] == \begin{bmatrix} T_{1j}U_{j1} & T_{1j}U_{j2} & T_{1j}U_{j3} \\ T_{2j}U_{j1} & T_{2j}U_{j2} & T_{2j}U_{j3} \\ T_{3j}U_{j1} & T_{3j}U_{j2} & T_{3j}U_{j3} \end{bmatrix}. \tag{1.5}$$

Or, even more compactly,

$$(\mathbf{TU})_{ij} = T_{ik}U_{kj}. \tag{1.6}$$

Note that we changed the summation index from j to k as j is used as a free index.

Trace of a tensor

The trace of a second order tensor is

$$tr\mathbf{A} = A_{ii}. \tag{1.7}$$

Tensor product of two vectors

In particular, if we have two vectors $\mathbf{a}, \mathbf{b}$, we can form a tensor with ij-components

$$a_i b_j.$$

It is simply called the tensor product (or dyadic product) of the two vectors. There are two notations for this in the literature. Either $\mathbf{a} \otimes \mathbf{b}$ or simply $\mathbf{ab}$ (note that there is no dot). In other words

$$(\mathbf{a} \otimes \mathbf{b})_{ij} = (\mathbf{ab})_{ij} = a_i b_j. \tag{1.8}$$

We can also form the tensor with ij-components

$$(\mathbf{b} \otimes \mathbf{a})_{ij} = (\mathbf{ba})_{ij} = b_i a_j = a_j b_i.$$

Exercise 2.2 *Show that*

$$(\mathbf{a} \otimes \mathbf{b})^T = \mathbf{b} \otimes \mathbf{a}.$$

Exercise 2.3 *Show that the trace of* $\mathbf{a} \otimes \mathbf{b}$ *is the scalar product* $\mathbf{a} \cdot \mathbf{b}$ *of the two vectors.*

This also gives us a possibility to define trace directly. The trace of a tensor is a linear scalar function of the tensor. When the tensor is a tensor product of two vectors, the trace is simply the scalar product of the vectors.

$$tr(\mathbf{a} \otimes \mathbf{b}) = \mathbf{a} \cdot \mathbf{b}. \tag{1.9}$$

Matrix multiplication and the summation convention

If we have a second order tensor T_{ij} and a vector a_i and consider

$$T_{ij} a_j$$

we can think of this in terms of matrix multiplication. If we write the tensor $\mathbf{T}$ and think of it as the matrix

$$[\mathbf{T}] = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}$$

and the vector as the column matrix

$$[\mathbf{a}] = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}.$$

$T_{ij} a_j$ is the i-component of the vector $\mathbf{Ta}$.

$$\begin{aligned} [\mathbf{Ta}] &= \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \\ &= \begin{bmatrix} T_{11}a_1 + T_{12}a_2 + T_{13}a_3 \\ T_{21}a_1 + T_{22}a_2 + T_{23}a_3 \\ T_{31}a_1 + T_{32}a_2 + T_{33}a_3 \end{bmatrix} = \begin{bmatrix} T_{1j}a_j \\ T_{2j}a_j \\ T_{3j}a_j \end{bmatrix}. \end{aligned}$$

In other words,

$$(\mathbf{Ta})_i = T_{ij}a_j.$$

Another way of defining a tensor $\mathbf{T}$ is that it is given by its action on an arbitrary vector $\mathbf{a}$, and this action is $\mathbf{Ta}$. So a tensor is nothing but a linear operator.

Transpose, symmetric and antisymmetric tensors

Transposing a tensor means transposing its matrix

$$(\mathbf{T}^T)_{ij} = (\mathbf{T})_{ji} = T_{ji}. \tag{1.10}$$

A symmetric tensor equals its transpose. A symmetric tensor has 6 independent components.

Let us now consider a_iT_{ij} . We note that there is one free index, in this case j. So clearly this is still a vector, the j-component of a vector.

$$\begin{aligned} a_iT_{ij} &= a_1T_{1j} + a_2T_{2j} + a_3T_{3j} \\ &= T_{1j}a_1 + T_{2j}a_2 + T_{3j}a_3 = T_{ij}a_i = (\mathbf{T}^T\mathbf{a})_j. \end{aligned}$$

Here,

$$\begin{aligned} \left[\mathbf{T}^T\mathbf{a}\right] &= \begin{bmatrix} T_{11} & T_{21} & T_{31} \\ T_{12} & T_{22} & T_{32} \\ T_{13} & T_{23} & T_{33} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \\ &= \begin{bmatrix} T_{11}a_1 + T_{21}a_2 + T_{31}a_3 \\ T_{12}a_1 + T_{22}a_2 + T_{32}a_3 \\ T_{13}a_1 + T_{23}a_2 + T_{33}a_3 \end{bmatrix} = \begin{bmatrix} T_{i1}a_i \\ T_{i2}a_i \\ T_{i3}a_i \end{bmatrix}. \end{aligned}$$

Exercise 2.4 $\mathbf{a}, \mathbf{b}$ *are two vectors and* $\mathbf{T}$ *is a tensor. What are the free indices of the expression* $a_iT_{ij}b_j$*?*

Answer: there is no free index, so this is a scalar.

Exercise 2.5 *Show that*

$$a_iT_{ij}b_j = \mathbf{a}\cdot(\mathbf{Tb}) = \mathbf{a}\cdot\mathbf{Tb} \tag{1.11}$$

If we remember first to calculate the vector $\mathbf{Tb}$*, we do not need the paranthesis.*

If in the foregoing formula, we take $\mathbf{a}$ to be the basis vector $\mathbf{e}_i$ and $\mathbf{b}$ to be the basis vector $\mathbf{e}_j$, we simply find the ij-matrix element of $\mathbf{T}$. To find the components of the basis vectors we use (2.19)

$$\mathbf{e}_i \cdot \mathbf{T}\mathbf{e}_j = (\mathbf{e}_i)_k T_{kl} (\mathbf{e}_j)_l = \delta_{ik} T_{kl} \delta_{jl} = T_{ij}.$$

Note that it was essential to change the dummy suffices here.

Exercise 2.6 *Show that (change dummy indices and note that* $(\mathbf{T}^T)_{ij} = (\mathbf{T})_{ji} = T_{ji}$*)*

$$\mathbf{a} \cdot \mathbf{T}\mathbf{b} = a_i T_{ij} b_j = b_j T_{ij} a_i = b_i T_{ji} a_j = \mathbf{b} \cdot \mathbf{T}^T \mathbf{a} \tag{1.12}$$

Eigenvectors and eigenvalues

An eigenvector $\mathbf{c}$ of a tensor $\mathbf{T}$ is defined by the equation

$$\mathbf{T}\mathbf{c} = \lambda \mathbf{c}. \tag{1.13}$$

λ is the eigenvalue. In components,

$$T_{ij} c_j = \lambda c_i. \tag{1.14}$$

We can rewrite this equation as

$$(T_{ij} - \lambda \delta_{ij}) c_j = 0.$$

Or as

$$(\mathbf{T} - \lambda \mathbf{1})\mathbf{c} = \mathbf{0}.$$

The eigenvalues are then found from

$$\det(\mathbf{T} - \lambda \mathbf{1}) = 0. \tag{1.15}$$

From linear algebra we know that a *symmetric* tensor always has three eigenvectors and that they are (or can be chosen) orthonormal. If we choose the eigenvectors as a basis, the matrix of the tensor is diagonal and the elements in the diagonal are the eigenvalues.

If we calculate the determinant in the preceding equation we find

$$\lambda^3 - I_1 \lambda^2 + I_2 \lambda - I_3 = 0, \tag{1.16}$$

where the principal invariants ot the tensor $\mathbf{T}$ are

$$\begin{aligned} I_1 &= tr(\mathbf{T}), \\ I_2 &= \frac{1}{2}[(tr(\mathbf{T}))^2 - tr(\mathbf{T}^2)], \\ I_3 &= \det(\mathbf{T}). \end{aligned} \tag{1.17}$$

Exercise 2.7 *Prove (1.17). Hint: Use the eigenvectors of* $\mathbf{T}$ *as a basis and calculate (1.15) to find the expressions for* I_1, I_2, I_3 *in terms of the eigenvalues of* $\mathbf{T}$. *Check that they agree with (1.17).*

An antisymmetric tensor satisfies

$$\begin{aligned} \mathbf{A}^T &= -\mathbf{A}; \\ A_{ij} &= -A_{ji}. \end{aligned}$$

An antisymmetric tensor has three independent components.

Any tensor can be written as a sum of a symmetric part and antisymmetric part

$$C_{ij} = C_{(ij)} + C_{[ij]}. \tag{1.18}$$

Here, the symmetric part is

$$C_{(ij)} = \frac{1}{2}(C_{ij} + C_{ji}). \tag{1.19}$$

and the antisymmetric part

$$C_{[ij]} = \frac{1}{2}(C_{ij} - C_{ji}). \tag{1.20}$$

The deviator of a symmetric tensor

For a symmetric tensor, we can write

$$S_{ij} = (S_{ij} - a\delta_{ij}) + a\delta_{ij}.$$

So far, a is arbitrary. But now we choose a so that the trace of the remaining part vanishes:

$$0 = S_{ii} - a\delta_{ii} = S_{ii} - 3a.$$

We obtain

$$S_{ij} = S_{<ij>} + \frac{1}{3} S_{kk} \delta_{ij}. \tag{1.21}$$

Here the (traceless) tensor

$$S_{<ij>} = S_{ij} - \frac{1}{3} S_{kk} \delta_{ij}. \tag{1.22}$$

is called the *deviator of* **S**. Let us explicitly write down that its trace vanishes

$$S_{<ii>} = 0. \tag{1.23}$$

$\frac{1}{3} S_{kk} \delta_{ij}$ is called the *spherical part* of the tensor **S**.

A symmetrical tensor has 6 independent components. The trace is one number, so the deviator has 5 independent components. The trace and the deviator of a tensor are independent of each other.

Exercise 2.8 *E_{ij} is a symmetric tensor. The tensor T_{ij} is given by*

$$T_{ij} = 2\mu E_{ij} + \lambda E_{kk} \delta_{ij}. \tag{1.24}$$

Show that

$$\frac{1}{2} T_{ij} E_{ij} = \mu E_{ij} E_{ij} + \frac{\lambda}{2} E_{kk} E_{ll} = \mu E_{ij} E_{ij} + \frac{\lambda}{2} (E_{kk})^2. \tag{1.25}$$

2.1 Orthonormal tensors

A tensor **R** is said to be *orthonormal* if it preserves scalar products

$$(\mathbf{Ra}) \cdot (\mathbf{Rb}) = \mathbf{a} \cdot \mathbf{b}$$

for any vectors $\mathbf{a}, \mathbf{b}$, or

$$\mathbf{R}^T \mathbf{R} = \mathbf{R} \mathbf{R}^T = \mathbf{1}. \tag{1.26}$$

Orthonormal tensors are the ones mapping cubes to cubes.

We find from the preceding equation that $(\det \mathbf{R})^2 = 1$. Rotations are orthonormal tensors that can be continously deformed to the unit tensor. This means that they must have $\det \mathbf{R} = 1$.

Exercise. Prove (1.26).

Exercise. Show that the only possible eigenvalues of an orthonormal tensor are 1 and -1. *Hint* Take the square of the eigenvalue equation

$$\mathbf{Rn} = r\mathbf{n} \ ,$$

using (1.26).

A 3×3 tensor has always at least one eigenvector. (The equation $\det(\mathbf{R} - r\mathbf{1}) = 0$ has three roots. Two of them are complex conjugated and the third one is real. Hence, there is at least one real eigenvalue and eigenvector.)

Exercise. Suppose $\mathbf{R}$ has an eigenvector with eigenvalue $+1$ and determinant 1 . Show that $\mathbf{R}$ is a rotation around $\mathbf{n}$. Hint. Show that the plane orthogonal to $\mathbf{n}$ is mapped onto itself. Then introduce a positively oriented basis with $\mathbf{e}_3 = \mathbf{n}$. Use the orthonormality of the resulting 2×2 matrix to write $R_{11} = \cos\vartheta; R_{12} = -\sin\vartheta$ to find that $R_{21} = \pm\sin\vartheta$; $R_{22} = \pm\cos\vartheta$. Determinant 1 gives the upper signs.

$$[\mathbf{R}] = \begin{bmatrix} \cos\vartheta & -\sin\vartheta & 0 \\ \sin\vartheta & \cos\vartheta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We can take one step further here. We write

$$\mathbf{R} = \cos\vartheta(\mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2) + \sin\vartheta(\mathbf{e}_2 \otimes \mathbf{e}_1 - \mathbf{e}_1 \otimes \mathbf{e}_2) + \mathbf{n} \otimes \mathbf{n}.$$

But

$$\mathbf{1} = \mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{n} \otimes \mathbf{n}$$

and

$$\mathbf{n}\times = \mathbf{e}_2 \otimes \mathbf{e}_1 - \mathbf{e}_1 \otimes \mathbf{e}_2$$

This formula can be checked by applying it to $\mathbf{e}_1, \mathbf{e}_2, \mathbf{n}$. Hence

$$\mathbf{R} = \cos\vartheta(\mathbf{1} - \mathbf{n} \otimes \mathbf{n}) + \sin\vartheta\mathbf{n}\times + \mathbf{n} \otimes \mathbf{n},$$

which only contains the axis of rotation and the angle ϑ of rotation.

Exercise. Show that if

$$\mathbf{R} = \mathbf{1} + \mathbf{A},$$

where $\mathbf{A}$ is infinitesimal, $\mathbf{R}$ is orthonormal if and only if $\mathbf{A}$ is antisymmetric.

Functions of symmetric tensors

Suppose we have symmetric tensor $\mathbf{S}$. If $f(x)$ is a function of a real variable we can define $f(\mathbf{S})$ as another symmetric tensor. It has the same eigenvectors $(\mathbf{e}_i)$ as $\mathbf{S}$ but the eigenvalues $f(S_i)$, where S_i are the eigenvalues of $\mathbf{S}$.

$$f(\mathbf{S})\mathbf{e}_i = f(S_i)\mathbf{e}_i. \tag{1.27}$$

Exercise. Show that for the function $f(x) = x^n$, where n is a natural number, $f(\mathbf{S}) = \mathbf{S}^n$.

Exercise. If $f(x) = 1/x$ then $f(\mathbf{S}) = \mathbf{S}^{-1}$, the inverse of $\mathbf{S}$.

For a symmetric positive semidefinite tensor $\mathbf{S}$ (a tensor with non-negative eigenvalues) we can in this way define $\sqrt{\mathbf{S}}$.

Exercise. Show that $\mathbf{T} = \sqrt{\mathbf{S}}$ is the only symmetric positive semidefinite solution of $\mathbf{T}^2 = \mathbf{S}$, if $\mathbf{S}$ is symmetric positive semidefinite.

Exercise . The matrix of a 2 dimensional tensor is

$$\begin{bmatrix} 26 & -10 \\ -10 & 26 \end{bmatrix}$$

Show that the tensor is positive definite and find its square root. *Answer*

$$\begin{bmatrix} 5 & -1 \\ -1 & 5 \end{bmatrix}.$$

3 Cauchy's polar decomposition

We will now show that any non-singular tensor can be written as a product of an orthonormal tensor and a positive definite symmetric tensor. Let us try to write the non-singular $\mathbf{F}$

$$\mathbf{F} = \mathbf{R}\mathbf{U}, \tag{1.28}$$

where $\mathbf{R}$ is orthonormal and $\mathbf{U}$ positive definite symmetric. We eliminate $\mathbf{R}$ through defining

$$\mathbf{C} = \mathbf{F}^T\mathbf{F} = \mathbf{U}^2. \tag{1.29}$$

For any non-zero vector $\mathbf{a}$ we have

$$\mathbf{a} \cdot \mathbf{Ca} = \mathbf{Fa} \cdot \mathbf{Fa} = |\mathbf{Fa}|^2 > 0 \tag{1.30}$$

as $\mathbf{F}$ is non-singular. This means that $\mathbf{C}$ is positive definite. We write its square root

$$\mathbf{U} = \sqrt{\mathbf{C}} = \sqrt{\mathbf{F}^T\mathbf{F}}, \tag{1.31}$$

which is also positive definite. $\mathbf{R}$ is given by (1.28).

$$\mathbf{R} = \mathbf{FU}^{-1},$$

so that

$$\mathbf{R}^T\mathbf{R} = \mathbf{U}^{-1}\mathbf{F}^T\mathbf{FU}^{-1} = \mathbf{U}^{-1}\mathbf{U}^2\mathbf{U}^{-1} = \mathbf{1}$$

We conclude that $\mathbf{R}$ is orthonormal.

We can just as well carry out the rotation first. In fact,

$$\mathbf{F} = \mathbf{RU} = \mathbf{RUR}^T\mathbf{R} = \mathbf{VR} \tag{1.32}$$

Here,

$$\mathbf{V} = \mathbf{RUR}^T \tag{1.33}$$

To find $\mathbf{V}$ we eliminate $\mathbf{R}$ from (1.32) and introduce $\mathbf{B}$

$$\mathbf{B} = \mathbf{FF}^T = \mathbf{V}^2. \tag{1.34}$$

In other words,

$$\mathbf{V} = \sqrt{\mathbf{B}} = \sqrt{\mathbf{FF}^T} \tag{1.35}$$

We can compare the decompositions (1.32) with the polar decomposition of a complex number

$$z = re^{i\vartheta}$$

$\mathbf{R}$ corresponds to $e^{i\vartheta}$ and $\mathbf{U}$ or $\mathbf{V}$ to r. As tensors do not commute ($\mathbf{AB} \neq \mathbf{BA}$ in general) there are two different polar decompositions for tensors but just one for complex numbers.

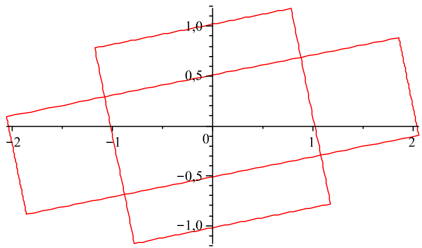


Figure 1.1: A symmetric tensor maps the cube into a rectangular box with parallel edges

If we have a positive definite symmetric tensor we can draw a cube from the eigenvectors of the tensor. The tensor maps the cube into a rectangular box with the same directions of the edges. For a general tensor there is also a special cube, which is mapped to a *rectangular* box. This means that the edges of the cube are mapped into orthogonal directions. - For $i \neq j$

$$0 = \mathbf{F}\mathbf{e}_i \cdot \mathbf{F}\mathbf{e}_j = \mathbf{e}_i \cdot \mathbf{F}^T\mathbf{F}\mathbf{e}_j = \mathbf{e}_i \cdot \mathbf{C}\mathbf{e}_j$$

Hence, $\mathbf{C}\mathbf{e}_i$ is orthogonal to the other basis vectors, i.e. parallel to $\mathbf{e}_i$. This means that $\mathbf{e}_i$ is an eigen vector of $\mathbf{C}$. We conclude that the preferred cube is given by the eigenvectors of $\mathbf{C}$ or of $\mathbf{U}$.The Cauchy polar decomposition $\mathbf{F} = \mathbf{R}\mathbf{U}$ can be pictured as

Exercise. The matrix of a tensor in 2 dimensions is

$$\frac{1}{4}\begin{bmatrix} -1+5\sqrt{3} & 5-\sqrt{3} \\ -5-\sqrt{3} & 1+5\sqrt{3} \end{bmatrix}$$

Show that the tensor is non-singular and calculate its Cauchy polar decompo-

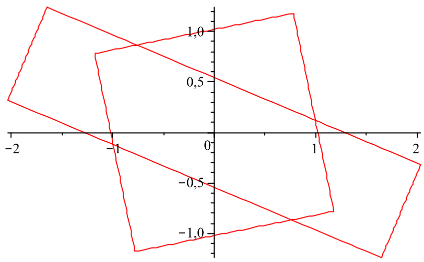


Figure 1.2: An arbitrary tensor maps the cube into a rectangular box with in general different directions

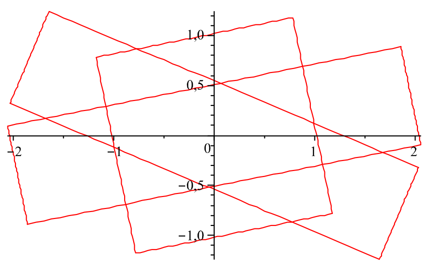


Figure 1.3: $\mathbf{F} = \mathbf{RU}$ is pictured as first a deformation of the cube into a rectangular box with parallel edges and then a rotation

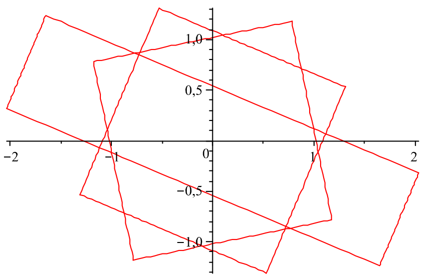


Figure 1.4: $\mathbf{F} = \mathbf{VR}$ is pictured as a rotation of the orginal cube followed by a deformation into a rectangular box that preserves the directions of the rotated cube.

sition, *i.e.* calculate $\mathbf{R}$, $\mathbf{U}$ and $\mathbf{V}$. *Answer*:

$$\begin{aligned}
[\mathbf{C}] &= \begin{bmatrix} 13/2 & -5/2 \\ -5/2 & 13/2 \end{bmatrix}, [\mathbf{U}] = \begin{bmatrix} 5/2 & -1/2 \\ -1/2 & 5/2 \end{bmatrix}, \\
[\mathbf{R}] &= \begin{bmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{bmatrix}, [\mathbf{V}] = \begin{bmatrix} -\sqrt{3}/4 + 5/2 & -1/4 \\ -1/4 & \sqrt{3}/4 + 5/2 \end{bmatrix}.
\end{aligned}$$

We obtain another pictorial realization of a positive definite symmetric second order tensor from the equation

$$|\mathbf{Fa}|^2 = |\mathbf{Ua}|^2 = \mathbf{a} \cdot \mathbf{Ca} = 1$$

The vectors $\mathbf{Fa}$ form a unit sphere. They are the map of the vectors $\mathbf{a}$. Let us for simplicity choose the eigenvectors of $\mathbf{U}$ (or $\mathbf{C}$) as the basis. Then

$$\mathbf{a} \cdot \mathbf{Ca} = \sum_{i=1}^{3} C_i a_i^2 = 1$$

The vectors $\mathbf{a}$ satisfying this equation form an ellipsoid. The half axes of the ellipsoid are $1/C_i$.

Chapter 2

Change of coordinates

1 Change of coordinates and change of basis

Let us now come to a change of Cartesian coordinates. Let us start by the vector d**r**. Its components are dx_i From the definition of differentials, we have

$$dx_{i'} = \frac{\partial x_{i'}}{\partial x_j} dx_j. \tag{2.1}$$

This formula is easy to remember. It immediately gives us the transformation formula for the components of an arbitrary vector

$$a_{i'} = \frac{\partial x_{i'}}{\partial x_j} a_j. \tag{2.2}$$

As I said, we are using Cartesian coordinates. It can be convenient to write the transformation matrix $\partial x_{i'}/\partial x_j$, to make it clear what index is the first one. But we know that for a change of Cartesian coordinates, the transformation matrix is *orthonormal* (another word for it is *orthogonal*). The meaning of this is that *the transpose of the matrix equals its inverse.* Inverse means shifting the prime from the denominator to the nominator. In other words,

$$\partial x_{i'}/\partial x_j = \partial x_j/\partial x_{i'} \tag{2.3}$$

So, the transformation formula can be written

$$a_{i'} = \frac{\partial x_{i'}}{\partial x_j} a_j = \frac{\partial x_j}{\partial x_{i'}} a_j. \tag{2.4}$$

You only have to see to it that *primed and unprimed indices match.*

Example 1 *The transformation formula for a rotation of the coordinate system the angle ϕ around the 3axis can be written*

$$\begin{aligned} x_{1'} &= x_1 \cos\phi + x_2 \sin\phi \\ x_{2'} &= -x_1 \sin\phi + x_2 \cos\phi \\ x_{3'} &= x_3 \end{aligned} \tag{2.5}$$

From this we find

$$[\partial x_{i'}/\partial x_j] = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.6}$$

Change of basis

We call the orthonormal basis vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. As the coordinate systems are Cartesian, the same transformation formula applies to them as to the components of a vector

$$\mathbf{e}_{i'} = \frac{\partial x_{i'}}{\partial x_j} \mathbf{e}_j, \tag{2.7}$$

Exercise 1.1 *We take an arbitrary vector* $\mathbf{a}$. *We know that the components of the vector can be written as scalar products with the corresponding basis vector*

$$a_i = \mathbf{e}_i \cdot \mathbf{a}$$

and similarly for another basis $\mathbf{e}_{1'}, \mathbf{e}_{2'}, \mathbf{e}_{3'}$.

$$a_{i'} = \mathbf{e}_{i'} \cdot \mathbf{a}$$

If we now use the transformation formula for the components of $\mathbf{a}$ *(2.2)*

$$\mathbf{e}_{i'} \cdot \mathbf{a} = \frac{\partial x_{i'}}{\partial x_j} \mathbf{e}_j \cdot \mathbf{a}.$$

As both sides are the scalar product with the arbitrary vector $\mathbf{a}$ *we conclude that the basis vectors satisfy (2.7), the same transformation formula as that of the components of a vector.*

Exercise 1.2 *Another way to check the correctness of (2.7) is to show that*

$$a_{i'}\mathbf{e}_{i'} = a_i\mathbf{e}_i = \mathbf{a}. \tag{2.8}$$

for any vector $\mathbf{a}$*. You need the transformation formula for the components and the fact that the transformation is orthonormal (2.3).*

$$\partial x_{i'}/\partial x_j = \mathbf{e}_{i'} \cdot \mathbf{e}_j = \cos(\mathbf{e}_{i'}, \mathbf{e}_j)$$

has a direct geometric interpretation: it is the cosine of the angle between $\mathbf{e}_i$ and $\mathbf{e}_{j'}$.

Now it is time to introduce tensors (of second order, to start with). A second order tensor is a set of 9 components which transform according to

$$T_{i'j'} = \frac{\partial x_{i'}}{\partial x_k}\frac{\partial x_{j'}}{\partial x_l}T_{kl}. \tag{2.9}$$

Exercise 1.3 *Show that the components of the tensor* $\mathbf{a} \otimes \mathbf{b}$ *have the right transformation property,*

$$(\mathbf{a} \otimes \mathbf{b})_{i'j'} = \frac{\partial x_{i'}}{\partial x_k}\frac{\partial x_{j'}}{\partial x_l}(\mathbf{a} \otimes \mathbf{b})_{kl}.$$

2 Rotation around an axis for a tensor. Mohr's circle

We now take an arbitrary tensor $\mathbf{T}$. We want to study in detail how its components transform under a rotation around the 3-axis. From (2.9) and (2.6) we immediately find

$$T_{3'3'} = T_{33}.$$

Let us now consider the components T_{i3} with $i \neq 3$. For them we find

$$\begin{aligned} T_{1'3'} &= \frac{\partial x_{1'}}{\partial x_k}\frac{\partial x_{3'}}{\partial x_l}T_{kl} = \frac{\partial x_{1'}}{\partial x_1}T_{13} + \frac{\partial x_{1'}}{\partial x_2}T_{23} = T_{13}\cos\phi + T_{23}\sin\phi \quad (2.10)\\ T_{2'3'} &= \frac{\partial x_{2'}}{\partial x_k}\frac{\partial x_{3'}}{\partial x_l}T_{kl} = \frac{\partial x_{2'}}{\partial x_1}T_{13} + \frac{\partial x_{2'}}{\partial x_3}T_{23} = -T_{13}\sin\phi + T_{23}\cos\phi \end{aligned}$$

They are the same as the transformation formulae for a vector orthogonal to the 3-axis.

Let us finally consider the components T_{ik}, where i and k take the values $1, 2$. For them we obtain

$$\begin{aligned} T_{1'1'} &= \frac{\partial x_{1'}}{\partial x_k}\frac{\partial x_{1'}}{\partial x_l}T_{kl} \quad (2.11)\\ &= \frac{\partial x_{1'}}{\partial x_1}\frac{\partial x_{1'}}{\partial x_1}T_{11} + \frac{\partial x_{1'}}{\partial x_1}\frac{\partial x_{1'}}{\partial x_2}T_{12} + \frac{\partial x_{1'}}{\partial x_2}\frac{\partial x_{1'}}{\partial x_1}T_{21} + \frac{\partial x_{1'}}{\partial x_2}\frac{\partial x_{1'}}{\partial x_2}T_{22}\\ &= T_{11}\cos^2\phi + (T_{12} + T_{21})\sin\phi\cos\phi + T_{22}\sin^2\phi. \end{aligned}$$

Exercise 2.1 *Show in the same way that*

$$T_{2'2'} = T_{11}\sin^2\phi - (T_{12} + T_{21})\sin\phi\cos\phi + T_{22}\cos^2\phi, \quad (2.12)$$

$$T_{1'2'} = (T_{22} - T_{11})\sin\phi\cos\phi + T_{12}\cos^2\phi - T_{21}\sin^2\phi. \quad (2.13)$$

If we take an arbitrary rotation, we realize that the calculation can be rather complicated. It can, of course, be rather easily carried out, using Maple.

Example 2 *Let us consider a homogeneous body in the shape of a rectangular prism with sides a, b, c and mass m. We choose a Cartesian coordinate system with origin in the center of mass of the body and axes along the sides of the body. The moment of inertia tensor $\mathbf{I}$ is a symmetric tensor. If the body has angular velocity $\boldsymbol{\omega}$ its angular momentum $\mathbf{L}$ is given as*

$$\mathbf{L} = \mathbf{I}\boldsymbol{\omega}.$$

With our choice of coordinate system, the moment of inertia tensor of the body is diagonal and

$$I_{xx} = \frac{m}{12}(b^2 + c^2), I_{yy} = \frac{m}{12}(a^2 + c^2), I_{zz} = \frac{m}{12}(a^2 + b^2).$$

Determine the moment of inertia for the axis, passing the center of mass, with polar directions θ, φ. - Let us call the axis $\mathbf{n}$. First of all you have to show that

$$\mathbf{n} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta). \quad (2.14)$$

So the moment of inertia for the axis $\mathbf{n}$ is given by

$$n_i I_{ij} n_j = \frac{m}{12}[(b^2 + c^2)\sin^2\theta\cos^2\varphi + (a^2 + c^2)\sin^2\theta\sin^2\varphi + (a^2 + b^2)\cos^2\theta)].$$

Mohr's circles for a symmetric tensor

Let us now consider a symmetric tensor. We choose axes along the eigenvectors of the tensor. The above formulas then simplify to

$$\begin{aligned} T_{1'1'} &= T_{11}\cos^2\theta + T_{22}\sin^2\phi, \\ T_{2'2'} &= T_{11}\sin^2\phi + T_{22}\cos^2\theta, \\ T_{1'2'} &= \alpha_{1i}\alpha_{2k}T_{ik} = (T_{22} - T_{11})\sin\phi\cos\phi. \end{aligned} \tag{2.15}$$

We rewrite this as

$$T_{1'1'} = \tfrac{1}{2}(T_{11} + T_{22}) + \tfrac{1}{2}(T_{11} - T_{22})\cos 2\phi, \tag{2.16}$$

$$T_{1'2'} = -\tfrac{1}{2}(T_{11} - T_{22})\sin 2\phi. \tag{2.17}$$

It is convenient to use complex notation

$$T_{1'1'} + iT_{1'2'} = \tfrac{1}{2}(T_{11} + T_{22}) + \tfrac{1}{2}(T_{11} - T_{22})e^{-i2\phi}. \tag{2.18}$$

From this formula it follows that $(T_{1'1'}, T_{1'2'})$ describe a circle with the angle 2ϕ in the negative sense when the coordinate system is turned ϕ in the positive sense. The centre of the circle is $((T_{11} + T_{22})/2, 0)$ and its radius is $\frac{1}{2}(T_{11} - T_{22})$. This is the so called *Mohr's circle*. We see that $T_{1'1'}$ varies between T_{11} and T_{22}. Further, that $T_{1'2'}$ varies between $\frac{1}{2}(T_{11} - T_{22})$ (which it takes on for $\phi = 3\pi/4$ and $7\pi/4$) and $-\frac{1}{2}(T_{11} - T_{22})$ (which it takes on for $\phi = \pi/4$ and $5\pi/4$).

The quotient theorem

We know that the components of a second order tensor have to transform according to (2.9). Suppose we have a set of components T_{ij} given in any coordinate system. Suppose also we know, that for any vector a_i

$$b_i = T_{ij}a_j$$

are the components of a vector. Then it follows that T_{ij} are the components of a tensor.

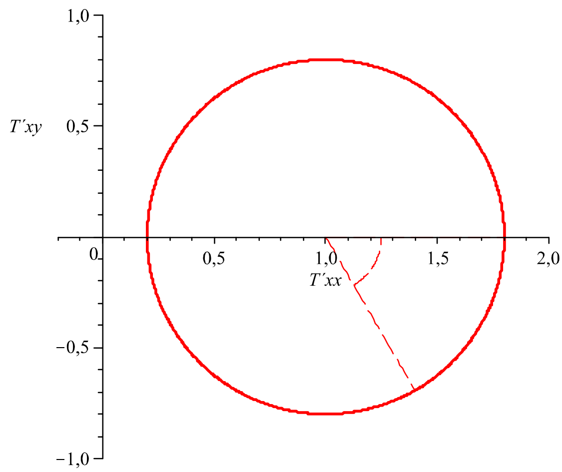


Figure 2.1: Mohr's circle. The angle is 2ϕ clockwise. In units such that the mean of the two stresses is 1.

Exercise 2.2 *Prove the quotient theorem. Hint. Write the preceding formula in a new coordinate system*

$$b_{i'} = T_{i'j'} a_{j'}.$$

Then use the transformation formula (2.4) for the vector a_i *as well as the vector* b_i *to obtain*

$$b_{i'} = \frac{\partial x_{i'}}{\partial x_j} b_j = \frac{\partial x_{i'}}{\partial x_j} T_{jk} a_k$$

$$b_{i'} = T_{i'j'} a_{j'} = T_{i'j'} \frac{\partial x_{j'}}{\partial x_k} a_k.$$

Here, a_i *is arbitrary. Conclude*

$$\frac{\partial x_{i'}}{\partial x_j} T_{jk} = T_{i'j'} \frac{\partial x_{j'}}{\partial x_k}.$$

Now multiply this with $\partial x_{l'}/\partial x_k = \partial x_k/\partial x_{l'}$, *see (2.3) to show that*

$$T_{i'l'} = \frac{\partial x_{i'}}{\partial x_j} \frac{\partial x_{l'}}{\partial x_k} T_{jk}.$$

The unit tensor - Kronecker delta

A very important second order tensor is the unit tensor **1**, with components

$$\delta_{ij}.$$

Let us calculate its components in the new coordinates $x_{i'}$. For the moment we call its components in the new coordinates $\delta'_{i'j'}$. We have

$$\delta'_{i'j'} = \frac{\partial x_{i'}}{\partial x_k} \frac{\partial x_{j'}}{\partial x_l} \delta_{kl}.$$

But using the orthonormality of the transformation we have

$$\delta'_{i'j'} = \frac{\partial x_{i'}}{\partial x_k} \frac{\partial x_l}{\partial x_{j'}} \delta_{kl} = \frac{\partial x_{i'}}{\partial x_k} \frac{\partial x_k}{\partial x_{j'}}$$

We have then used the Kronecker delta to sum over l. But now we use the chain rule of differentiation to find

$$\delta'_{i'j'} = \frac{\partial x_{i'}}{\partial x_k} \frac{\partial x_k}{\partial x_{j'}} = \frac{\partial x_{i'}}{\partial x_{j'}} = \delta_{i'j'}.$$

The last equality is immediate. So we conclude that $\delta'_{i'j'} = \delta_{i'j'}$.

So, the unit tensor, the components of which are δ_{ij}, has *the same components in all Cartesian frames*. Such a tensor is called *isotropic.*

The basis vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are given by

$$[\mathbf{e}_1] = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, [\mathbf{e}_2] = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, [\mathbf{e}_3] = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

The j-component of $\mathbf{e}_1$ is 1 for $j = 1$ and 0 for $j = 2$ and 3. In other words, the j-component of $\mathbf{e}_1$ is simply δ_{1j}. The j-component of $\mathbf{e}_2$ is δ_{2j} and similarly for $\mathbf{e}_3$. To summarize, we have found that the j-component of $\mathbf{e}_i$ is δ_{ij}.

$$(\mathbf{e}_i)_j = \delta_{ij}. \tag{2.19}$$

Chapter 3

Higher order tensors

1 Transformation formula for higher order tensors

Higher order tensors are defined analogously. A third order tensor A_{ijk} is a set of $3^3 = 27$ components transforming according to

$$A_{i'j'k'} = \frac{\partial x_{i'}}{\partial x_l} \frac{\partial x_{j'}}{\partial x_m} \frac{\partial x_{k'}}{\partial x_n} A_{lmn}. \tag{3.1}$$

2 The permutation symbol

Another very important object is the *permutation symbol* ϵ_{ijk}. It has the value 1 if ijk is an even permutation of 123 and the value -1 if it is an odd permutation. If two or more indices are equal, it vanishes. The permutation symbol is *totally antisymmetric.*

There is an important formula for the permutation symbol

$$\epsilon_{ijm}\,\epsilon_{klm} = \epsilon_{ijm}\,\epsilon_{mkl} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}. \tag{3.2}$$

If you think about it for a while you realize that the left hand side will be non-zero only when $ij = kl$ or $ij = lk$. This is also the case with the expression with the two Kronecker deltas. So you only have to check those two cases. The sum on the left hand side contains one term only, so that is easy.

Exercise 2.1 *Prove that*

$$\epsilon_{ilm}\,\epsilon_{klm} = 2\delta_{ik}$$

Exercise 2.2 *Prove that*

$$\epsilon_{klm}\,\epsilon_{klm} = 6 = 3! \tag{3.3}$$

The determinant

We can use the permutation symbol to write down the determinant of a matrix as

$$\det \mathbf{T} = \epsilon_{ijk} T_{1i} T_{2j} T_{3k}$$

Each term in this sum is a product of one element of the first row, one of the second row and one of the third row. This is then multiplied by the value of the permutation symbol. This is precisely the definition of a determinant.

A more elegant way to write the determinant is

$$\det \mathbf{T} = \frac{1}{3!}\epsilon_{lmn}\epsilon_{ijk} T_{li} T_{mj} T_{nk}. \tag{3.4}$$

In fact, let us take a look at

$$\epsilon_{ijk} T_{li} T_{mj} T_{nk}.$$

If lmn take the values 123 we already know that it is the determinant of $\mathbf{A}$. If instead lmn take the values 213 we realize that we have taken the original matrix and exchanged the first and second rows. The determinant then changes sign. If lmn take the values 113 the first two rows in the determinant are equal and the value is zero. We conclude that

$$\epsilon_{ijk} T_{li} T_{mj} T_{nk} = \epsilon_{lmn} \det \mathbf{T} \tag{3.5}$$

If we now use (3.3), we obtain (3.4).

Transformation properties of the permutation symbol

Let us define a tensor of third order, to have the components ϵ_{ijk} in one Cartesian frame. What are its components in a new frame? We use the tensor transformation formula

$$\epsilon'_{i'j'k'} = \frac{\partial x_{i'}}{\partial x_l}\frac{\partial x_{j'}}{\partial x_m}\frac{\partial x_{k'}}{\partial x_n}\epsilon_{lmn}$$

This we already know by (3.5). Note that the free indices here are primed.

$$\epsilon'_{i'j'k'} = g\epsilon_{i'j'k'}. \tag{3.6}$$

Here,

$$g = \begin{vmatrix} \partial x_{1'}/\partial x_1 & \partial x_{1'}/\partial x_2 & \partial x_{1'}/\partial x_3 \\ \partial x_{2'}/\partial x_1 & \partial x_{2'}/\partial x_2 & \partial x_{2'}/\partial x_3 \\ \partial x_{3'}/\partial x_1 & \partial x_{3'}/\partial x_2 & \partial x_{3'}/\partial x_3 \end{vmatrix} \tag{3.7}$$

is the determinant of the Jacobian of the coordinate transformation.

But the Jacobian matrix is orthonormal, so the determinant has to have the value 1 or -1. If the value is 1 the two Cartesian coordinate systems have the same orientation. This means that the tensor *has the same components ϵ_{ijk} in all Cartesian coordinates with the same orientation as the original one*, but the components $-\epsilon_{ijk}$ in coordinates systems with the opposite orientation. By the way, such a tensor is called *hemitropic.* Compare with isotropic (hemi means half in Greek). In the Cartesian coordinate systems with orientation opposite to the original one, it has the components $-\epsilon_{ijk}$.

Instead one often says that ϵ_{ijk} are the components of a pseudotensor. A pseudotensor has the same transformation properties as a tensor except for an extra ± 1 coming from the $\det(\partial x'/\partial x)$. So, the transformation formula for a pseudotensor is (3.6).

Cross product

The cross or vector product of two vectors can be expressed as

$$(\mathbf{a}\times\mathbf{b})_i = \epsilon_{ijk}a_j b_k. \tag{3.8}$$

This formula holds in *right-handed* Cartesian frame.

Double cross product formula Now we can use the double ε-formula (3.2) to show the formula for the double cross product

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}. \tag{3.9}$$

In fact,

$$[\mathbf{a}\times(\mathbf{b}\times\mathbf{c})]_i = \epsilon_{ijk}a_j(\mathbf{b}\times\mathbf{c})_k$$

Here, we insert (note that we cannot use j, k as summation indices)

$$(\mathbf{b}\times\mathbf{c})_k = \epsilon_{klm}b_l c_m$$

$$\begin{aligned}[\mathbf{a}\times(\mathbf{b}\times\mathbf{c})]_i &= \epsilon_{ijk}\epsilon_{klm}a_j b_l c_m \\ &= (\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl})a_j b_l c_m \\ &= a_j b_i c_j - a_j b_j c_i\end{aligned}$$

Connection between vectors and antisymmetric tensors

There is a simple relation between an antisymmetric tensor of second order and a (pseudo-)vector. Let us start by a vector $\mathbf{a}$. If we use the permutation symbol, we can write

$$A_{ik} = \varepsilon_{ijk}a_j. \tag{3.10}$$

It is clearly a tensor, which is antisymmetric. The order of the indices could seem a bit strange, but if we take an arbitrary vector $\mathbf{c}$ we have

$$(\mathbf{Ac})_i = A_{ik}c_k = \varepsilon_{ijk}a_j c_k = (\mathbf{a}\times\mathbf{c})_i.$$

Thus

$$\mathbf{Ac} = \mathbf{a}\times\mathbf{c}. \tag{3.11}$$

Using (3.2), we can also show that

$$a_i = \frac{1}{2}\varepsilon_{ijk}A_{kj}. \tag{3.12}$$

Explicitly

$$[\mathbf{A}] = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}. \tag{3.13}$$

Examples of pseudovectors

The angular velocity of a rigid body. The velocity field is given by

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}.$$

$$v_i = \varepsilon_{ijk} \omega_j x_k.$$

Position vector, velocity and acceleration are ordinary vectors. As there is a cross product, $\boldsymbol{\omega}$ is a pseudovector.

So is the case with the magnetic field $\mathbf{B}$. The Lorenz force on a particle with charge q and velocity $\mathbf{v}$ is given by the expression

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B}.$$

Or in components,

$$F_i = q\varepsilon_{ijk} v_j B_k.$$

3 Isotropic tensors

We have already encountered an isotropic tensor, the unit tensor $\mathbf{1}$ with components δ_{ij}. Isotropic means that it has the same components in all Cartesian frames. Now, one can show that all isotropic tensors of any order can be built from the Kronecker delta.

As an example we consider an isotropic fourth order tensor, with components C_{ijkl}. From δ_{ij} we can construct

$$\delta_{ij}\delta_{kl}, \delta_{ik}\delta_{jl}, \delta_{il}\delta_{jk}. \tag{3.14}$$

There is no other possibility. This means that C_{ijkl} has to be a linear combination of these:

$$C_{ijkl} = a\delta_{ij}\delta_{kl} + b\delta_{ik}\delta_{jl} + c\delta_{il}\delta_{jk}. \tag{3.15}$$

a, b, c are three scalars.

If C_{ijkl} is symmetric in the first pair of indices,

$$C_{ijkl} = C_{jikl}, \tag{3.16}$$

we have

$$a\delta_{ij}\delta_{kl} + b\delta_{ik}\delta_{jl} + c\delta_{il}\delta_{jk} = a\delta_{ij}\delta_{kl} + b\delta_{jk}\delta_{il} + c\delta_{jl}\delta_{ik}.$$

we conclude that $b = c$.

$$C_{ijkl} = a\delta_{ij}\delta_{kl} + b(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}). \tag{3.17}$$

We see that C_{ijkl} is automatically symmetric also in the second pair of indices:

$$C_{ijkl} = C_{ijlk}.$$

It also has one further symmetry: it does not change when the first pair and second pair are exchanged.

$$C_{ijkl} = C_{klij}. \tag{3.18}$$

Exercise 3.1 *E_{ij} is a symmetric tensor and (we have put $a = \lambda$ and $b = \mu$ in (3.17)*

$$C_{ijkl} = \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \lambda\delta_{ij}\delta_{kl}. \tag{3.19}$$

Calculate the second order tensor

$$T_{ij} = C_{ijkl}E_{kl}$$

Answer:

$$T_{ij} = 2\mu E_{ij} + \lambda E_{kk}\delta_{ij}.$$

Exercise 3.2 *Write E_{ij} as the sum of its deviator $E_{<ij>}$ and spherical part $\frac{1}{3}E_{kk}\delta_{ij}$.*

$$E_{ij} = E_{<ij>} + \frac{1}{3}E_{kk}\delta_{ij}.$$

We recall that the deviator has vanishing trace

$$E_{<ii>} = 0.$$

Show that

$$T_{ij} = 2\mu E_{<ij>} + (\frac{2\mu}{3} + \lambda)E_{kk}\delta_{ij}.$$

Find the deviator of T_{ij} and its spherical part. Answer

$$T_{<ij>} = 2\mu E_{<ij>};\ \frac{1}{3}T_{kk}\delta_{ij} = (\frac{2\mu}{3} + \lambda)E_{kk}\delta_{ij}.$$

Exercise 3.3 *For the same problem also calculate*

$$\frac{1}{2}C_{ijkl}E_{ij}E_{kl}.$$

Answer

$$\begin{aligned}\frac{1}{2}C_{ijkl}E_{ij}E_{kl} &= \mu E_{ij}E_{ij} + \frac{1}{2}\lambda E_{ii}E_{jj} \\ &= \mu E_{<ij>}E_{<ij>} + \frac{1}{2}(\frac{2\mu}{3} + \lambda)E_{ii}E_{jj}.\end{aligned}$$

Note that the second form has an advantage: the components $E_{<ij>}$ of the deviator can be chosen independently of the trace E_{ii}

There is no isotropic tensor of third order. - The only isotropic tensor of second order is

$$a\delta_{ij}.$$

Exercise 3.4 *The fourth order isotropic tensor*

$$I_{ijkl} = \delta_{ik}\delta_{jl}$$

has the property that when acting on a second order tensor T_{ij} it does not change it

$$I_{ijkl}T_{kl} = T_{ij}.$$

Exercise 3.5 *We take a fourth order isotropic tensor S_{ijkl}. We choose S_{ijkl} such that it for an arbitrary second order tensor T_{ij} projects out its symmetrical part.*

$$S_{ijkl}T_{kl} = T_{(kl)}.$$

We know that S_{ijkl} has to be of the form (3.15). Then

$$\begin{aligned}S_{ijkl}T_{kl} &= a\delta_{ij}\delta_{kl}T_{kl} + b\delta_{ik}\delta_{jl}T_{kl} + c\delta_{il}\delta_{jk}T_{kl} \\ &= a\delta_{ij}T_{kk} + bT_{ij} + cT_{ji}.\end{aligned}$$

For this to equal $T_{(ij)}$ we have to take $a = 0, b = c = 1/2$. We conclude that

$$S_{ijkl} = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}).$$

Exercise 3.6 *Show in the same way that the isotropic fourth order tensor A_{ijkl}, which projects out the antisymmetrical part*

$$A_{ijkl}T_{kl} = T_{[kl]}$$

is

$$A_{ijkl} = \frac{1}{2}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}).$$

Exercise 3.7 *Show that isotropic fourth order tensor s_{ijkl}, which projects out the spherical part*

$$s_{ijkl}T_{kl} = \frac{1}{3}T_{kk}\delta_{ij}$$

is

$$s_{ijkl} = \frac{1}{3}\delta_{ij}\delta_{kl}.$$

Exercise 3.8 *As the symmetrical part of a tensor is the sum of its deviator and spherical part, the projector d_{ijkl}, which projects out the deviator is $S_{ijkl} - s_{ijkl}$ or*

$$d_{ijkl} = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) - \frac{1}{3}\delta_{ij}\delta_{kl}.$$

Exercise 3.9 *Show that the general isotropic tensor of fourth order can be written*

$$\begin{aligned}
C_{ijkl} &= a\delta_{ij}\delta_{kl} + b\delta_{ik}\delta_{jl} + c\delta_{il}\delta_{jk} \\
&= \frac{a+b}{2}d_{ijkl} + \frac{a-b}{2}A_{ijkl} + (3c + \frac{a+b}{2})s_{ijkl}.
\end{aligned}$$

Chapter 4

Derivatives

1 Gradient of a scalar field

It is convenient to denote partial derivatives by a comma, for a scalar field f

$$f_{,i} = \frac{\partial f}{\partial x_i}. \tag{4.1}$$

Let us see that they are the components of a vector field, by calculating

$$\frac{\partial f}{\partial x_{i'}} = \frac{\partial f}{\partial x_j}\frac{\partial x_j}{\partial x_{i'}} = \frac{\partial x_{i'}}{\partial x_j}\frac{\partial f}{\partial x_j} \tag{4.2}$$

Here, we have used the chain rule. We have also used that the transformation matrix is orthonormal: $\partial x_{i'}/\partial x_j = \partial x_j/\partial x_{i'}$. This is (2.3).

What we have introduced is, of course the gradient of the scalar field.

1.1 The Nabla operator

The nabla operator is a vector operator. Its i-component is

$$(\boldsymbol{\nabla})_i = \frac{\partial}{\partial x_i}. \tag{4.3}$$

2 Gradient of a vector field

You have already in vector analysis encountered the rotation of a vector field. We can write it in component notation

$$(\boldsymbol{\nabla} \times \mathbf{a})_i = \varepsilon_{ijk}(\boldsymbol{\nabla})_j a_k = \varepsilon_{ijk} a_{k,j}. \tag{4.4}$$

You have also there encountered the divergence of a vector field

$$\boldsymbol{\nabla} \cdot \mathbf{a} = (\boldsymbol{\nabla})_i a_i = a_{i,i}. \tag{4.5}$$

But there are many more derivatives of the components of a vector field. In all there are 9 of them,

$$a_{i,j}. \tag{4.6}$$

They are the components of a second order tensor.

Exercise 2.1 *Fill in the details of the following proof.*

$$\frac{\partial a_{i'}}{\partial x_{j'}} = \frac{\partial a_{i'}}{\partial x_k}\frac{\partial x_k}{\partial x_{j'}} = \frac{\partial a_{i'}}{\partial x_k}\frac{\partial x_{j'}}{\partial x_k}.$$

Then use the transformation formula for the vector a_i to find

$$\frac{\partial a_{i'}}{\partial x_{j'}} = \frac{\partial x_{i'}}{\partial x_k}\frac{\partial x_{j'}}{\partial x_l}\frac{\partial a_k}{\partial x_l}. \tag{4.7}$$

Note that is essential in the proof, that the transformation matrix $\partial x_{i'}/\partial x_j$ is a constant.

Exercise 2.2 *Prove, for a vector field, that*

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{a}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{a}) - \triangle \mathbf{a}. \tag{4.8}$$

Fill in the details in the following proof.

$$\begin{aligned}
[\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{a})]_i &= \varepsilon_{ijk}\frac{\partial}{\partial x_j}(\boldsymbol{\nabla} \times \mathbf{a})_k \\
&= \varepsilon_{ijk}\frac{\partial}{\partial x_j}\varepsilon_{klm}\frac{\partial}{\partial x_l}a_m.
\end{aligned}$$

Then use the double ε-formula (3.2) to find

$$\begin{aligned}
[\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{a})]_i &= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})a_{m,jl} \\
&= a_{j,ji} - a_{i,jj}.
\end{aligned}$$

We already know that we can divide up any second order tensor into a symmetric and an antisymmetric part.

$$a_{i,j} = a_{(i,j)} + a_{[i,j]}. \tag{4.9}$$

If we use the formula connecting an antisymmetric tensor with a vector, we find that the i-component of the vector connected with $a_{[i,j]}$ is given by

$$\frac{1}{2}\varepsilon_{ijk}a_{[k,j]}.$$

Let us rewrite this expression. First of all we find that if we instead take the symmetric part $a_{(k,j)}$ in the formula, we obtain zero

$$\varepsilon_{ijk}a_{(k,j)} = 0.$$

We can show that the following way. First, we rename the dummy indices: to find

$$\varepsilon_{ijk}a_{(k,j)} = \varepsilon_{ikj}a_{(j,k)}.$$

Then we use that the first factor is antisymmetric in its indices, but the second factor is symmetric:

$$\varepsilon_{ikj}a_{(j,k)} = -\varepsilon_{ijk}a_{(j,k)} = -\varepsilon_{ijk}a_{(k,j)}.$$

Putting this together we have found that

$$\varepsilon_{ijk}a_{(k,j)} = -\varepsilon_{ijk}a_{(k,j)}.$$

This means that the expression has to vanish.

In other words, the i-component of the corresponding vector is

$$\frac{1}{2}\varepsilon_{ijk}\big(a_{[k,j]} + a_{(k,j)}\big) = \frac{1}{2}\varepsilon_{ijk}a_{k,j} = \frac{1}{2}\varepsilon_{ijk}\frac{\partial}{\partial x_j}a_k = \frac{1}{2}(\boldsymbol{\nabla}\times\mathbf{a})_i.$$

So *the vector corresponding to the antisymmetric part of the gradient of the vector field* $\mathbf{a}$ *is simply* $\frac{1}{2}(\boldsymbol{\nabla}\times\mathbf{a})$.

We can also write the symmetric part $a_{(i,j)}$ as the sum of the deviator

$$a_{<i,j>} = a_{(i,j)} - \frac{1}{3}a_{k,k}\delta_{ij}$$

and the spherical part

$$\frac{1}{3}a_{k,k}\delta_{ij}.$$

Chapter 5

Analysis of small deformations

1 Displacement vector

Present configuration and reference configuration

To be able to describe the deformation of a body, we need a *reference configuration*. For a solid body, the reference configuration is usually taken as stress-free. We compare the reference configuration with the present configuration. Let us call the position of a small element of the body in the reference configuration $\mathbf{X}$ and in the present configuration $\mathbf{x}$. The basic function we need is the mapping

$$\mathbf{x} = \mathbf{x}(\mathbf{X}). \tag{5.1}$$

The displacement vector

The displacement vector $\mathbf{u}$ is the vector from the position in the reference configuration to the position in the present configuration.

$$\mathbf{x} = \mathbf{X} + \mathbf{u}, \tag{5.2}$$
$$\mathbf{u}(\mathbf{X}) = \mathbf{x}(\mathbf{X}) - \mathbf{X}. \tag{5.3}$$

Small deformations

In this chapter we shall assume that the deformations (and rotations) are small. $\mathbf{u}$ has the dimension length. The derivatives of $\mathbf{u}$ are dimensionless, and they are assumed to be small

$$O(\varepsilon) = |\frac{\partial u_i}{\partial X_j}| << 1. \tag{5.4}$$

We shall shortly see the meaning of $\partial u_i / \partial X_j$. Differentiating (5.2) we find

$$\frac{\partial x_i}{\partial X_j} = \delta_{ij} + \frac{\partial u_i}{\partial X_j} = \delta_{ij} + O(\varepsilon).$$

We also have

$$\frac{\partial X_i}{\partial x_j} = \delta_{ij} - \frac{\partial u_i}{\partial X_j} + ... = \delta_{ij} + O(\varepsilon)$$

We can think of $\mathbf{u}$ as a function of the original position $\mathbf{X}$ or the present position $\mathbf{x}$.

$$\mathbf{u}(\mathbf{X}) = \mathbf{x}(\mathbf{X}) - \mathbf{X} = \widetilde{\mathbf{u}}(\mathbf{x}). \tag{5.5}$$

But we have that

$$\frac{\partial \widetilde{u}_i}{\partial x_j} = \frac{\partial u_i}{\partial X_k}\frac{\partial X_k}{\partial x_j} = \frac{\partial u_i}{\partial X_k}(\delta_{kj} + O(\varepsilon)) = \frac{\partial u_i}{\partial X_j} + O(\varepsilon^2)$$

It does not matter if we use the original position or the present position as argument in the displacement vector when we calculate its derivatives. Often $\widetilde{\mathbf{u}}(\mathbf{x})$ is simply written as $\mathbf{u}(\mathbf{x})$. So we have

$$u_{i,j} = \frac{\partial u_i}{\partial x_i} = \frac{\partial \widetilde{u}_i}{\partial x_i} \approx \frac{\partial u_i}{\partial X_j}$$

A small material vector

Let us now consider a small volume surrounding the point $\mathbf{X}$. The point $\mathbf{X}$ is mapped to $\mathbf{X} + \mathbf{u}(\mathbf{X})$ or in components

$$x_i = X_i + u_i(\mathbf{X}). \tag{5.6}$$

The neighbouring point $\mathbf{X}$+d$\mathbf{X}$ is mapped to $(\mathbf{X}+\mathrm{d}\mathbf{X}) + \mathbf{u}\,(\mathbf{X}+\mathrm{d}\mathbf{X})$, or in components

$$X_i + \mathrm{d}X_i + u_i(\mathbf{X}+\mathrm{d}\mathbf{X}).$$

Here we Taylor expand $u_i(\mathbf{X}+\mathrm{d}\mathbf{X})$

$$u_i(\mathbf{X}+\mathrm{d}\mathbf{X}) \approx u_i(\mathbf{X}) + \frac{\partial u_i}{\partial X_j}\mathrm{d}X_j.$$

to find that $\mathbf{X}+\mathrm{d}\mathbf{X}$ is mapped to the point with coordinates

$$X_i + \mathrm{d}X_i + u_i(\mathbf{X}) + \frac{\partial u_i}{\partial X_j}\mathrm{d}X_j = x_i + \mathrm{d}X_i + \frac{\partial u_i}{\partial X_j}\mathrm{d}X_j. \tag{5.7}$$

If we subtract the image $\mathbf{x}$ of the point $\mathbf{X}$ from this, we find the small vector with components

$$\mathrm{d}x_i = \mathrm{d}X_i + \frac{\partial u_i}{\partial X_j}\mathrm{d}X_j. \tag{5.8}$$

This is then the image in the present configuration of the vector $\mathrm{d}\mathbf{X}$ in the reference configuration.

We now divide up $\partial u_i/\partial X_j \approx u_{i,j}$ into symmetric and antisymmetric parts:

$$u_{i,j} = u_{(i,j)} + u_{[i,j]}.$$

2 The linear rotation vector

Let us take a look at the contribution from the antisymmetric term $u_{[i,j]}$. But

$$u_{[i,j]}\mathrm{d}X_j$$

is a well known expression for us, an antisymmetric tensor acting on the vector $\mathrm{d}\mathbf{X}$. We write it as a cross product. The vector corresponding to the antisymmetric tensor $u_{[i,j]}$ we call $\boldsymbol{\phi}$

$$u_{[i,j]}\mathrm{d}X_j = (\boldsymbol{\phi}\times\mathrm{d}\mathbf{X})_i. \tag{5.9}$$

The vector is given by

$$\boldsymbol{\phi} = \frac{1}{2}\mathrm{rot}\,\mathbf{u}. \tag{5.10}$$

To make life easier, let us for the moment assume that the symmetric part $u_{(i,j)}$ vanishes. So, a material vector, which in the reference configuration is d$\mathbf{X}$ is in the present configuration, see (5.8)

$$\mathrm{d}\mathbf{X} + \boldsymbol{\phi}\times\mathrm{d}\mathbf{X}. \tag{5.11}$$

To understand the meaning of this, let us take a coordinate system, with 3-axis along $\boldsymbol{\phi}$. The vector in the present configuration has the components

$$\begin{aligned} &\mathrm{d}X_1 - \phi\mathrm{d}X_2, \\ &\mathrm{d}X_2 + \phi\mathrm{d}X_1, \\ &\mathrm{d}X_3. \end{aligned} \tag{5.12}$$

d$\mathbf{X}$ has been rotated the small angle ϕ around the 3-axis.

3 The linear strain tensor

There remains one term in (5.8) to be interpreted, $u_{(i,j)}\mathrm{d}X_j$. Let us first of all introduce a notation for $u_{(i,j)}$

$$E_{ij} = u_{(i,j)}. \tag{5.13}$$

We know that as E_{ij} is a symmetric tensor, there alway exists three orthonormal *eigenvectors.* We call the corresponding eigenvalues E_1, E_2, E_3. Let us introduce a coordinate system, with axes along the eigenvectors of E_{ij}. We obtain

$$[\mathbf{E}] = \begin{bmatrix} E_1 & 0 & 0 \\ 0 & E_2 & 0 \\ 0 & 0 & E_3 \end{bmatrix}.$$

We now assume that the antisymmetric part $u_{[i,j]}$ vanishes. The vector with components $(\mathrm{d}X_1, 0, 0)$ is mapped to

$$(\mathrm{d}X_1 + E_1\mathrm{d}X_1, 0, 0).$$

This means that the relative increase of length in the 1-direction is simply the eigenvalue E_1.

E_{ij} is called the *linear strain tensor.* Its eigenvalues are called *principal strains.*

Change of volume

The present volume is

$$(1 + E_1)\mathrm{d}X_1(1 + E_2)\mathrm{d}X_2(1 + E_3)\mathrm{d}X_3 \approx [1 + (E_1 + E_2 + E_3)]\mathrm{d}X_1\mathrm{d}X_2\mathrm{d}X_3$$

So, the relative increase of volume is

$$E_1 + E_2 + E_3 = E_{ii} = \mathrm{tr}\,\mathbf{E} = u_{i,i} = \mathrm{div}\,\mathbf{u}. \tag{5.14}$$

Change of lengths

One can show that for a material vector, which in the reference configuration is along the unit vector $\mathbf{e}$, the relative increase of length is

$$\mathbf{e} \cdot \mathbf{Ee} = E_{ij}e_ie_j. \tag{5.15}$$

Exercise 3.1 *Show (5.15). Hint: Take the square of the vector (5.8).*

Change of angles

We consider two small material vectors, which in the reference configuration are along the orthogonal unit vectors $\mathbf{e}$ and $\mathbf{f}$. In the present configuration they will be slightly non-orthogonal and make an angle $\frac{\pi}{2} - \alpha$. From (5.8) we find that

$$\alpha = 2\mathbf{e} \cdot \mathbf{Ef} = 2e_iE_{ij}f_j. \tag{5.16}$$

Here we have used $\cos(\frac{\pi}{2} - \alpha) = \sin\alpha \approx \alpha$.

Exercise 3.2 *Show the preceding formula. Hint: Use (5.8) for the two small material vectors and take their scalar product.*

4 Examples

4.1 Simple shear

In a state of simple shear, choosing coordinates suitably, the displacement vector can be written

$$u_1 = u_1(X_2). \tag{5.17}$$

The only non-vanishing component of $u_{i,j}$ is

$$u_{1,2} = u^{'}$$

The non-vanishing components of the linear strain tensor are

$$E_{12} = E_{21} = \frac{1}{2}u^{'} \tag{5.18}$$

and those of the rotation vector

$$\phi_3 = -u_{[1,2]} = -\frac{1}{2}u^{'}. \tag{5.19}$$

$u^{'} = \mathrm{d}u_1/\mathrm{d}X_2$ is the angle of shear γ . Note that the condition for small deformations is that $\gamma = u^{'}$ is small compared with 1.

4.2 Torsion of circular cylinder

Let us consider a circular cylinder of radius a and height l. We choose cartesian coordinates with the X_3-axis along the axis of the cylinder. On the bottom and top of the cylinder ($X = 0$ and $X = l$) moments are applied. The cylindrical surface is free. A plane perpendicular to the axis is turned a small angle, the size of which is proportional to X_3. Let us write the angle βX_3. The displacement vector is hence

$$\begin{aligned} u_1 &= -\beta X_3 X_2, \\ u_2 &= \beta X_3 x_1, \\ u_3 &= 0. \end{aligned} \tag{5.20}$$

Its gradient is,

$$\begin{array}{lll} u_{1,1} = 0, & u_{1,2} = -\beta X_3, & u_{1,3} = -\beta X_2, \\ u_{2,1} = \beta X_3, & u_{2,2} = 0, & u_{2,3} = \beta X_1. \end{array}$$

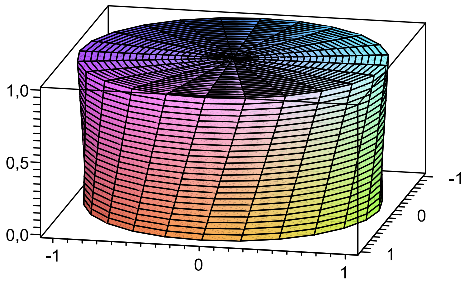


Figure 5.1: Torsion of circular cylinder

We note that the condition for small deformations, that $|u_{i,j}| << 1$ gives that the magnitudes of $\beta X_1, \beta X_2, \beta X_3$ all are small compared to one. For a cylinder with radius a and height l, we can write the conditions

$$|\beta|a << 1, |\beta|l << 1.$$

$\gamma \equiv \beta l$ is the total angle (in radians) of torsion, it has to be small compared to 1. Further, $\beta a = (a/l)\gamma$, so that the conditions for small deformations can be written

$$|\gamma| << 1, \frac{a}{l}|\gamma| << 1.$$

If the cylinder has a much larger height than radius, the only condition to be satisfied is that the torsion angle is small compared to 1. If, on the contrary, its radius is much larger than its height, the last condition has to be satisfied. It is not sufficient then that the torsion angle is small compared to 1.

The non-zero components of the linear strain tensor are

$$E_{13} = E_{31} = -\ \tfrac{1}{2}\beta X_2,\ E_{23} = E_{32} =\ \tfrac{1}{2}\beta X_1. \tag{5.21}$$

The linear rotation vector is

$$\phi_1 = -u_{[2,3]} = -\tfrac{1}{2}\beta X_1\ , \phi_2 = \ -u_{[3,1]} = -\tfrac{1}{2}\beta X_2\ , \phi_3 = -u_{[1,2]} = \beta X_3. \tag{5.22}$$

On the axis it is βX_3 as one would expect. But away from the axis, there is also a rotation around the $\boldsymbol{\rho}$-vector.

4.3 Bending of beam

We start with a straight beam. Its cross-section is arbitrary. The center of area of the cross-sections is on the X-axis. The X-axis is along the axis of the beam. Shear forces in the positive y-direction are applied to the end surfaces of the beam and as a result, it bends. The other sides are free. One can then show that the displacement vector is given by

$$\begin{aligned} u_x &= -\frac{1}{R}XY, \\ u_y &= \frac{1}{2R}X^2 + \frac{\nu}{2R}(Y^2 - Z^2), \\ u_z &= \frac{\nu}{R}YZ. \end{aligned} \tag{5.23}$$

Here, R and ν are constants. The X-axis $(Y = Z = 0)$ is mapped into $x = X,\ y = u_y = X^2/2R,\ z = 0$. Hence

$$x^2 + (y - R)^2 = X^2 + u_y^2 - 2Ru_y + R^2 \approx X^2 - 2Ru_y + R^2 = R^2$$

which is the equation for a circle with radius R and center at $y = R$ on the y-axis. ν is a dimensionless material constant, the Poisson ratio. Also calculate the gradient of the displacement vector, $u_{i,j}$.

$$[u_{i,j}] = \begin{bmatrix} -\frac{Y}{R} & -\frac{X}{R} & 0 \\ \frac{X}{R} & \frac{\nu Y}{R} & -\frac{\nu Z}{R} \\ 0 & \frac{\nu Z}{R} & \frac{\nu Y}{R} \end{bmatrix}. \tag{5.24}$$

As ν is of the order of 1 conclude that the condition for small deformations is that

$$|X| << R, |Y| << R, |Z| << R.$$

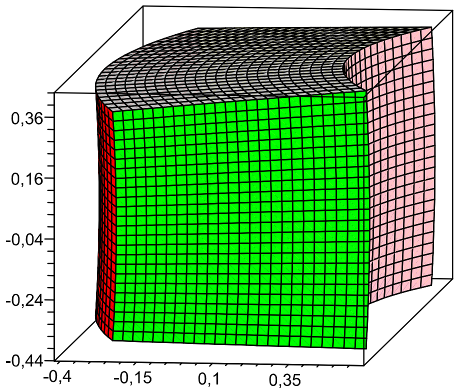


Figure 5.2: Bent beam.

Find from this the nonvanishing components of the linear strain tensor as

$$\begin{aligned} E_{xx} &= -\frac{Y}{R}, \\ E_{yy} &= E_{zz} = -\nu E_{xx}. \end{aligned} \tag{5.25}$$

and calculate the linear rotation vector

$$\phi_x = -u_{[y,z]} = \frac{\nu Z}{R}\,, \phi_y = \ -u_{[z,x]} = 0\,, \phi_z = -u_{[x,y]} = \frac{X}{R}. \tag{5.26}$$

4.4 When the linear strain vanishes identically the mapping is rigid

Suppose that the linear strain tensor vanishes identically,

$$\begin{aligned} E_{ij} &= 0, \\ u_{i,j} &= -u_{j,i}. \end{aligned} \tag{5.27}$$

Let us show that all second derivatives $u_{i,jk}$ vanish identically. We have

$$u_{i,jk} = -u_{j,ik} = -u_{j,ki} = u_{k,ji} = u_{k,ij} = -u_{i,kj} = -u_{i,jk}.$$

Here we have used first (5.27), then that derivatives commute, then again (5.27) and so on. We conclude that the second derivatives vanish, which means that the first derivatives are constants. In other words,

$$u_i(\mathbf{X}) = u_i(\mathbf{X}_0) + u_{i,j}(X_j - X_{0j}) = u_i(\mathbf{X}_0) + u_{[i,j]}X_j,$$

or, using (5.9) which applies to any vector,

$$\mathbf{u}(\mathbf{X}) = \mathbf{u}(\mathbf{X}_0) + \boldsymbol{\phi} \times (\mathbf{X} - \mathbf{X}_0).$$

The constant $\mathbf{u}(\mathbf{X}_0)$ is a translation, followed by a small rotation $\boldsymbol{\phi} \times (\mathbf{X} - \mathbf{X}_0)$. In other words a small rigid transformation.

4.5 Spherical symmetry

When $\mathbf{u} = u(r)\mathbf{e}_r$, we can write

$$\mathbf{u} = \frac{u(r)}{r}\mathbf{r}$$

Or in Cartesian components

$$u_i = \frac{u(r)}{r}x_i. \tag{5.28}$$

Let us calculate the derivatives. We then have to use

$$\frac{\partial r}{\partial x_i} = \frac{x_i}{r}. \tag{5.29}$$

Exercise 4.1 *Starting from*

$$r^2 = x_j x_j$$

show (5.29)

Exercise 4.2 *Show that*

$$\begin{aligned} u_{i,j} &= \frac{u(r)}{r}\delta_{ij} + [r\frac{d}{dr}(\frac{u(r)}{r})]\frac{x_i x_j}{r^2} \\ &= \frac{du}{dr}\frac{x_i x_j}{r^2} + \frac{u(r)}{r}(\delta_{ij} - \frac{x_i x_j}{r^2}). \end{aligned} \tag{5.30}$$

First of all we see that $u_{i,j}$ is symmetric. This means that

$$\nabla \times \mathbf{u} = \mathbf{0}. \tag{5.31}$$

So there is *no rotation*. This also means that the linear strain tensor is simply

$$E_{ij} = u_{i,j}.$$

Now we use the symmetry of the problem to consider a point on the positive x_1-axis. This means that

$$x_2 = x_3 = 0.$$

Exercise 4.3 *Show that the only nonvanishing components of the linear strain tensor are now*

$$\begin{aligned} E_{11} &= \frac{du}{dr}, \\ E_{22} &= E_{33} = \frac{u}{r}. \end{aligned}$$

If we introduce spherical coordinates, the r-direction coincides with the x_1-direction. So we can equally well write

$$\begin{aligned} E_{rr} &= \frac{du}{dr}, \\ E_{\theta\theta} &= E_{\varphi\varphi} = \frac{u}{r}. \end{aligned} \tag{5.32}$$

We also find the trace of the linear strain tensor

$$E_{ii} = \frac{du}{dr} + 2\frac{u}{r} = \frac{1}{r^2}\frac{d}{dr}(r^2 u).$$

So from

$$E_{ii} = u_{i,i}$$

we recover the well-known expression for the divergence for a spherically symmetric vector field

$$E_{ii} = u_{i,i} = \frac{1}{r^2}\frac{d}{dr}(r^2 u). \tag{5.33}$$

4.6 Cylindrical symmetry

For cylindrical symmetry, we write the coordinates x_1, x_2, x_3, with x_3 along the axis of symmetry. We have

$$\mathbf{u} = (\frac{x_1}{r}, \frac{x_2}{r}, 0)u(r), \tag{5.34}$$

where r now is the distance to the axis,

$$r = \sqrt{x_1^2 + x_2^2}.$$

Exercise 4.4 *Show*

$$\frac{\partial r}{\partial x_1} = \frac{x_1}{r}, \frac{\partial r}{\partial x_2} = \frac{x_2}{r}, \frac{\partial r}{\partial x_3} = 0. \tag{5.35}$$

Exercise 4.5 *Show that when i, j take the values $1, 2$*

$$\begin{aligned} u_{i,j} &= \frac{u(r)}{r}\delta_{ij} + [r\frac{d}{dr}(\frac{u(r)}{r})]\frac{x_i x_j}{r^2} \qquad (5.36) \\ &= \frac{du}{dr}\frac{x_i x_j}{r^2} + \frac{u(r)}{r}(\delta_{ij} - \frac{x_i x_j}{r^2}). \end{aligned}$$

and that the remaining components vanish.

First of all we see that $u_{i,j}$ is symmetric. This means that

$$\boldsymbol{\nabla} \times \mathbf{u} = \mathbf{0}.$$

So there is *no rotation.* This also means that the linear strain tensor is simply

$$E_{ij} = u_{i,j}.$$

Now we use the symmetry of the problem to consider a point on the positive x_1-axis. This means that

$$x_1 = r;\ x_2 = x_3 = 0.$$

Exercise 4.6 *Show that in this case the linear strain tensor is given by (the other components vanish)*

$$\begin{aligned} E_{11} &= \frac{du}{dr}, \\ E_{22} &= \frac{u}{r}, \\ E_{33} &= 0. \end{aligned}$$

We now introduce cylindrical coordinates, the r-direction coincides with the x_1-direction and $Z = x_3$. So our result can be written

$$\begin{aligned} E_{rr} &= \frac{du}{dr}, \\ E_{\theta\theta} &= \frac{u}{r} \\ E_{zz} &= 0. \end{aligned} \tag{5.37}$$

We also find the trace of the linear strain tensor

$$u_{i,i} = E_{ii} = \frac{du}{dr} + \frac{u}{r} = \frac{1}{r}\frac{d}{dr}(ru). \tag{5.38}$$

Chapter 6

Kinematics

1 Velocity and acceleration

To be able to describe the motion of a body, we now allow the basic mapping (5.1) to be timedependent.

$$\mathbf{x} = \mathbf{x}(\mathbf{X},t). \tag{6.1}$$

When we follow a given particle, $\mathbf{X}$ is fixed. The *velocity* is then the derivative of this function, with $\mathbf{X}$ fixed

$$\mathbf{v} = \frac{\partial}{\partial t}\mathbf{x}(\mathbf{X},t).$$

We shall not use this notation, but instead write

$$\mathbf{v} = \frac{D\mathbf{x}}{Dt}. \tag{6.2}$$

$Subs\tan tial$ **derivative**

D/Dt is called the *substantial derivative*. Substantial means that we are following a given particle in its motion.

The acceleration, then is

$$\mathbf{a} = \frac{D\mathbf{v}}{Dt} = \frac{D^2\mathbf{x}}{Dt^2}. \tag{6.3}$$

Lagrangian and Eulerian descriptions

Using $\mathbf{X}$,t as independent variables is often very useful. Such a description is called *material* (or *Lagrangian).* In the present course, we shall, however instead use $\mathbf{x}$,t as independent variables. Such a description is called *spatial* or *local* (or *Eulerian*). We consider a scalar field $f(\mathbf{x}, t)$. Following a particle, $\mathbf{x}$ will be a function of time,

$$\mathbf{x} = \mathbf{x}(t). \tag{6.4}$$

This is the same as (6.1), where we have suppressed $\mathbf{X}$.

$$f(t) = f(x_j(t), t).$$

Differentiating with respect to t we find the substantial derivative (as we are following a given particle)

$$\frac{Df}{Dt} = \frac{\partial f}{\partial x_j}\frac{Dx_j}{Dt} + \frac{\partial f}{\partial t} = v_j\frac{\partial f}{\partial x_j} + \frac{\partial f}{\partial t} = \mathbf{v}\cdot\boldsymbol{\nabla} f + \frac{\partial f}{\partial t}. \tag{6.5}$$

We can use the same formula for a component of the velocity field, to obtain

$$\begin{aligned} \frac{Dv_i}{Dt} &= v_j\frac{\partial v_i}{\partial x_j} + \frac{\partial v_i}{\partial t} = (v_j\frac{\partial}{\partial x_j})v_i + \frac{\partial v_i}{\partial t} \\ &= v_j v_{i,j} + \frac{\partial v_i}{\partial t}. \end{aligned} \tag{6.6}$$

But

$$v_j\frac{\partial}{\partial x_j} = \mathbf{v}\cdot\boldsymbol{\nabla},$$

so that in vector notation

$$\frac{D\mathbf{v}}{Dt} = (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v} + \frac{\partial \mathbf{v}}{\partial t}. \tag{6.7}$$

In the local description, acceleration is a non-linear expression in the velocity field. This non-linearity is a source of much of the difficulties and richness of problems of continuum mechanics. Using a material description we have a linear expression for the acceleration, but in general non-linearities will appear elsewhere.

1.1 Small deformations

If we introduce the displacement vector and write

$$\mathbf{x} = \mathbf{X} + \mathbf{u}, \tag{6.8}$$

we find that

$$\mathbf{v} = \frac{D\mathbf{u}}{Dt}, \tag{6.9}$$

as $\mathbf{X}$ is constant (we are following a given particle). We also have

$$\mathbf{v} = \frac{D\mathbf{u}}{Dt} = (\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{u} + \frac{\partial \mathbf{u}}{\partial t}. \tag{6.10}$$

In components,

$$\frac{Du_i}{Dt} = v_j u_{i,j} + \frac{\partial u_i}{\partial t}.$$

For small deformations, $u_{i,j}$ are small, see (5.4), so that

$$\mathbf{v} \approx \frac{\partial \mathbf{u}}{\partial t}. \tag{6.11}$$

Similarly, for small deformations, the acceleration is

$$\mathbf{a} \approx \frac{\partial^2 \mathbf{u}}{\partial t^2}. \tag{6.12}$$

In the following exercise we have a simple kinematical model of an explosion.

Example 3 *The velocity field of rigid rotation about the origin is*

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}. \tag{6.13}$$

We assume that the angular velocity $\boldsymbol{\omega}(t)$ is a function of time. We find the local derivative $\partial \mathbf{v}/\partial t$ and the substantial derivative $D\mathbf{v}/\mathbf{D}t = \mathbf{a}$ of the velocity field to be given by

$$\frac{\partial \mathbf{v}}{\partial t} = \frac{d\boldsymbol{\omega}}{dt} \times \mathbf{r}.$$

$$\begin{aligned} \mathbf{a} &= \frac{D\mathbf{v}}{Dt} = \frac{d\boldsymbol{\omega}}{dt} \times \mathbf{r} + \boldsymbol{\omega} \times \frac{D\mathbf{r}}{Dt} \\ &= \frac{d\boldsymbol{\omega}}{dt} \times \mathbf{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}). \end{aligned} \tag{6.14}$$

Exercise 1.1 *Exercise. A velocity field is given by*

$$\mathbf{v} = t^{-1}\mathbf{r}.$$

Sketch the velocity field. It is a simple model of an explosion taking place at time $t = 0$. Show that the local time derivative of the velocity is

$$\frac{\partial \mathbf{v}}{\partial t} = -t^{-2}\mathbf{r}.$$

Also calculate the acceleration and show that it vanishes. -The small volume elements (or particles) of the body move like free particles after the explosion.

2 Deformation rate and vorticity

Let us choose the configuration at time t as reference configuration for the moment. We consider the displacement vector $\mathbf{u}$ to the position at time $t + \mathrm{d}t$. When $\mathrm{d}t$ is small enough, this is a small deformation (of the configuration at time t), so that the velocity field is

$$\mathbf{v} \approx \frac{\partial \mathbf{u}}{\partial t}.$$

This means that we can write

$$\mathbf{u} \approx \mathbf{v}\mathrm{d}t$$

We can analyze this displacement field as a small displacement and write

$$u_{i,j} \approx v_{i,j}\mathrm{d}t = (v_{(i,j)} + v_{[i,j]})\mathrm{d}t$$

We find the linear strain tensor for the displacement in time $\mathrm{d}t$

$$E_{ij} = v_{(i,j)}\mathrm{d}t.$$

Dividing the linear strain tensor by $\mathrm{d}t$, we find the rate of linear strain, which is called the *rate of deformation tensor D_{ij}*

$$D_{ij} = v_{(i,j)}. \tag{6.15}$$

Similarly, we find the linear rotation vector, or small angle of rotation in time $\mathrm{d}t$ as

$$\boldsymbol{\phi} = \frac{1}{2}\mathrm{rot}\,\mathbf{u} = \frac{1}{2}\mathrm{rot}\,\mathbf{v}\,\mathrm{d}t$$

If we divide this with dt we find the angular velocity of a small material volume as

$$\frac{1}{2}\operatorname{rot}\mathbf{v}. \tag{6.16}$$

The vector

$$\boldsymbol{\zeta} = \operatorname{rot}\mathbf{v} \tag{6.17}$$

is called *the vorticity.*

For small deformations from a given initial configuration (not necessarily the one at time t as in the above reasoning), we have that

$$\mathbf{v} \approx \frac{\partial \mathbf{u}}{\partial t},$$

see (6.11). This gives us the rate of deformation tensor

$$D_{ij} = v_{(i,j)} \approx \frac{\partial}{\partial t} u_{(i,j)} \tag{6.18}$$

and the vorticity

$$\boldsymbol{\zeta} = \operatorname{rot}\frac{\partial \mathbf{u}}{\partial t} = \frac{\partial}{\partial t}\operatorname{rot}\mathbf{u}. \tag{6.19}$$

Chapter 7

The dynamic equations of continuum mechanics

1 Integral over material domain

We consider a material domain. Material means that it contains the same particles. This means that the domain will be a function of time. We also consider a field, f and the integral

$$\int f \mathrm{d}v.$$

We need to calculate the time-derivative of this integral. There are two contributions to its change. The first is that f changes locally. This change in time $\mathrm{d}t$ is given by

$$\int (\frac{\partial f}{\partial t} \mathrm{d}t) \mathrm{d}v.$$

But the domain is also changing with time. We consider a small surface element $\mathrm{d}\mathbf{s}$. The points of this small surface element will move $\mathbf{v}\mathrm{d}t$. This means that the surface element will sweep out a cylinder with volume $\mathrm{d}\mathbf{s} \cdot \mathbf{v}\mathrm{d}t$. So the integral will have the change

$$\int f(\mathrm{d}\mathbf{s} \cdot \mathbf{v})\mathrm{d}t.$$

Adding together the contribution from local change of the function f and the change of the domain, we have the total change. Dividing it with dt we have the time-derivative

$$\frac{d}{dt}\int f\mathrm{d}v = \int \frac{\partial f}{\partial t}\mathrm{d}v + \int f\mathrm{d}\mathbf{s}\cdot\mathbf{v}.$$

We can apply Gauss's theorem on the surface integral. We obtain

$$\frac{d}{dt}\int f\mathrm{d}v = \int [\frac{\partial f}{\partial t} + \boldsymbol{\nabla}\cdot(f\mathbf{v})]\mathrm{d}v.$$

This equation can be rewritten. We recall that the substantial derivative is given by

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + (\mathbf{v}\cdot\boldsymbol{\nabla})f.$$

This means that the time-derivative of the integral over a material domain can be written

$$\frac{d}{dt}\int f\mathrm{d}v = \int [\frac{Df}{Dt} + f(\boldsymbol{\nabla}\cdot\mathbf{v})]\mathrm{d}v.$$

2 Mass

The mass of the domain is given by

$$\int \rho\mathrm{d}v,$$

ρ is the density. The mass does not change with time. We conclude that conservation of mass is expressed by the *equation of continuity*

$$\frac{D\rho}{Dt} + \rho\,\mathrm{div}\mathbf{v} = 0. \tag{7.1}$$

Alternatively we can write the equation of continuity

$$\rho_{,t} + \mathrm{div}\,(\rho\mathbf{v}) = 0 \tag{7.2}$$

Or in integral form

$$\frac{\partial}{\partial t}\int \rho \mathrm{d}v = -\int \rho \mathbf{v}\cdot \mathrm{d}\mathbf{s} \tag{7.3}$$

The partial time derivative signifies that the volume of integration is fixed in space and not a material volume.

Example 4 *Gas flow in a tube with slowly varying cross section.For a stationary flow,*

$$\int \rho \mathbf{v}\cdot d\mathbf{s} = 0.$$

As the cross section is slowly varying we can consider the flow as one-dimensional. Choosing as volume of integration the region between two cross sections, we find

$$\rho_1 v_1 A_1 = \rho_2 v_2 A_2. \tag{7.4}$$

A stands for the area of the cross section. We conclude that $\rho A v$ is constant in the tube.

We can now simplify the integral. If we write the function f as $g\rho$ we have

$$\frac{d}{dt}\int g\rho \mathrm{d}v = \int \frac{Dg}{Dt}\rho \mathrm{d}v. \tag{7.5}$$

This follows from the following

Exercise 2.1 *Show that*

$$\frac{D}{Dt}(g\rho) + g\rho(\boldsymbol{\nabla}\cdot \mathbf{v}) = \rho\frac{Dg}{Dt}.$$

3 Momentum

The momentum of a material part of the body is given by

$$\int \mathbf{v}\rho \mathrm{d}v$$

The time derivative of the momentum equals the total force. Let urs first write down the time derivative.

$$\frac{d}{dt}\int \mathbf{v}\rho \mathrm{d}v = \int \frac{D\mathbf{v}}{Dt}\rho \mathrm{d}v = \int \mathbf{a}\mathrm{d}m. \tag{7.6}$$

3.1 The stress vector

According to the law of momentum, the time-derivative of the momentum equals the force. The inner forces in the material have a very short range. Hence, the force on a part of a body can be represented as a force acting on its surface. Let us write the outward directed area vector

$$\mathrm{d}\mathbf{s} = \mathbf{n}\mathrm{d}s. \tag{7.7}$$

We denote the force on d$\mathbf{s}$ by

$$\mathbf{t}\mathrm{d}s\ . \tag{7.8}$$

The surface force is proportional to the area of the small surface. It is the force on the side from which $\mathbf{n}$ is pointing. It is a function not only of $\mathbf{x}$ but also of the normal $\mathbf{n}$.[1] The force per area, $\mathbf{t}(\mathbf{n})$, is called the *stress vector*.

Balance of momentum (Newton's second law) is thus expressed by

$$\frac{d}{dt}\int \mathbf{v}\rho \mathrm{d}v = \int \frac{D\mathbf{v}}{Dt}\rho \mathrm{d}v = \int \mathbf{t}(\mathbf{n})\mathrm{d}s + \int \mathbf{f}\rho \mathrm{d}v. \tag{7.9}$$

3.2 The stress tensor

If we look at a small region, with a typical length l, its area is of the order of l^2, but its volume of the order l^3. In (7.9) there is just one surface integral. According to the equation it is equal to a volume integral, hence must be of the order of l^3 rather than l^2. Hence, it must vanish to order l^2

To be more precise, let us choose an arbitrary vector $\mathbf{n}$. We then choose a tetrahedron having one surface with $\mathbf{n}$ as outward normal. The three remaining surfaces are chosen with outwards normals -$\mathbf{e}_1, -\mathbf{e}_2, -\mathbf{e}_3$, along the negative axes. If the area of the first surface is ds, the areas of the other surfaces are $(\mathbf{n}\cdot\mathbf{e}_i)\mathrm{d}s = n_i\mathrm{d}s$. The total surface force is thus

$$\begin{aligned}\int t_i(\mathbf{n}')\mathrm{d}s' &= [t_i(\mathbf{n}) + t_i(-\mathbf{e}_1)n_1 + t_i(-\mathbf{e}_2)n_2 + t_i(-\mathbf{e}_3)n_3]\mathrm{d}s \\ &\quad + O((ds)\times(ds)^{\frac{1}{2}})\end{aligned}$$

[1] This is often referred to as the Cauchy stress hypothesis

The error term is of the order of the volume of the tetrahedron. As the sum is equal to a volume integral, the term proportional to ds has to vanish. Using (7.8) we obtain

$$t_i(\mathbf{n}) = -t_i(-\mathbf{e}_1)n_1 - t_i(-\mathbf{e}_2)n_2 - t_i(-\mathbf{e}_3)n_3.$$

Let us, first of all, choose $\mathbf{n}$ as $\mathbf{e}_1$, so that the body is flat. We find

$$t_i(\mathbf{e}_1) = -t_i(-\mathbf{e}_1).$$

This is really Newton's third law of action and reaction. The same, of course, applies to the other two coordinate directions. So we can take away the minus signs to obtain

$$t_i(\mathbf{n}) = t_i(\mathbf{e}_1)n_1 + t_i(\mathbf{e}_2)n_2 + t_i(\mathbf{e}_3)n_3.$$

If we write

$$T_{ij} = t_i(\mathbf{e}_j), \tag{7.10}$$

we have

$$t_i(\mathbf{n}) = T_{ij}n_j. \tag{7.11}$$

But this formula applies to any vector $\mathbf{n}$. This means that we can apply the quotient theorem and conclude that T_{ij} are the components of a tensor, the *stress tensor.*

We can now express balance of momentum (7.9) as

$$\int_v \frac{Dv_i}{Dt}\rho \mathrm{d}v = \int T_{ij}\mathrm{d}s_j + \int f_i \rho \mathrm{d}v \ . \tag{7.12}$$

Here, the i- component of the surface force can be written, using Gauss's theorem,

$$\int T_{ij}\mathrm{d}s_j = \int T_{ij,j}\mathrm{d}v \ .$$

We conclude that balance of momentum is expressed by the local equation

$$\rho\frac{D\mathbf{v}}{Dt} = \mathrm{div}\ \mathbf{T} + \rho\mathbf{f}; \tag{7.13}$$

$$\rho\frac{Dv_i}{Dt} = T_{ij,j} + \rho f_i. \tag{7.14}$$

This equation and the equation of continuity are the basic dynamic equations of continuum mechanics.

For an arbitrary unit vector $\mathbf{n}$, we write the stress vector

$$\mathbf{t}(\mathbf{n}) = t_n\mathbf{n} + \mathbf{t}_t \tag{7.15}$$

where the scalar $t_{(n)}$ is the *normal stress* and the vector $\mathbf{t}_t$, which is tangent to the surface, is the *shear stress.* We have

$$t_n = \mathbf{n} \cdot \mathbf{T}\mathbf{n} = n_i T_{ij} n_j. \tag{7.16}$$

Note that t_n is the same for $\mathbf{n}$ and $-\mathbf{n}$.

Exercise 3.1 *Introduce the tensor*

$$P_{ij} = \delta_{ij} - n_i n_j. \tag{7.17}$$

Show that the components $t_{(t)i}$ of the shear stress $\mathbf{t}_t$ are given by

$$t_{(t)i} = P_{ik} T_{kj} n_j \ .$$

Conditions at a surface of discontinuity

Let us consider a surface where two different materials meet, *e.g.* the surface of a liquid, where the liquid meets the gas. We apply (7.9) to a cylinder of small height at the surface. Using the second equality sign in the equation, we note that all terms are volume integrals except the contribution from the stresses. If we now let the height of the cylinder go to zero, only the stress term remains. If the cylinder is small enough we can write the stress term

$$[\mathbf{t}_{(2)}(\mathbf{n}) + \mathbf{t}_{(1)}(-\mathbf{n})]\mathrm{d}s,$$

which thus has to vanish. We conclude that the *stress vector is continuous at the bounding surface*, or

$$T_{(1)ij} n_j = T_{(2)ij} n_j \tag{7.18}$$

The stress *tensor* is however not in general continuous.. In particular for the surface of a liquid. In the gas there is just a pressure. So the condition at the bounding surface is that the normal stress in the liquid equals minus the pressure in the gas

$$t_n = -p \tag{7.19}$$

Further, the shear stress vanishes.

Maximum shear stress

We take two orthogonal unit vectors and consider the shear stress

$$\tau = \mathbf{m} \cdot \mathbf{Tn}. \tag{7.20}$$

We look for its maxima and minima. We have

$$d\tau = \mathbf{m} \cdot \mathbf{T}d\mathbf{n} + d\mathbf{m} \cdot \mathbf{Tn} = \mathbf{m} \cdot \mathbf{T}d\mathbf{n} + \mathbf{n} \cdot \mathbf{T}d\mathbf{m}.$$

The vectors $d\mathbf{m}$,$d\mathbf{n}$ are however not arbitrary. $\mathbf{m}, \mathbf{n}$ satisfy

$$\mathbf{m} \cdot \mathbf{m} = 1, \ \mathbf{n} \cdot \mathbf{n} = 1, \ \mathbf{m} \cdot \mathbf{n} = 0.$$

Hence $d\mathbf{m}$,$d\mathbf{n}$ have to satisfy

$$\begin{aligned} \mathbf{m} \cdot d\mathbf{m} &= 0, \ \mathbf{n} \cdot d\mathbf{n} = 0, \\ \mathbf{m} \cdot d\mathbf{n} + \mathbf{n} \cdot d\mathbf{m} &= 0. \end{aligned} \tag{7.21}$$

Let us introduce a third unit vector $\mathbf{p}$ orthogonal to $\mathbf{m}, \mathbf{n}$. We can then write

$$\begin{aligned} d\mathbf{m} &= dm_n \mathbf{n} + dm_p \mathbf{p}, \\ d\mathbf{n} &= dn_m \mathbf{m} + dn_p \mathbf{p}. \end{aligned}$$

The condition $(7.21)_2$ gives

$$dn_m = -dm_n.$$

Now we have three differentials dm_n, dm_p, dn_p which are independent. We find

$$d\tau = (\mathbf{n} \cdot \mathbf{Tn} - \mathbf{m} \cdot \mathbf{Tm}) dm_n + \mathbf{m} \cdot \mathbf{Tp} dn_p + \mathbf{n} \cdot \mathbf{Tp} dm_p. \tag{7.22}$$

The last two terms give

$$\mathbf{m} \cdot \mathbf{Tp} = \mathbf{n} \cdot \mathbf{Tp} = 0.$$

$\mathbf{Tp}$ is thus parallel to $\mathbf{p}$, so that $\mathbf{p}$ is an eigenvector. $\mathbf{m}, \mathbf{n}$ lie in the plane spanned by the other two eigenvectors. The first term in (7.22) gives

$$\mathbf{n} \cdot \mathbf{Tn} - \mathbf{m} \cdot \mathbf{Tm} = 0. \tag{7.23}$$

If we introduce the orthonormal vectors

$$\mathbf{e}_1 = \frac{1}{\sqrt{2}}(\mathbf{m} + \mathbf{n}), \ \mathbf{e}_2 = \frac{1}{\sqrt{2}}(\mathbf{m} - \mathbf{n})$$

from (7.23) we find that

$$\mathbf{e}_2 \cdot \mathbf{T}\mathbf{e}_1 = 0.$$

$\mathbf{T}\mathbf{e}_1$ is parallel to $\mathbf{e}_1$. This means that $\mathbf{e}_1$ is an eigenvector and so is $\mathbf{e}_2$. The maximum shear stress occurs at $\pi/4$ with the eigenvectors.

4 Angular momentum

Let us now consider the balance of angular momentum. The angular momentum with respect to $\mathbf{x}_{(0)}$ is given by

$$\int (\mathbf{x} - \mathbf{x}_{(0)}) \times \mathbf{v}\rho dv.$$

The i-component of the moment of the surface forces is (we use Gauss's theorem)

$$\int \epsilon_{ijk}(x_j - x_{(0)i})T_{kl}\mathrm{d}s_l = \int [\epsilon_{ijk}(x_j - x_{(0)i})T_{kl,l} + \epsilon_{ijk}T_{kj}]\mathrm{d}v$$

We conclude that balance of angular momentum is expressed by

$$\epsilon_{ijk}T_{kj} = 0$$

which means that the stress tensor is *symmetric*.[2]

Exercise 4.1 *Fill in the details of the preceding argument.*

5 Balance laws and conservation laws

Using the equation of continuity, we can write (in the last equality we have used the equation of continuity)

[2] Some materials have a small *internal* angular momentum in addition to usual term. For these materials the stress tensor is not symmetric. Such materials are called *polar*.

$$\begin{aligned}\frac{Dg}{Dt}\rho &= (g_{,t} + v_i g_{,i})\rho \\ &= (\rho g)_{,t} - g\rho_{,t} + (\rho g v_k)_{,k} - g(\rho v_k)_{,k} = (\rho g)_{,t} + (\rho g v_k)_{,k}\end{aligned}$$

We conclude that

$$\frac{Dg}{Dt}\rho = (\rho g)_{,t} + (\rho g v_j)_{,j} \tag{7.24}$$

Hence, we can write the equation of momentum (7.13) in the following alternative form

$$(\rho v_i)_{,t} + (\rho v_i v_j - T_{ij})_{,j} = \rho f_i. \tag{7.25}$$

This is most easily seen in Cartesian coordinates, putting $g = v_i$ in (7.24). We compare (7.25) with the equation of continuity. In components, ρv_i is the density of i-component of momentum. $\rho v_i v_k - T_{ik}$ is the k-component of the current density of i-component of momentum. $\rho\mathbf{f}$ is the momentum supplied by the volume forces.

We can also express this in integral form

$$\frac{\partial}{\partial t}\int \rho v_i \mathrm{d}v = -\int (\rho v_i v_j - T_{ij})\mathrm{d}s_j + \int f_i \rho \mathrm{d}v \tag{7.26}$$

The partial time derivative here expresses that the volume of integration is independent of time. *Cf* (7.12), where balance of momentum is expressed for a material volume.

Example 5 *The force on a bent tube. We now consider the stationary flow in a bent tube. Momentum balance is expressed by*

$$\int (\rho v_i v_j - T_{ij})\mathrm{d}s_j = 0.$$

The flow is assumed incompressible and friction negligible except in a small boundary layer at the wall. The cross section area of the tube is A, the same before (1) and after (2) the bend. According to the Bernoulli equation the pressure *is the same at* 1 *and* 2*. At* 1 *and* 2 *there is just pressure, so that,* $T_{ij}= -p\delta_{ij}$ *. Thus*

$$\int_{1+2} T_{ij}\mathrm{d}s_j = -\int_{1+2} p\mathrm{d}s_i$$

and

$$\int_{1+2} p\mathrm{d}\mathbf{s} = pA(\mathbf{e}_2 - \mathbf{e}_1) = p\sqrt{2}A\mathbf{n}.$$

Further

$$\int_{1+2} \rho\mathbf{v}(\mathbf{v}\cdot\mathrm{d}\mathbf{s}) = \rho v^2 A(\mathbf{e}_2 - \mathbf{e}_1) = \rho v^2\sqrt{2}A\mathbf{n}.$$

The remaining integration is over the wall only. Here, the velocity vanishes. The force that the wall exerts on the fluid is

$$\int_{wall} T_{ij}\mathrm{d}s_j = -F_i.$$

According to Newton's third law the force on the tube exerted by the fluid is $\mathbf{F}$*. Adding all this together, we have*

$$-\mathbf{F}+(p+\rho v^2)\sqrt{2}A\mathbf{n} = \mathbf{0}.$$

The force exerted by the fluid on the tube is thus

$$\mathbf{F} = (p+\rho v^2)\sqrt{2}A\mathbf{n}.$$

Chapter 8

Elastic materials

1 Elastic energy

In an elastic material, fluid or solid, the stress tensor only depends on the deformation gradient $\mathbf{F}$, but not on the velocity and its derivatives. We assume that there is an elastic energy ϵ per mass. The total energy in a part of the body is then the sum of kinetic and elastic energy

$$E = \int (\frac{\mathbf{v}^2}{2} + \epsilon) \mathrm{d}m$$

The rate of change of E is given by the power exerted on the part of the body by volume and surface forces, *i.e.*

$$\dot{E} = \int v_i \cdot T_{ij} \mathrm{d}s_j + \int v_i f_i \mathrm{d}m \tag{8.1}$$

Energy balance expressed locally

We have

$$\dot{E} = \int (v_i \frac{Dv_i}{Dt} + \frac{D\epsilon}{Dt}) \mathrm{d}m$$

Let us now apply Gauss's theorem to transform the power of the surface forces into a volume integral.

$$(v_i T_{ij})_{,j} = v_{i,j} T_{ij} + v_i T_{ij,j} = D_{ij} T_{ij} + v_i T_{ij,j}. \tag{8.2}$$

In the expression for balance of energy most terms now cancel out due to the equation for balance of momentum. There remains

$$\rho \frac{D\epsilon}{Dt} = D_{ij} T_{ij} \tag{8.3}$$

which is the local expression of energy balance.

The elastic energy function

In linear elasticity, the elastic energy is assumed to be a function of the linear strain tensor $\mathbf{E}$.

2 Galilean invariance

We recall, that the linear strain tensor is

$$E_{ij} = u_{(i,j)}.$$

What about the antisymmetric part of $u_{i,j}$. Could the elastic energy depend on $u_{[i,j]}$? We remember, that this antisymmetric part gives the rotation angle

$$\boldsymbol{\phi} = \frac{1}{2} \boldsymbol{\nabla} \times \mathbf{u}$$

But all directions in space are equal. So a superposed rotation cannot change the energy. We conclude, that it is not possible for the elastic energy to depend on the antisymmetric part $u_{[i,j]}$. This is a special case of *Galilean invariance.*

We conclude that due to Galilean invariance the elastic energy is a function

$$\epsilon = \epsilon(\mathbf{E}). \tag{8.4}$$

The reference configuration is assumed to be stress-free. This means that the elastic energy has to be quadratic in the linear strain tensor:

$$\epsilon = \frac{1}{2\rho_0} C_{ijkl} E_{ij} E_{kl}. \tag{8.5}$$

C_{ijkl} is called the *elasticity tensor* and ρ_0 is the density in the reference configuration. It is clear that we can choose the elasticity tensor to satisfy the symmetries

$$C_{ijkl} = C_{jikl} = C_{klij}. \tag{8.6}$$

There are 6 independent components in a symmetric tensor. So, the first pair of indices in the elasticity tensor C_{ijkl} can take 6 independent values and so can the second pair. As C_{ijkl} is symmetric also in the two pairs, there are in all at most $(6 \cdot 7)/2 = 21$ independent components in the elasticity tensor.

Exercise 2.1 *Fill in the details in the following reasoning*

$$\begin{aligned} C_{ijkl}E_{ij}E_{kl} &= C_{jikl}E_{ji}E_{kl} = C_{jikl}E_{ij}E_{kl} \\ &= \frac{1}{2}(C_{ijkl} + C_{jikl})E_{ij}E_{kl}. \end{aligned}$$

This proves the first equality (8.6). Prove also the second equality.

Stress tensor as derivative of elastic energy

For linear elasticity we can in the equation (8.3) approximate ρ by ρ_0, replace the substantial derivative by a partial time derivative. We can also use the approximation $\mathbf{D} \approx \partial \mathbf{E}/\partial t$,

$$\rho_0 \frac{\partial \epsilon}{\partial t} = T_{ij} \frac{\partial E_{ij}}{\partial t}. \tag{8.7}$$

Using (8.5) we find

$$\begin{aligned} \rho_0 \frac{\partial \epsilon}{\partial E_{ij}} \frac{\partial E_{ij}}{\partial t} &= T_{ij} \frac{\partial E_{ij}}{\partial t}, \\ C_{ijkl}E_{kl} \frac{\partial E_{ij}}{\partial t} &= T_{ij} \frac{\partial E_{ij}}{\partial t}. \end{aligned}$$

As $\partial E_{ij}/\partial t$ is an arbitrary symmetric tensor, this gives us

$$T_{ij} = \rho_0 \frac{\partial \epsilon}{\partial E_{ij}} = C_{ijkl}E_{kl}. \tag{8.8}$$

We have now completely explored the energy balance equation. It is an identity, which is expressed in (8.8). It is important to realize that in the elastic case, we have just two dynamic equations: the equation of continuity (7.1) and the equation of momentum (7.13). The equation of energy we have just studied is precisely equivalent to the preceding equation, giving the stress as a derivative of the elastic energy.

3 Material symmetry

3.1 Isotropic materials

We have already in (3.19) found the most general isotropic tensor with the symmetries of the elasticity tensor,

$$C_{ijkl} = \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \lambda\delta_{ij}\delta_{kl}. \tag{8.9}$$

We have also calculated the stress tensor in (1.24)

$$T_{ij} = 2\mu E_{ij} + \lambda E_{kk}\delta_{ij}. \tag{8.10}$$

We have even calculated the elastic energy in (1.25),

$$\rho\epsilon = \mu E_{ij}E_{ij} + \frac{\lambda}{2}(E_{kk})^2. \tag{8.11}$$

Elastic fluids

For a *fluid*, the elastic energy is just a function of the density.

$$\epsilon = \epsilon(\rho). \tag{8.12}$$

From the equation of continuity,

$$\frac{D\rho}{Dt} + \rho\,\mathrm{div}\,\mathbf{v} = 0.$$

But

$$\mathrm{div}\,\mathbf{v} = v_{i,i} = D_{ii} = D_{ij}\delta_{ij}.$$

We conclude that

$$\frac{D\rho}{Dt} = -\rho\,\delta_{ij}D_{ij}.$$

The Cauchy stress tensor is, see (8.8)

$$T_{ij} = -\rho^2 \frac{\partial \epsilon}{\partial \rho} \delta_{ij}.$$

$$\mathbf{T} = -\rho^2 \frac{\partial \epsilon}{\partial \rho} \mathbf{1} = -p\mathbf{1}. \tag{8.13}$$

In other words, the stress tensor in an elastic fluid reduces to just a pressure: an elastic fluid is the same as an *ideal fluid*.

$$p = \rho^2 \frac{\partial \epsilon}{\partial \rho} = -\frac{\partial \epsilon}{\partial v}$$

where

$$v = \frac{1}{\rho}$$

is the volume per mass. We can write

$$\mathrm{d}\epsilon = -p\mathrm{d}v.$$

Connection between linearly elastic solid and a fluid

For small deformations, the relative increase of volume is $E_{ii} = u_{i,i}$. In other words,

$$\frac{v}{v_0} \approx 1 + E_{ii}.$$

The deviator $E_{<ij>}$ does not enter the elastic energy. This would correspond to vanishing of the shear modulus μ. So in this sense, a fluid is a special case of an elastic solid. There is however a difference in that there is no stressfree state in a fluid: the pressure is always positive. So, if we develop a theory of linearly elastic solid, but with a reference configuration, which is deformed, we can specialize this to small deformations of a fluid, by putting the value of the shear modulus to zero. This applies to sound waves, in particular.

Chapter 9

Isotropic linearly elastic solids

1 Navier's equations

We recall that the stress tensor for an isotropic elastic solid in small deformations from a stress free state is given by

$$\mathbf{T} = 2\mu\mathbf{E} + \lambda(\,\mathrm{tr}\mathbf{E})\mathbf{1}. \tag{9.1}$$

The equation of equilibrium is

$$\mathrm{div}\ \mathbf{T} + \rho\mathbf{f} = \mathbf{0}. \tag{9.2}$$

For a homogeneous material, this is explicitly

$$\mu \triangle \mathbf{u} + (\mu+\lambda) grad\ div \mathbf{u} + \rho\mathbf{f} = 0. \tag{9.3}$$

These are the so-called *Navier's equations.* In cartesian coordinates,

$$\mu u_{i,kk} + (\mu+\lambda)u_{k,ki} + \rho f_i = 0. \tag{9.4}$$

Example 6 *Let us start with a simple example. We consider a rectangular solid body, standing on the ground and deforming elastically under its own weight* *and air pressure. So we are comparing the shape of the body out in free space and on the earth. Let us choose the z-axis vertical and upwards, $z = 0$ corresponding to the ground . The lengths of the edges are called a, b, c respectively. Let us assume that the displacement is in the vertical direction only. From the x- and y-components of Navier's equations, we realize that $E_{zz} = u_{z,z}$ is a function of z only. Hence, u_z is a function of z plus a function of x, y. But clearly u_z has to vanish at $z = 0$, which excludes the function of x, y. The z-component of Navier's equation is then*

$$(2\mu + \lambda)u_{z,zz} - \rho g = 0.$$

Consequently,

$$u_z = \frac{\rho g}{2\mu + \lambda}\left(\tfrac{1}{2}z^2 + Az + B\right),$$

where A and B are constants. The boundary condition at $z = 0$ gives $B = 0$. On the top surface $z = c$ the stress vector has to reduce to the air pressure (as the air pressure is negligible in comparison . We have T_{zz}

$$T_{zz} = (2\mu + \lambda)u_{z,z} = \rho g(z + A).$$

For $z = c$ this has to reduce to $-p_0$, giving

$$A = -c - \frac{p_0}{\varrho g}.$$

We then obtain

$$T_{zz} = -[p_0 + \rho g(c - z)].$$

The displacement field becomes

$$u_z = \frac{z}{2\mu + \lambda}[\rho g(\frac{1}{2}z - c) - p_0].$$

We also have normal stresses in the other directions

$$T_{xx} = T_{yy} = \lambda u_{z,z} = \frac{\lambda}{2\mu + \lambda}T_{zz} = \frac{\nu}{1 - \nu}T_{zz}.$$

Consequently, a pressure

$$p = \frac{\nu}{1 - \nu}[p_0 + \rho g(c - z)]$$

has to be applied to the vertical sides of the body for the displacement to be purely vertical. When those sides are free, acted on by the air pressure only, the displacement is not just in the vertical direction.

2 Homogeneous deformations

In homogeneous deformations, **E** is constant, so that **T** is constant. The equation of equilibrium is then clearly satisfied if there is no volume force.

2.1 Shear and shear modulus

In a state of pure shear, choosing coordinates suitably, the displacement vector can be written

$$\mathbf{u} = u(x_2)\mathbf{e}_1.$$

The only non-vanishing components of the linear strain tensor are

$$E_{12} = E_{21} = \frac{1}{2}\frac{\mathrm{d}u}{\mathrm{d}x_2},$$

$\mathrm{d}u/\mathrm{d}x_2$ is the angle of shear γ .

The non-vanishing components of the stress tensor are

$$T_{12} = T_{21} = \mu\frac{du}{dx_2} = \mu\gamma. \tag{9.5}$$

The equation of equilibrium is just $d^2u/dx_2^2 = 0$, which means that the angle of shear is a constant.

Let us consider a cube, with edges parallel to the basis vectors. On the surfaces, normal to the x$_1$- and x_2- axis shear stresses $\mu\ \gamma$ have to be applied. μ is called the *shear modulus.*

2.2 Isotropic expansion and the bulk modulus

In an *isotropic expansion* **E** must be proportional to **1** , in other words, $\mathbf{E} = \frac{1}{3}(\text{ tr }\mathbf{E})\mathbf{1}$.

The stress tensor is thus

$$\mathbf{T} = (\lambda + \tfrac{2}{3}\mu)(tr\ \mathbf{E})\mathbf{I},$$

It is also isotropic, and contains just a pressure $p = -(\lambda + \frac{2}{3}\mu)\,\mathrm{tr}\mathbf{E}$. $\mathrm{tr}\mathbf{E}$ is the relative increase of volume. The coefficient (quotient between pressure and relative decrease of volume

$$\kappa = (\lambda + \tfrac{2}{3}\mu) \tag{9.6}$$

is called *bulk modulus.*

We can now divide up the linear strain tensor in its spherical part and deviator,

$$E_{ij} = E_{<ij>} + \frac{1}{3}E_{kk}\delta_{ij}. \tag{9.7}$$

Here, the *deviator* has vanishing trace,

$$E_{<ii>} = 0.$$

The stress tensor can then be written

$$T_{ij} = 2\mu E_{<ij>} + \kappa E_{kk}\delta_{ij}.$$

We have

$$\begin{aligned} T_{<ij>} &= 2\mu E_{<ij>}, \\ T_{kk} &= 3\kappa E_{kk}. \end{aligned} \tag{9.8}$$

From this we easily invert the relation

$$\begin{aligned} E_{<ij>} &= \frac{1}{2\mu}T_{<ij>}, \\ E_{kk} &= \frac{1}{3\kappa}T_{kk}, \end{aligned} \tag{9.9}$$

giving us

$$E_{ij} = \frac{1}{2\mu}T_{<ij>} + \frac{1}{9\kappa}T_{kk}\delta_{ij}. \tag{9.10}$$

2.3 Uniaxial tension and the modulus of elasticity

In a tensile test of a thin cylinder we have a uniaxial tension. A normal tension is applied in the direction of the axis of the cylinder. We choose the axis of the cylinder as x_1-axis. T_{11} is the only non-zero component of the stress tensor. The boundary conditions are then satisfied. In particular, the surface with normal perpendicular to the axis of the cylinder is stress free. From the preceding equation (9.10) we obtain when T_{11} is the only nonzero stress compontent the relative increase of length in the direction of the applied tension

$$E_{11} = \frac{1}{2\mu}(T_{11} - \frac{1}{3}T_{11}) + \frac{1}{9\kappa}T_{11} = (\frac{1}{3\mu} + \frac{1}{9\kappa})T_{11}.$$

Or

$$T_{11} = E_Y E_{11},$$

where

$$\frac{1}{E_Y} = \frac{1}{3\mu} + \frac{1}{9\kappa}, \tag{9.11}$$

$$E_Y = \mu \frac{9\kappa}{3\kappa + \mu}. \tag{9.12}$$

E_Y is the *modulus of elasticity* or *Young's modulus*

The cylinder also contracts in the orthogonal directions. We have

$$E_{22} = E_{33} = -\frac{1}{6\mu} T_{11} + \frac{1}{9\kappa} T_{11} = -(\frac{1}{6\mu} - \frac{1}{9\kappa}) T_{11} = -(\frac{1}{6\mu} - \frac{1}{9\kappa}) E_Y E_{11}.$$

The coefficient here is called *Poisson's ratio* ν

$$E_{22} = E_{33} = -\nu E_{11}.$$

We obtain

$$\frac{\nu}{E_Y} = \frac{1}{6\mu} - \frac{1}{9\kappa}, \tag{9.13}$$

$$\nu = \frac{1}{2} \frac{3\kappa - 2\mu}{3\kappa + \mu}. \tag{9.14}$$

The relative increase of volume is

$$\mathrm{tr}\, \mathbf{E} = (1 - 2\nu) E_{11}.$$

For metals ν is usually around 0.3, so that volume is increasing under a tensile test. For rubber, however, ν is close to 0.5. Its volume changes very little in a tensile test. Rubber is an almost *incompressible* material.

Let us also express μ, κ in terms of E_Y and ν. We have already found

$$\frac{1}{E_Y} = \frac{1}{3\mu} + \frac{1}{9\kappa},$$

$$\frac{\nu}{E_Y} = \frac{1}{6\mu} - \frac{1}{9\kappa}.$$

We use these as a linear system of equations for $1/\mu$ and $1/\kappa$. We find

$$\begin{aligned} \frac{1}{\mu} &= 2(\nu+1)\frac{1}{E_Y}, \\ \frac{1}{\kappa} &= 3(1-2\nu)\frac{1}{E_Y}. \end{aligned} \tag{9.15}$$

This also gives us

$$\begin{aligned} E_{<ij>} &= (\nu+1)\frac{1}{E_Y}T_{<ij>}, \\ E_{kk} &= (1-2\nu)\frac{1}{E_Y}T_{kk}. \end{aligned} \tag{9.16}$$

and

$$E_{ij} = \frac{1}{E_Y}[(\nu+1)T_{ij} - \nu T_{kk}\delta_{ij}]. \tag{9.17}$$

3 Stability of unstressed state

The unstressed state is stable. Hence, the elastic energy has a minimum there. The elastic energy is a quadratic form in (the components of) **E** which thus has to be positive definite. We divide up the linear strain tensor and the stress tensor in a spherical part and the deviator. For the linear strain tensor,

$$E_{ij} = E_{<ij>} + \frac{1}{3}E_{kk}\delta_{ij}. \tag{9.18}$$

Here, the *deviator* has vanishing trace. The important point is that the deviator and the trace are independent of each other.

Exercise 3.1 *Show that the elastic energy is*

$$\rho\epsilon = \mu E_{<ij>}E_{<ij>} + \frac{1}{2}\kappa(E_{ii})^2 + \frac{1}{2}(\kappa + \frac{2\mu}{3})E_{kk}\delta_{ij}E_{<ij>}.$$

Also show that the last term here vanishes.

We conclude that the elastic energy is

$$\rho\epsilon = \mu E_{<ij>}E_{<ij>} + \frac{\kappa}{2}(E_{ii})^2. \tag{9.19}$$

But the two terms here are independent. The elastic energy has to increase when the body is deformed. Otherwise, the undeformed state would not be stable. We conclude that the elastic constants have to satisfy

$$\mu \geq 0, \ \lambda + \frac{2\mu}{3} = \kappa \geq 0. \tag{9.20}$$

The *shear modulus is positive. The bulk modulus is positive.*

The latter condition could also be expressed in terms of Poisson's number ν. *Poisson's number is in the range between -1 and 1/2.* In practice, ν is, however, negative in very special materials only.

Exercise 3.2 *Show from*

$$\nu = \frac{1}{2}\frac{3\kappa - 2\mu}{3\kappa + \mu}$$

that Poisson's number has to lie in the range between -1 *and* $1/2$.

4 Uniqueness

Stability of the unstressed state also implies that the solution of the static problem in linear elasticity is unique.

We assume that on the boundary of the body either the displacement vector u_i is given or the stress vector

$$T_{ij}n_j = C_{ijkl}n_j u_{k,l}$$

is given. $\mathbf{n}$ is the outward normal. In the body the equation of equilibrium (9.2) is satisfied. Solving the problem consists in finding the displacement vector field $\mathbf{u}$.

Suppose there are two solutions of this problem. The *difference* between the solutions is now denoted $\mathbf{u}$. Similarly, the difference between the linear strain tensors is denoted E_{ij} and the difference between the stress tensors in the two solutions is denoted T_{ij}. Due to the linearity of the stress-strain relation, (9.1) also holds for the differences. On the boundary of the body either the displacement vector is given, so that

$$u_i = 0,$$

or the stress vector is given, so that

$$T_{ij}n_j = C_{ijkl}n_j u_{k,l} = 0.$$

In the body

$$T_{ij,j} = 0.$$

Let us now consider the following surface integral (formally a surface work)

$$\int u_i T_{ij} \mathrm{d}s_j.$$

This integral vanishes according to the boundary conditions, as one or the other of the two factors in the integral vanishes. We now appeal to Gauss's theorem.

$$\int u_i T_{ij} \mathrm{d}s_j = \int (u_i T_{ij})_{,j} \mathrm{d}v.$$

But

$$(u_i T_{ij})_{,j} = u_i T_{ij,j} + u_{i,j} T_{ij} = u_i T_{ij,j} + u_{(i,j)} T_{ij} = u_i T_{ij,j} + E_{ij} T_{ij}.$$

The last equality follows from the fact that T_{ij} is symmetric. But it also has vanishing divergence. We conclude that

$$\int u_i T_{ij} \mathrm{d}s_j = \int E_{ij} T_{ij} \mathrm{d}v.$$

The integrand here is twice the elastic energy per volune, so that

$$0 = \int u_i T_{ij} \mathrm{d}\mathbf{s}_j = 2 \int \epsilon \rho dv.$$

ϵ is, however, positive definite form in (the components of) E_{ij}. Consequently, the linear strain tensor $\mathbf{E}$ vanishes. This means that $\mathbf{u}$ is at most a rigid displacement. If the displacement vector is given at the boundary, $\mathbf{u}$, being the difference between the displacement vector for the two solutions, has to vanish there. This means that the solution is unique. If, instead, the stress vector is given on the boundary, this means that the difference between the two solutions is a rigid displacement. So the solution is then unique up to a rigid displacement.

5 Torsion of a cylinder

5.1 Circular cylinder

Let us first consider a *circular cylinder* of radius a and height l. We choose cartesian coordinates with the x_3-axis along the axis of the cylinder. On the bottom and top of the cylinder ($x = 0$ and $x = l$) moments are applied. The cylindrical surface is free. We guess, that a plane perpendicular to the axis is turned a small angle, the size of which is proportional to x_3. Let us write the angle βx_3. The displacement vector is hence

$$\begin{aligned} u_1 &= -\beta x_3 x_2, \\ u_2 &= \beta x_3 x_1, \\ u_3 &= 0. \end{aligned} \tag{9.21}$$

We have already calculated the non-zero components of the linear strain tensor

$$E_{13} = E_{31} = -\tfrac{1}{2}\beta x_2,\ E_{23} = E_{32} = \tfrac{1}{2}\beta x_1.$$

Clearly, $E_{ii} = 0$, so in this case, simply

$$T_{ij} = 2\mu E_{ij}.$$

The non-vanishing components of the stress tensor are thus

$$T_{13} = T_{31} = -\mu\beta x_2,\ T_{23} = T_{32} = \mu\beta x_1. \tag{9.22}$$

The stress tensor has clearly vanishing divergence. Let us finally consider the boundary conditions. It is convenient to use the symmetry of the problem, so choose a points in the x_1x_3-plane, so that $x_2 = 0$ and with positive x_1-value. The only nonvanishing stress component is then

$$T_{23} = T_{32} = \mu\beta x_1.$$

The outward normal of the cylindrical surface at a point where $x_2 = 0$ is the basis vector $\mathbf{e}_1$. The stress vector has thus the components $T_{i1} = 0$. So this surface is free, as it should be. On $z = h$ the outward normal is $\mathbf{e}_3$. Hence the stress vector is T_{i3}, with a non-vanishing component in the 2-direction. The corresponding torque is

$$x_1 T_{23} = \mu\beta x_1^2 = \mu\beta r^2$$

$r = x_1$ is the distance from the axis. The area of the small surface element is $rdrd\varphi$.

The torque on the surface $x_3 = h$ about the axis is given by

$$\int_0^a \int_0^{2\pi} \rho t_\varphi \rho d\rho d\varphi = \mu\beta 2\pi \int_0^a \rho^3 d\rho = \frac{\pi\mu\beta a^4}{2}. \tag{9.23}$$

Measurement of the torsion angle and torque can be used to give the shear modulus of a material.

5.2 Cylinder of arbitrary cross section

For the simple displacement field (9.21) the stress tensor is given by (9.22). The corresponding stress vector vanishes for $\mathbf{n} = \mathbf{e}_\rho$ only. This displacement field is thus just valid for a circular cylinder.

For a non-circular cylinder we have to generalize (5.20)

$$\begin{aligned} u_1 &= -\beta x_3 x_2, & (9.24)\\ u_2 &= \beta x_3 x_1, & (9.25)\\ u_3 &= \beta\Phi(x_1, x_2). \end{aligned}$$

The planes orthogonal to the axis will be distorted.

The linear strain tensor is now

$$\begin{aligned} E_{13} &= E_{31} = \frac{\beta}{2}[\Phi_{,1} - x_2],\\ E_{23} &= E_{32} = \frac{\beta}{2}[\Phi_{,2} + x_1]. \end{aligned}$$

So, also in this case E_{ii} vanishes. There is thus no change of volume. The stress tensor is

$$\begin{aligned} T_{13} &= T_{31} = \mu\beta[\Phi_{,1} - x_2],\\ T_{23} &= T_{32} = \mu\beta[\Phi_{,2} + x_1]. \end{aligned}$$

Navier's equation gives

$$\triangle\Phi = 0. \tag{9.26}$$

Φ is thus a *harmonic function.*

Let us consider the boundary conditions. Let us denote the outward normal of the mantle surface by **n** . The only non-vanishing component of the stress vector is a shear stress

$$t_3 = \mu\beta[n_i\Phi_{,i} - (n_1x_2 - n_2x_1)].$$

The boundary condition is that t_3 vanishes

$$n_1\Phi_{,1} + n_2\Phi_{,2} = (n_1x_2 - n_2x_1). \tag{9.27}$$

(9.26) with (9.27) is a Neumann problem for Laplace's equation. It is well-known that this problem has precisely one solution.

5.3 Elliptical cylinder

For an elliptical cylinder (9.26) and (9.27) have an exact solution. The harmonic function

$$\Phi = Ax_1x_2 \tag{9.28}$$

gives

$$n_1\Phi_{,1} + n_2\Phi_{,2} = A(n_1x_2 + n_2x_1).$$

The equation of the ellipse is (a and b are the semi-major axis and the semi-minor axes, respectively)

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1.$$

Taking the gradient of this expression, we find that the normal vector **n** is proportional to $(x_1/a^2, x_2/b^2)$. (9.27) can thus be written

$$A\left(\frac{1}{a^2} + \frac{1}{b^2}\right) = \frac{1}{a^2} - \frac{1}{b^2}.$$

We then find

$$A = \frac{b^2 - a^2}{a^2 + b^2} = -\frac{e^2}{2 - e^2}. \tag{9.29}$$

e is the *eccentricity* of the ellipse ($e^2a^2 = a^2 - b^2$).

The stress vector on the surface $x_3 = l$, with outward normal $\mathbf{e}_3$ is

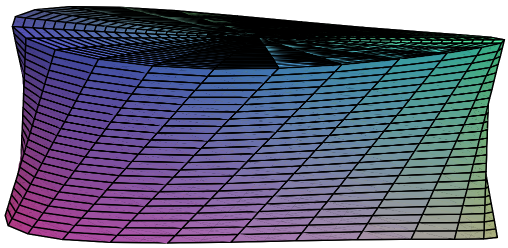


Figure 9.1: Torsion of elliptical cylinder

$$t_1 = - \frac{2\mu\beta a^2}{a^2+b^2}x_2,$$
$$t_2 = \frac{2\mu\beta b^2}{a^2+b^2}x_1.$$

The moment about the axis is thus

$$\int\int (x_1t_2 - x_2t_1)dx_1dx_2$$

$$= 2\mu\beta[a^2+b^2]^{-1}\left[a^2\int\int x_2^2dx_1dx_2 + b^2\int\int x_1^2dx_1dx_2\right].$$

Using

$$x_1 = ay_1,\ x_2 = by_2,$$

where (y_1, y_2) describes a circle with radius 1,we easily calculate the integrals. We conclude that the moment is

$$\frac{\pi\mu\beta a^3 b^3}{a^2+b^2}. \tag{9.30}$$

Homogeneous deformations have a $\mathbf{u}$ which is a polynomial of first degree in the coordinates. In the torsion of circular and elliptical cylinder have second degree polynomial solutions in x_i.

6 Spherically symmetric deformations

When $\mathbf{u} = u(r)\mathbf{e}_r$, we can write

$$\mathbf{u} = \frac{u(r)}{r}\mathbf{r}$$

Or in Cartesian components

$$u_i = \frac{u(r)}{r}x_i.$$

We have already shown that

$$\frac{\partial r}{\partial x_i} = \frac{x_i}{r}. \tag{9.31}$$

Further,

$$\begin{aligned} u_{i,j} &= \frac{u(r)}{r}\delta_{ij} + [r\frac{d}{dr}(\frac{u(r)}{r})]\frac{x_i x_j}{r^2} \\ &= \frac{du}{dr}\frac{x_i x_j}{r^2} + \frac{u(r)}{r}(\delta_{ij} - \frac{x_i x_j}{r^2}). \end{aligned}$$

$u_{i,j}$ is symmetric. This means that

$$\boldsymbol{\nabla} \times \mathbf{u} = \mathbf{0}.$$

So there is *no rotation*. The linear strain tensor is thus

$$E_{ij} = u_{i,j}.$$

If we introduce spherical coordinates, we have

$$\begin{aligned} E_{rr} &= \frac{du}{dr}, \\ E_{\theta\theta} &= E_{\varphi\varphi} = \frac{u}{r}. \end{aligned}$$

The trace of the linear strain tensor

$$u_{i,i} = E_{ii} = \frac{du}{dr} + 2\frac{u}{r} = \frac{1}{r^2}\frac{d}{dr}(r^2 u).$$

Let us now calculate the non-zero components of the stress tensor as

$$\begin{aligned} T_{rr} &= (2\mu+\lambda)\frac{du}{dr} + 2\lambda\frac{u}{r}; \\ T_{\theta\theta} &= T_{\varphi\varphi} = \lambda\frac{du}{dr} + 2(\mu+\lambda)\frac{u}{r}. \end{aligned}$$

As $\boldsymbol{\nabla} \times \mathbf{u}$ vanishes, we find from the formula (4.8) that

$$\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{u}) = \triangle\mathbf{u}.$$

As a consequence, the Navier equations of equilibrium can be written, if the exterior force is given as $\rho\mathbf{f}$,

$$(2\mu+\lambda)\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{u}) + \rho\mathbf{f} = \mathbf{0}. \tag{9.32}$$

Now we assume that there is no external force. We then find that

$$\boldsymbol{\nabla}\cdot\mathbf{u} = 3A,$$

where A is a constant. Using the expression for $\boldsymbol{\nabla}\cdot\mathbf{u}$ in spherical coordinates we find

$$\frac{1}{r^2}\frac{d}{dr}(r^2 u) = 3A$$

Integrating, we obtain (where B is a new constant of integration)

$$u = Ar + B\frac{1}{r^2}.$$

The constants A and B are given by the boundary conditions. - If the body has no hole around the origin, B has to vanish.

Exercise 6.1 *A pressure p is applied to the surface of an elastic sphere of radius R. Determine the displacement vector field. Answer:*

$$u = -\frac{1}{2\mu + 3\lambda} r = -\frac{p}{3k} r.$$

This is the same solution as we have already found by other methods in the section on homogeneous deformations.

Exercise 6.2 *A hollow sphere, with inner radius R_1 and outer radius R_2, is elastic. A pressure p is applied at the inner radius, whereas the pressure at the outer surface is negligible. Determine the displacement field. Answer:*

$$u = \frac{p}{(\frac{R_2}{R_1})^3 - 1}(\frac{1}{3k} r + \frac{R_2^3}{4\mu}\frac{1}{r^2}).$$

We now consider the case with a volume force. As we have spherical symmetry, there is always a potential Ω such that

$$\mathbf{f} = -\boldsymbol{\nabla}\Omega.$$

The equation of equilibrium is thus

$$(2\mu + \lambda)\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{u}) - \rho\boldsymbol{\nabla}\Omega = \mathbf{0}. \tag{9.33}$$

This can be integrated to (A is a constant)

$$(2\mu + \lambda)(\boldsymbol{\nabla} \cdot \mathbf{u}) - \rho\Omega = 3A.$$

Let us also consider a heavy self-gravitating elastic body. Neutron stars are composed of solid elastic material. The gravitational force is calculated from the undeformed sphere. The density is ρ and the force per mass is $f = -(4\pi G\rho/3)r$. G is the Newtonian gravitation constant. $\mathbf{f}$ has a potential and we have

$$\rho\mathbf{f} = -\boldsymbol{\nabla}[(4\pi G\rho^2/3)\frac{r^2}{2}]$$

The equation of equilibrium takes the form

$$(2\mu + \lambda)\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{u}) - \boldsymbol{\nabla}[(4\pi G\rho^2/3)\frac{r^2}{2}] = \mathbf{0}.$$

Exercise 6.3 *Show that (A is a constant)*

$$(2\mu + \lambda)(\boldsymbol{\nabla} \cdot \mathbf{u}) - (4\pi G\rho^2/3)\frac{r^2}{2} = 3A.$$

Integrate this equation, to find the general solution

$$u = \frac{2\pi G\rho^2}{15(2\mu + \lambda)}r^3 + \frac{A}{2\mu + \lambda}r + B\frac{1}{r^2}.$$

Exercise 6.4 *Calculate A and B for a sphere of undeformed radius R. Answer:*

$$u = \frac{2\pi G\rho^2}{15(2\mu + \lambda)}(r^3 - \frac{6\mu + 5\lambda}{2\mu + 3\lambda}R^2 r).$$

We have here been able to work with spherical polar coordinates, by using cartesian coordinates in intermediate steps. This worked because we had spherical symmetry.

Chapter 10

Compatibility and Plane elasticity

1 Compatibility relations

For many problems it turns out to be convenient to consider the equation of equilibrium as an equation for the linear strain tensor $\mathbf{E}$ (or the stress tensor $\mathbf{T}$) rather than the displacement vector $\mathbf{u}$. Suppose we have found a symmetric tensor field $\mathbf{E}$, can we then from this field find the displacement vector $\mathbf{u}$? The answer in general is no. A simple argument is that $\mathbf{E}$ has six independent components but $\mathbf{u}$ three only. Consequently, the $\mathbf{E}$ field has to satisfy certain conditions of integrability for it to be possible as a linear strain tensor, derived from a displacement field. Those conditions are often called *compatibility relations*. The linear strain tensor is

$$E_{ij} = u_{(i,j)}. \tag{10.1}$$

If we use this formula, we find

$$\begin{aligned} E_{ij,kl} + E_{kl,ij} &= \frac{1}{2}(u_{i,jkl} + u_{j,ikl}) + \frac{1}{2}(u_{k,lij} + u_{l,kij}) \\ &= \frac{1}{2}(u_{i,klj} + u_{k,ilj}) + \frac{1}{2}(u_{l,jik} + u_{j,lik}) = E_{ik,lj} + E_{lj,ik}. \end{aligned}$$

We conclude that

$$E_{ij,kl} + E_{kl,ij} = E_{ik,lj} + E_{lj,ik}. \tag{10.2}$$

These are the *compatibility relations.* They are necessary conditions on a symmetric tensor field E_{ij} for the existence of a displacement vector u_i. One can also show that they are sufficient conditions, if there are no topological complications. In three dimensions they are sufficient if the region looks like a sphere topologically. In two dimensions if the region is simply connected, so that there are no holes.

Of the compatibility relations there are six independent ones. In the plane case, there is only one compatibility relation

$$E_{xx,yy} + E_{yy,xx} = 2E_{xy,xy} \ . \tag{10.3}$$

This is the only one we shall use in the sequel.

Exercise 1.1 *Show that there are two analogous compatibility relations:*

$$E_{yy,zz} + E_{zz,yy} = 2E_{yz,yz} \ ; \tag{10.4}$$
$$E_{zz,xx} + E_{xx,zz} = 2E_{zx,zx} \ . \tag{10.5}$$

All of these relations involve just two indices.

There are also three compatibility relations involving three indices.

Exercise 1.2 *Show that there are also the following three compatibility relations involving three indices.*

$$E_{xx,yz} = (E_{xy,z} + E_{xz,y} - E_{yz,x})_{,x} \ ; \tag{10.6}$$
$$E_{yy,zx} = (E_{yz,x} + E_{yx,z} - E_{zx,y})_{,y} \ ; \tag{10.7}$$
$$E_{zz,xy} = (E_{zx,y} + E_{zy,x} - E_{xy,z})_{,z} \ . \tag{10.8}$$

Also show that there are no others.

In *two dimensions* (plane strain) we have found that there is just one relation of compatibility, namely (10.3).

2 Plane elasticity

We now consider the plane case, where all quantities are independent of the coordinate x_3. Let us write down the equations of equilibrium in the plane. They are

$$T_{11,1} + T_{12,2} = 0 \ , \ T_{21,1} + T_{22,2} = 0. \tag{10.9}$$

There are two cases of interest. Plane strain, where u_3 vanishes and plane stress, where T_{3i} vanishes. For plane strain, it is obvious that the remaining equation $T_{3i,i} = 0$ is satisfied. For plane strain

$$T_{3j,j} = \mu u_{3,jj} + (\mu + \lambda) u_{j,j3} = 0.$$

Airy stress function

Before analyzing these equations, we consider a plane vector field $\mathbf{v}$ with vanishing divergence

$$v_{1,1} + v_{2,2} = 0. \tag{10.10}$$

This can be the velocity field of an incompressible fluid. It is clear, that if there is a function Ψ such that

$$v_1 = \Psi_{,2}, v_2 = -\Psi_{,1}, \tag{10.11}$$

the divergence of $\mathbf{v}$ identically vanishes. - In the dynamics of incompressible fluids, Ψ is called the *Stokes stream function* of $\mathbf{v}$.

Exercise 2.1 *Show that if* $\mathbf{v}$ *has vanishing divergence, there always exists a Stokes stream function* Ψ*. You could define* Ψ *by solving the first of the equations (10.11) by integration and then see that the function* Ψ *also satisfies the second of the equations. You have to choose the constants of integration properly.*

Let us now go back to the equilibrium equations (10.9). Basically the same trick as for the divergence-free vector field applies here. In fact, let us try the following *Ansatz* for the components of the stress tensor in the plane

$$T_{11} = \phi_{,22} \ , \ T_{12} = -\phi_{,12} \ , \ T_{22} = \phi_{,11}. \tag{10.12}$$

We immediately find that the equations of equilibrium are satisfied.

It is not so difficult to show, that conversely, every solution of (10.9) is of the form (10.12) for some function ϕ.

Exercise 2.2 *Carry out the details in the foregoing proof. The idea of the proof is to solve the first of (10.12) for ϕ, which will be found by twice integrating T_{11} with respect to x_2. There is also an undetermined part, which is of the form $f + gx_2$, where f and g are functions of x_1 only. This result is substituted in the second of the equations to determine f and g. There remains the third of the equations.*

We conclude, that the equations of equilibrium (*i.e.* that the two-dimensional stress tensor is divergence-free) are precisely solved by (10.12). The potential is called the *Airy stress function.*

Biharmonic equation

So far, so good. We have solved the equations of equilibrium for the stress tensor. But in actual fact, the equations of equilibrium are equations for the displacement vector $\mathbf{u}$, so we have not yet solved them completely.

An important case is that of *plane strain.* The meaning of plane strain is that in a suitably chosen Cartesian coordinate system, u_3 vanishes and u_1 and u_2 are independent of x_3.

This means that $E_{33} = 0$. From (9.17) we find

$$\lambda(u_{1,1} + u_{2,2}) = T_{33} = \nu(T_{11} + T_{22}). \tag{10.13}$$

So there is a normal stress in the 3-direction. This means that such a normal stress has to be applied to the boundary of the body, to make it undergo plane strain. If not it will expand or contract also in the 3-direction.

Then we obtain from (9.17)

$$\begin{aligned} E_{11} &= \frac{1+\nu}{E_Y}[(1-\nu)T_{11} - \nu T_{22}], \\ E_{22} &= \frac{1+\nu}{E_Y}[(1-\nu)T_{22} - \nu T_{11}], \\ E_{12} &= \frac{1+\nu}{E_Y}T_{12}. \end{aligned} \tag{10.14}$$

This means that we have the linear strain tensor in terms of the Airy stress function.

$$\begin{aligned} E_{11} &= \frac{1+\nu}{E_Y}[\phi_{,22} - \nu \triangle \phi], \\ E_{22} &= \frac{1+\nu}{E_Y}[\phi_{,11} - \nu \triangle \phi], \\ E_{12} &= -\frac{1+\nu}{E_Y}\phi_{,12}. \end{aligned} \tag{10.15}$$

But a given linear strain tensor corresponds to a possible deformation of a body if and only if the equations of compatibility are satisfied. In this two-dimensional case, there is only one non-trivial equation of compatibility, which is given by two indices 1 and two indices 2 and is

$$E_{11,22} + E_{22,11} = 2E_{12,12}.$$

It follows that the condition of compatibility is

$$\triangle^2 \phi = 0. \tag{10.16}$$

We conclude that for plane strain the equations of equilibrium are equivalent to the biharmonic equation for the Airy stress function and that the stress tensor is given by (10.12) and the linear strain tensor by (10.14). From (10.13) we have

$$T_{33} = \nu \triangle \phi \tag{10.17}$$

Unless the divergence of **u** vanishes, a *normal stress* has to be applied in the 3-direction to produce the plane strain. No shear stress is needed on the 3-planes.

Example 7 *An isotropic linearly elastic body is in a state of plane strain. The intersection of the body with the xy−plane is in the shape of a rectangle with* $-a \leq x \leq a$ *and* $-b \leq y \leq b$*. On its surface there are no shear stresses, but there are normal stresses, which grow quadratically with the distance from the axes. We make the Ansatz that the shear stress is identically zero in the body and that the normal stresses are given everywhere by*

$$T_{xx} = \alpha y^2, T_{yy} = \beta x^2.$$

Determine the Airy stress function φ and show that

$$\alpha = -\beta.$$

Then calculate the displacement field in the body. Plot the deformation of the body.

The relative change of area in the $xy-$plane is the same as the relative change of volume. Why? Calulate this relative change of area and show in the plot where the area is increasing and decreasing.

Also calculate the total force (per length in the z-direction) on each of the four bounding surfaces and the torques around axes around the midpoint of the surface and direction parallel to the z-axis.

From (10.14) we find

$$\begin{aligned} u_{x,x} &= E_{xx} = \frac{\alpha(1+\nu)}{E_Y}[(1-\nu)y^2 + \nu x^2], \\ u_{y,y} &= E_{yy} = \frac{\alpha(1+\nu)}{E_Y}[-(1-\nu)x^2 - \nu y^2], \\ u_{x,y} + u_{y,x} &= 2E_{xy} = 0. \end{aligned} \tag{10.18}$$

The first two equations given

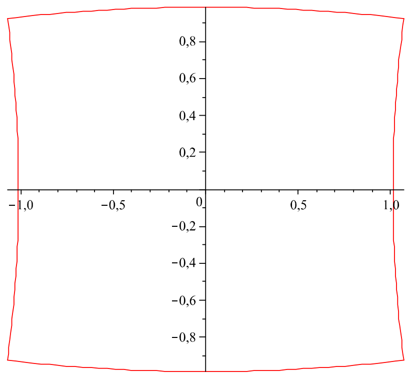
$$\begin{aligned} u_x &= \frac{\alpha(1+\nu)}{E_Y}[(1-\nu)y^2 x + \nu\frac{x^3}{3} + f(y)], \\ u_y &= \frac{\alpha(1+\nu)}{E_Y}[-(1-\nu)x^2 y - \nu\frac{y^3}{3} + g(x)] \end{aligned}$$

We insert these into the third equation to obtain

$$(1-\nu)2yx + f'(y) - (1-\nu)2xy + g'(x) = 0$$

One solution is $f = g = 0$. Hence

$$\begin{aligned} u_x &= \frac{\alpha(1+\nu)}{E_Y}[(1-\nu)y^2 x + \nu\frac{x^3}{3}], \\ u_y &= \frac{\alpha(1+\nu)}{E_Y}[-(1-\nu)x^2 y - \nu\frac{y^3}{3}] \end{aligned}$$



The force F_x on the side $x = a$ and the force on the side $y = b$ are

$$F_x = \alpha \int_{-b}^{b} y^2 dy = \alpha \frac{2b^3}{3},$$
$$F_y = -\alpha \int_{-b}^{b} x^2 dx = -\alpha \frac{2a^3}{3}.$$

The torque around and axis through $(a, 0)$ and parallel with the z-axis is

$$-\int_{-b}^{b} y\alpha y^2 dy = 0.$$

The torque around and axis through $(0, b)$ and parallel with the z-axis is

$$-\int_{-b}^{b} x\alpha x^2 dy = 0.$$

2.1 Rectangular beam with perpendicular load

Let us, as an example of plane elasticity, consider a beam, which is clamped at $x = 0$ and has, a transverse load P (per length) at the free end $x = l$. Its two other sides in the plane are $y = \pm d$, which are assumed free. Clearly, there has to be a shearing stress T_{xy} at $x = l$, which vanishes at $y = \pm d$. The simplest such function is

$$T_{xy}(l,y) = A(d^2 - y^2).$$

Let us try this for an *Ansatz* all through the beam, so that T_{xy} is independent of x. This gives us the stress function

$$\phi = A(\frac{xy^3}{3} - d^2xy) + f(x) + g(y).$$

Let us note that this is a biharmonic function if f and g are of the third order at most. We now obtain

$$T_{xx} = A2xy + g''(y)$$

This has to vanish at $x = l$, leaving us with $g = -Aly^3/3$. (A constant or a first-order term will not contribute to the stress, can hence be discarded.) Finally, we have

$$T_{yy} = f''(x),$$

which has to vanish at $x = l$, so that $f = 0$. - Let us also calculate the value of A.

$$P = \int_{-d}^{d} T_{xy}(l,y)dy = \frac{4}{3}Ad^3.$$

We conclude, that

$$\phi = \frac{3P}{4d^3}(\frac{xy^3}{3} - d^2xy - \frac{ly^3}{3}).$$

Let us also write down the final expression for the components of the stress tensor

$$T_{xx} = \frac{3P}{2d^3}(x-l)y \ , \ T_{xy} = \frac{3P}{4d^3}(d^2 - y^2) \ , \ T_{yy} = 0.$$

We shall now calculate the displacement vector. First of all we need the linear strain tensor in terms of the stress tensor. We use (10.14), assuming plane strain. This gives us the set of equations

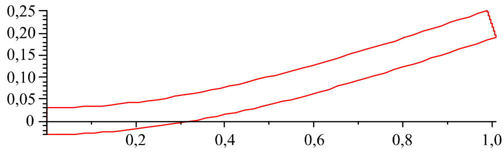


Figure 10.1: Beam with perpendicular load. Plane strain

$$u_{x,x} = \frac{3P}{2E_Y d^3}(\nu+1)(1-\nu)(xy-ly),$$

$$u_{y,y} = -\frac{3P}{2E_Y d^3}(\nu+1)\nu(xy-ly),$$

$$u_{x,y}+u_{y,x} = \frac{3P}{2E_Y d^3}(\nu+1)(d^2-y^2).$$

From the first two equations clearly

$$u_x = \frac{3P}{2E_Y d^3}(\nu+1)[(1-\nu)(\frac{1}{2}x^2y-lxy)+f(y)],$$

$$u_y = \frac{3P}{2E_Y d^3}(\nu+1)[\nu(-\frac{1}{2}xy^2+ly^2)+g(x)].$$

The functions f and g are determined from the remaining equation.

$$\begin{aligned} & u_{x,y} + u_{y,x} - \frac{3P}{2E_Y d^3}(\nu+1)(d^2 - y^2) \\ = & \frac{3P}{2E_Y d^3}(\nu+1)\{[(1-\nu)(\frac{1}{2}x^2 - lx) + g'(x)] \\ & +[-d^2 + (1-\frac{\nu}{2})y^2 + f'(y)]\}. \end{aligned}$$

giving (α is a constant yet to be determined)

$$\begin{aligned} (1-\nu)(\frac{1}{2}x^2 - lx) + g'(x) &= \alpha, \\ -d^2 + (1-\frac{\nu}{2})y^2 + f'(y) &= -\alpha \end{aligned}$$

$$\begin{aligned} f(y) &= -(1-\frac{\nu}{2})\frac{y^3}{3} + (d^2-\alpha)y \ , \\ g(x) &= (1-\nu)(\frac{lx^2}{2} - \frac{x^3}{6}) + \alpha x \ . \end{aligned}$$

Now we have to consider the boundary at the wall. There

$$\begin{aligned} u_x &= \frac{3P}{2E_Y d^3}(\nu+1)[-(1-\frac{\nu}{2})\frac{y^3}{3} + (d^2-\alpha)y] \\ u_y &= \frac{3P}{2E_Y d^3}(\nu+1)\nu l y^2 \ . \end{aligned}$$

If we choose $\alpha = d^2$, u_x vanishes up to third order in the width of the beam and u_y to second order, but none of them vanish exactly. So the solution is not very good at the wall, but presumably is a good approximation far from the wall.

Hence,

$$\begin{aligned} u_x &= \frac{3P}{2E_Y d^3}(\nu+1)[(1-\nu)(\frac{1}{2}x^2 y - lxy) - (1-\frac{\nu}{2})\frac{y^3}{3}] \qquad (10.19) \\ u_y &= \frac{3P}{2E_Y d^3}(\nu+1)[\nu(-\frac{1}{2}xy^2 + ly^2) + (1-\nu)(\frac{lx^2}{2} - \frac{x^3}{6}) + d^2 x] \ . \end{aligned}$$

Exercise 2.3 *The same problem, when a longitudinal force F is applied at the free end. Answer:*

$$T_{xx} = \frac{F}{2d}\ ,\ T_{xy} = T_{yy} = 0.$$

3 Bending of a beam

We start with a straight beam. Its cross-section is arbitrary. The center of area of the cross-sections is on the x-axis. The x-axis is along the axis of the beam. Moments are applied to the beginning and end surfaces of the beam and as a result, it bends. The other sides are free. Let us guess the stress distribution in the beam. As all components of the stress tensor except T_{11} have to vanish on the surface of the beam, the simplest *Ansatz* is to assume that the same applies throughout the beam, so that the stress is uni-axial. We can then invert the stress- strain relation and find that the only non-vanishing components of the linear strain tensor are the diagonal ones and that

$$E_{22} = E_{33} = -\nu E_{11}. \tag{10.20}$$

Let us also note that the equation of equilibrium reduces to $T_{11,1} = 0$. Hence, T_{11} is independent of x_1.

We can now appeal to the compatibility conditions, using the index-combination $11ab$, where a and b take the values 2 and 3. As all x_1-derivatives vanish, there will only be one term left in the equation, namely

$$E_{11,ab} = 0$$

Consequently, E_{11} is a linear function of y and z. If we choose the coordinate system conveniently, we can write

$$E_{11} = -\frac{y}{R},$$

where R is a constant.[1]

With this expression for the linear strain tensor, we obtain the normal stress as

[1] This is an example of plane stress.

$$T_{11} = -E_Y \frac{y}{R}. \tag{10.21}$$

The moment at the end of the bar is

$$M_z = -\int yT_{11}dydz = \frac{E_Y}{R}\int y^2 dydz = \frac{E_Y I_z}{R}, \tag{10.22}$$

$$M_y = \int zT_{11}dydz = -\frac{E_Y}{R}\int xydydz = \frac{E_Y I_{xy}}{R}. \tag{10.23}$$

Let us also calculate the elastic energy of the beam. We know that it is given by

$$\epsilon\rho_\kappa = \mu tr(\mathbf{E}^2) + \frac{\lambda}{2}(tr\mathbf{E})^2 = \ \tfrac{1}{2}tr(\mathbf{TE}) = \ \tfrac{1}{2}T_{11}E_{11} = \frac{E_Y y^2}{2R^2}.$$

The total energy per length of the beam is hence

$$\frac{E_Y I_z}{2R^2} = \frac{M_z}{2R}. \tag{10.24}$$

We shall now solve for the displacement vector $\mathbf{u}$. As the linear strain tensor is linear in the coordinates, $\mathbf{u}$ should be quadratic. As x is in the direction of the bar and y points towards the center of curvature, we can appeal to symmetry. Changing x to $-x$ should change u_x to $-u_x$ and leave the other components unchanged. Similarly, changing z to $-z$ should change u_z to $-u_z$ and leave the other components unchanged. We conclude that (α... are constants to be determined)

$$\begin{aligned} u_x &= \alpha xy \\ u_y &= \beta x^2 + \gamma y^2 + \delta z^2 \\ u_z &= \epsilon xz \end{aligned}$$

Inserting this into

$$E_{ij} = u_{(i,j)}$$

we find

$$
\begin{aligned}
u_x &= -\frac{1}{R}xy, \\
u_y &= \frac{1}{2R}x^2 + \frac{\nu}{2R}(y^2 - z^2), \\
u_z &= \frac{\nu}{R}yz.
\end{aligned} \tag{10.25}
$$

Looking at the expression for u_y, putting $y = z = 0$ for the center of the beam, we find that it is bent into a circle with radius R. - If the argument for finding $\mathbf{u}$ appears to loose, we can appeal to the theorem of uniqueness: the displacement field (10.25) does satisfy the equation of equilibrium and the boundary values and hence is the solution.

This is an example of plane stress, not plane strain. The only non-vanishing component of the stress tensor is T_{11} and it is a function of y alone.

4 Elastic waves

For small deformations we have

$$\rho\frac{D\mathbf{v}}{Dt} = \rho\frac{\partial^2\mathbf{u}}{\partial t^2} \approx \rho_0\frac{\partial^2\mathbf{u}}{\partial t^2}.$$

Assuming μ and κ to be constants we have

$$T_{ij,j} = \mu v_{i,jj} + (\kappa + \frac{\mu}{3})v_{j,ji}$$

or

$$\rho_0\frac{\partial^2\mathbf{u}}{\partial t^2} = \mu\triangle\mathbf{u} + (\kappa + \frac{\mu}{3})\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{u})$$

This equation is the Navier equation.

As the equations have constant coefficients we look for solutions of the form

$$\mathbf{u} = \mathbf{a}\exp[i(\mathbf{k}\cdot\mathbf{r} - \omega t].$$

We find

$$\rho_0\omega^2\mathbf{a} = \mu k^2\mathbf{a} + (\kappa + \frac{\mu}{3})\mathbf{k}(\mathbf{k}\cdot\mathbf{a}).$$

In a transverse wave, $\mathbf{k}$ and $\mathbf{a}$ are orthogonal, so that

$$\rho_0\omega^2 = \mu k^2,$$

which gives a phase velocity

$$c_T = \frac{\omega}{k} = \sqrt{\frac{\mu}{\rho_0}}.$$

In a longitudinal wave, **k** and **a** are parallel, so that

$$\rho_0 \omega^2 = (\kappa + \frac{4\mu}{3})k^2.$$

This gives a phase velocity

$$c_L = \frac{\omega}{k} = \sqrt{\frac{\kappa + \frac{4\mu}{3}}{\rho_0}}.$$

The longitudinal waves are faster than the transverse waves. As $\kappa \geq 0$ we have

$$\frac{c_T}{c_L} \leq \frac{\sqrt{3}}{2} = 0.8660...$$

Chapter 11

Variational principles

A very compact and powerful formulation of an equation is that of a variational principle. In this chapter we shall restrict ourselves to time independent equations. But it is important to realize that there are more general variational principles applying to timedependent equations. We start with the most basic equation.

1 Laplace's equation

Let us take a look at the following integral over some domain

$$I_1 = \int \frac{1}{2}(\boldsymbol{\nabla}\phi)^2 \mathrm{d}v. \tag{11.1}$$

We think of it as a function of the function $\boldsymbol{\phi}$, in other words as a *functional*. What happens, if we change the function ϕ slightly, to $\phi + \delta\phi$?

$$I_1 = \int \frac{1}{2}(\boldsymbol{\nabla}\phi + \boldsymbol{\nabla\delta}\phi)^2 \mathrm{d}v = \int \frac{1}{2}(\boldsymbol{\nabla}\phi)^2 \mathrm{d}v + \int \boldsymbol{\nabla}\delta\phi \cdot \boldsymbol{\nabla}\phi \mathrm{d}v + \int \frac{1}{2}(\boldsymbol{\nabla}\delta\phi)^2 \mathrm{d}v.$$

We neglect the last term and write the first order change of I_1 as δI_1

$$\delta I_1 = \int \boldsymbol{\nabla}\delta\phi \cdot \boldsymbol{\nabla}\phi \mathrm{d}v. \tag{11.2}$$

It is important to realize, that δI_1 is a *functional of two functions*. It depends on the original function $\boldsymbol{\phi}$ as well as the small variation $\delta\phi$ of it. It is *linear in $\delta\phi$*.

Now we want to get rid of the $\boldsymbol{\nabla}$ acting on $\boldsymbol{\delta\phi}$ in (11.2). We shall come back to that point later. Gauss's theorem offers a possibility. In fact, we can write

$$\boldsymbol{\nabla}\delta\phi\cdot\boldsymbol{\nabla}\phi = \boldsymbol{\nabla}\cdot(\delta\phi\boldsymbol{\nabla}\phi) - \delta\phi\triangle\phi \tag{11.3}$$

This is perhaps more clear in components notation

$$\delta\phi_{,i}\phi_{,i} = (\delta\phi\phi_{,i})_{,i} - \delta\phi\phi_{,ii}.$$

We now obtain

$$\begin{aligned}\delta I_1 &= \int \boldsymbol{\nabla}\cdot(\delta\phi\boldsymbol{\nabla}\phi)\mathrm{d}v - \int \delta\phi\triangle\phi\mathrm{d}v \\ &= \int \delta\phi\boldsymbol{\nabla}\phi\cdot\mathbf{n}\mathrm{d}s - \int \delta\phi\triangle\phi\mathrm{d}v.\end{aligned}$$

So, we have been able to get rid of the $\boldsymbol{\nabla}$ acting on $\delta\phi$.

We could pose the following problem: find the function ϕ, which minimizes the integral (11.1)? It is clear that for all small changes from such a function, δI has to vanish. Suppose δI did not vanish for the change $\delta\phi$. As ϕ gives a minimum, δI then must be positive. But if we take $-\delta\phi$ instead of $\delta\phi$ then δI changes sign, which is impossible.

But $\delta\phi$ is arbitrary, this means that the function ϕ has to satisfy

$$\triangle\phi = 0 \tag{11.4}$$

in the interior of the domain, and

$$\boldsymbol{\nabla}\phi\cdot\mathbf{n} = \frac{\partial\phi}{\partial n} = 0. \tag{11.5}$$

So we have found that the function, which minimizes the integral (11.1) has to satisfy Laplace's equation, with a homogeneous Neumann boundary condition. We recall that a Neumann boundary condition is to give the normal derivative of ϕ on the boundary

$$\frac{\partial\phi}{\partial n} = f. \tag{11.6}$$

Here, f is a given function.

This seems promising! What we have found is that for ϕ to satisfy

$$\delta \int \frac{1}{2} (\boldsymbol{\nabla}\phi)^2 \mathrm{d}v = 0. \tag{11.7}$$

for any $\delta\phi$ it is necessary and sufficient that ϕ satisfies Laplace's equation (11.4) together with the homogeneous Neumann boundary condition (11.5) on the boundary.

We can change to arbitrary Neumann boundary conditions, by adding the term

$$-\int f\phi \mathrm{d}s,$$

so that we consider the variation of

$$I_2 = \int \frac{1}{2} (\boldsymbol{\nabla}\phi)^2 \mathrm{d}v - \int f\phi \mathrm{d}s. \tag{11.8}$$

We now consider functions ϕ satisfying the Neumann boundary condition (11.6) only. We have

$$\delta \int f\phi \mathrm{d}s = \int f\delta\phi \mathrm{d}s,$$

so that

$$\delta I_2 = \int \delta\phi (\boldsymbol{\nabla}\phi \cdot \mathbf{n} - f) \mathrm{d}s - \int \delta\phi \triangle\phi \mathrm{d}v.$$

But as ϕ satisfies the Neumann boundary condition (11.6), the surface integral vanishes, and we find the the necessary and sufficient condition for δI_2 to vanish is that ϕ satisfies Laplace's equation.

Let us also consider Dirichlet boundary conditions.

$$\phi = g$$

is then given at the boundary. As a consequence, at the boundary

$$\delta\phi = 0.$$

So we consider the original functional (11.1)

$$I_1 = \int \frac{1}{2} (\boldsymbol{\nabla}\phi)^2 \mathrm{d}v. \tag{11.9}$$

but only use functions ϕ satisfying Dirichlet boundary conditions. We also now obtain the variation

$$\delta I_1 = \int \delta\phi \boldsymbol{\nabla}\phi \cdot \mathbf{n} \mathrm{d}s - \int \delta\phi \triangle \phi \mathrm{d}v.$$

Now, the surface integral vanishes, as $\delta\phi$ vanishes there. We conclude that δI_1 vanishes if and only if ϕ satisfies Laplace's equation.

2 Elasticity

In elasticity we have the elastic energy and an equilibrium state is a state of minimum energy. So we minimize

$$E = \int \rho\varepsilon \mathrm{d}v = \int \frac{1}{2} C_{ijkl} E_{ij} E_{kl} \mathrm{d}v.$$

$$\delta E = \int \rho \frac{\partial \varepsilon}{\partial E_{ij}} \delta E_{ij} \mathrm{d}v$$

But

$$T_{ij} = \rho \frac{\partial \varepsilon}{\partial E_{ij}}$$

so that

$$\delta E = \int T_{ij} \delta E_{ij} \mathrm{d}v$$

Let us use

$$\delta E_{ij} = \frac{1}{2}(\delta u_{i,j} + \delta u_{j,i}).$$

From the symmetry of T_{ij} we have

$$T_{ij}\delta u_{i,j} = T_{ij}\delta u_{j,i}$$

We can thus write

$$T_{ij}\delta E_{ij} = T_{ij}\delta u_{i,j} = (T_{ij}\delta u_i)_{,j} - T_{ij,j}\delta u_j.$$

This gives us

$$\begin{aligned} \delta E &= \int [(T_{ij}\delta u_i)_{,j} - T_{ij,j}\delta u_j] \mathrm{d}v \\ &= \int \delta u_i T_{ij} n_j \mathrm{d}s - \int T_{ij,j}\delta u_j \mathrm{d}v \end{aligned}$$

From this we have two results

1) Either we consider a body, with *free surfaces*,

$$T_{ij}n_j = 0.$$

We see that the surface contribution to the variation $\delta \int \rho\varepsilon \mathrm{d}v$ vanishes. Then the configuration giving minimum total energy has to satisfy the equation of equilibrium

$$T_{ij,j} = 0.$$

2) Or we consider a body with *fixed boundary*, so that $\mathbf{u}$ is given on the boundary. This means that

$$\delta u_i = 0$$

on the boundary. Also in this case, the surface contribution vanishes, and the configuration giving minimum total energy has to satisfy the equation of equilibrium.

Physically, it is clear that no work is done on the body through its boundary in these two cases.

But if we, instead, consider the case where the force on the surface is given, that is

$$T_{ij}n_j = t_i,$$

where the stress vector t_i is given on the boundary, things are different. In this case, the force on the boundary does work on the body. The force on $\mathrm{d}s$ is $t_i\mathrm{d}s$. It is a given force. So, when the point on the boundary is displaced the vector u_i, the work

$$\mathbf{u} \cdot \mathbf{t}\mathrm{d}s = u_i t_i \mathrm{d}s.$$

We know that the work done is minus the difference in potential energy. So we have the potential energy for the total boundary

$$-\int u_i t_i \mathrm{d}s = -\int u_i t_i \mathrm{d}s$$

This means that the total energy in this case is

$$E = \int \rho\varepsilon \mathrm{d}v - \int u_i t_i \mathrm{d}s.$$

We have already calculated the variation of the first integral. For the second term we have simply

$$\delta \int u_i t_i \mathrm{d}s = \int (\delta u_i) t_i \mathrm{d}s.$$

So now we have

$$\delta E = \int \delta u_i [T_{ij} n_j - t_i] \mathrm{d}s - \int T_{ij,j} \delta u_j \mathrm{d}v$$

But from the boundary condition, we find that the surface integral vanishes. And we conclude also in this case that the configuration has to satisfy the equilibrium equations.

Chapter 12

Elastic fluids

1 Euler's equations

For fluids the stress tensor is, see (8.13)

$$\mathbf{T} = -\rho^2 \frac{\partial \epsilon}{\partial \rho} \mathbf{1} = -p\mathbf{1}. \tag{12.1}$$

For an elastic fluid, there is just a pressure, p. As a result, we obtain Euler's equations

$$\rho \frac{D\mathbf{v}}{Dt} = -\operatorname{grad} p + \rho \mathbf{f}. \tag{12.2}$$

In the spatial description, we have to add the equation of continuity, to make the system complete

$$\dot{\rho} + \rho \operatorname{div} \mathbf{v} = 0. \tag{12.3}$$

2 Statics of fluids

Let us, to start with, consider statics of fluids in gravity. This means, that $\mathbf{f} = \mathbf{g} = -g\mathbf{e}_z$. The equation of equilibrium is

$$-\operatorname{grad} p + \rho \mathbf{f} = \mathbf{0}. \tag{12.4}$$

Let us calculate the total force due to the pressure on a submerged body.

$$\int (-p d\mathbf{s}) = -\int \operatorname{grad} p \, dv = \int \rho dv \mathbf{g} = m\mathbf{g}.$$

If the body is taken away, m is the mass of the fluid occupying its volume. This is the *principle of Archimedes*. It applies to compressible fluids as well as to incompressible ones. Actually, we have cheated a little here. The equation we have used only applies outside of the submerged body. The result is nevertheless true. The formal solution of Euler's equation gives a formal pressure inside the submerged body , which is the right one at the surface of the body and outside of it. The value of the integral only depends on the pressure at the surface and there it is the correct one.

If the fluid is *incompressible*, we find directly from (12.4)

$$p = p_0 - \rho g z \tag{12.5}$$

.

Let us also consider the *equilibrium of the atmosphere*. Now we have to take into account the compressibility of air. We make the simplifying assumption, that temperature is independent of height. The equation of state is the gas law $pV = NkT$, where V is the volume, N the number of molecules, k the Boltzmann constant and T the temperature. We can rewrite this as

$$p = \frac{kT}{m}\rho, \tag{12.6}$$

where m is the mean molecular mass. Hence, the equation of equilibrium reduces to

$$\frac{dp}{dz} + \frac{mg}{kT}p = 0$$

with the solution

$$p = p_0 exp(-\frac{mgz}{kT}),$$

which is the *barometric formula*.

3 Linear sound waves

When a fluid is slightly disturbed from its state of rest, the equations of motion can be linearized. So, let us assume that $\mathbf{v}$ is small and that $\rho = \rho_0 + \rho'$, where ρ' is small. In the pressure gradient, we write

$$\mathrm{grad}\, p = \frac{\partial p}{\partial \rho}\mathrm{grad}\, \rho'.$$

We introduce

$$c^2 = \frac{\partial p}{\partial \rho}. \tag{12.7}$$

c has the dimension of velocity.

We now have

$$\dot{\mathbf{v}} \approx \mathbf{v}_{,t} \quad , \quad \dot{\rho} \approx \rho_{,t} = \rho'_{,t}.$$

This leaves us with

$$\rho_0 \mathbf{v}_{,t} = -c^2 \mathrm{grad}\, \rho', \tag{12.8}$$

$$\rho'_{,t} + \rho_0 \mathrm{div}\, \mathbf{v} = 0. \tag{12.9}$$

If we eliminate $\mathbf{v}$, we find that ρ' satisfies a wave equation, with wave speed c. So, c is the *speed of sound* in the fluid. Let us point out that for most frequencies, small disturbances propagate isentropically, so that the relationship between p and ρ should be taken as the isentropic one. Only waves at very low frequencies propagate isothermally. In the range in between, the situation is more complex.

Exercise 3.1 *Eliminate instead ρ', to obtain an equation for $\mathbf{v}$. Show that it is*

$$\mathbf{v}_{,tt} = c^2 \mathrm{grad}\, \mathrm{div}\, \mathbf{v}.$$

Now consider a harmonic plane wave

$$\mathbf{v} = \mathbf{a} \exp[i(\mathbf{k} \cdot \mathbf{x} - \boldsymbol{\omega} t)].$$

Show that if the disturbance is transversal*, so that*

$$\mathbf{k} \cdot \mathbf{a} = 0,$$

ω vanishes, which means that the disturbance does not propagate *at all. Also show from the equation (12.9) that there are no density perturbations in this case. - On the other hand, if the disturbance is* longitudinal*, so that* $\mathbf{a}$ *and* $\mathbf{k}$ *are parallel*

$$\mathbf{a} = a\mathbf{e}, \mathbf{k} = k\mathbf{e},$$

show that $\mathbf{v}$ *satisfies the wave equation with wavespeed c.*

Sound waves are usually irrotational, so that there is a velocity potential, $\mathbf{v} = \mathrm{grad}\ \phi$. *The velocity potential is determined up to an arbitrary function of time.* One can then show that, choosing the arbitrary function of time properly, the velocity potential satisfies the wave equation

$$\triangle\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = 0. \tag{12.10}$$

Exercise 3.2 *Carry out the details in the preceding argument. Hint: Consider the wave equation for* $\mathbf{v}$ *and insert* $\mathbf{v} = \mathrm{grad}\,\phi$*. Conclude that the gradient of (12.10) vanishes, so that (12.10) is a function of time only. Change the velocity potential appropriately, so that it satisfies the wave equation (12.10).*

Chapter 13

Newtonian fluids

1 Viscous stresses

In a fluid there are usually viscous stresses as well as the pressure. In a Newtonian fluid, the viscous stresses are given by the velocity gradient and are linear in this,

$$T_{ij} = -p\delta_{ij} + A_{ijkl}v_{k,l}.$$

A Newtonian fluid is isotropic. This means that if a new velocity field is set up which is given by a rotation of the old one, the stress tensor will be rotated in the same way. This means that the tensor A_{ijkl} has to be isotropic. It has to be of the form

$$A_{ijkl} = \eta_1\delta_{ij}\delta_{kl} + \eta_2\delta_{ik}\delta_{jl} + \gamma\delta_{ij}\delta_{kl}$$

But the stress tensor is symmetric, hence $A_{ijkl} = A_{jikl}$. This means that $\eta_1 = \eta_2 = \eta$.We find

$$T_{ij} = -p\delta_{ij} + \eta(v_{i,j} + v_{j,i}) + \gamma v_{k,k}\delta_{ij}.$$

We introduce the deviator and write

$$T_{ij} = -p\delta_{ij} + 2\eta v_{<i,j>} + \zeta v_{k,k}\delta_{ij}. \tag{13.1}$$

We now assume the visosities to be constants and write down the momentum equation

$$\rho[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\mathbf{\nabla})\mathbf{v}] = -\mathbf{\nabla}p + \eta\triangle\mathbf{v} + (\zeta + \frac{\eta}{3})\mathbf{\nabla}(\mathbf{\nabla}\cdot\mathbf{v}) + \rho\mathbf{f}.$$

They are the Navier-Stokes equations.

For an incompressible fluid the equation of continuity gives us

$$\frac{D\rho}{Dt} + \rho(\boldsymbol{\nabla} \cdot \mathbf{v}) = 0,$$

$$\boldsymbol{\nabla} \cdot \mathbf{v} = 0. \tag{13.2}$$

The Navier-Stokes equations then simplify to

$$\rho[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{v}] = -\boldsymbol{\nabla} p + \eta \triangle \mathbf{v} + \rho \mathbf{f}. \tag{13.3}$$

1.1 Shear and bulk viscosities are nonnegative

For the kinetic energy we derive

$$\frac{d}{dt} \int \frac{1}{2} v^2 \rho dv = \int v_i T_{ij} ds_j + \int v_i f_i \rho dv - \int v_{(i,j)} T_{ij} dv. \tag{13.4}$$

Just like in particle mechanics, the rate of change of the kinetic energy is given by the power of the forces. The first term is the power of the forces acting on the boundary of the body, the second term is the power of the volume. The remaining term has to be the power of the internal forces, the stresses.

In a Newtonian fluid we have

$$T_{ij} = -p\delta_{ij} + 2\eta v_{<i,j>} + \zeta v_{k,k}\delta_{ij}$$

The viscous stresses always decrease the kinetic energy, so that

$$\begin{aligned} 0 &\leq v_{(i,j)}(2\eta v_{<i,j>} + \zeta v_{k,k}\delta_{ij}) \\ &= (v_{<i,j>} + \frac{1}{3} v_{l,l}\delta_{ij})(2\eta v_{<i,j>} + \zeta v_{k,k}\delta_{ij}) \\ &= 2\eta v_{<i,j>} v_{<i,j>} + \zeta v_{k,k} v_{l,l}. \end{aligned}$$

We conclude that η and ζ have to be nonnegative.

A more careful consideration needs to take thermodynamics into account and use that the entropy production is nonnegative.

2 Flows where the nonlinear term vanishes

The nonlinear term $(\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{v}$ disappears when

$$\mathbf{v} = v(y, z, t)\mathbf{e}_x \tag{13.5}$$

The resulting linear equation is much simpler to solve (we have assumed the volume force to be gravity)

$$\rho \frac{\partial v}{\partial t}\mathbf{e}_x = -\boldsymbol{\nabla} p + \eta \triangle v \mathbf{e}_x + \rho \mathbf{g}$$

We can introduce

$$\widetilde{p} = p + \rho \mathbf{g} \cdot \mathbf{r}$$

to simplify the equation to

$$\rho \frac{\partial v}{\partial t}\mathbf{e}_x = -\boldsymbol{\nabla} \widetilde{p} + \eta \triangle v \mathbf{e}_x.$$

In the static case, the equation reduces to

$$\boldsymbol{\nabla} \widetilde{p} = \mathbf{0}$$

so that $\widetilde{p}$ is a constant or

$$p = p_0 - \rho \mathbf{g} \cdot \mathbf{r}.$$

In the stationary case we have the equation

$$\mathbf{0} = -\boldsymbol{\nabla} \widetilde{p} + \eta \triangle v \mathbf{e}_x.$$

In components

$$\begin{aligned} 0 &= -\frac{\partial \widetilde{p}}{\partial x} + \eta \triangle v(y, z), \\ \frac{\partial p}{\partial y} &= \frac{\partial \widetilde{p}}{\partial z} = 0 \end{aligned}$$

The last two equations tell us that $\widetilde{p} = \widetilde{p}(x)$. The first equation then tells us that $d\widetilde{p}/dx$ is a constant.

2.1 Plane Couette flow

We first assume that $dp/dx = 0$ and that v is a function of y alone.

$$\begin{aligned} \frac{d^2v}{dy^2} &= 0, \\ v &= Ay + B \end{aligned}$$

We also assume that the flow is limited to the region between two planes at $y = 0$ and $y = a$. The non-slip boundary condition gives $B = A = 0$. There will be no flow. To obtain a flow we need to put one of the planes into motion. Suppose the plane at $y = a$ has the velocity V. Then we find that $B = 0$ and $Aa = V$, so that

$$V = \frac{y}{a}V. \tag{13.6}$$

The velocity profile is linear, this is plane Couette flow.

We also calculate the shear stress

$$T_{xy} = \eta\frac{dv}{dy} = \eta\frac{V}{a} \tag{13.7}$$

A shear stress has to be applied to the plane at $y = a$ in the positive x-direction and one to the plane $y = 0$ in the negative y-direction to produce the flow.

2.2 Plane Poiseuille flow

We now consider the case with a non-zero pressure gradient. The pressure gradient can then drive the flow so the boundaries can be fixed. We now take the planes to be at $y = -a$ and $y = a$. We assume the velocity also now to be a function of y only. The equation to be solved is then

$$-\frac{dp}{dx} + \eta\frac{d^2v}{dx^2} = 0$$

$$v = \frac{1}{\eta}\frac{dp}{dx}(\frac{y^2}{2} + Ay + B)$$

The boundary conditions are

$$\begin{aligned} \frac{a^2}{2} + Aa + B &= 0, \\ \frac{a^2}{2} - Aa + B &= 0 \end{aligned}$$

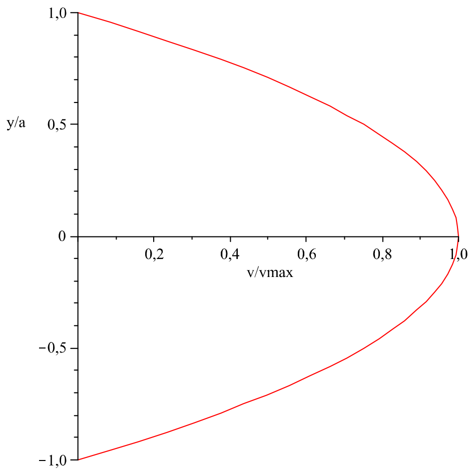


Figure 13.1: Plane Pouseuille flow.

or $A = 0$ and $B = -a^2/2$.

$$v = -\frac{1}{2\eta}\frac{dp}{dx}(a^2 - y^2) \tag{13.8}$$

The flow is in the direction of decreasing pressure.

The shear stress is

$$T_{xy} = \eta\frac{dp}{dx}y \tag{13.9}$$

This means shear stresses in the negative direction of the flow need to be applied to the two planes in order to keep them not to be dragged along with the flow. This is Plane Poiseuille flow.

2.3 Poiseuille flow in a circular pipe

Let us also consider Poiseuille flow in a circular pipe with radius R. We then have cylindrical symmetry. Let us take the direction of the pipe as the z-axis. Then

$$\begin{aligned} \mathbf{v} &= v(r)\mathbf{e}_z, \\ r &= \sqrt{x^2 + y^2} \end{aligned}$$

We have

$$\begin{aligned} r^2 &= x^2 + y^2, \\ r\frac{\partial r}{\partial x} &= x, \\ \frac{\partial r}{\partial x} &= \frac{x}{r}. \end{aligned}$$

Hence

$$\frac{\partial v}{\partial x} = \frac{dv}{dr}\frac{x}{r} = x(\frac{1}{r}\frac{dv}{dr})$$

and

$$\begin{aligned} \frac{\partial^2 v}{\partial x^2} &= (\frac{1}{r}\frac{dv}{dr}) + x\frac{d}{dr}(\frac{1}{r}\frac{dv}{dr})\frac{x}{r} \\ &= \frac{x^2}{r^2}\frac{d^2 v}{dr^2} + (1 - \frac{x^2}{r^2})\frac{dv}{dr} \end{aligned}$$

We also obtain

$$\frac{\partial^2 v}{\partial y^2} = \frac{y^2}{r^2}\frac{d^2 v}{dr^2} + (1 - \frac{y^2}{r^2})\frac{dv}{dr},$$

so that

$$\triangle v = \frac{d^2 v}{dr^2} + \frac{1}{r}\frac{dv}{dr} = \frac{1}{r}\frac{d}{dr}(r\frac{dv}{dr}). \tag{13.10}$$

The equation we need to solve is then

$$-\frac{dp}{dz} + \eta\frac{1}{r}\frac{d}{dr}(r\frac{dv}{dr}) = 0$$

We find

$$\begin{aligned} \frac{d}{dr}(r\frac{dv}{dr}) &= \frac{1}{\eta}\frac{dp}{dz}r, \\ r\frac{dv}{dr} &= \frac{1}{2\eta}\frac{dp}{dz}r^2 + A, \\ \frac{dv}{dr} &= \frac{1}{2\eta}\frac{dp}{dz}r + A\frac{1}{r}, \\ v &= \frac{1}{4\eta}\frac{dp}{dz}r^2 + A\ln r + B \end{aligned}$$

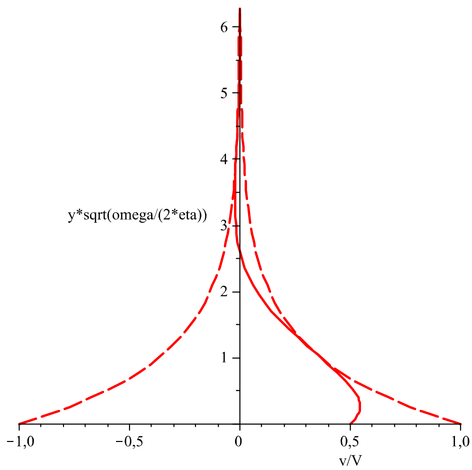


Figure 13.2: Flow field outside oscillating plane. The fully drawn line is v normalized by the velocity of the plane.

Here, we have to put $A = 0$ not to have a singularity on the axis of the tube. Further, v has to vanish at $r = R$, which gives us the values of B.

$$v = -\frac{1}{4\eta}\frac{dp}{dz}(R^2 - r^2) \tag{13.11}$$

The total volume flux is

$$\begin{aligned} \int v(r) 2\pi r dr &= -\frac{1}{4\eta}\frac{dp}{dz}2\pi(R^2\frac{R^2}{2} - \frac{R^4}{4}) \\ &= -\frac{\pi}{8\eta}\frac{dp}{dz}R^4. \end{aligned} \tag{13.12}$$

2.4 Flow generated by an oscillating plane

We assume that the plane $y = 0$ has the velocity

$$\mathbf{v} = A\cos(\omega t))\mathbf{e}_x = \mathrm{Re}(A\exp(-i\omega t)\mathbf{e}_x)$$

When the flow has been established we expect it to have the same frequency as the oscillating plane, so that

$$\mathbf{v} = w(y)\exp(-i\omega t)\mathbf{e}_x.$$

Here we understand the real part. We assume that there is no pressure gradient
The equation to be solved is now

$$-i\omega w(y) = \eta \frac{d^2 w}{dy^2}$$

The solution is

$$w = B\exp(Kx) + C\exp(-Kx).$$

Here

$$\begin{aligned} K^2 &= -i\frac{\omega}{\eta}, \\ K &= \sqrt{\frac{\omega}{2\eta}}(1-i). \end{aligned}$$

We need to take $B = 0$ to avoid the exponentiall growing solution. The boundary condition on the plane gives $C = A$. Hence,

$$v = A\exp(-\sqrt{\frac{\omega}{2\eta}}y)\cos[\sqrt{\frac{\omega}{2\eta}}y - \omega t] \tag{13.13}$$

The resulting flow is a damped wave that is propagating away from the plane.

3 The meaning of bulk viscosity

To find the meaning of the bulk viscosity, we consider the compressible case and we take an expanding flow

$$\begin{aligned} \mathbf{v} &= a\mathbf{r}, \\ v_i &= a x_i \end{aligned} \tag{13.14}$$

We have

$$\begin{aligned} v_{i,j} &= a\delta_{ij}, \\ v_{i,i} &= 3a, \\ v_{<i,j>} &= v_{(i,j)} - \frac{1}{3}v_{k,k}\delta_{ij} = 0 \end{aligned}$$

The equation of continuity gives us

$$v_{i,i} = -\frac{1}{\rho}\frac{D\rho}{Dt}$$

$$T_{ij} = -p\delta_{ij} + \zeta v_{k,k}\delta_{ij} = -(p + \zeta\frac{1}{\rho}\frac{D\rho}{Dt})\delta_{ij}$$

Hence,

$$T_{rr} = -(p + \zeta\frac{1}{\rho}\frac{D\rho}{Dt}) \tag{13.15}$$

To compress the fluid, a larger negative normal stress has to be applied than the static pressure.