Part II General Relativity

Lecture Notes

Abstract

These notes represent the material covered in the Part II lecture *General Relativity* (GR). While the course is largely self-contained and some aspects of Newtonian Gravity and Special Relativity will be reviewed, it assumed that readers will already be familiar with these topics. Also, calculus in N dimensions and Linear Algebra will be used extensively without being introduced.

There is wide range of books available on the topic and these notes have found inspiration in several of these. Likewise, these notes benefit considerably from other lecture notes used for this course or its Part III extension in previous years. Readers may find it helpful to consult any of these as alternative sources for the material, although the goal of these notes is to make this an optional rather than a necessary procedure for following the material. We note in particular the lecture notes for Part III GR by Harvey Reall [36], and the Part II GR notes by Gary Gibbons [37] and Stephen Siklos [38].

The content of these notes is too comprehensive to be put on the blackboard in verbatim fashion. A condensed version mirroring with high precision the blackboard content will be generated at some later stage.

A subset of the wealth of literature on Einstein's theory is given as follows.

- S. M. Carroll: "Spacetime and Geometry: An Introduction to General Relativity" [8] ; cf. also [7] .
- R. d'Inverno: "Introducing Einstein's Relativity" [9] .
- J. B. Hartle: "Gravity, An Introduction to Einstein's General Relativity" [11] .
- L. P. Hughston & K. P. Tod: "An Introduction to General Relativity" [15] .
- C. W. Misner, K. S. Thorne & J. A. Wheeler: "Gravitation" [17] .
- W. Rindler: "Relativity: Special, General, and Cosmological" [20] .
- L. Ryder: "Introduction to General Relativity" [21] .
- B. Schutz, "A first course in general relativity" [24] .
- H. Stephani: "An Introduction to Special and General Relativity" [27] .
- R. M. Wald: "General Relativity" [30] .
- S. Weinberg: "Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity" [31] .

I have not read all of these books, but will attempt here to give my two cents on guidance based on what I have read. Schutz' book is an excellent very first reading of general relativity. I also enjoyed Carroll's book a lot (on top of a good compromise between mathematical foundation and physics, I enjoyed his sense of humor). I found d'Inverno amazingly readable especially given that it goes quite a bit beyond the standard material

on several occasions. I may be biased, but certainly enjoyed a lot how much material of his book I found of high value in numerical relativity. (Note besides: it's German translation, while equally readable has a good chunk of typos in its first edition – the one I know). Misner, Thorne & Wheeler is often referred to as "The Bible of GR" and you will quickly find out why (starting when carrying it home). It was my first introduction to the geometrical foundation of relativity and it is simply breathtaking at providing the reader with a visual idea of curved geometry and it's mathematical toolkit. Weinberg is also a classic, but focuses more on the field theoretical side rather than geometric images. I enjoyed the Cosmology part most. I have frequently used Ryder and Wald for selected chapters but have not read them from the beginning (simply because I only knew about them at a later stage when reading books from the beginning had become an unaffordable luxury). Ryder seems a great introduction while Wald is rightfully famous for considerable mathematical rigor and depth (if you like, a good stepping stone towards Hawking & Ellis [13]). I have heard good things about Hartle's book but haven't got a hand on it myself yet. It goes without saying that these are merely my own humble opinions. As usual with textbooks, the recommendation is to have a look yourself and find your optimal selection. Chocolate is a wonderful thing in my opinion, but I know people who just don't happen to like it...

Example sheets will be pointed to at some later stage, probably on

`http://www.damtp.cam.ac.uk/user/examples`

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Contents

A Preliminaries

A.1 Units and constants of nature

The units we use for measuring things in our day-to-day lives are naturally adjusted to the magnitude of the size or mass of ourselves and the objects we tend to deal with. It does not matter here, whether you prefer Imperial or SI units; on a good pub outing, you will very likely consume of the order $\mathcal{O}(1)$ pints or liters of beer rather than, say, $\mathcal{O}(10^{-2})$ or $\mathcal{O}(10^2)$. When dealing with the wide range of objects in physics, these units are often not most suitable because we have essentially no intuitive understanding of numbers such as 2×10^{30} – the mass of the sun in kg. Here lies one reason why physicists often introduce units other than those used in supermarkets. It is not the only reason, however.
A second, and more profound, reason arises from the seeming constancy of certain values in nature. While we cannot be absolutely certain that the speed of light, Planck's $\hbar$ or Newton's gravitational constant are genuinely constant over all of space and time, experiments and observations made so far suggest that they are, and we will follow in this course the working hypothesis that this is indeed the case.
Constants of nature have two prominent implications: (i) they relate what previously appeared to be different fundamental physical dimensions and (ii) they give us an intuitive notion about the regime of validity of a physical theory. In this section, we will discuss these two phenomena for the speed of light c, Newton's constant G and Planck's constant $\hbar$.

Speed of light: In SI units, the speed of light is

$$c = 299\,792\,458 \text{ m/s} \approx 3 \times 10^8 \text{ m/s}\,. \tag{A.1}$$

Its constancy, of course, was one of the key ingredients in Einstein's derivation of the theory of special relativity. It turns out very convenient in these notes and, indeed, in much of research in relativity, to measure all velocities in units of the speed of light, i.e. set $c = 1$. This is to be applied quite literally to Eq. (A.1), so that

$$\begin{aligned} c = 3.00 \times 10^8 \text{ m/s} &\overset{!}{=} 1 \\ \Rightarrow \quad 1 \text{ s} = 3.00 \times 10^8 \text{ m}\,. \end{aligned} \tag{A.2}$$

Note that we really mean that 1 second *is* the same as 3.00×10^8 meters. This notion is most familiar from the use of "light years" for astrophysical distance,

$$1 \text{ yr} = 365.25 \,\frac{\text{days}}{\text{year}} \times 86\,400\, \frac{\text{seconds}}{\text{day}} \times 2.9979 \times 10^8\, \frac{\text{m}}{\text{s}} = 9.4607 \times 10^{15} \text{ m}\,. \tag{A.3}$$

It is a testament to the intuitive potential of this concept that the light year is frequently used in public presentations of astrophysical results and in science fiction, whereas astrophysicists at work tend to use the unit *parsec* instead. A parsec, 1 pc ≈ 3.26 lightyears, is the distance at which a celestial body undergoes a parallax of 1 arcsecond while the Earth orbits once around the sun – parsec = short for parallax second.

The speed of light thus gives us a natural unit for velocities and establishes a fundamental link between time and spatial distance. It furthermore tells us when a velocity is *large* in an absolute sense, namely in terms of a dimensionless number. Absolute numbers in physics only give us a real sense of the magnitude of something when that number is dimensionless. Often, such numbers also suggest when a physical theory hits the limits of its regime of validity. For instance, for velocities $v \ll c$, the Galileo transformations give us an exquisitely accurate rule for transforming from one coordinate frame to another moving with v relative to the first. For $v \approx c$, however, we know that this rule breaks down and we need to use Lorentz transformations instead. In fact, Galileo transformations turn out to be the leading order Taylor expansion of Lorentz transformations around $v = 0$. Likewise, the Newtonian expression for kinetic energy $mv^2/2$ is the leading-order approximation obtained from Taylor expanding the relativistic $E^2 = p^2c^2 + m^2c^4$ around $v = 0$. We have here a first warning that a theory that is practically used only in the limit of a small dimensionless number may turn out to be merely a leading order approximation of a more fundamental theory. This may also be the case of General Relativity itself.

Gravitational constant: In SI units, Newton's constant is

$$G = 6.67408 \times 10^{-11} \frac{\mathrm{m}^3}{\mathrm{kg\,s}^2}\,. \tag{A.4}$$

Note that G is known with significantly less accuracy than pretty much all other constants of nature; gravity is a very weak effect between laboratory test masses and therefore hard to measure with a high level of precision. The gravitational force of Earth is strong enough, but we would need an independent estimate of the Earth's mass; that, however, we obtain from the Earth's gravitational field.
As before, we set the constant to unity and also use $c = 1$, so that

$$\begin{aligned} & G = 1 = c \\ \Rightarrow\quad & 6.6741 \times 10^{-11}\ \mathrm{m}^3 = 1\ \mathrm{kg\,s}^2 = 1\ \mathrm{kg} \times (2.9979 \times 10^8\ \mathrm{m})^2 \\ \Rightarrow\quad & 1\ \mathrm{m} = 1.3466 \times 10^{27}\ \mathrm{kg} \quad \text{or} \quad 1\ \mathrm{s} = 4.0370 \times 10^{35}\ \mathrm{kg}\,. \end{aligned} \tag{A.5}$$

For comparison, the solar mass is

$$M_\odot = 1.4771\ \mathrm{km} = 4.9269\ \mu\mathrm{s}\,. \tag{A.6}$$

These are quite useful values to bear in mind when it comes to applications of GR. For instance, natural units using $G = 1 = c$ give us a good estimate of the size of the event horizon associated with a specific object; if a mass M is compressed inside a sphere of about the size of its mass expressed in meters or kilometers, it becomes a black hole. The mass expressed in seconds is a little less intuitive, but gives a measure of the oscillation periods of a black holes. Solar oscillations take much longer than a micro second, but if the sun were compressed to a black hole, that would oscillate on such a time scale.
Again, the *natural* units $G = 1 = c$ tell us when we have strong effects and a theory, here Newtonian gravity, reaches its limits. Objects with

$$\frac{G}{c^2}\frac{M}{R} = \frac{M}{R} \approx 1\,, \tag{A.7}$$

in general behave quite differently than Newtonian theory would predict; we need general relativity for their modeling. The sun has a radius $R_\odot = 6.957 \times 10^5$ km, so

$$\frac{M_\odot}{R_\odot} \ll 1\,. \tag{A.8}$$

Solar dynamics are accurately modelled using Newtonian gravity. For example, the relativistic effects of light bending near the solar surface are very small and require rather high-precision measurements to become detectable. We will return to this in Sec. D.3.2 below.
Note that in many physical systems, the regime of high velocity and strong gravity overlap. For example, the velocity of a test mass in spherical orbital of radius r around a spherically symmetric body of mass M is (using Newtonian theory) given by

$$\frac{v^2}{c^2} = \frac{G}{c^2}\frac{M}{r}\,, \tag{A.9}$$

and the escape velocity from the surface of a spherically symmetric body of mass M and radius R is

$$v_e = \sqrt{\frac{2GM}{R}}\,. \tag{A.10}$$

So we have $v^2 \sim M/R$ when the velocity is determined by gravitational effects and the regime $v \approx 1$ coincides with the regime $M/R \approx 1$. Post-Newtonian theory is a whole branch of gravitational research concerned with expanding general relativity around Newtonian gravity in terms of a power series of a dimensionless parameter $\epsilon = v^2 = M/R$ [5]. If, on the other hand, large velocities are of non-gravitational origin, special relativity provides a satisfactory description. This applies, for example, to collisions experiments at particle colliders.

Planck's constant: Planck's constant in SI units is given by

$$h = 6.62607004 \times 10^{-34}\,\frac{\mathrm{kg\,m^2}}{\mathrm{s}} \quad \Leftrightarrow \quad \hbar := \frac{h}{2\pi} = 1.0545718 \times 10^{-34}\,\frac{\mathrm{kg\,m^2}}{\mathrm{s}}\,. \tag{A.11}$$

Here, we will use $\hbar$ and also set the speed of light to unity.
We start by setting

$$\begin{aligned} 1 &= \frac{\hbar}{c^2} = 1.17 \times 10^{-51}\ \mathrm{kg\,s} \\ \Rightarrow \quad 1\ \mathrm{kg} &= 8.5223 \times 10^{50}\ \mathrm{Hz} \qquad \text{or} \qquad 1\ \mathrm{m} = \frac{1}{3.51767288 \times 10^{-43}\ \mathrm{kg}}\,. \end{aligned} \tag{A.12}$$

So we identify the mass of a particle with a frequency or, as we shall discuss a bit further below, the Compton wavelength with the inverse of a particle's mass. We can therefore construct a dimensionless quantity from the quotient of mass and frequency and again we will find that the breakdown of a theory is signaled when this parameter approaches unity. Consider for example that we use photons of frequency ω to explore the structure of a body of mass m. The photon energy is $\hbar\omega$ and the mass-energy of the body is mc^2. If

$$\frac{\hbar\omega}{mc^2} = \frac{\omega}{m} \approx 1\,, \tag{A.13}$$

classical physics break down and we have entered the realm of quantum mechanics. For example, we can safely track the sun using optical light ($\nu \sim 5 \times 10^{14}$ Hz), since

$$\frac{\hbar\omega}{M_\odot c^2} = \frac{\omega}{M_\odot} = \frac{2\pi \times 5 \times 10^{14}\ \text{Hz}}{1.989 \times 10^{30}\ \text{kg} \times 8.5223 \times 10^{50}\ \frac{\text{Hz}}{\text{kg}}} \approx 1.85 \times 10^{-66} \ll 1\,. \tag{A.14}$$

Life doesn't get much more classical than that. How about tracking protons? Using the proton mass in SI units, $m_p = 1.6726219 \times 10^{-27}$ kg, we obtain

$$\frac{\omega}{m_p} = \frac{2\pi \times 5 \times 10^{14}\ \text{Hz}}{1.6726 \times 10^{-27}\ \text{kg} \times 8.5223 \times 10^{50}\ \frac{\text{Hz}}{\text{kg}}} \approx 2.2 \times 10^{-9} \ll 1\,, \tag{A.15}$$

which is still ok. For instance, we can safely trace the trajectory of protons in bubble chambers. Next let us consider energy levels in atoms. For this purpose recall that the energy difference between different electron states in an atom is of the order of electron volts and that

$$1\ \text{eV} = 1.602176565 \times 10^{-19}\ \text{J} = 1.602176565 \times 10^{-19}\ \frac{\text{kg m}^2}{\text{s}^2}\,.$$

$$\Rightarrow \quad m_{\text{eV}} = \frac{1\ \text{eV}}{c^2} = 1.78269 \times 10^{-36}\ \text{kg}\,. \tag{A.16}$$

If we wish to probe energy levels in atoms using optical light, we have

$$\frac{\hbar\omega}{m_{\text{eV}} c^2} = \frac{\omega}{m_{\text{eV}}} = \frac{2\pi \times 5 \times 10^{14}\ \text{Hz}}{1.78269 \times 10^{-36}\ \text{kg} \times 8.5223 \times 10^{50}\ \frac{\text{Hz}}{\text{kg}}} \approx 2 = \mathcal{O}(1)\,, \tag{A.17}$$

and have definitely reached the quantum regime. The light thrown at the atoms is manifestly perturbing the very energy levels we are interested in studying.
An alternative way to look at the unity of Planck's constant is to consider the Compton wavelength

$$\lambda\!\!\!^{-} = \frac{\hbar}{mc} = \frac{1}{m}\,, \tag{A.18}$$

so in natural units a particle's mass is merely the inverse of its Compton wavelength. The dimensionless quantity then is the ratio of the Compton wavelength of the object to its size or the characteristic length scale of its available volume. Macroscopic objects are much larger than their Compton wavelength. For the sun, for instance, we obtain the absurdly small value

$$\lambda\!\!\!^{-}_\odot = \frac{\hbar}{M_\odot c} = 0.5028 \times 10^{-30}\ \text{kg}^{-1} = 0.177 \times 10^{-72}\ \text{m}\,, \tag{A.19}$$

and clearly $\lambda\!\!\!^{-}_\odot / R_\odot \ll 1$. The sun as a compound object is a classical object through and through. Of course, quantum effects play a very important role for the behaviour of the sun's constituent matter, but not for the sun as a lump object. For a proton, on the other hand, the Compton wavelength is

$$\lambda\!\!\!^{-}_p = \frac{\hbar}{m_p c} = 2.10268 \times 10^{-16}\ \text{m} = 0.210268\ \text{fm}\,. \tag{A.20}$$

The radius of atomic nuclei ranges from $\mathcal{O}(1)$ to $\mathcal{O}(10)$ fm, so the available volume is comparable to the proton wavelength and quantum effects are important.

In summary, we have the following three dimensionless quantities that mark the onset of the need for new physics when they approach values of the order of unity.

(1) $\frac{v}{c} \approx 1 \quad \Rightarrow$ Galileo transformations are no longer valid and we need special relativity.

(2) $\frac{G}{c^2}\frac{M}{R} \approx 1 \quad \Rightarrow$ Newtonian gravity breaks down and we need general relativity.

(3) $\frac{\lambda\!\!\!^{-}}{R} = \frac{\hbar}{McR} \approx 1 \quad \Rightarrow$ Classical physics break down and we need quantum theory.

We conclude this discussion with the question of the overlap between the three regimes. We already discussed this issue for the first two items: we may have large velocities without strong gravity which is well described by Einstein's theory of special relativity. General relativity fully includes special relativity, on the other hand, so when we have strong gravity, we automatically have relativistic effects. The most intriguing overlap is that between general relativity and quantum theory and it remains one of the great unknowns of contemporary physics. This overlap regime is characterized by having

$$\frac{G}{c^2}\frac{M}{R} = 1 \qquad \text{and} \qquad \frac{\hbar}{McR} = 1$$
$$\Rightarrow \quad M^2 = \frac{\hbar c}{G}\,. \tag{A.21}$$

This scale is called the Planck mass, length or time defined by

$$\text{Planck mass} \quad M_{\text{Pl}} = \sqrt{\frac{\hbar c}{G}} = 2.18 \times 10^{-8}\ \text{kg} = 1.22 \times 10^{19}\ \text{GeV} \tag{A.22}$$

$$\text{Planck length} \quad L_{\text{Pl}} = \frac{G}{c^2} M_{\text{Pl}} = 1.61 \times 10^{-35}\ \text{m} \tag{A.23}$$

$$\text{Planck time} \quad T_{\text{Pl}} = \frac{1}{c} L_{\text{Pl}} = 5.37 \times 10^{-44}\ \text{s}\,. \tag{A.24}$$

This is the regime where we need a new theory: quantum gravity.

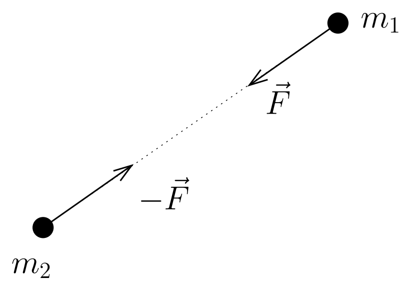


Figure 1: Illustration of the Newtonian two-body problem.

A.2 Newtonian gravity

A.2.1 A tale of three masses

Let us start by considering two point masses located at position vectors $\vec{r}_1$ and $\vec{r}_2$; cf. Fig. 1. According to Newton's law the gravitational force $\vec{F}_{\rm 1on2}$ exerted by particle 1 on particle 2 is given by

$$\vec{F}_{\rm 1on2} = G m_{1a} m_{2p} \frac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|^3} \stackrel{!}{=} m_{2i} \ddot{\vec{r}}_2 \,, \tag{A.25}$$

and gives rise to an acceleration $\ddot{\vec{r}}_2$ of the second body. Here, a dot denotes a time derivative and the additional labels 'a', 'p' and 'i' stand for the following three types of mass:

active mass:	the source of the gravitational field,
passive mass:	the sensitivity to gravitational fields generated by other sources,
inertial mass:	a body's resistant to change motion when exposed to forces.

According to Newton's 3rd law of motion, for every action force, there is a reaction force equal in magnitude and pointing in the opposite direction. In consequence, the second body reacts on the first with a force $\vec{F}_{\rm 2on1}$ given by

$$\vec{F}_{\rm 2on1} = G m_{1p} m_{2a} \frac{\vec{r}_2 - \vec{r}_1}{|\vec{r}_2 - \vec{r}_1|^3} \stackrel{!}{=} -\vec{F}_{\rm 1on2} = G m_{1a} m_{2p} \frac{\vec{r}_2 - \vec{r}_1}{|\vec{r}_2 - \vec{r}_1|^3} \,. \tag{A.26}$$

This equality holds for arbitrary position vectors $\vec{r}_1$, $\vec{r}_2$, so that

$$m_{1p} m_{2a} = m_{1a} m_{2p} \tag{A.27}$$

$$\Rightarrow \quad \frac{m_{1p}}{m_{1a}} = \frac{m_{2p}}{m_{2a}} \,. \tag{A.28}$$

So for every body, the ratio of passive to active mass is the same and with a convenient choice of units, we can set it to unity,

$$m_p = m_a \,. \tag{A.29}$$

Note that this is not a special feature of gravity. For instance, we also have equality of passive and active charge in electromagnetism.

How about the inertial mass then? This has been studied throughout a good part of history in a variety of experiments. An incomplete list is as follows.

(1) $\sim$ **500 AD: Philoponus** observes that two weights differing from each other by a wide measure fall in times whose ratio differs much less than the ratio of their weights.

(2) $\sim$ **1590: Galileo** studies balls rolling down a slope and measures that irrespective of the balls' weight, they require for this an amount of time equal to within about 2 %.

(3) $\approx$ **1686: Newton** finds the oscillation period of pendulums of different matter types equal to within $\sim 10^{-3}$.

(4) 1922: Eötvös uses a torsion balance with arms of different material to check for a torque exerted by the sun's gravity. He finds none to within $\sim 5 \times 10^{-9}$.

(5) 1964: Dicke *et al* perform a refined version of Eötvös' experiment and observe no torque to within $\sim 10^{-11}$.

More experiments have been carried out since to search for signs of inequality between the inertial and the gravitational mass, all compatible to within error bars with the universality of free fall. If we denote the gravitational field by $\vec{g}$, a freely falling particle in this field follows

$$m_i \ddot{\vec{r}} = m_p \vec{g}(\vec{r}, t)\,, \tag{A.30}$$

and the universality of motion implies that all objects have the same ratio m_i/m_p which, again, we set to unity without loss of generality. Note that gravity differs in this regard from all other interactions: inertial mass is identical to the gravitational "charge" of a body, but has no relation to its electric charge or the body's coupling to the weak and strong nuclear forces.

A.2.2 Equivalence principles

The observation that all particles fall the same way has led to the formulation of so-called equivalence principles. It is common to distinguish between three versions.

Weak Equivalence Principle (WEP): Freely falling small bodies with negligible gravitational self interaction follow the same path if they have the same initial velocity and position.

The WEP summarizes the observations reviewed in the previous subsection. You may wonder at this stage why this version excludes gravitational self interaction. We will return to this question shortly, but first introduce Einstein's version which promotes the principle to a more general status. For this purpose, we need the following definition.

Def. : A "local inertial frame" is a coordinate frame $(t,\, x,\, y,\, z)$ defined by a freely falling observer in the same way as an inertial frame is defined in Minkowski spacetime. In this context, "local" is defined to mean small compared with the length scale of variations in the gravitational field $\vec{g}$.

The word "local" marks the key difference from inertial frames in special relativity. This constraint is necessary to avoid effects such as tidal forces. As illustrated in Fig. 2, tidal forces in an oversized laboratory give rise to a relative acceleration of two falling particles relative to

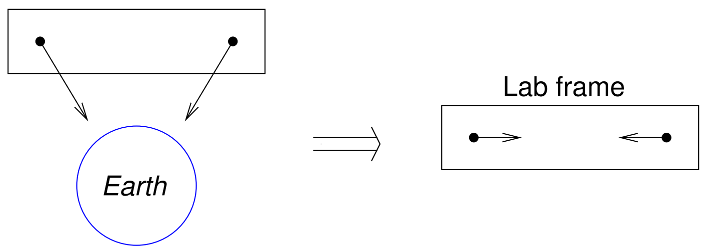


Figure 2: If an observer's frame is too large, inhomogeneities in the gravitational field lead to relative acceleration of particles when viewed inside this frame. Both particles fall towards the Earth's center. In a large freely falling laboratory, the different horizontal components of $\vec{g}$ make the particles appear accelerating towards each other without apparent cause.

each other. Local inertial frames are central to Einstein's version of the equivalence principle.

Einstein equivalence principle (EEP): In a local inertial frame, the results of all non-gravitational experiments are indistinguishable from those of the same experiment performed in an inertial frame in Minkowski spacetime.

In the 1960s, Schiff [23] conjectured that the WEP implies the EEP. The idea is that matter is composed of particles (quarks, electrons etc.), that the binding energy merely forms a contribution to the particle's masses and that the overall interactions in any experiment can thus be reduced to point particle motion that obeys the WEP. Intriguing though this idea may be, it remains an unproven conjecture. That leaves the strong equivalence principle which is undoubtedly a stronger requirement than the WEP.

Strong equivalence principle (SEP): The gravitational motion of a small test body (that may have gravitational self interaction) depends only on its initial velocity and position but not on its constitution.

We conclude this discussion with the following remarks.

- As already indicated, the SEP implies the WEP, but not the other way round.
- In the SEP, we require the test body to be small, so that tidal effects are negligible: Over sufficiently short times, the motion of the Earth and Moon about each other is well approximated by the motion of point masses. On very long time scales, however, the tidal interaction transfers angular momentum from the Earth's spin to the lunar orbit causing a slow increase in the Earth-Moon distance. This effect would not be present in a system of genuine point particles.
- The SEP is related to the equality of active and passive mass. Let us consider again the Earth-Moon system and let us assume Earth and Moon had a different ratio of passive versus active mass, for example due to differences in the contribution of the gravitational binding energy to the passive mass. In that case, the Earth and the Moon would fall differently in the Sun's gravitational field. This would lead to a distortion of the geocentric lunar orbit, an effect known as the "Nortvedt effect". Lunar laser ranging experiments

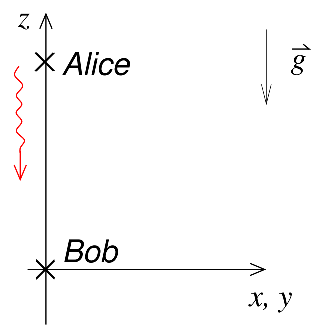


Figure 3: Two observers, Alice and Bob, are located at different height in a uniform gravitational field $\vec{g}$. Alice sends light to Bob that undergoes a change in frequency.

rule out this effect to within 3 ± 4 cm.

- The SEP implies that Newton's gravitational constant G is the same everywhere in the universe, and suggests that gravity is entirely of geometrical nature. Otherwise, the gravitational binding energy of an extended object would depend on its position and we would obtain the Nordtvedt effect.
- General relativity satisfies all three equivalence principles.
- Some modifications of general relativity satisfy the WEP and the EEP, but evoke fields additional to the spacetime metric to mediate the force of gravity. These additional fields lead to violations of the SEP.

A.2.3 Gravitational redshift

The equivalence principle allows us to make predictions for the gravitational redshift even in the absence of a fully developed theory. Consider for this purpose standard Cartesian coordinates (x, y, z) and a uniform gravitational field

$$\vec{g} = (0,\, 0,\, -g)\,, \qquad g = \text{const}\,. \tag{A.31}$$

Let Alice and Bob be located at $x = y = 0$ and $z = h$ and $z = 0$, respectively; cf. Fig. 3. According to the EEP, we can describe this scenario using the laws of special relativity in a freely falling frame, i.e. a frame accelerated with $\vec{g}$ relative to the rest frame with gravitational field displayed in Fig. 3.

For simplification, we assume the velocity of both Alice and Bob to be much smaller than the speed of light, $v \ll c$, so that we can ignore $(v/c)^2$ and higher order special relativistic terms. The trajectories of Alice and Bob in the freely falling frame are then

$$z_A(t) = h + \frac{1}{2} g t^2\,, \qquad z_B(t) = \frac{1}{2} g t^2\,, \qquad v_A = v_B = gt \overset{!}{\ll} c\,. \tag{A.32}$$

The calculation then proceeds as follows.

1. Alice emits a first signal at $t = t_1$. The trajectory of this signal is

$$z_1(t) = z_A(t_1) - c(t - t_1) = h + \frac{1}{2}gt_1^2 - c(t - t_1)\,. \tag{A.33}$$

2. This signal reaches Bob at $t = T_1$, where T_1 is given by $z_1(T_1) = z_B(T_1)$, i.e.

$$h + \frac{1}{2}gt_1^2 - c(T_1 - t_1) = \frac{1}{2}gT_1^2\,. \tag{A.34}$$

3. Alice emits a second signal at $t_2 = t_1 + \Delta\tau_A$ and this signal follows the trajectory (A.33) merely with t_1 replaced by t_2 on the right-hand side. The signal reaches Bob at $T_2 = T_1 + \Delta\tau_B$, where $z_2(T_2) = z_B(T_2)$, i.e.

$$h + \frac{1}{2}g(t_1 + \Delta\tau_A)^2 - c(T_1 + \Delta\tau_B - t_1 - \Delta\tau_A) = \frac{1}{2}g(T_1 + \Delta\tau_B)^2\,. \tag{A.35}$$

Subtracting Eq. (A.34) gives

$$c(\Delta\tau_A - \Delta\tau_B) + \frac{1}{2}g\Delta\tau_A(2t_1 + \Delta\tau_A) = \frac{1}{2}g\Delta\tau_B(2T_1 + \Delta\tau_B)\,. \tag{A.36}$$

4. We now assume that $\Delta\tau_A$, $\Delta\tau_B \ll T_1 - t_1$. For example, the two signals may be two consecuitve crests in a light wave where $\Delta\tau = \mathcal{O}(10^{-15})$ s which is much smaller than the travel time $T_1 - t_1$ in all practical experiments. We can then ignore the terms $\Delta\tau_A^2$ and $\Delta\tau_B^2$ in (A.36) and obtain

$$\begin{aligned} & c(\Delta\tau_A - \Delta\tau_B) + g\Delta\tau_A\, t_1 = g\Delta\tau_B\, T_1 \\ \Rightarrow\quad & \Delta\tau_B(gT_1 + c) = \Delta\tau_A(gt_1 + c) \\ \Rightarrow\quad & \Delta\tau_B = \left(1 + \frac{gT_1}{c}\right)^{-1}\left(1 + \frac{gt_1}{c}\right)\Delta\tau_A \approx \left[1 - \frac{g(T_1 - t_1)}{c}\right]\Delta\tau_A\,, \end{aligned} \tag{A.37}$$

where we have used $gt/c \ll 1$ in the last step.

5. Next, we reshuffle terms in Eq. (A.34), so that

$$\frac{h}{c} - (T_1 - t_1) = \frac{g}{2c}(T_1^2 - t_1^2) = \frac{1}{2}\underbrace{\frac{g}{c}(T_1 + t_1)}_{\ll 1}\underbrace{(T_1 - t_1)}_{\approx \frac{h}{c}} \approx 0\,.$$

$$\Rightarrow\quad T_1 - t_1 = \frac{h}{c} \quad \text{to leading order.} \tag{A.38}$$

6. Using this expression in Eq. (A.37) for the redshift gives

$$\boxed{\Delta\tau_B \approx \left(1 - \frac{gh}{c^2}\right)\Delta\tau_A \overset{!}{<} \Delta\tau_A}\,. \tag{A.39}$$

The signal appears blue shifted to Bob: in terms of the wavelength λ we have

$$c\Delta\tau_B = \lambda_B \approx \left(1 - \frac{gh}{c^2}\right)\lambda_A\,. \tag{A.40}$$

This prediction was verified to within about 10 % by Pound & Rebka [19] in 1959 at Havard's Jefferson Laboratory. With a height difference of about 22.5 m, the quantity $gh/c \approx 7 \times 10^{-7}$ m/s $\ll 1$ satisfies our simplifying assumption exquisitely. The fractional change in energy of a photon (i.e. its frequency) was $\mathcal{O}(10^{-15})$ in this experiment. Later similar experiments, all compatible with the equivalence principle, refined the accuracy by several orders of magnitude.
Anticipating material that we will develop further down the road of this course, we can generalize the result (A.39) to gravitational fields with non-uniform fields. The invariant special relativistic interval

$$c^2 \Delta\tau^2 = -c^2\,\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2\,, \tag{A.41}$$

generalizes in the case of a weak and time independent Newtonian gravitational potential $\phi(x, y, z)$ to

$$c^2 d\tau^2 = \left[1 + \frac{2\phi(x,y,z)}{c^2}\right] c^2\,dt^2 - \left[1 - \frac{2\phi(x,y,z)}{c^2}\right](dx^2 + dy^2 + dz^2)\,, \quad \frac{\phi}{c^2} \ll 1\,. \tag{A.42}$$

In Sec. G.3, we will recover this expression as the spacetime metric of general relativity in the Newtonian limit. Note that the interval is infinitesimally small in contrast to the special relativistic (A.41). Let Alice and Bob now be located at fixed positions $\vec{x}_A$ and $\vec{x}_B$. We calculate the redshift from the invariant (A.42) as follows.

1. Alice emits signals at t_A and $t_A + \Delta t$. Let t_B denote the time when Bob receives the first signal. When does Bob receive the second?
2. Because the spacetime is static (ϕ does not depend on t), the two signals travel on identical trajectories, merely shifted in time. Bob therefore receives the second signal at $t_B + \Delta t$.
3. The time measured by Alice's and Bob's clocks, however, is given by the proper times τ at their respective positions. These are

$$\begin{aligned}
& \Delta\tau_A^2 = \left(1 + \frac{2\phi_A}{c^2}\right)\Delta t^2\,, \qquad && \Delta\tau_B^2 = \left(1 + \frac{2\phi_B}{c^2}\right)\Delta t^2 \\
\Rightarrow\quad & \Delta\tau_A \approx \left(1 + \frac{\phi_A}{c^2}\right)\Delta t\,, && \Delta\tau_B \approx \left(1 + \frac{\phi_B}{c^2}\right)\Delta t \\
\Rightarrow\quad & \Delta\tau_B \approx \left(1 + \frac{\phi_B}{c^2}\right)\left(1 + \frac{\phi_A}{c^2}\right)^{-1}\Delta\tau_A \\
\Rightarrow\quad & \boxed{\Delta\tau_B \approx \left(1 + \frac{\phi_B - \phi_A}{c^2}\right)\Delta\tau_A}\,.
\end{aligned} \tag{A.43}$$

The redshift depends only on the potential difference between the point of emission and the point of absorption.
The equivalence principles played an important role in the development of general relativity. If the response of a body's motion to gravitational forces is independent of the properties of the body, it suggests that the gravitational force is not a feature of the body but exclusively of the spacetime in which it moves. To be more precise, gravity is a feature of the spacetime's geometry.

A.2.4 An index based formulation of Newtonian Gravity

We have so far concentrated on Newtonian gravity acting on point masses and only qualitatively considered the effect on bodies of finite extent. In this section, we will discuss the Newtonian field equations more generally and also introduce an index notation for their formulation. This serves two purposes. First, it enables us to introduce index notation in a familiar environment. Second, this formulation will emphasize more clearly the analogy between Newtonian and general relativistic laws when we discuss the latter further below.

Index notation is a way to write vectorial and matrix valued quantities in terms of their components. For instance, we can represent a vector $\vec{v}$ in terms of its components v_i where the index i runs over the components in a specific coordinate system. If we use Cartesian coordinates (x, y, z), for instance, we can write

$$v_i = (v_x, v_y,\, v_z)\,. \tag{A.44}$$

In view of things to come later when we discuss relativity, we will not equate the vector $\vec{v}$ with its components. Our hesitation in this regard will become clearer further below. In contrast to general relativity, we will also not distinguish between upstairs and downstairs indices, but only use the latter. Again, the difference between the index positions will be clarified when we discuss tensors in general relativity. In the example of a vector, we have one index, for example for the components of a velocity. A quantity may have more indices, however. An example would be the moment of inertia tensor which is matrix valued and has two indices. We will encounter further examples as we move along.

The following rules will govern our index notation.

(1) Repeated indices in a product are summed over. For example

$$A_{ij} v_j := \sum_{j=1}^{3} A_{ij} v_j\,. \tag{A.45}$$

Repeated indices appear exactly twice. More than two identical indices in one term do not give a meaningful expression.

(2) Indices over which a summation is performed may be renamed as long as no conflict with other indices arises. So,

$$A_{ij} v_j = A_{ik} v_k\,, \tag{A.46}$$

really are the same. The j may not be replaced with an i in this case, however, since $A_{ii} v_i$ is not a well defined expression.

(3) In an equation, free (i.e. not repeated) indices must match on both sides and in added terms. For example, $w_i + A_{ij} v_j = 0$ is a valid equation but $w_k = A_{ij} v_j$ is not.

(4) Coordinates can also be written in index form. We often use the letter x for this purpose. For example, Cartesian coordinates can be written as $x_i = (x, y, z)$. We may also denote spherical coordinates in this way, $x_i = (r, \theta, \varphi)$. Some expressions are valid in all coordinate systems, others may only hold for specific coordinates. In the latter case, we will make clear which coordinates we are using.

(5) The partial derivative with respect to the coordinate x_i is sometimes denoted by $\partial_i :=$ $\partial/\partial x_i$. Sometimes, we also use a comma for this purpose as for example in

$$v_{i,j} := \partial_j v_i := \frac{\partial v_i}{\partial x_j}\,. \tag{A.47}$$

Let us start using the index notation in the already familiar case of the motion of a point mass. Consider, for this purpose, Cartesian coordinates

$$x_i = (x, y, z)\,, \tag{A.48}$$

and the time coordinate t. Let $\vec{g}(t, \vec{x})$ be the gravitational field and m the mass of a freely falling particle. The equation of motion for the particle is then

$$m_i \ddot{\vec{x}} = m_g \vec{g}(\vec{x}, t) \qquad \Big| \quad m_i = m_g \tag{A.49}$$

$$\Rightarrow \quad \ddot{\vec{x}} = \partial_t \partial_t \vec{x} = \vec{g}(\vec{x}, t)\,, \tag{A.50}$$

where a dot denotes ∂_t and we assumed equality of inertial and gravitational mass. In index notation, this becomes

$$\ddot{x}_i = g_i(x_j, t)\,, \tag{A.51}$$

where the j index on the right hand side merely denotes the coordinate labels. It is *not* a free index in the sense of requiring an analog on the left hand side.
We can now introduce a non-inertial coordinate system $\tilde{x}_i$ by

$$\tilde{x}_i = x_i - b_i(t)\,. \tag{A.52}$$

and Eq. (A.51) becomes in this new coordinate system

$$\ddot{\tilde{x}}_i = \tilde{g}_i(\tilde{x}_j, t) = g_i(\tilde{x}_j, t) - \ddot{b}_i(t)\,. \tag{A.53}$$

Comments: 1) If g_i is uniform (independent of x_j), we can choose b_i such that $\tilde{g}_i = 0$.

2) If g_i is not uniform, we can only achieve that locally. The frame $\tilde{x}_i$ is then a freely falling frame.

We have already seen that in too large a laboratory, tidal effects will give rise to non-inertial phenomena; cf. Fig. 2. We calculate the tidal forces by considering two particles located at $\vec{x}$ and $\vec{x} + \delta\vec{x}$. The two particles' motion follows

$$\frac{d^2}{dt^2} x_i = g_i(x_j, t)\,, \qquad \frac{d^2}{dt^2}(x_i + \delta x_i) = g_i(x_j + \delta x_j, t) \tag{A.54}$$

$$\Rightarrow \quad \frac{d^2}{dt^2}\delta x_i = (\delta\vec{x} \cdot \vec{\nabla}) g_i + \mathcal{O}(|\delta\vec{x}|^2) \tag{A.55}$$

$$\Rightarrow \quad \frac{d^2}{dt^2}\delta x_i = \delta x_k \partial_k g_i + \mathcal{O}(\delta x_j^2)\,, \tag{A.56}$$

where we introduced gradient $\vec{\nabla}$ and dropped higher-order terms in the δx_j. We now define the tidal tensor as (the minus sign is merely a convention)

$$E_{ij} := -\partial_j g_i \,, \tag{A.57}$$

and write the tidal effect on the particle's relative motion as the equation of *geodesic deviation* (the name will become clear when we consider the general relativistic analog)

$$\boxed{\frac{d^2}{\partial t^2}\delta x_i + E_{ij}\delta x_j = 0} \tag{A.58}$$

The gravitational field in Newtonian theory is curl free, i.e. $\vec{\nabla} \times \vec{g} = 0$, so that there exists a potential ϕ such that

$$\boxed{\vec{g} = -\vec{\nabla}\phi \quad \Leftrightarrow \quad g_i = -\partial_i \phi} \tag{A.59}$$

It follows that $E_{ij} = +\partial_j \partial_i \phi$ and, since partial derivatives commute, that the tidal tensor is symmetric,

$$\boxed{E_{ji} = E_{ij}}\,. \tag{A.60}$$

Generic matter distributions are described in terms of an energy density field $\rho(\vec{x}, t)$ and source a gravitational field according to Poisson's equation

$$\vec{\nabla} \cdot \vec{g} = -4\pi G \rho \quad \Rightarrow \quad \boxed{\vec{\nabla}^2 \phi = \partial_i \partial_i \phi = 4\pi G \rho}\,. \tag{A.61}$$

Note that we can write this equation equivalently as

$$E_{ii} = 4\pi G \rho \,, \tag{A.62}$$

which, as we shall see, bears considerable resemblance to the general relativistic version of the field equations.
Finally, we note that the definition $E_{ij} = -\partial_j g_i$ implies

$$\partial_k E_{ij} = -\partial_k \partial_j g_i = \partial_j E_{ki} \tag{A.63}$$

$$\Rightarrow \quad E_{i[j,k]} := \frac{1}{2}(E_{ij,k} - E_{ik,j}) = 0 \,, \tag{A.64}$$

where we used brackets to denote anti symmetrization over the enclosed indices (the factor 1/2 in front is merely a convention). Again, we will encounter a similar equation in general relativity that goes under the name of *Bianchi Identities*. In summary, we have the following main relations:

(1) The geodesic deviation equation

$$\frac{d^2}{dt^2}\delta x_i + E_{ij}\,\delta x_j = 0 \,. \tag{A.65}$$

(2) The field equation

$$E_{ii} = 4\pi G\rho\,. \tag{A.66}$$

(3) The integrability conditions

$$E_{ij} = E_{ji}\,, \tag{A.67}$$

$$E_{i[j,k]} = 0\,. \tag{A.68}$$

A.2.5 The need for general relativity

Unlike special relativity, the general theory of relativity was not urgently required to reconcile observation with theory. Even though observational puzzles existed, as for example the anomalous perihelion advance of mercury, effects of this kind had been satisfactorily explained in the form of dark matter before: irregularities in the orbit of Uranus lead to the prediction, by Le Verrier of the location of a further planet whose existence was duly confirmed by Galle in 1846. Possibly spurred on by the tremendous success of his predictions for Neptune, Le Verrier conjectured yet another planet to explain mercury's abnormal orbital motion. This planet was dubbed *Vulcan*, though not in anticipation of future televisional fiction, but because its seeming proximity to the sun sparked fiery visions (Vulcan is the ancient Roman god of fire). Of course, Vulcan was never found and Mercury's abnormal motion found a perfectly satisfactory explanation in the form of modified gravity (general relativity as opposed to Newtonian gravity). Nevertheless, the idea of another planet seemed far from grotesque at the time and Mercury's perihelion precession did not constitute a fatal observational paradox analogous to the Michelson-Morley experiment's contradiction of the Galilean/Newtonian concept of relativity.

The need for general relativity instead arose more from theoretical arguments. Newtonian gravity is Galileo invariant and therefore incompatible with special relativity. Furthermore the equivalence principals pointed towards a geometric nature of gravity. At least in hindsight the extension from special to general relativity along the same lines flat Euclidean geometry had been generalized to curved Riemannian geometry looks natural. At the time, of course, this concept was as revolutionary as it was conceptually beautiful.

A.3 A review of special relativity

A.3.1 Notation and metric

In relativity, we introduce two new ingredients to our index notation.

(1) We now distinguish between upstairs and downstairs indices, so in general $v^i \neq v_i$. Below in Sec. B.1 we will see that this distinction arises from the concept of vectors and co-vectors (or one-forms) which are defined as maps from vectors to real numbers. Summation over repeated indices is now only performed if one index is upstairs and the other is downstairs. So

$$v^j u_j := \sum_{j=1}^{3} v^j u_j\,. \tag{A.69}$$

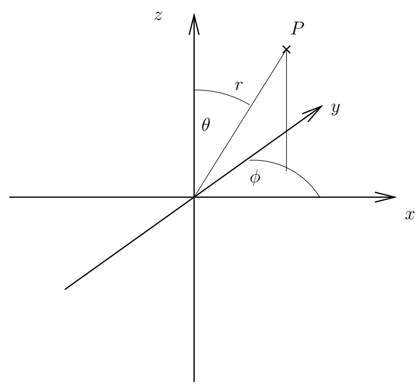


Figure 4: Spherical coordinates (r, θ, ϕ).

Coordinates will from now on be denoted with an upstair index. Again, this choice will be motivated below when we introduce differential geometry and tensors.

(2) We introduce Greek indices α, β, ... which run from 0 to 3 and include $x^0 = t$ as the time coordinate. We will keep the notation that middle Latin indices i, j, ... run from 1 to 3 and will occasionally write $x^\alpha = (x^0,\, x^i)$ or $u_\beta = (u_0,\, u_j)$ etc.

We also introduce a *metric* as a generalization of Pythagoras' theorem familiar from the $\mathbb{R}^2$ or $\mathbb{R}^3$. In Euclidean geometry in $\mathbb{R}^3$, Pythagoras gives us the distance between two points $x^i = (x\,, y\,, z)$ and $x^i + \Delta x^i = (x + \Delta x,\, y + \Delta y,\, z + \Delta z)$ as

$$\Delta s^2 = \Delta x^2 + \Delta y^2 + \Delta z^2 = \delta_{ij}\Delta x^i \Delta x^j\,, \tag{A.70}$$

where

$$\delta_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{A.71}$$

is the Kronecker delta. In curvilinear coordinates, we can use chain rule to obtain the separation between neighboring points, but the result will in general only apply to infinitesimally close points. Les us consider this for spherical coordinates (r, θ, ϕ), defined through (see Fig. 4)

$$\begin{aligned} x &= r\sin\theta\,\cos\phi\,, \\ y &= r\sin\theta\,\sin\phi\,, \\ z &= r\cos\theta\,. \end{aligned} \tag{A.72}$$

Using

$$\begin{aligned} dx &= \frac{\partial x}{\partial r}dr + \frac{\partial x}{\partial \theta}d\theta + \frac{\partial x}{\partial \phi}d\phi\,, \\ dy &= \frac{\partial y}{\partial r}dr + \frac{\partial y}{\partial \theta}d\theta + \frac{\partial y}{\partial \phi}d\phi\,, \\ dz &= \frac{\partial z}{\partial r}dr + \frac{\partial z}{\partial \theta}d\theta + \frac{\partial z}{\partial \phi}d\phi\,, \end{aligned} \tag{A.73}$$

we obtain

$$\begin{aligned} ds^2 &= dx^2 + dy^2 + dz^2 \\ &= dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2 \, . \end{aligned} \tag{A.74}$$

The second equality, however, only holds in the limit of infinitesimally small separation. In fact, this is the general case; the only situation where we are allowed to apply the distance calculation to finite separations Δx^i is that of flat, Euclidean geometry in Cartesian coordinates. Again, it is customary to write the second equality of (A.74) in index notation as

$$ds^2 = g_{ij} d\tilde{x}^i d\tilde{x}^j \, , \tag{A.75}$$

where $\tilde{x}^i = (r, \theta, \phi)$ and

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\theta \end{pmatrix} \, . \tag{A.76}$$

The Kronecker delta (A.71) is therefore only one specific metric, namely the metric in Euclidean geometry with Cartesian coordinates. In the remainder of this section (but this section only!), the coordinates x^i will be assumed to be Cartesian unless otherwise stated. We will also set the speed of light $c = 1$, i.e. use natural units as described in Sec. A.1.

A.3.2 Lorentz transformations

In special relativity, space and time join together to form a spacetime continuum. If x^α denote the coordinates of an inertial frame, then two spacetime events $(t,\, x,\, y,\, z)$ and $(t + \Delta t,\, x + \Delta x,\, y + \Delta y,\, z + \Delta z)$ are separated by a proper distance

$$\Delta s^2 = -\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2 \, , \tag{A.77}$$

Of course, this relation remains valid in the limit of infinitesimally close points; we then merely replace all 'Δ' with 'd'.
According to the theory of special relativity, however, no inertial frame is preferred over another. If we denote by $\tilde{x}^{\tilde{\alpha}}$ the coordinate system of another inertial frame, Eq. (A.77) also holds in this frame, i.e.

$$\Delta s^2 = -\Delta \tilde{t}^2 + \Delta \tilde{x}^2 + \Delta \tilde{y}^2 + \Delta \tilde{z}^2 \, . \tag{A.78}$$

Note that this implies, in particular, that $\Delta s = 0$ for events connected by a light ray and all inertial observers will therefore agree on the value of the speed of light (unity in our coordinates). Switching again to index notation, we can write Eqs. (A.77), (A.78) as

$$\Delta s^2 = \eta_{\alpha\beta} \Delta x^\alpha x^\beta = \eta_{\tilde{\alpha}\tilde{\beta}} \Delta \tilde{x}^{\tilde{\alpha}} \Delta \tilde{x}^{\tilde{\beta}} \, . \tag{A.79}$$

Here Greek indices with a tilde also run from 0 to 3; the tilde has merely been introduced to mark that this index is related to the new coordinate system $\tilde{x}^{\tilde{\alpha}}$. Normally, we will not introduce the tilde on the index letters, since the tilde on the x already signifies different coordinates. We mark the index as well here because it will help us below to distinguish between the Lorentz transformation and its inverse. In Eq. (A.79), we have also introduced the *Minkowski metric* whose components are

$$\eta_{\alpha\beta} = \eta_{\tilde{\alpha}\tilde{\beta}} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \Leftrightarrow \qquad \eta^{\alpha\beta} = \eta^{\tilde{\alpha}\tilde{\beta}} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{A.80}$$

where $\eta^{\alpha\beta}$ is defined as the inverse matrix of $\eta_{\alpha\beta}$ and has exactly the same components in this case. There now remains the task of identifying the coordinate transformations that ensure the invariance of Δs^2. Inertial frames move with constant velocity relative to each other, so that their coordinates are related by linear transformations of the kind

$$\tilde{x}^{\tilde{\alpha}} = \Lambda^{\tilde{\alpha}}{}_{\mu} x^{\mu} + x_0^{\mu}\,, \tag{A.81}$$

where the $\Lambda^{\tilde{\alpha}}{}_{\mu} = \text{const}$. The translation given by the constant x_0^{μ} has no impact on the following calculations and we can set $x_0^{\mu} = 0$ without loss of generality. Equation (A.79) together with the transformation (A.81) implies

$$\eta_{\tilde{\alpha}\tilde{\beta}} \Delta\tilde{x}^{\tilde{\alpha}} \Delta\tilde{x}^{\tilde{\beta}} = \eta_{\tilde{\alpha}\tilde{\beta}} \Lambda^{\tilde{\alpha}}{}_{\mu} \Delta x^{\mu}\, \Lambda^{\tilde{\beta}}{}_{\nu} \Delta x^{\nu} \stackrel{!}{=} \eta_{\mu\nu} \Delta x^{\mu}\, \Delta x^{\nu}\,. \tag{A.82}$$

This condition holds for arbitrary Δx^{μ}, $\Delta\tilde{x}^{\tilde{\alpha}}$, so that we require

$$\eta_{\mu\nu} = \Lambda^{\tilde{\alpha}}{}_{\mu} \Lambda^{\tilde{\beta}}{}_{\nu} \eta_{\tilde{\alpha}\tilde{\beta}}\,, \tag{A.83}$$

or, written as a matrix multiplication,

$$\boldsymbol{\eta} = \boldsymbol{\Lambda}^T \boldsymbol{\eta} \boldsymbol{\Lambda}\,, \tag{A.84}$$

where now the "T" denotes the transpose of a matrix. The class of transformations satisfying this condition are the *Lorentz transformations*

$$\Lambda^{\tilde{\alpha}}{}_{\mu} = \left(\begin{array}{c|c} \gamma & -\gamma v_j \\ \hline -\gamma v^i & \delta^i{}_j + (\gamma - 1)\frac{v^i v_j}{|\vec{v}|^2} \end{array}\right) \qquad \Leftrightarrow \qquad \Lambda^{\mu}{}_{\tilde{\alpha}} = \left(\begin{array}{c|c} \gamma & \gamma v_j \\ \hline \gamma v^i & \delta^i{}_j + (\gamma - 1)\frac{v^i v_j}{|\vec{v}|^2} \end{array}\right), \tag{A.85}$$

where the Kronecker delta $\delta^i{}_j$ with one index raised has the same components as δ_{ij} in Eq. (A.71), v^i is the velocity (see Fig. 5) of the frame $(\tilde{x}^{\tilde{\alpha}})$ relative to the frame (x^{μ}), $|\vec{v}|^2 := \delta_{ij} v^i v^j$ is the norm of this velocity, and $\gamma = 1/\sqrt{1 - |\vec{v}|^2}$ is the Lorentz boost factor. As one would expect, the inverse transformation $\Lambda^{\mu}{}_{\tilde{\alpha}}$ to get back from $(\tilde{x}^{\alpha})$ to the original frame x^{μ} is given by merely inverting the sign of the velocity vector. One straightforwardly shows that

$$\Lambda^{\tilde{\alpha}}{}_{\mu} \Lambda^{\mu}{}_{\tilde{\beta}} = \delta^{\tilde{\alpha}}{}_{\tilde{\beta}}\,, \qquad \Lambda^{\mu}{}_{\tilde{\alpha}} \Lambda^{\tilde{\alpha}}{}_{\nu} = \delta^{\mu}{}_{\nu}\,, \tag{A.86}$$

where $\delta^{\mu}{}_{\nu} = \text{diag}(1, 1, 1, 1)$ is the four-dimensional Kronecker delta. In practice, one can often choose the relative velocity v^i to point in the direction of one coordinate axis. Choosing, for instance, the x direction simplifies Eq. (A.85) to

$$\Lambda^{\tilde{\alpha}}{}_{\mu} = \left(\begin{array}{c|ccc} \gamma & -\gamma v & 0 & 0 \\ \hline -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) \qquad \Leftrightarrow \qquad \Lambda^{\mu}{}_{\tilde{\alpha}} = \left(\begin{array}{c|ccc} \gamma & \gamma v & 0 & 0 \\ \hline \gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right). \tag{A.87}$$

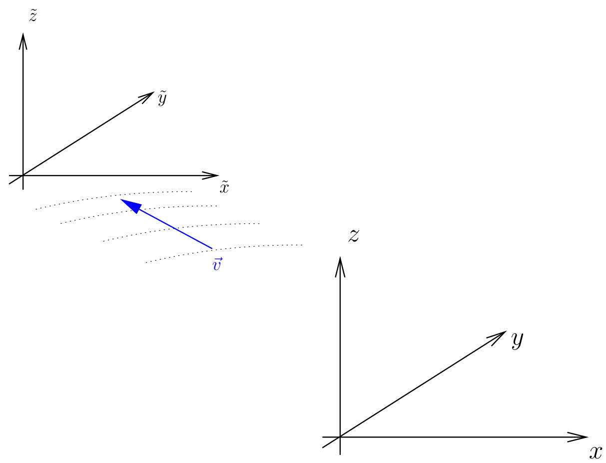


Figure 5: An inertial frame $(\tilde{x}^{\tilde{\alpha}})$ moves with constant velocity v^i relative to the frame (x^μ).

A.3.3 World lines and the four velocity

The invariance of the proper distance between spacetime events allows us to make the following definition.

Def.: The interval between two spacetime events x^α and $x^\alpha + \Delta x^\alpha$ is called

timelike $:\Leftrightarrow \quad \eta_{\mu\nu}\Delta x^\mu\,\Delta x^\nu < 0$

null $:\Leftrightarrow \quad \eta_{\mu\nu}\Delta x^\mu\,\Delta x^\nu = 0$

spacelike $:\Leftrightarrow \quad \eta_{\mu\nu}\Delta x^\mu\,\Delta x^\nu > 0\,.$

For timelike intervals, we often use the proper time

$$\Delta\tau^2 := -\Delta s^2 = \Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2\,. \tag{A.88}$$

Using the proper time, we can state the **Clock postulate** of special relativity:

Postulate: A clock moving on a world line $x^\alpha(\lambda)\,,\ \lambda \in \mathbb{R}$, that is in every point timelike or null, measures the proper time along this world line

$$\tau := \int_{\lambda_1}^{\lambda_2} \sqrt{-\eta_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\mu}{d\lambda}}d\lambda\,. \tag{A.89}$$

The requirement that the curve be everywhere timelike or null implies that for all $\lambda \in [\lambda_1, \lambda_2]$, we have $\eta_{\mu\nu}\dfrac{dx^\mu}{d\lambda}\dfrac{dx^\nu}{d\lambda} \le 0$. Note that the expression (A.89) is invariant under a reparameterization $\lambda \to \mu(\lambda)$ of the world line and that such a parameterization does not alter the local timelike or null character of the curve.
It is often convenient to parameterize a timelike curve by the proper time, i.e. use $\lambda = \tau$. From

Eq. (A.89), we then obtain

$$d\tau = \sqrt{-\eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}}d\tau$$

$$\Rightarrow \quad \eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu := \eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = -1\,. \tag{A.90}$$

We define

Def.: The four velocity along a timelike curve is

$$u^\alpha := \frac{dx^\alpha}{d\tau}\,. \tag{A.91}$$

From Eq. (A.90) we find that the four-velocity satisfies by definition

$$\eta_{\mu\nu}u^\mu u^\nu = -1\,. \tag{A.92}$$

By chain rule, the four velocity changes under a coordinate transformation $(x^\mu) \to (\tilde{x}^{\tilde{\alpha}})$ according to

$$\tilde{u}^\alpha = \Lambda^{\tilde{\alpha}}{}_\mu u^\mu\,. \tag{A.93}$$

Its norm is therefore manifestly invariant under Lorentz transformations,

$$\begin{aligned}
\eta_{\tilde{\alpha}\tilde{\beta}}\tilde{u}^{\tilde{\alpha}}\tilde{u}^{\tilde{\beta}} &= \Lambda^\mu{}_{\tilde{\alpha}}\Lambda^\nu{}_{\tilde{\beta}}\eta_{\mu\nu}\,\Lambda^{\tilde{\alpha}}{}_\rho u^\rho\,\Lambda^{\tilde{\beta}}{}_\sigma u^\sigma \\
&= \delta^\mu{}_\rho\delta^\nu{}_\sigma\eta_{\mu\nu}u^\rho u^\sigma && \text{(A.94)}\\
&= \eta_{\mu\nu}u^\mu u^\nu\,, && \text{(A.95)}
\end{aligned}$$

where we used Eq. (A.82) for the transformation rule of the Minkowski metric and Eq. (A.86) for the product of the Lorentz transformation matrix with its inverse. Note that we also used the property of the Kronecker delta to replace indices according to

$$\delta^\mu{}_\rho u^\rho = u^\mu\,, \tag{A.96}$$

which directly follows from the definition of $\delta^\mu{}_\rho$ and will be frequently used in the remainder of these notes.

A special class of curves are the **Geodesics**. We will introduce geodesics in terms of a variational principle. For this purpose, we use the action for timelike curves

$$\mathcal{S}[x^\alpha(\lambda)] = \int \underbrace{\sqrt{-\eta_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda}}}_{=:\mathcal{L}}d\lambda\,, \tag{A.97}$$

which we identify as the proper time along the curve $x^\alpha(\lambda)$; cf. Eq. (A.89). Timelike geodesics are then defined as the curves that extremize this action. This is an Euler-Lagrange variation problem and the solutions are obtained from the Euler-Lagrange equation

$$\frac{d}{d\lambda}\frac{\partial\mathcal{L}}{\partial\dot{x}^\mu} = \frac{\partial\mathcal{L}}{\partial x^\mu}\,, \tag{A.98}$$

where $\dot{x}^\mu := dx^\mu/d\lambda$. With the Lagrangian $\mathcal{L}$ from Eq. (A.97) we obtain

$$\frac{\partial\mathcal{L}}{\partial x^\mu} = 0\,, \qquad \frac{\partial\mathcal{L}}{\partial\dot{x}^\mu} = \frac{1}{2\sqrt{-\eta_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta}}\left(-\eta_{\alpha\mu}\dot{x}^\alpha - \eta_{\mu\beta}\dot{x}^\beta\right)\,. \tag{A.99}$$

The definition of $\mathcal{L}$ in Eq. (A.97) implies $\mathcal{L} = d\tau/d\lambda$, so that

$$\frac{d}{d\lambda}\frac{\partial\mathcal{L}}{\partial\dot{x}^\mu} = \frac{d}{d\lambda}\frac{d\lambda}{d\tau}\left(-\eta_{\mu\beta}\frac{dx^\beta}{d\lambda}\right) = -\eta_{\mu\beta}\frac{d}{d\lambda}\frac{dx^\beta}{d\tau} \qquad \Big| \quad \times\frac{-\eta^{\mu\alpha}}{\mathcal{L}}$$

$$\Rightarrow \quad \frac{d^2x^\alpha}{d\tau^2} = 0\,. \tag{A.100}$$

The same equation can be derived for spacelike and null geodesics; cf. Sec. B.3 below. With this result, we can formulate the **geodesic postulate** of special relativity.

Postulate: Free massive (massless) particles in special relativity move on straight timelike (null) curves,

$$\frac{d^2x^\alpha}{d\tau^2} = 0\,. \tag{A.101}$$

Note that τ denotes the proper time only along timelike geodesics. For null geodesics it merely parameterizes the curve.

A.3.4 Time dilation and Lorentz contraction

Special relativity is infamous for the apparent paradoxes that have been constructed out of its sometimes counter intuitive predictions. All of these can be resolved by properly calculating and interpreting the results, but it requires care at times. In this subsection, we will discuss two of the most infamous predictions of special relativity that also feature prominently in the aforementioned paradoxes: *time dilation* and *Lorentz contraction*.

Time dilation: Let $\mathcal{O}$ and $\tilde{\mathcal{O}}$ be two inertial observers using coordinates x^μ and $\tilde{x}^{\tilde{\alpha}}$, respectively, in their rest frames and let $\tilde{\mathcal{O}}$ move with velocity v^i relative to the frame $\mathcal{O}$. Our goal is to find the relation between the proper time measured along world lines at rest in the respective frames.

We consider for this purpose a world line at rest in the frame $\tilde{\mathcal{O}}$. The four-velocity tangential to this world line in coordinates $\tilde{x}^\alpha$ is

$$\tilde{u}^{\tilde{\alpha}} = \left(\frac{d\tilde{t}}{d\tau},\, 0,\, 0,\, 0\right)\,. \tag{A.102}$$

The norm of the four-velocity is -1 from which we find

$$-1 = \eta_{\tilde{\alpha}\tilde{\beta}}\tilde{u}^{\tilde{\alpha}}\tilde{u}^{\tilde{\beta}} = -\left(\frac{d\tilde{t}}{d\tau}\right)^2 \quad \Rightarrow \quad d\tilde{t} = d\tau\,, \tag{A.103}$$

where the sign of $d\tilde{t}$ follows from assuming that both $\tilde{t}$ and τ are future oriented.
In the frame $\mathcal{O}$, this world line is not at rest and the four velocity expressed in coordinates x^μ is

$$u^\mu = \left(\frac{dt}{d\tau}, \frac{dx^i}{d\tau}\right) \stackrel{!}{=} \Lambda^\mu{}_{\tilde{\alpha}}\tilde{u}^{\tilde{\alpha}} = \left(\gamma\frac{d\tilde{t}}{d\tau}, \gamma v^i\frac{d\tilde{t}}{d\tau}\right)\,. \tag{A.104}$$

Let us first consider the time component of this equation. We find

$$\frac{dt}{d\tau} = \gamma\frac{d\tilde{t}}{d\tau} \quad \Rightarrow \quad \frac{dt}{d\tilde{t}} = \gamma \quad \Rightarrow \quad dt = \gamma d\tilde{t}\,. \tag{A.105}$$

With the result (A.103) and the definition of γ, we can write this result as

$$dt = \frac{d\tilde{t}}{\sqrt{1-|\vec{v}|^2}} = \frac{d\tau}{\sqrt{1-|\vec{v}|^2}}\,. \tag{A.106}$$

So while the moving observer ages by an amount $d\tau$, observer $\mathcal{O}$ sees a larger amount of time $dt = d\tau/\sqrt{1-|v|^2}$ elapse in his/her own frame. The moving observer $\tilde{\mathcal{O}}$ ages more slowly than his twin $\mathcal{O}$ remaining at rest. The argument is entirely symmetric: as viewed from the rest frame of $\tilde{\mathcal{O}}$, the aging of $\mathcal{O}$ is slower. This is not a paradox, since the two observers cannot return to one another to compare their two clocks without undergoing acceleration at some point. This accelerated phase of their motion requires additional calculation which resolves the seeming paradox. The interested reader is referred to Sec. 1.13 of Schutz [24].
It is instructive to also consider the spatial components of Eq. (A.104) which gives us

$$\frac{dx^i}{d\tau} = \gamma v^i \underbrace{\frac{d\tilde{t}}{d\tau}}_{=1} = \gamma v^i \quad \Rightarrow \quad v^i = \frac{dx^i}{\gamma d\tilde{t}} = \frac{dx^i}{dt}\,, \tag{A.107}$$

so that the velocity v^i denotes the coordinate velocity of frame $\tilde{\mathcal{O}}$ as seen in frame $\mathcal{O}$.

Lorentz contraction: We have defined the measure of time by clocks but still need the proper size of an object. We define this concept through the length of a rod, which generalizes obviously to the extent of an object in more than one direction.

Def.: The length in a reference frame $\mathcal{O}$ of a rod is defined as the proper distance Δs between two events $\mathcal{A}$ and $\mathcal{B}$, where $x^i_{\mathcal{A}}$ is the position of the rod's tail at a specified time $t_{\mathcal{A}} = t_0$ and $x^i_{\mathcal{B}}$ is the position of the rod's head at the same time $t_{\mathcal{B}} = t_0$. Denoting $x^i_{\mathcal{B}} - x^i_{\mathcal{A}} = \Delta x^i$, the length is given by

$$\Delta s = \sqrt{\eta_{\alpha\beta}\Delta x^\alpha \Delta x^\beta} = \sqrt{\delta_{ij}\Delta x^i \Delta x^j}\,. \tag{A.108}$$

Note, that the length of the rod is by this definition frame dependent. We could define a preferred measure for the rod's length by applying the above definition in a special frame, e.g. the frame comoving with the rod.
Let us now consider an observer $\mathcal{O}$ who is comoving with the rod and therefore measures its length ℓ as given by (A.108). A second observer $\tilde{\mathcal{O}}$ is moving with velocity v^i relative to the rod. What length $\tilde{\ell}$ does this observer measure? Of course, both observers will agree with the proper distance between the two events we called $\mathcal{A}$ and $\mathcal{B}$ in the above definition; Δs^2 is Lorentz invariant. What they will not agree upon is whether these two events are simultaneous. We start by considering the world lines x^μ of the tail and y^μ of the head of the rod in the system $\mathcal{O}$. They are

$$x^\mu = (t_{\text{tail}}, x_0^i)\,, \qquad y^\mu = (t_{\text{head}}, x_0^i + \Delta x^i)\,, \tag{A.109}$$

where x_0^i, $\Delta x^i = \text{const}$ and t_{tail} and t_{head} are coordinate time which we use as parameters along the respective world lines. Observer $\mathcal{O}$ will pick two simultaneous events by setting $t_{\text{tail}} = t_{\text{head}}$ evaluate the length of the rod as

$$\ell^2 = \Delta s^2 = \delta_{ij} \Delta x^i \, \Delta x^j\,. \tag{A.110}$$

In the frame of the moving observer $\tilde{\mathcal{O}}$, the world lines of the rod's head and tail are given by

$$\begin{aligned} (\tilde{t}_{\text{tail}}, \tilde{x}^i) = \tilde{x}^{\tilde{\alpha}} &= \Lambda^{\tilde{\alpha}}{}_\mu x^\mu\,, \\ (\tilde{t}_{\text{head}}, \tilde{y}^i) = \tilde{y}^{\tilde{\alpha}} &= \Lambda^{\tilde{\alpha}}{}_\mu y^\mu = \Lambda^{\tilde{\alpha}}{}_\mu (x^\mu + \Delta x^\mu)\,. \end{aligned} \tag{A.111}$$

Note that here, $\tilde{x}^i$ and $\tilde{y}^i$ are not constant; the rod is moving in this frame. In order to measure the length of the rod, observer $\tilde{\mathcal{O}}$ will choose two events $\tilde{\mathcal{A}}$ and $\tilde{\mathcal{B}}$, one respectively on the tail's and the head's world line, that are simultaneous in her/his frame. This means setting

$$\begin{aligned} & \tilde{t}_{\text{tail}} = \tilde{t}_{\text{head}} \\ \Rightarrow \quad & \Lambda^{\tilde{0}}{}_0 t_{\text{tail}} + \Lambda^{\tilde{0}}{}_i x_0^i = \Lambda^{\tilde{0}}{}_0 t_{\text{head}} + \Lambda^{\tilde{0}}{}_i (x_0^i + \Delta x^i) \\ \Rightarrow \quad & \Lambda^{\tilde{0}}{}_0 (t_{\text{tail}} - t_{\text{head}}) = \Lambda^{\tilde{0}}{}_i \, \Delta x^i \\ \Rightarrow \quad & t_{\text{tail}} = t_{\text{head}} + \frac{\Lambda^{\tilde{0}}{}_i \, \Delta x^i}{\Lambda^{\tilde{0}}{}_0} = t_{\text{head}} + v_i \Delta x^i\,. \end{aligned} \tag{A.112}$$

We see here explicitly how the mixing of time and spatial components in the Lorentz transformation matrix alters the meaning of simultaneity from one observer to another.
All that is left to do is to evaluate the proper distance between the two events $\tilde{\mathcal{A}}$ and $\tilde{\mathcal{B}}$ that observer $\tilde{\mathcal{O}}$ sees as simultaneously representing tail and head, respectively, of the rod. This proper separation will be independent of which frame, $\mathcal{O}$ or $\tilde{\mathcal{O}}$, we choose to evaluate it in. We choose the former frame $\mathcal{O}$ because it makes the comparison with the rod's length in its own rest frame easier. In the frame $\mathcal{O}$, the coordinates of the two events are

$$x^\mu_{\tilde{\mathcal{A}}} = (t_{\text{head}} + v_i \Delta x^i, x_0^i)\,, \qquad x^\mu_{\tilde{\mathcal{B}}} = (t_{\text{head}}, x_0^i + \Delta x^i)\,, \tag{A.113}$$

and the length of the rod as viewed in the frame $\tilde{\mathcal{O}}$ is

$$\begin{aligned}\tilde{\ell}^2 &= \Delta s^2_{\tilde{\mathcal{A}}\tilde{\mathcal{B}}} = \eta_{\mu\nu}(x^\mu_{\tilde{\mathcal{B}}} - x^\mu_{\tilde{\mathcal{A}}})(x^\nu_{\tilde{\mathcal{B}}} - x^\nu_{\tilde{\mathcal{A}}}) \\ &= -(v_i \Delta x^i)^2 + \delta_{ij}\Delta x^i\,\Delta x^j\,. \end{aligned} \tag{A.114}$$

The length is positive by definition, so that in both Eqs. (A.110) and (A.114), we take the positive square root. Without loss of generality, we can orient our coordinates so that the rod is aligned with, say, the x coordinate axis. Then we have

$$\ell = \Delta x\,, \qquad \tilde{\ell} = \sqrt{1 - v_x^2}\Delta x\,. \tag{A.115}$$

This is the famous Lorentz contraction: Relative to its length in the rest frame, the rod is shorter by a factor $\sqrt{1-v^2}$ as viewed by an observer moving relative to the rod with a velocity component v parallel to the rod. Note that (i) the sign of the velocity component (moving tail-to-head or the other way round) does not affect the result, and (ii) motion perpendicular to the rod does *not* contribute to the Lorentz contraction.

A.3.5 Four momentum and Doppler shift

For timelike curves, the four-velocity is a unit vector tangential to the curve. For particles traveling on such a curve, we define

Def.: The four momentum of a particle of rest mass m is

$$p^\alpha = m u^\alpha\,. \tag{A.116}$$

Because the four velocity is a vector of length -1, we immediately obtain the frame invariant relation

$$\eta_{\mu\nu} p^\mu p^\nu = -m^2\,. \tag{A.117}$$

Let us again consider two inertial observers $\mathcal{O}$ and $\tilde{\mathcal{O}}$, where $\tilde{\mathcal{O}}$ is moving with velocity v^i in the frame $\mathcal{O}$. A particle at rest in the frame $\tilde{\mathcal{O}}$ has a four momentum with components in this frame given by

$$\tilde{p}^{\tilde{\alpha}} = (m,\, 0,\, 0,\, 0)\,. \tag{A.118}$$

Relative to the frame $\mathcal{O}$, the particle moves with velocity v^i, and the four momentum components in this frame are obtained from the Lorentz transformation (A.85),

$$p^\mu = \Lambda^\mu{}_{\tilde{\alpha}} \tilde{p}^{\tilde{\alpha}} = \gamma m (1,\, v^i)\,. \tag{A.119}$$

Here, γm is the total relativistic mass-energy and $\gamma m v^i$ is the linear momentum of the particle as measured in the frame $\mathcal{O}$. The components of the four momentum can therefore be written as

$$p^\mu = (E\,, p^i)\,. \tag{A.120}$$

From the norm of the four momentum, we obtain the special relativistic energy formula

$$\begin{aligned}
& \eta_{\mu\nu}p^\mu p^\nu = -E^2 + |\vec{p}|^2 \overset{!}{=} -m^2 \\
\Rightarrow \quad & E^2 = m^2 + |\vec{p}|^2 \\
\Rightarrow \quad & E^2 = m^2c^4 + |\vec{p}|^2c^2 \,,
\end{aligned} \tag{A.121}$$

where in the last line we restored factors of c by using dimensional arguments.
According to the geodesic postulate, free massless particles move along null geodesics. For null curves, we cannot define the four velocity, since proper time vanishes along these curves. The curves still have tangent vectors, but they all have zero magnitude, so that we cannot define a tangent vector of unit length. The four momentum, however, is not a vector of unit length. For massless particles, it satisfies $\eta_{\mu\nu}p^\mu p^\nu = 0$ and therefore is indeed a null vector. The components are obtained from Eq. (A.120), recalling that the energy of a massless particle, e.g. a photon, is $E = h\nu$ and the momentum $p = h/\lambda$, where ν and λ are frequency and wavelength, related by $c = \lambda\nu$. Setting the speed of light $c = 1$, we can thus write the four momentum of a massless particle is

$$p^\alpha = h\nu(1,\, n^i)\,, \tag{A.122}$$

where n^i is a unit vector.
The redshift can be calculated directly from the Lorentz transformation. Let us consider our usual frames $\mathcal{O}$ and $\tilde{\mathcal{O}}$, the latter moving with v^i relative to the former. Without loss of generality we orient the frame $\mathcal{O}$ such that the photon momentum points in the $+x$ direction. The four momentum of the photon in this frame can then be written as

$$p^\alpha = (E,\, E,\, 0,\, 0)\,. \tag{A.123}$$

Next we assume that observer $\tilde{\mathcal{O}}$ is moving with velocity $\vec{v} = (v,\, 0,\, 0)$ relative to $\mathcal{O}$. The four-momenta $\tilde{p}^{\tilde{\alpha}}$ and p^α of the particle in the two frames are then related by a Lorentz transformation according to

$$\tilde{p}^{\tilde{\alpha}} = \Lambda^{\tilde{\alpha}}{}_\mu p^\mu = \big(\gamma E - \gamma v E,\ -\gamma v E + \gamma E,\ 0,\ 0\big) =: \big(\tilde{E},\ \tilde{E},\ 0,\ 0\big)\,. \tag{A.124}$$

The redshift is obtained from the ratio $\tilde{E}/E$,

$$\frac{\tilde{\nu}}{\nu} = \frac{\tilde{E}}{E} = \gamma - \gamma v = \frac{1-v}{\sqrt{1-v^2}} = \sqrt{\frac{1-v}{1+v}} = 1 - v + \mathcal{O}(v^2)\,. \tag{A.125}$$

As expected, the photon is redshifted if the frame $\tilde{\mathcal{O}}$ moves in the same direction, i.e. "tries to run away from the photon", but is blue-shifted if $v^x < 0$, i.e. $\tilde{\mathcal{O}}$ moves towards the photon. There is also a so-called *transverse Doppler effect* arising from velocity components of observer $\tilde{\mathcal{O}}$ in the y or z directions (i.e. transverse to the propagation of the photon). The calculation of this transverse effect proceeds along similar lines, but requires some care: The general Lorentz transformation would mix x components with y or z components, so that we would first have to decide whether the photon propagation proceeds in the x direction in the frame $\mathcal{O}$ or in the frame $\tilde{\mathcal{O}}$. These two cases represent different physical scenarios and would lead to different redshift factors.

B Differential geometry

Differential geometry is the mathematical formulation of the properties of curved manifolds, i.e. the extension of flat, Euclidean geometry. Some of the observations we have made so far suggest that the generalization of special relativity to encapsulate gravitation will follow a similar path like that from Euclidean to curved geometry. A full discussion of differential geometry is beyond the scope of these lectures. On the other hand the geometric view of Einstein's general relativity is constructive for the understanding of the theory. We will therefore pursue a middle path in these notes; while not dealing with all aspects in full mathematical rigor, we will introduce the main concepts as necessary to form a geometrical picture of the theory. Readers who wish to delve deeper into the topic are referred to DAMTP's Part III course on general relativity, the corresponding lecture notes [36] and the books by Stewart [28], Hawking & Ellis [13] and, especially for an intuitive pictorial introduction, Misner, Thorne & Wheeler [17].
From now on, we will extensively use Einstein's summation convention in the same way as introduced in Sec. A.3. We only make two additional remarks.

(1) In the literature, you will sometimes find upstairs indices referred to as *contravariant* and downstairs indices as *covariant*. We will not use this terminology, but it is good to bear these names in mind.

(2) An upstairs index appearing in the denominator of an expression counts as a downstairs index. Likewise a downstairs index appearing in a denominator counts as an upstairs index. Typically, we encounter this phenomenon when we take partial derivatives with respect to a coordinate. We therefore use the notation

$$\partial_\mu = \frac{\partial}{\partial x^\mu}\,, \tag{B.1}$$

which makes it manifest that the index really is downstairs.

B.1 Manifolds and tensors

Our starting point is a manifold on which we will, step by step, develop all the structure required to describe its geometrical properties. We introduce a **manifold** without full mathematical rigor as follows; cf. Fig. 6.

Def.: An n dimensional manifold $\mathcal{M}$ is a set of points that locally resembles Euclidean space $\mathbb{R}^n$ at each point. For our purposes, this means that there exists a one-to-one and onto map

$$\phi : \mathcal{M} \to U \subset \mathbb{R}^n\,, \qquad p \in \mathcal{M} \mapsto x^\alpha \in U \subset \mathbb{R}^n\,, \quad \alpha = 0,\,\ldots,\,n-1\,, \tag{B.2}$$

where U is an open subset of $\mathbb{R}^n$.

A few comments are in order.

- It is not strictly required that we have one map ϕ that globally covers the entire manifold $\mathcal{M}$. Instead, it is sufficient if we can chop up the manifold into subsets and find

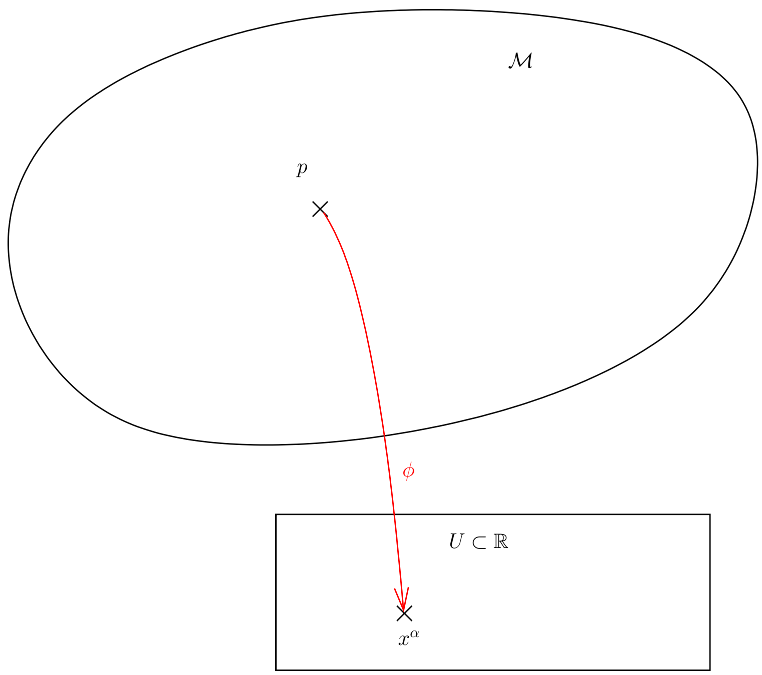


Figure 6: Illustration of the mapping from points in a manifold $\mathcal{M}$ to coordinates in the $\mathbb{R}^n$.

a coordinate map for each of them. Wherever the subsets of $\mathcal{M}$ overlap, we then have multiple coordinate charts and require that these are smoothly related to each other. In most practical applications, this subtlety is not required and one instead works with one or more coordinate systems covering the entire manifold. We will therefore assume in the rest of this work that we do not need to subdivide the manifold. The results we will obtain are valid either for a global chart or for a collection of local coordinate charts.

- As we have already seen in the discussion of special relativity, there does not exist one unique coordinate system, but an infinite number of different coordinate systems. The coordinates serve us in labeling points and in translating operations on the manifold into operations in the $\mathbb{R}^n$, where we are already familiar with, for example, taking derivatives. As we will discuss in more detail further below, the objects in the manifold remain invariant under the choice of coordinates. A convenient way to think about coordinates is the use of house numbers in a street. They are convenient, but a relabeling of houses does not affect the physical structure of the houses or the street.

- The operations (e.g. taking derivatives) and objects (e.g. functions) that we will be dealing with, really all live in the manifold $\mathcal{M}$, not in the coordinate space U. Because the mapping $\phi : \mathcal{M} \to U$ is one-to-one, however, this distinction is often blurred and we will not always rigorously distinguish between operating on the manifold or in coordinate space.

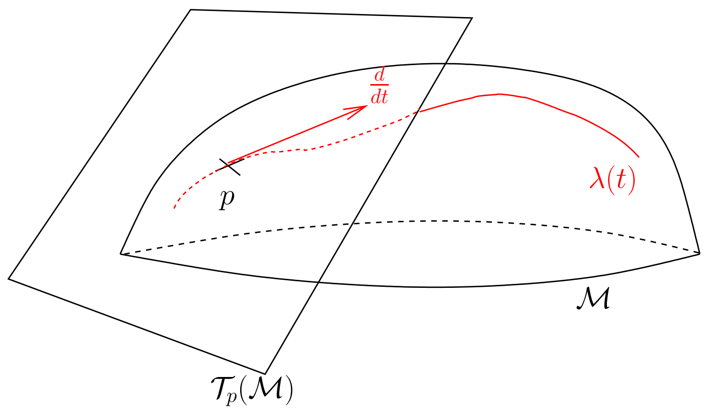


Figure 7: Illustration of defining a vector as the derivative operator along a curve. $\mathcal{T}_p(\mathcal{M})$ is the space of all vectors at point p.

B.1.1 Functions and curves

Def.: A function on the manifold is a map

$$f : \mathcal{M} \to \mathbb{R}\,. \tag{B.3}$$

The function is smooth iff for any coordinate system x^α on the manifold, $f(x^\alpha)$ is a smooth function from $\mathbb{R}^n$ to $\mathbb{R}$. If a function is invariant under a change of coordinates, it is also called a *scalar*.

Def.: A curve is a map

$$\lambda : I \subset \mathbb{R} \to \mathcal{M}\,, \tag{B.4}$$

where I is an open interval. The curve is smooth iff for all coordinate systems x^α on $\mathcal{M}$, the map $x^\alpha \circ \lambda : I \to \mathbb{R}^n$ is a smooth function.

B.1.2 Vectors

Def.: Let $\mathcal{C}^\infty$ be the space of all smooth functions $f : \mathcal{M} \to \mathbb{R}$, λ be a smooth curve and $p \equiv \lambda(0) \in \mathcal{M}$. The tangent vector to the curve λ at $p \in \mathcal{M}$ is the map

$$\boldsymbol{V} : \mathcal{C}^\infty \to \mathbb{R}\,, \qquad f \mapsto \boldsymbol{V}(f) = \frac{d}{dt} f\big(\lambda(t)\big)\Big|_{t=0}\,. \tag{B.5}$$

A vector is thus defined as the directional derivative operator along a curve at a specific point of that curve; for an illustration see Fig. 7. Note that vectors inherit the following properties from derivative operators.

(i) Linearity: For constant α, $\beta \in \mathbb{R}$ and smooth functions f, g,

$$\boldsymbol{V}(\alpha f + \beta g) = \alpha \boldsymbol{V}(f) + \beta \boldsymbol{V}(g)\,. \tag{B.6}$$

(ii) Leibniz rule: For two smooth functions f and g,

$$\boldsymbol{V}(f\,g) = \boldsymbol{V}(f)\,g(p) + f(p)\,\boldsymbol{V}(g)\,. \tag{B.7}$$

We next consider the choice of a convenient basis of the vector space $\mathcal{T}_p(\mathcal{M})$. Let x^α be a coordinate system on the manifold $\mathcal{M}$. Using chain rule, we can write

$$\boldsymbol{V}(f) = \frac{d}{dt} f\big(x^\mu\big(\lambda(t)\big)\big) = \left.\frac{dx^\mu}{dt}\right|_\lambda \frac{\partial}{\partial x^\mu}\, f(x^\alpha)\,. \tag{B.8}$$

$$\nearrow \qquad\qquad\qquad \uparrow \qquad\qquad \nwarrow$$

vector components basis vectors

It can indeed be shown that $\mathcal{T}_p(\mathcal{M})$ is a vector space of dimension n and that the n partial derivative operators $\partial_\mu = \partial/\partial x^\mu$ define a basis of this vector space. We denote the basis vectors by either of

$$\mathbf{e}_\mu = \partial_\mu = \frac{\partial}{\partial x^\mu}\,. \tag{B.9}$$

The components of the vector $\boldsymbol{V}$ are then

$$V^\mu = \left.\frac{dx^\mu}{dt}\right|_\lambda = \frac{dx^\mu}{dt}\,, \tag{B.10}$$

where we often drop the explicit reference to the curve λ. We can then expand the vector in terms of the basis according to any of the following combinations,

$$\boxed{\boldsymbol{V} = V^\mu \mathbf{e}_\mu = V^\mu \partial_\mu = \frac{dx^\mu}{dt}\frac{\partial}{\partial x^\mu} = \frac{d}{dt}}\,. \tag{B.11}$$

Note that the vector components V^μ and the basis vectors $\partial/\partial x^\mu$ both change when we transform from one coordinate system (x^μ) to another $(\tilde{x}^\alpha)$. More specifically they change according to chain rule,

$$\mathbf{e}_\mu = \frac{\partial}{\partial x^\mu} \quad \rightarrow \quad \tilde{\mathbf{e}}_\alpha = \frac{\partial}{\partial \tilde{x}^\alpha} = \frac{\partial x^\mu}{\partial \tilde{x}^\alpha}\frac{\partial}{\partial x^\mu} = \frac{\partial x^\mu}{\partial \tilde{x}^\alpha}\mathbf{e}_\mu\,, \tag{B.12}$$

$$V^\mu = \frac{dx^\mu}{dt} \quad \rightarrow \quad \tilde{V}^\alpha = \frac{d\tilde{x}^\alpha}{dt} = \frac{\partial \tilde{x}^\alpha}{\partial x^\nu}\frac{dx^\nu}{dt} = \frac{\partial \tilde{x}^\alpha}{\partial x^\nu}V^\nu \tag{B.13}$$

While the components of the vector change under a coordinate transformation according to

$$\boxed{\tilde{V}^\mu = \frac{\partial \tilde{x}^\mu}{\partial x^\alpha}V^\alpha}\,, \tag{B.14}$$

the vector $\boldsymbol{V}$ transforms as

$$\boldsymbol{V} = V^\mu \mathbf{e}_\mu \qquad \rightarrow \qquad \tilde{V}^\alpha \tilde{\mathbf{e}}_\alpha = \frac{\partial \tilde{x}^\alpha}{\partial x^\nu} V^\nu \frac{\partial x^\mu}{\partial \tilde{x}^\alpha} \mathbf{e}_\mu = \delta^\mu{}_\nu V^\nu \mathbf{e}_\mu = V^\mu \mathbf{e}_\mu \stackrel{!}{=} \boldsymbol{V}\,, \tag{B.15}$$

i.e. the vector is invariant under coordinate transformations! This is an important point. The components and the basis depend on the coordinates, but the vector is an invariant object.
The specific type of basis vectors $\mathbf{e}_\mu = \partial_\mu$ form a so-called *coordinate basis*. This is not the only possibility for a basis and for some other choices one can even show that there exist no coordinates y^α such that the basis vectors are partial derivatives $\partial/\partial y^\alpha$. For most applications (inside and outside of this course), however, coordinate bases will do fine. Furthermore the statements we will make in this work hold for coordinate as well as non-coordinate bases unless we explicitly state otherwise. We shall therefore use coordinate bases throughout the remainder of these notes.

B.1.3 Covectors / one-forms

Def.: A covector or one-form (the two terms are synonymous and we shall be using both) is a linear [cf. item (i) just below] map

$$\boldsymbol{\eta} : \mathcal{T}_p(\mathcal{M}) \to \mathbb{R}\,, \qquad \boldsymbol{V} \mapsto \boldsymbol{\eta}(\boldsymbol{V})\,. \tag{B.16}$$

The space of all covectors at a point $p \in \mathcal{M}$ is called the *cotangent space* $\mathcal{T}_p^*(\mathcal{M})$ and can be shown to be an n dimensional vector space, just like $\mathcal{T}_p(\mathcal{M})$. If $\mathbf{e}_\mu$ be a basis for the tangent space $\mathcal{T}_p(\mathcal{M})$, we define the components of a covector $\boldsymbol{\eta}$ as

$$\eta_\mu := \boldsymbol{\eta}(\mathbf{e}_\mu)\,, \tag{B.17}$$

i.e. we plug in the μ^{th} basis vector.

Covectors have the following properties.

(i) Linearity: Let $\alpha,\ \beta \in \mathbb{R}$ and $\boldsymbol{V},\ \boldsymbol{W} \in \mathcal{T}_p(\mathcal{M})$. A covector $\boldsymbol{\eta}$ obeys [cf. Eq. (B.6) for vectors]

$$\boldsymbol{\eta}(\alpha \boldsymbol{V} + \beta \boldsymbol{W}) = \alpha \boldsymbol{\eta}(\boldsymbol{V}) + \beta \boldsymbol{\eta}(\boldsymbol{W})\,. \tag{B.18}$$

(ii) Components: With the definition (B.17) we therefore obtain for an arbitrary vector and covector

$$\boldsymbol{\eta}(\boldsymbol{V}) = \boldsymbol{\eta}(V^\mu \mathbf{e}_\mu) = V^\mu \boldsymbol{\eta}(\mathbf{e}_\mu) \qquad \Big| \quad \text{Because } \boldsymbol{\eta} \text{ is linear} \tag{B.19}$$

$$\Rightarrow \quad \boldsymbol{\eta}(\boldsymbol{V}) = V^\mu \eta_\mu\,. \tag{B.20}$$

We require $\boldsymbol{\eta}(\boldsymbol{V}) \in \mathbb{R}$ to be a scalar, i.e. invariant under coordinate transformations.

(iii) Transformation rule: The coordinate invariance of $\boldsymbol{\eta}(\boldsymbol{V})$ determines the behaviour of the components η_μ under a change of coordinates. Let us transform from x^μ to new coordinates $\tilde{x}^\alpha$. We already know the transformation rule (B.13) for the components of a vector, so that for any $\boldsymbol{V} \in \mathcal{T}_p(\mathcal{M})$

$$\boldsymbol{\eta}(\boldsymbol{V}) = \eta_\mu V^\mu \stackrel{!}{=} \tilde{\eta}_\alpha \tilde{V}^\alpha = \tilde{\eta}_\alpha \frac{\partial \tilde{x}^\alpha}{\partial x^\mu} V^\mu$$

$$\Rightarrow \quad \eta_\mu = \frac{\partial \tilde{x}^\alpha}{\partial x^\mu} \tilde{\eta}_\alpha \qquad \Big| \cdot \frac{\partial x^\mu}{\partial \tilde{x}^\beta} \tag{B.21}$$

$$\Rightarrow \quad \boxed{\tilde{\eta}_\beta = \frac{\partial x^\mu}{\partial \tilde{x}^\beta} \eta_\mu} . \tag{B.22}$$

We illustrate the concept of covectors with the following example.

Def.: The gradient $\mathbf{d}f$ of a smooth function f is the map

$$\mathbf{d}f : \mathcal{T}_p(\mathcal{M}) \to \mathbb{R} , \qquad \frac{d}{dt} \mapsto \frac{df}{dt} . \tag{B.23}$$

Recall that a vector is the derivative operator d/dt along a curve λ. If we denote this vector by $\boldsymbol{V} = d/dt$, we write synonymously

$$\mathbf{d}f(\boldsymbol{V}) = \boldsymbol{V}(f) = \frac{df}{dt} . \tag{B.24}$$

In particular, we can regard the coordinates x^α as functions on the manifold. Setting $f = x^\alpha$ for some fixed $\alpha \in \{1, 2, \ldots, n\}$, we obtain

$$\mathbf{d}x^\alpha(\mathbf{e}_\beta) = \mathbf{d}x^\alpha \left(\frac{\partial}{\partial x^\beta} \right) = \frac{\partial x^\alpha}{\partial x^\beta} = \delta^\alpha{}_\beta . \tag{B.25}$$

Recalling Eq. (B.17) for the components of a covector, we conclude the following relation for any vector $\boldsymbol{V}$,

$$\eta_\alpha \mathbf{d}x^\alpha(\boldsymbol{V}) = \eta_\alpha \mathbf{d}x^\alpha(V^\beta \partial_\beta) = \eta_\alpha V^\beta \mathbf{d}x^\alpha(\partial_\beta) = \eta_\alpha V^\beta \delta^\alpha{}_\beta = \eta_\alpha V^\alpha = \boldsymbol{\eta}(\boldsymbol{V}) , \tag{B.26}$$

so that $\eta_\alpha \mathbf{d}x^\alpha$ and $\boldsymbol{\eta}$ are the same one-form. The coordinate gradients $\mathbf{d}x^\alpha$ therefore form a basis of the cotangent space $\mathcal{T}_p^*(\mathcal{M})$, the *dual basis* of the vector basis ∂_μ. We thus have the basis expansion of a one-form $\boldsymbol{\eta}$,

$$\boxed{\boldsymbol{\eta} = \eta_\alpha \mathbf{d}x^\alpha} . \tag{B.27}$$

B.1.4 Tensors

Now that we have defined vectors and covectors, we can define general *tensors* which include the former two and also scalars as special cases.

Def. : A tensor $\boldsymbol{T}$ at $p \in \mathcal{M}$ of rank $\binom{r}{s}$, $r,\, s \in \mathbb{N}_0$, is a multilinear map

$$T : \underbrace{\mathcal{T}_p^*(\mathcal{M}) \times \ldots \times \mathcal{T}_p^*(\mathcal{M})}_{r \text{ factors}} \times \underbrace{\mathcal{T}_p(\mathcal{M}) \times \ldots \times \mathcal{T}_p(\mathcal{M})}_{s \text{ factors}} \to \mathbb{R}\,. \tag{B.28}$$

Put bluntly, a tensor is a machine into which one plugs r one-forms and s vectors and out pops a real number.

We illustrate this with a few examples.

1) A covector $\boldsymbol{\eta}$ is a tensor of rank $\binom{0}{1}$; we plug in one vector $\boldsymbol{V}$ and obtain the number $\boldsymbol{\eta}(\boldsymbol{V})$.

2) A vector $\boldsymbol{V}$ defines the following linear map

$$\boldsymbol{V} : \mathcal{T}_p^*(\mathcal{M}) \to \mathbb{R}\,, \qquad \boldsymbol{\eta} \mapsto \boldsymbol{\eta}(\boldsymbol{V})\,. \tag{B.29}$$

A vector can therefore be regarded as a tensor of rank $\binom{1}{0}$. This view also gives us a convenient way to obtain the components of a vector. From the basis expansion of a one-form (B.27), we have

$$\boldsymbol{\eta}(\boldsymbol{V}) = \eta_\alpha \mathbf{d}x^\alpha(\boldsymbol{V}) = \eta_\alpha V^\alpha$$

$$\Rightarrow \quad V^\alpha = \mathbf{d}x^\alpha(\boldsymbol{V}) = \boldsymbol{V}(\mathbf{d}x^\alpha)\,. \tag{B.30}$$

Just as we obtained the components η_α of a covector $\boldsymbol{\eta}$ in (B.17) by filling its slot with the basis vector $\mathbf{e}_\alpha$, we obtain the components of a vector by filling its slot with the basis one-form $\mathbf{d}x^\alpha$.

This rule holds for tensors in general: the components of a tensor $\boldsymbol{T}$ of rank $\binom{r}{s}$ are obtained by filling its slots with the respective basis one-forms and basis vectors:

$$T^{\alpha_1\ldots\alpha_r}{}_{\beta_1\ldots\beta_s} = \boldsymbol{T}(\mathbf{d}x^{\alpha_1},\, \ldots,\, \mathbf{d}x^{\alpha_r},\, \mathbf{e}_{\beta_1},\, \ldots,\, \mathbf{e}_{\beta_s})\,. \tag{B.31}$$

3) We define the $\binom{1}{1}$ tensor $\boldsymbol{\delta}$ through

$$\boldsymbol{\delta} : \mathcal{T}_p^*(\mathcal{M}) \times \mathcal{T}_p(\mathcal{M}) \to \mathbb{R}\,, \qquad (\boldsymbol{\eta}, \boldsymbol{V}) \mapsto \boldsymbol{\eta}(\boldsymbol{V}) \quad \forall \;\; \boldsymbol{\eta} \in \mathcal{T}_p^*(\mathcal{M})\,, \quad \boldsymbol{V} \in \mathcal{T}_p(\mathcal{M})\,. \tag{B.32}$$

From Eq. (B.31), we obtain its components

$$\delta^\alpha{}_\beta = \boldsymbol{\delta}(\mathbf{d}x^\alpha, \partial_\beta) = \mathbf{d}x^\alpha(\partial_\beta) = \frac{\partial x^\alpha}{\partial x^\beta} = \delta^\alpha{}_\beta\,, \tag{B.33}$$

as the Kronecker delta.

It can be shown that the tensors of rank $\binom{r}{s}$ form a vector space of dimension n^{r+s}. The transformation properties of the components of a tensor are determined by requiring that the number obtained by filling all its slots with one-forms and vectors is a scalar, i.e. invariant under coordinate transformations. A straightforward calculation shows that transforming from coordinates x^μ to $\tilde{x}^\alpha$ changes the components of a tensor of rank $\binom{r}{s}$ according to

$$\boxed{\tilde{T}^{\alpha_1\ldots\alpha_r}{}_{\beta_1\ldots\beta_s} = \frac{\partial \tilde{x}^{\alpha_1}}{\partial x^{\mu_1}} \ldots \frac{\partial \tilde{x}^{\alpha_r}}{\partial x^{\mu_r}} \frac{\partial x^{\nu_1}}{\partial \tilde{x}^{\beta_1}} \ldots \frac{\partial x^{\nu_s}}{\partial \tilde{x}^{\beta_s}} T^{\mu_1\ldots\mu_r}{}_{\nu_1\ldots\nu_s}}\,. \tag{B.34}$$

Note the simple rule underlying this lengthy expression: one factor $\partial\tilde{x}^\alpha/\partial x^\mu$ for each upstairs index of the tensor and one factor $\partial x^\nu/\partial\tilde{x}^\beta$ for each downstairs index. The transformation rules (B.14) and (B.22) are merely special cases of this rule for $\binom{1}{0}$ and $\binom{0}{1}$ tensors.

B.1.5 Tensor operations

There are several ways how we can construct new tensors out of existing ones. We summarize them as follows.

(1) Tensors can be **added** together and **multiplied with numbers** by correspondingly combining their output numbers. For example, we define for two tensors $\boldsymbol{S}$, $\boldsymbol{T}$ of rank $\binom{1}{1}$ and two numbers c_1, $c_2 \in \mathbb{R}$ the new tensor

$$c_1\boldsymbol{S} + c_2\boldsymbol{T} : \mathcal{T}_p^*(\mathcal{M}) \times \mathcal{T}_p(\mathcal{M}) \to \mathbb{R}\,, \qquad \boldsymbol{\eta},\, \boldsymbol{V} \mapsto c_1\boldsymbol{S}(\boldsymbol{\eta}, \boldsymbol{V}) + c_2\boldsymbol{T}(\boldsymbol{\eta}, \boldsymbol{V})\,. \tag{B.35}$$

(2) A special case of adding and scalar-multiplying tensors is the **symmetrization** and **anti-symmetrization**. For a tensor $\boldsymbol{T}$ of rank $\binom{0}{2}$, we define

$$\text{its symmetric part} \qquad S_{\alpha\beta} := \frac{1}{2}(T_{\alpha\beta} + T_{\beta\alpha}) =: T_{(\alpha\beta)}\,, \tag{B.36}$$

$$\text{its anti-symmetric part} \qquad A_{\alpha\beta} := \frac{1}{2}(T_{\alpha\beta} - T_{\beta\alpha}) =: T_{[\alpha\beta]}\,. \tag{B.37}$$

This operation can be applied over a subset of indices of tensors of higher rank, as for example in

$$T^{(\alpha\beta)\gamma}{}_\delta := \frac{1}{2}(T^{\alpha\beta\gamma}{}_\delta + T^{\beta\alpha\gamma}{}_\delta)\,. \tag{B.38}$$

For (anti-)symmetrizing over non-adjacent indices, we use the | symbol as a delimiter between the indices we operate on and those we do not. For example,

$$T_{(\alpha|\beta\gamma|\delta)} := \frac{1}{2}(T_{\alpha\beta\gamma\delta} - T_{\delta\beta\gamma\alpha})\,. \tag{B.39}$$

We can also (anti-)symmetrize over more than two indices. This is done as follows.

- Sum over all permutations of the indices we (anti-)symmetrize over.
- For antisymmetrization, each of these terms is multiplied by the sign of its permutation.
- Divide by $n!$ (n factorial).

For example, this procedure gives us

$$T^\alpha{}_{[\beta\gamma\delta]} = \frac{1}{3!}(T^\alpha{}_{\beta\gamma\delta} + T^\alpha{}_{\gamma\delta\beta} + T^\alpha{}_{\delta\beta\gamma} - T^\alpha{}_{\beta\delta\gamma} - T^\alpha{}_{\delta\gamma\beta} - T^\alpha{}_{\gamma\beta\delta})\,. \tag{B.40}$$

(3) The **contraction** of a tensor $\boldsymbol{T}$ of rank $\binom{r}{s}$ is the $\binom{r-1}{s-1}$ tensor obtained by filling one of the "upstairs" slots with the basis one-form $\mathbf{d}x^\alpha$ and one of the "downstairs" slots with the basis vector ∂_α (with the same index α!). For example, let $\boldsymbol{T}$ be a $\binom{3}{2}$ tensor, $\boldsymbol{\omega}$ and $\boldsymbol{\eta}$ two covectors and $\boldsymbol{V}$ a vector. Then a $\binom{2}{1}$ tensor $\boldsymbol{S}$ is defined through contraction of $\boldsymbol{T}$ by

$$\boldsymbol{S}(\boldsymbol{\omega}, \boldsymbol{\eta}, \boldsymbol{V}) \coloneqq \boldsymbol{T}(\mathbf{d}x^\alpha, \boldsymbol{\omega}, \boldsymbol{\eta}, \partial_\alpha, \boldsymbol{V}) . \tag{B.41}$$

The definition is invariant under a change of coordinates, since

$$\boldsymbol{T}\Big(\mathbf{d}\tilde{x}^\mu, \boldsymbol{\omega}, \boldsymbol{\eta}, \frac{\partial}{\partial \tilde{x}^\mu}, \boldsymbol{V}\Big) = \underbrace{\frac{\partial \tilde{x}^\mu}{\partial x^\alpha}\frac{\partial x^\beta}{\partial \tilde{x}^\mu}}_{=\delta^\beta{}_\alpha} \boldsymbol{T}\Big(\mathbf{d}x^\alpha, \boldsymbol{\omega}, \boldsymbol{\eta}, \frac{\partial}{\partial x^\beta}, \boldsymbol{V}\Big) = \boldsymbol{T}(\mathbf{d}x^\alpha, \boldsymbol{\omega}, \boldsymbol{\eta}, \partial_\alpha, \boldsymbol{V}) . \tag{B.42}$$

Note that the derivatives $\partial\tilde{x}^\mu/\partial x^\alpha$ and $\partial x^\beta/\partial\tilde{x}^\mu$ are merely numbers and can therefore be pulled out of the argument of $\boldsymbol{T}$; $\boldsymbol{T}$ is linear in its vector and covector arguments!

The components of the contracted tensor are obtained from Eq. (B.31),

$$S^{\mu\nu}{}_\rho = \boldsymbol{S}(\mathbf{d}x^\mu, \mathbf{d}x^\nu, \mathbf{e}_\rho) = \boldsymbol{T}(\mathbf{d}x^\alpha, \mathbf{d}x^\mu, \mathbf{d}x^\nu, \mathbf{e}_\alpha, \mathbf{e}_\rho) = T^{\alpha\mu\nu}{}_{\alpha\rho} . \tag{B.43}$$

Note the following properties of contractions.

- It matters over which of the slots of the tensor we contract. In general

$$T^{\alpha\mu\nu}{}_{\alpha\rho} \neq T^{\mu\alpha\nu}{}_{\alpha\rho} . \tag{B.44}$$

- Often the same letter is used for the tensor and its contraction, as for example in $T^{\mu\nu}{}_\rho = T^{\alpha\mu\nu}{}_{\alpha\rho}$. This is not strictly wrong, but in index free notation, it will be confusing if the same letter is used for different tensors.

(4) The **outer product** of a $\binom{p}{q}$ tensor $\boldsymbol{S}$ end a $\binom{r}{s}$ tensor $\boldsymbol{T}$ is the $\binom{p+r}{q+s}$ tensor $\boldsymbol{S}\otimes\boldsymbol{T}$ defined through

$$\boldsymbol{S}\otimes\boldsymbol{T}(\boldsymbol{\omega}_1, \ldots, \boldsymbol{\omega}_p, \boldsymbol{\eta}_1, \ldots, \boldsymbol{\eta}_r, \boldsymbol{V}_1, \ldots, \boldsymbol{V}_q, \boldsymbol{W}_1, \ldots, \boldsymbol{W}_s) \tag{B.45}$$

$$= \boldsymbol{S}(\boldsymbol{\omega}_1, \ldots, \boldsymbol{\omega}_p, \boldsymbol{V}_1, \ldots, \boldsymbol{V}_q)\, \boldsymbol{T}(\boldsymbol{\eta}_1, \ldots, \boldsymbol{\eta}_r, \boldsymbol{W}_1, \ldots, \boldsymbol{W}_s) , \tag{B.46}$$

where $\boldsymbol{\omega}_1, \ldots, \boldsymbol{\omega}_p, \boldsymbol{\eta}_1, \ldots, \boldsymbol{\eta}_r$ are covectors and $\boldsymbol{V}_1, \ldots, \boldsymbol{V}_p, \boldsymbol{W}_1, \ldots, \boldsymbol{W}_s$ are vectors.

By plugging in the basis vectors and one-forms into all slots of the tensor product, we obtain for its components

$$(\boldsymbol{S}\otimes\boldsymbol{T})^{\alpha_1\ldots\alpha_p\beta_1\ldots\beta_r}{}_{\mu_1\ldots\mu_q\nu_1\ldots\nu_s} = S^{\alpha_1\ldots\alpha_p}{}_{\mu_1\ldots\mu_q} T^{\beta_1\ldots\beta_r}{}_{\nu_1\ldots\nu_s} . \tag{B.47}$$

One can furthermore show that an arbitrary tensor $\boldsymbol{T}$ of rank $\binom{r}{s}$ can be expanded in terms of the basis vectors and one-forms according to

$$\boldsymbol{T} = T^{\alpha_1\ldots\alpha_r}{}_{\beta_1\ldots\beta_s} \mathbf{e}_{\alpha_1} \otimes \ldots \otimes \mathbf{e}_{\alpha_r} \otimes \mathbf{d}x^{\beta_1} \otimes \ldots \otimes \mathbf{d}x^{\beta_s} . \tag{B.48}$$

The outer products $\mathbf{e}_{\alpha_1} \otimes \ldots \otimes \mathbf{e}_{\alpha_r} \otimes \mathbf{d}x^{\beta_1} \otimes \ldots \otimes \mathbf{d}x^{\beta_s}$ thus form a basis of the vector space of $\binom{r}{s}$ tensors.

B.1.6 Tensor fields

So far, we have only considered tensors at a specific point $p \in \mathcal{M}$. Einstein's theory of general relativity is formulated in terms of tensor fields. A rigorous definition of tensor fields requires the concept of *fibre bundles* which is beyond the scope of this course; for the interested reader, we recommend Hawking & Ellis [13] for a more in-depth discussion. Here, we loosely define fields as follows.

Def.: A tensor field of rank $\binom{r}{s}$ is a collection of $\binom{r}{s}$ tensors at each point. We can regard the tensor field as a map that associates with every point p a tensor $\boldsymbol{T}_p$ of rank $\binom{r}{s}$. The tensor field is smooth $:\Leftrightarrow$ its components in a coordinate basis are smooth functions.

The distinction between a tensor and a tensor field will often be clear from the context. Sometimes, however, we will use an index p to distinguish a vector $\boldsymbol{X}_p$ at $p \in \mathcal{M}$ from the vector field $\boldsymbol{X}$. As an example, we consider a vector field

$$\boldsymbol{X} : \mathcal{M} \to \mathcal{T}_p(\mathcal{M}) \,, \qquad p \mapsto \boldsymbol{X}_p \,. \tag{B.49}$$

If $f : \mathcal{M} \to \mathbb{R}$ is a smooth function on the manifold, the vector field $\boldsymbol{X}$ defines a new function $\boldsymbol{X}(f)$ through

$$\boldsymbol{X}(f) : \mathcal{M} \to \mathbb{R} \,, \qquad p \mapsto \boldsymbol{X}_p(f) \,, \tag{B.50}$$

i.e. at any point p, the function $\boldsymbol{X}(f)$ returns the directional derivative df/dt along the curve that defines the vector at that point. For a vector field, we can define smoothness in a conceptually different but fully equivalent way to the above smoothness criterion for tensors.

Def.: The vector field $\boldsymbol{X}$ is smooth $\quad :\Leftrightarrow \quad \boldsymbol{X}(f)$ is a smooth function for every smooth f.

For illustration, let x^α be a coordinate system on the manifold and consider the vector field defined by the coordinate basis vector ∂_μ at every point. For a function f, the vector field generates the new function

$$\partial_\mu(f) : \mathcal{M} \to \mathbb{R} \,, \qquad p \mapsto \frac{\partial f}{\partial x^\mu} \,. \tag{B.51}$$

The vector field ∂_μ is clearly smooth, since for every smooth function f, its partial derivative $\partial f/\partial x^\mu$ is also a smooth function. We now see why the two definitions of smoothness for a vector field are equivalent: we merely expand a vector field in the coordinate basis and obtain smoothness of all individual terms in the expansion iff the vector field's components are smooth. As a final example, we consider a smooth vector field $\boldsymbol{V}$ and a smooth covector field $\boldsymbol{\eta}$. Then

$$\boldsymbol{\eta}(\boldsymbol{V}) : \mathcal{M} \to \mathbb{R} \,, \qquad p \mapsto \boldsymbol{\eta}_p(\boldsymbol{V}_p) \,, \tag{B.52}$$

is a smooth function because $\boldsymbol{\eta}(\boldsymbol{V}) = \eta_\mu V^\mu$ and the components are smooth. Throughout the remainder of this work, we will assume all tensors to be smooth.

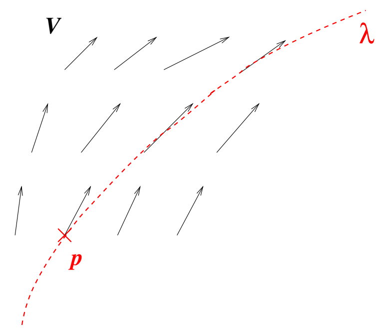


Figure 8: Illustration of the integral curve λ of a vector field $\boldsymbol{V}$ through the point $p \in \mathcal{M}$.

B.1.7 Integral curves

In Sec. B.1.2 we introduced vectors as directional derivatives along curves. A vector field, in turn, defines curves in a unique manner.

Def.: The integral curve of a vector field $\boldsymbol{V}$ through a point $p \in \mathcal{M}$ is defined as the curve through p whose tangent at every point q along the curve is $\boldsymbol{V}_q$.

An integral curve λ of a vector field $\boldsymbol{V}$ through $p \in \mathcal{M}$ can be parametrized without loss of generality such that $\lambda(0) = p$. If we let x^α be a coordinate system on the manifold, the condition for an integral curve can be written in terms of the vector components as

$$\left.\frac{d}{dt}\right|_\lambda = \boldsymbol{V} \qquad \Rightarrow \qquad \frac{dx^\mu\big(\lambda(t)\big)}{dt} = V^\mu(x^\alpha)\,, \tag{B.53}$$

with the boundary condition $x^\mu\big(\lambda(0)\big) = x^\mu(p)$. The theory of ordinary differential equations ensures that Eq. (B.53) has a unique solution. A vector field $\boldsymbol{X}$ therefore has a unique integral curve through point $p \in \mathcal{M}$; for illustration see Fig. 8.

B.2 The metric tensor

B.2.1 Metrics

In Sec. A.3.1 we introduced the *metric* as a generalization of Pythagoras' theorem to measure the distance between infinitesimally close points labeled in not necessarily Cartesian coordinates. In Sec. A.3.2, we further generalized the idea of a metric to include the time coordinate through a negative metric component. In both cases, we had some idea of what the distance of the points should be, for instance from Pythagoras' theorem in $\mathbb{R}^3$ or the invariant proper separation (A.77).

In this section, we will reverse this point of view. While we have established a layer of structure on our manifold (coordinates, curves, tensors), this manifold remains amorphous. There is nothing in what we have said so far that tells us anything about the curvature of the manifold $\mathcal{M}$ or, equivalently, the distance of neighboring points. Recall that we likened coordinates to house numbers who are convenient for labeling houses in a street but not for giving us a precise measure of how far apart they are. We will now define the metric tensor in such a general manner that it accommodates the description of spacetimes as different as those containing multiple black holes, describing open and closed universes or the gravitational collapse of stellar cores in supernova explosions.

Def.: A metric is a tensor field $\boldsymbol{g}$ of rank $\binom{0}{2}$ with the following properties.

(i) $\boldsymbol{g}$ is symmetric: $\boldsymbol{g}(\boldsymbol{V},\boldsymbol{W}) = \boldsymbol{g}(\boldsymbol{W},\boldsymbol{V}) \quad \forall\ \boldsymbol{V},\ \boldsymbol{W} \in \mathcal{T}_p(\mathcal{M})$

or, equivalently, $g_{\alpha\beta} = g_{\beta\alpha}$.

(ii) $\boldsymbol{g}$ is non-degenerate: $\boldsymbol{g}(\boldsymbol{V},\boldsymbol{W}) = 0 \ \ \forall\ \ \boldsymbol{W} \in \mathcal{T}_p(\mathcal{M})$ if and only if $\boldsymbol{V} = 0$.

According to Eq. (B.48), we can expand the metric in terms of basis one-forms as

$$\boldsymbol{g} = g_{\alpha\beta}\mathbf{d}x^{\alpha} \otimes \mathbf{d}x^{\beta}\,, \qquad g_{\mu\nu} = \boldsymbol{g}(\partial_{\mu},\partial_{\nu})\,. \tag{B.54}$$

This relation is reminiscent of the more common notation for the line element

$$ds^2 = g_{\alpha\beta}dx^{\alpha}dx^{\beta}\,. \tag{B.55}$$

Note, however, that the two relations express mathematically very different objects, the former a tensor on a manifold, the latter a differential. Combining Eq. (B.34) for the transformation of tensors under a change of coordinates with chain rule, we directly obtain the invariance of the line element (B.55),

$$d\tilde{s}^2 = \tilde{g}_{\mu\nu}d\tilde{x}^{\mu}\tilde{x}^{\nu} = \frac{\partial x^{\alpha}}{\partial\tilde{x}^{\mu}}\frac{\partial x^{\beta}}{\partial\tilde{x}^{\nu}}g_{\alpha\beta}\frac{\partial\tilde{x}^{\mu}}{\partial x^{\rho}}dx^{\rho}\frac{\partial\tilde{x}^{\nu}}{\partial x^{\sigma}}dx^{\sigma} = \delta^{\alpha}{}_{\rho}\delta^{\beta}{}_{\sigma}g_{\alpha\beta}dx^{\rho}dx^{\sigma} = g_{\alpha\beta}dx^{\alpha}dx^{\beta} = ds^2\,. \tag{B.56}$$

A metric introduces an isomorphism between vectors and one-forms,

$$\boldsymbol{V} \mapsto \underline{\boldsymbol{V}} \coloneqq \boldsymbol{g}(\boldsymbol{V},\,.\,)\,, \tag{B.57}$$

i.e. $\underline{\boldsymbol{V}}$ is a one-form defined through

$$\underline{\boldsymbol{V}} : \mathcal{T}_p(\mathcal{M}) \to \mathbb{R}\,, \qquad \boldsymbol{W} \mapsto \underline{\boldsymbol{V}}(\boldsymbol{W}) \coloneqq \boldsymbol{g}(\boldsymbol{V},\boldsymbol{W})\,. \tag{B.58}$$

The components of $\underline{\boldsymbol{V}}$ are obtained by expanding all involved vectors and covectors in the coordinate basis,

$$\begin{aligned} &\boldsymbol{W} = W^{\alpha}\partial_{\alpha}\,, \qquad \boldsymbol{V} = V^{\alpha}\partial_{\alpha}\,, \qquad \underline{\boldsymbol{V}} = \underline{V}_{\alpha}\mathbf{d}x^{\alpha} \\ \Rightarrow\ \ &\underline{\boldsymbol{V}}(\boldsymbol{W}) = \underline{V}_{\alpha}\mathbf{d}x^{\alpha}(W^{\mu}\partial_{\mu}) = \underline{V}_{\alpha}W^{\mu}\delta^{\alpha}{}_{\mu} = \underline{V}_{\mu}W^{\mu}\,. \end{aligned} \tag{B.59}$$

Furthermore,

$$\begin{aligned} &\boldsymbol{g}(\boldsymbol{V},\boldsymbol{W}) = g_{\alpha\beta}V^{\alpha}W^{\beta} \\ \Rightarrow\ \ &\underline{V}_{\mu} = g_{\mu\nu}V^{\nu}\,. \end{aligned} \tag{B.60}$$

In the following, we will drop the underbar in the covector and write $V_\mu = \underline{V}_\mu$. The index position makes clear whether we have a vector or a one-form. In index free notation, the distinction will often be clear from the context. In those rare cases where it is not, we will explicitly state what type of tensor we are dealing with.

Since the metric $\boldsymbol{g}$ is non-degenerate, it can be inverted. We define

Def.: The inverse metric $\boldsymbol{g}^{-1}$ is a symmetric tensor of rank $\binom{2}{0}$ with

$$(g^{-1})^{\alpha\mu} g_{\mu\beta} = \delta^\alpha{}_\beta \,. \tag{B.61}$$

From now on, we will drop the exponent -1 when we write the components of the inverse metric and merely distinguish it from the metric by the position of the indices.

Example: The line element on the unit sphere, $x^2+y^2+z^2=1$ in $\mathbb{R}^3$, is $ds^2 = d\theta^2 + \sin^2\theta\, d\phi^2$, so that

$$g_{\alpha\beta} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix} \qquad \text{and hence} \qquad g^{\alpha\beta} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\sin^2\theta} \end{pmatrix} . \tag{B.62}$$

Just as the metric defines a mapping from vectors to covectors, the inverse metric defines a map in the other direction. If $\boldsymbol{\eta}$ is a one-form, a tensor of rank $\binom{1}{0}$, i.e. a vector, is defined through

$$\boldsymbol{g}^{-1}(\boldsymbol{\eta},\,.\,) : \mathcal{T}^*_p(\mathcal{M}) \to \mathbb{R} \qquad \boldsymbol{\omega} \mapsto \boldsymbol{g}^{-1}(\boldsymbol{\eta},\boldsymbol{\omega})\,. \tag{B.63}$$

In components,

$$\eta^\alpha = g^{\alpha\mu}\eta_\mu \,. \tag{B.64}$$

The two isomorphisms defined by the metric and the inverse metric through Eqs. (B.58), (B.63) are inverses of each other,

$$\boldsymbol{g}^{-1}\big(\boldsymbol{g}(\boldsymbol{V},\,.\,)\,.\big) = \boldsymbol{V}\,, \qquad \boldsymbol{g}\big(\boldsymbol{g}^{-1}(\boldsymbol{\eta},\,.\,),\,.\,\big) = \boldsymbol{\eta}\,. \tag{B.65}$$

In analogy to Eq. (B.64), we can raise and lower any number of indices in a tensor with the metric or its inverse. For example, if $\boldsymbol{T}$ is a tensor of rank $\binom{3}{2}$, we obtain a tensor of rank $\binom{4}{1}$ through

$$T^\alpha{}_\beta{}^{\gamma\delta\epsilon} = g_{\beta\lambda} g^{\delta\mu} g^{\epsilon\nu} T^{\alpha\lambda\gamma}{}_{\mu\nu} \,. \tag{B.66}$$

Because these mappings between tensors of different rank are isomorphims, we usually use the same letter, here T, for the object, irrespective of the positions of the indices.

B.2.2 Lorentzian signature

The symmetry and non-degeneracy of the metric has an important consequence that we state here without proof.

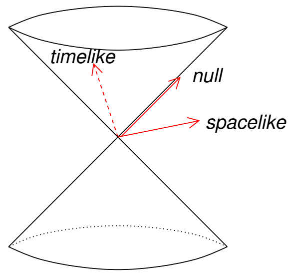


Figure 9: Light cone structure for vectors at a point $p \in \mathcal{M}$.

Lemma: For every point $p \in \mathcal{M}$, there exists a coordinate system y^α such that at p the components $g_{\alpha\beta}$ are (i) non-zero only on the diagonal, i.e. for $\alpha = \beta$, and (ii) that these non-zero components are $+1$ or -1. "Sylvester's law" furthermore states that the number of such $+1$ or -1 entries is invariant under any coordinate change that preserves the requirements (i) and (ii).

This fact allows us to make the following definition.

Def.: The signature σ of a metric $g_{\alpha\beta}$ on an n-dimensional manifold $\mathcal{M}$ is the sum over the $+1$ and -1 entries over all diagonal elements. A metric with signature $\sigma = n$ is called a "Riemannian metric" and a metric with signature $\sigma = n - 2$ is called "Lorentzian".

For example, the four-dimensional Minkowski metric $\eta_{\alpha\beta} = \text{diag}(-1, +1, +1, +1)$ has signature $\sigma = 2$ and we define spacetimes accordingly in general relativity.

Def.: An n dimensional spacetime, or Lorentzian manifold, is defined as a smooth n-dimensional manifold $\mathcal{M}$ equipped with a metric of signature $n - 2$. Many (though not all) applications of general relativity are concerned with $n = 4$ dimensional spacetimes and we shall assume $n = 4$ from now on unless stated otherwise.

We emphasize, that the signature is convention dependent; some authors write the Minkowski metric as $\eta_{\alpha\beta} = (+1, -1, -1 - 1)$, and correspondingly use metrics of signature -2 in general relativity.
We will discuss the construction of the particular coordinates later in Sec. B.7, but already make one important comment here. For metrics of signature 2, we can transform the metric locally to the Minkowski metric. This is only possible locally, since at points $q \neq p$, the metric will in general not be Minkowskian in this coordinate system. This is strikingly reminiscent of the Einstein equivalence principle: locally, we have the Minkowski metric of special relativity and the corresponding coordinate frame is a freely falling frame. Furthermore, we know that the Minkowski metric is invariant under the Lorentz transformations (A.82). The coordinate system that locally transforms the metric to $\eta_{\alpha\beta} = \text{diag}(-1, +1, +1, +1)$ is therefore not unique, but all coordinates with this property are related by Lorentz transformations.
Locally at a point of the manifold, we thus recover the laws of special relativity. This also includes the light cone structure discussed in Sec. A.3.3 which we define for vectors on a

Lorentzian manifold as follows; cf. also Fig. 9.

Def.: Let $(\mathcal{M}, \boldsymbol{g})$ be a Lorentzian manifold, $\boldsymbol{V} \in \mathcal{T}_p(\mathcal{M})$, $\boldsymbol{V} \neq 0$. $\boldsymbol{V}$ is

timelike $:\Leftrightarrow$ $\boldsymbol{g}(\boldsymbol{V}, \boldsymbol{V}) < 0$

null $:\Leftrightarrow$ $\boldsymbol{g}(\boldsymbol{V}, \boldsymbol{V}) = 0$

spacelike $:\Leftrightarrow$ $\boldsymbol{g}(\boldsymbol{V}, \boldsymbol{V}) > 0$.

For spacelike vectors $\boldsymbol{V}$, $\boldsymbol{W}$, we can further define norm and angles.

Def.: The norm of a spacelike vector $\boldsymbol{V} \in \mathcal{T}_p(\mathcal{M})$ is $|\boldsymbol{V}| := \sqrt{\boldsymbol{g}(\boldsymbol{V}, \boldsymbol{V})}$.

The angle between spacelike $\boldsymbol{V}$, $\boldsymbol{W} \in \mathcal{T}_p(\mathcal{M})$ is $\theta := \arccos\left(\frac{\boldsymbol{g}(\boldsymbol{V},\boldsymbol{W})}{|\boldsymbol{V}|\,|\boldsymbol{W}|}\right)$.

B.3 Geodesics

B.3.1 Curves revisited

On a manifold with Lorentzian metric, we can distinguish between timelike, null and spacelike vectors according to the above definition. This property is directly transferred to curves.

Def.: A curve is timelike (null, spacelike) at a point $p \in \mathcal{M}$
$:\Leftrightarrow$ its tangent vector at that point is timelike (null, spacelike).

Note that in general, the null, time or spacelike character of a curve can change along the curve. For curves or segments of curves that are timelike or spacelike throughout, we can define the following measures.

Def.: The length of a spacelike curve (segment) is

$$s := \int_{t_0}^{t_1} \sqrt{\boldsymbol{g}(\boldsymbol{V}, \boldsymbol{V})|_{\lambda(t)}}\,dt\,, \tag{B.67}$$

where $V = \dfrac{d}{dt}$ is the tangent vector of the curve λ. In components, this becomes

$$s := \int_{t_0}^{t_1} \sqrt{g_{\alpha\beta}\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt}}\,dt\,, \tag{B.68}$$

which, by differentiation, also justifies our notation $ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta$ for the line element.

Def.: For timelike curves, we define the proper time along a curve as

$$\tau(t_1) := \int_{t_0}^{t_1} \sqrt{-\, \boldsymbol{g}(\boldsymbol{V},\boldsymbol{V})|_{\lambda(t)}}\; dt = \int_{t_0}^{t_1} \sqrt{-g_{\alpha\beta}\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt}}\; dt\,, \tag{B.69}$$

where again, $\boldsymbol{V} = d/dt$ is the tangent vector along the curve.

For timelike curves, we define the four-velocity as through Eq. (A.91) in special relativity:

Def.: The four-velocity along a timelike curve λ is the tangent vector to that curve parametrized by proper time τ,

$$u^\mu := \left.\frac{dx^\mu}{d\tau}\right|_{\lambda(\tau)}\,. \tag{B.70}$$

According to Eq. (B.69) the proper time along this curve is

$$\begin{aligned}
\tau &= \int_{\tau_0}^{\tau} \sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\tilde{\tau}}\frac{dx^\nu}{d\tilde{\tau}}}\; d\tilde{\tau} = \int_{\tau_0}^{\tau} \sqrt{-g_{\mu\nu}u^\mu u^\nu}\, d\tilde{\tau} \qquad \Big|\ \frac{d}{d\tau} \\
\Rightarrow\quad 1 &= \sqrt{-g_{\mu\nu}u^\mu u^\nu} \\
\Rightarrow\quad & g_{\mu\nu}u^\mu u^\nu = -1\,.
\end{aligned} \tag{B.71}$$

Just as in special relativity, the four-velocity of a timelike curve is a unit vector.

B.3.2 Geodesic curves defined by a variational principle: Version 1

Geodesics are the analog in differential geometry to straight lines in Euclidean geometry. Even though we have an intuitive idea of what a straight line is in flat geometry (think of a ruler, for instance), a cleaner mathematical definition is more suitable for generalization to curved manifolds: A straight line is the curve of minimal length between two points A and B. In Sec. A.3.3, we defined timelike geodesics in special relativity as curves that extremize the action (A.97), i.e. the proper time along the curve. We shall now do the same for curves in generic Lorentzian manifolds.
First, however, we recall Noether's theorem which is of outstanding value in many calculations involving geodesics. Noether's theorem really consists of two parts and we will apply both in the remainder of this work. Let us consider for this purpose a Lagrangian L that depends on generalized coordinates q_k and $\dot{q}_k$, where $\dot{q}_k := dq_k/d\lambda$, and λ denotes the parameter along the possible paths $q_k(\lambda)$ of a physical system. The action

$$\mathcal{S} = \int \mathcal{L}(q_k, \dot{q}_k, \lambda)\, d\lambda\,, \tag{B.72}$$

is extremized by curves satisfying the Euler-Lagrange equation

$$\frac{d}{d\lambda}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_k}\right) = \frac{\partial \mathcal{L}}{\partial q_k}\,. \tag{B.73}$$

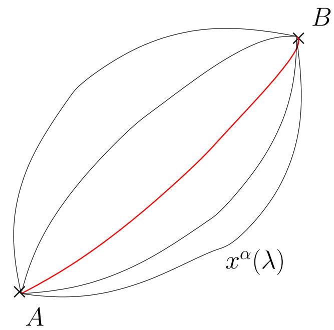


Figure 10: Graphical illustration of varying curves from A to B such that the action (B.77) is extremal.

We may then obtain integrals of motion as follows.

Noether's theorem: (i) If $\mathcal{L}$ does not explicitly depend on q_k, then

$$p_k := \frac{\partial \mathcal{L}}{\partial \dot{q}_k}\,, \tag{B.74}$$

is a first integral of motion, i.e. is conserved along the path that extremizes the action $\mathcal{S}$.

(ii) If $\mathcal{L}$ does not explicitly depend on the parameter λ, then

$$I := \dot{q}_k \frac{\partial \mathcal{L}}{\partial \dot{q}_k} - \mathcal{L}\,, \tag{B.75}$$

is a first integral of motion.

Proof: Part (i) follows directly from the Euler-Lagrange equation (B.73). For part (ii), we start by differentiating Eq. (B.75),

$$\begin{aligned}
\frac{d}{d\lambda}\left(\dot{q}_k \frac{\partial \mathcal{L}}{\partial \dot{q}_k} - \mathcal{L}\right) &= \ddot{q}_k \frac{\partial \mathcal{L}}{\partial \dot{q}_k} + \dot{q}_k \frac{d}{d\lambda}\frac{\partial \mathcal{L}}{\partial \dot{q}_k} - \frac{d\mathcal{L}}{d\lambda} \\
&= \ddot{q}_k \frac{\partial \mathcal{L}}{\partial \dot{q}_k} + \dot{q}_k \frac{d}{d\lambda}\frac{\partial \mathcal{L}}{\partial \dot{q}_k} - \Big(\underbrace{\frac{\partial \mathcal{L}}{\partial \lambda}}_{=0} + \dot{q}_k \frac{\partial \mathcal{L}}{\partial q_k} + \ddot{q}_k \frac{\partial \mathcal{L}}{\partial \dot{q}_k}\Big) \\
&= \dot{q}_k \left(\frac{d}{d\lambda}\frac{\partial \mathcal{L}}{\partial \dot{q}_k} - \frac{\partial \mathcal{L}}{\partial q_k}\right) = 0 \qquad \text{by Eq. (B.73).}
\end{aligned} \tag{B.76}$$

In the study of geodesics, this result will turn out particularly valuable if $\dot{q}_k \neq 0$.

Let us then extremize proper time for timelike curves. More specifically, we consider curves $x^\alpha(\lambda)$ connecting points A and B of the manifold; cf. Fig. 10. Without loss of generality, we choose the parameter λ such that $\lambda = 0$ corresponds to point A and $\lambda = 1$ to point B. We wish to extremize the proper time between the points which gives us the action [cf. Eq. (B.69)]

$$\mathcal{S} = \int_0^1 \mathcal{L} d\lambda\,, \qquad \mathcal{L} = \sqrt{-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}\,. \tag{B.77}$$

Note that $\mathcal{S}$ is invariant under a reparametrization of the curve. For example, we can introduce a new parameter κ required only to be a monotonic function of λ, i.e. $d\kappa/d\lambda > 0$. Then we have chain rule

$$\begin{aligned}
\mathcal{S} &= \int_0^1 \sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}} d\lambda \\
&= \int_{\kappa(0)}^{\kappa(1)} \sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\kappa}\frac{dx^\nu}{d\kappa}}\;\frac{d\kappa}{d\lambda}\left(d\kappa\,\frac{d\lambda}{d\kappa}\right) \\
&= \int_{\kappa(0)}^{\kappa(1)} \sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\kappa}\frac{dx^\nu}{d\kappa}}\; d\kappa\,.
\end{aligned} \tag{B.78}$$

We now apply the Euler-Lagrange equation (B.73) to the action (B.77). The derivatives of the Lagrangian are (a dot denotes $d/d\lambda$)

$$\frac{\partial \mathcal{L}}{\partial \dot{x}^\alpha} = \frac{1}{2\mathcal{L}}\left(-g_{\mu\nu}\delta^\mu{}_\alpha \dot{x}^\nu - g_{\mu\nu}\dot{x}^\mu \delta^\nu{}_\alpha\right) = -\frac{g_{\mu\alpha}\dot{x}^\mu}{\mathcal{L}}\,, \tag{B.79}$$

$$\frac{\partial \mathcal{L}}{\partial x^\alpha} = \frac{1}{2\mathcal{L}}\left(-\dot{x}^\mu \dot{x}^\nu\, \partial_\alpha g_{\mu\nu}\right)\,, \tag{B.80}$$

so that the Euler-Lagrange equation becomes

$$\frac{d}{d\lambda}\left(-\frac{g_{\mu\alpha}\dot{x}^\mu}{\mathcal{L}}\right) + \frac{\dot{x}^\mu \dot{x}^\nu\, \partial_\alpha g_{\mu\nu}}{2\mathcal{L}} = 0\,. \tag{B.81}$$

If you haven't got a social life, like me, you might want to go ahead and evaluate the λ derivatives. But there is an easier way: we reparametrize the curve using proper time

$$\begin{aligned}
\tau(\lambda) &= \int_0^\lambda \sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\tilde{\lambda}}\frac{dx^\nu}{d\tilde{\lambda}}}\, d\tilde{\lambda} \\
\Rightarrow \quad \left(\frac{d\tau}{d\lambda}\right)^2 &= -g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = \mathcal{L}^2 \\
\Rightarrow \quad \frac{d\tau}{d\lambda} &= \mathcal{L} \\
\Rightarrow \quad \frac{d}{d\tau} &= \frac{d\lambda}{d\tau}\frac{d}{d\lambda} = \frac{1}{\mathcal{L}}\frac{d}{d\lambda}\,,
\end{aligned} \tag{B.82}$$

where we assumed in the third line that $d\tau/d\lambda > 0$, i.e. both parameters are future oriented. Inserting this result into Eqs. (B.81) gives

$$-\mathcal{L}\frac{d}{d\tau}\left(g_{\mu\alpha}\frac{dx^\mu}{d\tau}\right) + \frac{\mathcal{L}}{2}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\partial_\alpha g_{\mu\nu} = 0$$

$$\Rightarrow \quad \frac{d^2x^\mu}{d\tau^2}g_{\mu\alpha} + \partial_\nu g_{\mu\alpha}\frac{dx^\nu}{d\tau}\frac{dx^\mu}{d\tau} - \frac{1}{2}\partial_\alpha g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0 \tag{B.83}$$

$$\Rightarrow \quad g_{\mu\alpha}\frac{d^2x^\mu}{d\tau^2} + \frac{1}{2}\left(\partial_\nu g_{\mu\alpha} + \partial_\mu g_{\nu\alpha} - \partial_\alpha g_{\mu\nu}\right)\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0 \qquad \Big| \cdot g^{\beta\alpha}$$

$$\Rightarrow \quad \frac{d^2x^\beta}{d\tau^2} + \frac{1}{2}g^{\beta\alpha}\left(\partial_\nu g_{\alpha\mu} + \partial_\mu g_{\nu\alpha} - \partial_\alpha g_{\mu\nu}\right)\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\,. \tag{B.84}$$

In view of this result we make the following definition.

<u>Def.:</u> The Christoffel symbols are

$$\boxed{\left\{{}^{\;\alpha}_{\beta\;\gamma}\right\} := \frac{1}{2}g^{\alpha\mu}\left(\partial_\beta g_{\gamma\mu} + \partial_\gamma g_{\mu\beta} - \partial_\mu g_{\beta\gamma}\right)}\,. \tag{B.85}$$

By construction, they are symmetric in their two downstairs indices.

The geodesic equation (B.84) can then be written as

$$\boxed{\frac{d^2x^\alpha}{d\tau^2} + \left\{{}^{\;\alpha}_{\beta\;\gamma}\right\}\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau} = 0}\,. \tag{B.86}$$

For spacelike geodesics, we can perform an analogous calculation, merely starting with the action

$$\tilde{\mathcal{S}} = \int_0^1 \tilde{\mathcal{L}}d\lambda\,, \qquad \tilde{\mathcal{L}} = \sqrt{g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}\,, \tag{B.87}$$

in place of Eq. (B.77) and then reparametrizing from λ to the proper length s according to

$$\frac{ds}{d\lambda} = \tilde{\mathcal{L}}\,. \tag{B.88}$$

We then obtain

$$\boxed{\frac{d^2x^\alpha}{ds^2} + \left\{{}^{\;\alpha}_{\beta\;\gamma}\right\}\frac{dx^\beta}{ds}\frac{dx^\gamma}{ds} = 0}\,. \tag{B.89}$$

B.3.3 Geodesic curves defined by a variational principle: Version 2

It is instructive for a number of reasons to derive the geodesic equation also from a slightly different action. Instead of varying the action (B.77), we now start with

$$\hat{\mathcal{S}} = \int_A^B \hat{\mathcal{L}}d\lambda\,, \qquad \hat{\mathcal{L}} = g_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda}\,. \tag{B.90}$$

The difference to our first Lagrangian (B.77) is (i) that we do not take the square root and (ii) that we do not restrict the discussion to timelike or spacelike or null curves. For this reason, we need not worry about the overall sign of $g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta$ and, just for convenience, choose to not put a minus in front.
The variation of (B.90) is straightforward. The derivatives of $\hat{\mathcal{L}}$ are

$$\frac{\partial\hat{\mathcal{L}}}{\partial\dot{x}^\mu} = g_{\alpha\beta}\dot{x}^\beta\delta^\alpha{}_\mu + g_{\alpha\beta}\dot{x}^\alpha\delta^\beta{}_\mu = 2g_{\mu\beta}\dot{x}^\beta\,, \tag{B.91}$$

$$\frac{\partial\hat{\mathcal{L}}}{\partial x^\mu} = \dot{x}^\alpha\dot{x}^\beta\,\partial_\mu g_{\alpha\beta}\,, \tag{B.92}$$

and the Euler-Lagrange equation gives us

$$\begin{aligned}
&\frac{d}{d\lambda}\frac{\partial\hat{\mathcal{L}}}{\partial\dot{x}^\mu} - \frac{\partial\hat{\mathcal{L}}}{\partial x^\mu} = 2g_{\mu\beta}\ddot{x}^\beta + 2\dot{x}^\beta(\partial_\nu g_{\mu\beta})\,\dot{x}^\nu - \dot{x}^\alpha\dot{x}^\beta\,\partial_\mu g_{\alpha\beta} = 0\\
\Rightarrow\quad & 2g_{\mu\beta}\ddot{x}^\beta + 2\dot{x}^\nu\dot{x}^\beta\partial_\nu g_{\mu\beta} - \dot{x}^\nu\dot{x}^\beta\partial_\mu g_{\nu\beta} = 0\\
\Rightarrow\quad & g_{\mu\beta}\ddot{x}^\beta + \frac{1}{2}\dot{x}^\nu\dot{x}^\beta\left(\partial_\nu g_{\mu\beta} + \partial_\beta g_{\mu\nu} - \partial_\mu g_{\nu\beta}\right) = 0 \qquad \Big|\ \cdot g^{\alpha\mu}\\
\Rightarrow\quad & \ddot{x}^\alpha + \left\{{}^{\ \alpha}_{\nu\ \beta}\right\}\dot{x}^\nu\dot{x}^\beta = 0\,.
\end{aligned} \tag{B.93}$$

Aside from the fact that we have the more general parameter λ instead of proper time or length, this equation looks exactly like Eqs. (B.86), (B.89) derived above for time and spacelike geodesics and all seems fine. But it is not quite as simple as that.
Let us consider, for example timelike geodesics and choose a parameter λ related to proper time by

$$\lambda = e^\tau \quad\Rightarrow\quad \frac{d}{d\tau} = \frac{d\lambda}{d\tau}\frac{d}{d\lambda} = \lambda\frac{d}{d\lambda} \quad\Rightarrow\quad \frac{d^2}{d\tau^2} = \frac{d}{d\tau}\left(e^\tau\frac{d}{d\lambda}\right) = \lambda\frac{d}{d\lambda} + \lambda^2\frac{d^2}{d\lambda^2}\,. \tag{B.94}$$

We have demonstrated above that the action (B.77) is invariant under any reparametrization, so its variation proceeds the same way for any λ and the geodesic equations (B.86), (B.89) still are the correct results. Rewritten in terms of $\lambda = e^\tau$, however, (B.86) becomes [using (B.94)]

$$\ddot{x}^\alpha + \left\{{}^{\ \alpha}_{\nu\ \beta}\right\}\dot{x}^\nu\dot{x}^\beta = -\frac{1}{\lambda}\frac{dx^\alpha}{d\lambda}\,. \tag{B.95}$$

This is clearly not compatible with Eq. (B.93). So which one is correct and what is going on? The answer arises from the fact that the action (B.90) is *not* invariant under a change of the parameter λ. If we change the parameter, say from λ_1 to λ_2, we are not necessarily extremizing the same action and should not be surprised that the result of this exercise, namely Eq. (B.93), gives us a different curve when choosing parameter λ_1 than for choosing parameter λ_2. So, for our particular choice $\lambda = e^\tau$, Eq. (B.95) gives us geodesics and Eq. (B.93) does not.

On the other hand, if we set $\lambda = \tau$, Eq. (B.93) agrees with (B.86) and gives us geodesics. The question then remains to figure out for which choices of the parameter λ, Eq. (B.93) is correct. Let us first consider timelike geodesics which are given by Eq. (B.86). Let λ and τ be monotonically increasing and, thus, invertible functions of each other: $d\tau/d\lambda > 0$. Then

$$\frac{d}{d\tau} = \frac{d\lambda}{d\tau}\frac{d}{d\lambda} \quad \Rightarrow \quad \frac{d^2}{d\tau^2} = \frac{d}{d\tau}\left(\frac{d\lambda}{d\tau}\frac{d}{d\lambda}\right) = \frac{d^2\lambda}{d\tau^2}\frac{d}{d\lambda} + \left(\frac{d\lambda}{d\tau}\right)^2 \frac{d^2}{d\lambda^2}\,, \tag{B.96}$$

and Eq. (B.86), reparametrized with λ, becomes

$$\frac{d^2x^\alpha}{d\lambda^2} + \left\{ {}^{\,\alpha}_{\nu\ \beta} \right\} \frac{dx^\nu}{d\lambda}\frac{dx^\beta}{d\lambda} = -\left(\frac{d\lambda}{d\tau}\right)^{-2} \frac{d^2\lambda}{d\tau^2}\frac{dx^\alpha}{d\lambda} \quad \propto \quad \frac{dx^\alpha}{d\lambda}\,. \tag{B.97}$$

This agrees with Eq. (B.93) if the right-hand side vanishes which is only achieved for

$$\frac{d^2\lambda}{d\tau^2} = 0 \qquad \Leftrightarrow \qquad \lambda = c_1\tau + c_2\,, \quad c_1,\, c_2 = \text{const} \in \mathbb{R}\,, \tag{B.98}$$

i.e. if λ and τ are linearly related. We likewise find that (B.93) defines spacelike geodesic if the parameter λ is linearly related to the proper distance s. This leads to the definition of *affine parameters.*

Def.: The parameter λ along a timelike (spacelike) curve is called an affine parameter if it is linearly related to the proper time (proper distance) along this curve: $\lambda = c_1\tau + c_2$ ($\lambda = c_1 s + c_2$).
For an affine parameter, timelike and spacelike geodesics are determined by Eq. (B.93).
If we choose a non-affine parameter instead, geodesics are given by Eq. (B.97).

In this discussion, we have so far gracefully ignored null geodesics. Null geodesics are special in the sense that they do not have a natural affine parameter analogous to proper time or proper distance. Nevertheless, null geodesics are honorable curves that can be parametrized just like other curves. We can even define affine parameters as follow.

Def.: If a null curve

$$\mathcal{C} : I \subset \mathbb{R} \to \mathcal{M}\,, \qquad \lambda \mapsto x^\alpha(\lambda)\,, \tag{B.99}$$

satisfies Eq. (B.93) it is a geodesic and its parameter λ is called an affine parameter. If the curve satisfies Eq. (B.97) with a non-zero right-hand side, it is a geodesic and its parameter is non-affine. If the curve satisfies neither (B.93) nor (B.97), it is not a geodesic. This definition holds as well for timelike and spacelike geodesics.

In general relativity we define test particles as sufficiently small bodies that generate negligible gravitational fields. Their motion is governed by a geodesic postulate analogous to Eq. (A.101) in special relativity.

Geodesic postulate: Test particles with positive (zero) rest mass move on timelike (null) geodesics.

The geodesic equation, either in the form (B.93) for an affine parameter or (B.97) for a non-affine parameter, is a second-order ordinary differential equation. The uniqueness theorems of the theory of ordinary differential equations ensure that a unique solution exists for specified position $x^\alpha(\lambda)$ and velocity $\dot{x}^\alpha(\lambda)$ at some $\lambda = \lambda_0$.
Aside from demonstrating the difference between affine and non-affine parameters, the variational method discussed in this section also serves a practical purpose: it gives us a convenient method to calculate the Christoffel symbols without grinding through its definition (B.85). This method is best illustrated using an example. As will be shown below in Sec. D.1, the Schwarzschild metric for a static black hole can be written in spherical coordinates as

$$ds^2 = -f(r)\,dt^2 + \frac{1}{f(r)}\,dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2\,, \qquad f(r) = 1 - \frac{2M}{r}\,, \tag{B.100}$$

where the constant M denotes the mass of the black hole. For an affine parameter λ, the geodesic equation is then given by (B.93) if we know the Christoffel symbols. Viewed the other way round, however, we can use Eq. (B.93) to extract the Christoffel symbols if we know the geodesic equation. And for reasonably simple metrics like (B.100), the geodesic equation is quite easily obtained by directly varying the Lagrangian $\hat{\mathcal{L}}$ of Eq. (B.90). For the Schwarzschild metric (B.100), the Lagrangian is

$$\hat{\mathcal{L}} = -f\,\dot{t}^2 + f^{-1}\,\dot{r}^2 + r^2\,\dot{\theta}^2 + r^2\sin^2\theta\;\dot{\phi}^2\,. \tag{B.101}$$

The t component of the Euler Lagrange equation is obtained from

$$\frac{\partial\hat{\mathcal{L}}}{\partial\dot{t}} = -2f\dot{t}\,, \qquad \frac{\partial\hat{\mathcal{L}}}{\partial t} = 0\,, \tag{B.102}$$

leading to

$$\begin{aligned} & \frac{d}{d\lambda}(-2f\dot{t}) = 0 \\ \Rightarrow\quad & \frac{d^2t}{d\lambda^2} + f^{-1}\frac{df}{dr}\dot{r}\dot{t} = 0 \\ \Rightarrow\quad & \left\{ {}^{\;t}_{t\;r} \right\} = \left\{ {}^{\;t}_{r\;t} \right\} = \frac{1}{2f}\frac{df}{dr} = \frac{2M}{r(r-2M)}\,, \qquad \left\{ {}^{\;t}_{\mu\;\nu} \right\} = 0 \;\; \text{otherwise}\,. \end{aligned} \tag{B.103}$$

Note the factor $1/2$ that arises for Christoffel symbols with mixed downstairs indices which are equal and thus appear twice in the summation $\left\{ {}^{\;t}_{\mu\;\nu} \right\}\dot{x}^\mu\dot{x}^\nu$ in the geodesic equation.

B.4 The covariant derivative

Physical laws typically involve derivatives which requires us to compare mathematical objects at nearby points. In general relativity, the relevant objects are tensors and that presents us with a difficulty: vectors at different points $p,\ q \in \mathcal{M}$, for example, live in different vector

spaces, namely $\mathcal{T}_p(\mathcal{M})$ and $\mathcal{T}_q(\mathcal{M})$. We can therefore not take the difference between them. So how can we calculate their derivative?
The answer is to construct the so-called *covariant derivative*. We will do this in steps, first for scalars, then for vectors and finally for arbitrary tensors. For scalars, this is trivial since they are the only class of tensors for which the problem just mentioned does not arise; we can just subtract the scalar at one point from that at another.

Def.: The covariant derivative ∇f of a function f is a map

$$\nabla f : \mathcal{T}_p(\mathcal{M}) \to \mathbb{R}\,, \qquad \boldsymbol{V} \mapsto \nabla_{\boldsymbol{V}} f := \boldsymbol{V}(f)\,. \tag{B.104}$$

By definition, ∇f is therefore a tensor of rank $\binom{0}{1}$. In components, we write

$$\nabla_\alpha f := (\nabla f)_\alpha = \partial_\alpha f\,. \tag{B.105}$$

Recall that $\boldsymbol{V}(f)$ is the derivative of f defined by (B.5). Covariantly differentiating vector fields is a bit more complicated.

Def.: The covariant derivative $\nabla \boldsymbol{V}$ of a vector field $\boldsymbol{V}$ is a map

$$\nabla \boldsymbol{V} : \mathcal{T}_p(\mathcal{M}) \to \mathcal{T}_p(\mathcal{M})\,, \qquad \boldsymbol{X} \mapsto \nabla_{\boldsymbol{X}} \boldsymbol{V}\,, \tag{B.106}$$

with the following properties (f, g are functions and $\boldsymbol{X}$, $\boldsymbol{Y}$, $\boldsymbol{V}$, $\boldsymbol{W}$ are vector fields)

(1) $\nabla_{f\boldsymbol{X}+g\boldsymbol{Y}} \boldsymbol{V} = f\nabla_{\boldsymbol{X}} \boldsymbol{V} + g\nabla_{\boldsymbol{Y}} \boldsymbol{V}$,

(2) $\nabla_{\boldsymbol{X}}(\boldsymbol{V} + \boldsymbol{W}) = \nabla_{\boldsymbol{X}} \boldsymbol{V} + \nabla_{\boldsymbol{X}} \boldsymbol{W}$

(3) $\nabla_{\boldsymbol{X}}(f\boldsymbol{V}) = f\nabla_{\boldsymbol{X}} \boldsymbol{V} + (\nabla_{\boldsymbol{X}} f)\boldsymbol{V}$ (Leibniz rule)

Note that we can also define $\nabla \boldsymbol{V}$ as the following type of map, which is completely equivalent to (B.106),

$$\nabla \boldsymbol{V} : \mathcal{T}_p^*(\mathcal{M}) \times \mathcal{T}_p(\mathcal{M}) \to \mathbb{R}\,, \qquad (\boldsymbol{\eta}, \boldsymbol{X}) \mapsto \boldsymbol{\eta}(\nabla_{\boldsymbol{X}} \boldsymbol{V})\,. \tag{B.107}$$

In this form, the tensor rank $\binom{1}{1}$ of $\nabla \boldsymbol{V}$ is manifest. In components, we use the following notations

$$V^\alpha{}_{;\beta} := \nabla_\beta V^\alpha := (\nabla \boldsymbol{V})^\alpha{}_\beta\,. \tag{B.108}$$

You may wonder at this stage that this definition is all nice and fine, but how do we actually calculate the covariant derivative of a vector? Patience, we will come to that. First we define another level of structure on the manifold.

Def.: Let $(\mathbf{e}_\mu)$ be a basis of the tangent space $\mathcal{T}_p(\mathcal{M})$. The connection coefficients $\Gamma^\rho_{\mu\nu}$ are defined through

$$\nabla_\nu \mathbf{e}_\mu := \nabla_{\mathbf{e}_\nu} \mathbf{e}_\mu = \Gamma^\rho_{\mu\nu} \mathbf{e}_\rho\,. \tag{B.109}$$

Some comments are in order.

- In these notes, we only consider coordinate basis vectors $\mathbf{e}_\mu = \partial_\mu$. This definition of the connection coefficients, however, is general and also holds for non-coordinate bases.
- Note that $\nabla_{\mathbf{e}_\nu}\mathbf{e}_\mu$ is a vector by construction; cf. Eq. (B.106). The $\Gamma^\rho_{\mu\nu}$ therefore are the expansion coefficients of this vector in the basis $(\mathbf{e}_\rho)$.
- The word *connection* arises from the fact that through Eq. (B.109) we "connect" the tangent spaces at different points $p,\ q \in \mathcal{M}$. Specifically, the connection coefficients give us the rate of change of the basis vector $\mathbf{e}_\mu$ in the direction of the basis vector $\mathbf{e}_\nu$. This is the key element we were missing above when we wondered how we can compare vectors in different tangent spaces $\mathcal{T}_p(\mathcal{M})$ and $\mathcal{T}_q(\mathcal{M})$.
- Here we use the convention that the second downstairs index of the connection, i.e. ν in Eq. (B.109), denotes the direction in which the derivative is taken. The first index denotes the basis vector we are considering. There is bad news and good news about this. The bad news is that this convention is not ubiquitous; some people have the downstairs indices in Γ the other way round. The good news is that in general relativity, the connection turns out to be symmetric in its downstairs indices, so that this convention does not really matter. Beware, however, that there are instances where one uses another connection that is not symmetric in the two indices. For example, this can happen in studies of modifications of general relativity. We will not do this in these notes, but we recommend that you stay aware of which convention you use, whether you follow these notes or choose the opposite. It is a good idea, in general, to write down your conventions unless your choice is evident.

With the definition (B.109), we obtain a more concrete expression for the covariant derivative. Let $\boldsymbol{V} = V^\mu\mathbf{e}_\mu$ and $\boldsymbol{W} = W^\mu\mathbf{e}_\mu$ be two vector fields. Then we have

$$\begin{aligned}
\nabla_{\boldsymbol{V}}\boldsymbol{W} &= \nabla_{\boldsymbol{V}}(W^\mu\mathbf{e}_\mu) = \boldsymbol{V}(W^\mu)\mathbf{e}_\mu + W^\mu\nabla_{\boldsymbol{V}}(\mathbf{e}_\mu) \quad &&\Big|\ \text{by Leibnitz rule} \\
&= V^\nu\mathbf{e}_\nu(W^\mu)\,\mathbf{e}_\mu + W^\mu\nabla_{V^\nu\mathbf{e}_\nu}(\mathbf{e}_\mu) \\
&= V^\nu\mathbf{e}_\nu(W^\mu)\,\mathbf{e}_\mu + V^\nu W^\mu\nabla_{\mathbf{e}_\nu}e_\mu &&\Big|\ \text{by item (1) of the definition of } \nabla \\
&= V^\nu\left[\partial_\nu W^\rho + W^\mu\Gamma^\rho_{\mu\nu}\right]\mathbf{e}_\rho, &&\text{(B.110)} \\
\Rightarrow\quad (\nabla_{\boldsymbol{V}}\boldsymbol{W})^\rho &= V^\nu\partial_\nu W^\rho + \Gamma^\rho_{\mu\nu}W^\mu V^\nu\,, &&\text{(B.111)}
\end{aligned}$$

where in the last but one line, we used $\mathbf{e}_\nu(f) = \partial_\nu f$ for $f = W^\mu$, renamed the summation index μ to ρ in the first term and inserted the connection through its definition (B.109). Since the vector $\boldsymbol{V}$ is arbitrary in Eq. (B.111), we can rewrite this result, also defining standard notation, in the form

$$\boxed{W^\rho{}_{;\nu} := \nabla_\nu W^\rho := (\nabla\boldsymbol{W})^\rho{}_\nu = \partial_\nu W^\rho + \Gamma^\rho_{\mu\nu}W^\mu}\,. \tag{B.112}$$

So we now have a perfectly nice expression for the covariant derivative of a vector field *provided* we know the connection. Before you conclude that we are just kicking the can down the road,

we will get to that point in due course. But first we deal with a couple of other important points concerning Eq. (B.112).
First, we would like to check how it changes under a transformation of coordinates from x^α to $\tilde{x}^\mu$. For the connection, we start with its definition (B.109) and replace $\mathbf{e}_\alpha = \partial_\alpha$ which makes it easier to spot where to apply chain rule. Transformed to coordinates $\tilde{x}^\mu$, this equation becomes (we denote $\tilde{\partial}_\alpha := \partial/\partial\tilde{x}^\alpha$)

$$\begin{aligned}
\tilde{\Gamma}^\sigma_{\mu\nu}\tilde{\partial}_\sigma &= \nabla_{\tilde{\partial}_\nu}\tilde{\partial}_\mu = \frac{\partial x^\alpha}{\partial\tilde{x}^\nu}\nabla_{\partial_\alpha}\left(\frac{\partial x^\beta}{\partial\tilde{x}^\mu}\tilde{\partial}_\beta\right) \\
&= \frac{\partial x^\alpha}{\partial\tilde{x}^\nu}\frac{\partial^2 x^\beta}{\partial x^\alpha\partial\tilde{x}^\mu}\partial_\beta + \frac{\partial x^\alpha}{\partial\tilde{x}^\nu}\frac{\partial x^\beta}{\partial\tilde{x}^\mu}\nabla_{\partial_\alpha}(\partial_\beta) \\
&= \frac{\partial^2 x^\beta}{\partial\tilde{x}^\nu\partial\tilde{x}^\mu}\partial_\beta + \frac{\partial x^\alpha}{\partial\tilde{x}^\nu}\frac{\partial x^\beta}{\partial\tilde{x}^\mu}\Gamma^\rho_{\beta\alpha}\partial_\rho \\
&= \left[\frac{\partial^2 x^\rho}{\partial\tilde{x}^\nu\partial\tilde{x}^\mu} + \frac{\partial x^\alpha}{\partial\tilde{x}^\nu}\frac{\partial x^\beta}{\partial\tilde{x}^\mu}\Gamma^\rho_{\beta\alpha}\right]\frac{\partial\tilde{x}^\sigma}{\partial x^\rho}\tilde{\partial}_\sigma \\
\Rightarrow \tilde{\Gamma}^\sigma_{\mu\nu} &= \frac{\partial\tilde{x}^\sigma}{\partial x^\rho}\frac{\partial x^\alpha}{\partial\tilde{x}^\nu}\frac{\partial x^\beta}{\partial\tilde{x}^\mu}\Gamma^\rho_{\beta\alpha} + \frac{\partial\tilde{x}^\sigma}{\partial x^\rho}\frac{\partial^2 x^\rho}{\partial\tilde{x}^\nu\partial\tilde{x}^\mu}\,. \qquad \text{(B.113)}
\end{aligned}$$

The first term on the right-hand side of the last line corresponds to the transformation properties of a tensor of rank $\binom{1}{2}$, but the second term spoils the transformation; $\Gamma^\rho_{\alpha\beta}$ is *not* a tensor. Note, however, that the second term is independent of the connection itself. So the difference of two connections is a tensor and it leads to the following definition.

Def.: Let $\mathcal{M}$ be a manifold with connection $\Gamma^\lambda_{\mu\nu}$. The torsion tensor is defined as

$$T_{\mu\nu}{}^\lambda := \Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu}\,. \qquad \text{(B.114)}$$

The connection Γ is called torsion free $\quad :\Leftrightarrow T_{\mu\nu}{}^\lambda = 0$.

The difference between two connections often appears naturally in perturbation theory where we study small deviations from the connection of a background spacetime. This deviation is indeed a tensor.
With Eq. (B.113), we have all tools at hand to check how the covariant derivative of a vector transforms. Let us first consider the partial derivative on the right-hand side of Eq. (B.112),

$$\frac{\partial\tilde{W}^\rho}{\partial\tilde{x}^\nu} = \frac{\partial x^\alpha}{\partial\tilde{x}^\nu}\frac{\partial}{\partial x^\alpha}\left(\frac{\partial\tilde{x}^\rho}{\partial x^\beta}W^\beta\right) = \frac{\partial x^\alpha}{\partial\tilde{x}^\nu}\frac{\partial\tilde{x}^\rho}{\partial x^\beta}\partial_\alpha W^\beta + \frac{\partial x^\alpha}{\partial\tilde{x}^\nu}W^\beta\frac{\partial^2\tilde{x}^\rho}{\partial x^\alpha\partial x^\beta}\,. \qquad \text{(B.115)}$$

Again, the first term on the right hand side would give us a tensor transformation law, this time for a tensor of rank $\binom{1}{1}$, but the second term spoils the transformation. It looks suspiciously similar to the extra term we obtained in Eq. (B.113) and you probably guess where this is heading. Combining Eqs. (B.113) and (B.115), we obtain the transformation of the covariant derivative $\nabla_\alpha W^\beta$,

$$
\begin{aligned}
\tilde{\nabla}_\nu \tilde{W}^\rho &= \tilde{\partial}_\nu \tilde{W}^\rho + \tilde{\Gamma}^\rho_{\mu\nu}\tilde{W}^\mu \\
&= \frac{\partial x^\alpha}{\partial \tilde{x}^\nu}\frac{\partial \tilde{x}^\rho}{\partial x^\beta}\partial_\alpha W^\beta + \frac{\partial x^\alpha}{\partial \tilde{x}^\nu} W^\beta \partial_\alpha\left(\frac{\partial \tilde{x}^\rho}{\partial x^\beta}\right) \\
&\quad + \frac{\partial x^\alpha}{\partial \tilde{x}^\mu}\frac{\partial x^\beta}{\partial \tilde{x}^\nu}\frac{\partial \tilde{x}^\rho}{\partial x^\gamma}\Gamma^\gamma_{\alpha\beta}\frac{\partial \tilde{x}^\mu}{\partial x^\delta}W^\delta + \frac{\partial \tilde{x}^\rho}{\partial x^\lambda}\tilde{\partial}_\nu\left(\frac{\partial x^\lambda}{\partial \tilde{x}^\mu}\right)\frac{\partial \tilde{x}^\mu}{\partial x^\beta}W^\beta \\
&= \frac{\partial x^\alpha}{\partial \tilde{x}^\nu}\frac{\partial \tilde{x}^\rho}{\partial x^\beta}\partial_\alpha W^\beta + W^\beta\tilde{\partial}_\nu\left(\frac{\partial \tilde{x}^\rho}{\partial x^\beta}\right) + \frac{\partial x^\beta}{\partial \tilde{x}^\nu}\frac{\partial \tilde{x}^\rho}{\partial x^\gamma}\Gamma^\gamma_{\alpha\beta}W^\alpha + W^\beta\frac{\partial \tilde{x}^\rho}{\partial x^\lambda}\frac{\partial \tilde{x}^\mu}{\partial x^\beta}\tilde{\partial}_\nu\left(\frac{\partial x^\lambda}{\partial \tilde{x}^\mu}\right).
\end{aligned}
\tag{B.116}
$$

We simplify the very last term using

$$
\begin{aligned}
&\frac{\partial \tilde{x}^\mu}{\partial x^\beta}\frac{\partial x^\lambda}{\partial \tilde{x}^\mu} = \delta^\lambda{}_\beta \qquad \Rightarrow \qquad \frac{\partial \tilde{x}^\mu}{\partial x^\beta}\tilde{\partial}_\nu\left(\frac{\partial x^\lambda}{\partial \tilde{x}^\mu}\right) = -\frac{\partial x^\lambda}{\partial \tilde{x}^\mu}\tilde{\partial}_\nu\left(\frac{\partial \tilde{x}^\mu}{\partial x^\beta}\right) \\
\Rightarrow\ & W^\beta\frac{\partial \tilde{x}^\rho}{\partial x^\lambda}\frac{\partial \tilde{x}^\mu}{\partial x^\beta}\tilde{\partial}_\nu\left(\frac{\partial x^\lambda}{\partial \tilde{x}^\mu}\right) = -W^\beta\frac{\partial \tilde{x}^\rho}{\partial x^\lambda}\frac{\partial x^\lambda}{\partial \tilde{x}^\mu}\tilde{\partial}_\nu\left(\frac{\partial \tilde{x}^\mu}{\partial x^\beta}\right) = -W^\beta\tilde{\partial}_\nu\left(\frac{\partial \tilde{x}^\rho}{\partial x^\beta}\right),
\end{aligned}
\tag{B.117}
$$

which exactly cancels the second term in the last line of Eq. (B.116), so that

$$
\tilde{\nabla}_\nu \tilde{W}^\rho = \frac{\partial x^\alpha}{\partial \tilde{x}^\nu}\frac{\partial \tilde{x}^\rho}{\partial x^\beta}\partial_\alpha W^\beta + \frac{\partial x^\alpha}{\partial \tilde{x}^\nu}\frac{\partial \tilde{x}^\rho}{\partial x^\beta}\Gamma^\beta_{\gamma\alpha}W^\gamma = \frac{\partial x^\alpha}{\partial \tilde{x}^\nu}\frac{\partial \tilde{x}^\rho}{\partial x^\beta}\nabla_\alpha W^\beta\,. \tag{B.118}
$$

The covariant derivative $\nabla_\alpha W^\beta$ transforms as a tensor as it had better do since we defined it as a tensor in the first place. But we now see that the extra term that spoiled the tensor transformation law for the connection in (B.113) and the partial derivative in (B.115) cancel each other; the extra term involving the connection in the covariant derivative (B.112) adjusts the partial derivative such that we obtain a tensorial derivative for vectors. Another viewpoint regarding the extra term on the right-hand side of Eq. (B.112) evokes Leibnitz rule. The vector is $\boldsymbol{W} = W^\rho \mathbf{e}_\rho$. The partial derivative $\partial_\nu W^\rho$ only takes care of changes in the vector's component, but not in the basis vectors. Leibnitz rule gives us two terms in the derivative of the produce $W^\rho \mathbf{e}_\rho$ and the second term accounting for the rate of change of the basis vector involves the connection which has just been defined to measure this change.

We have dealt with the transformation properties for the covariant derivative of vectors in much detail here because the construction of covariant derivatives of arbitrary tensors proceeds along very similar lines. This construction is straightforward but involves a good deal of lengthy calculations which are not particularly enlightening. Having introduced all the key ideas in the special case of vectors, we therefore feel more comfortable skipping those detailed manipulations and focus on the results instead.

Before moving on to other tensors, we mention a subtle point about the notation that has some potential for confusion but is nonetheless used almost ubiquitously in the field. concerns the component functions of tensor fields; for example the components W^μ of a vector $\boldsymbol{W}$. Strictly speaking, these are merely functions on the manifold. We have treated them as such, for instance, in the derivation (B.110) where we regarded $\mathbf{e}_\nu(W^\mu)$ as the derivative $\partial_\nu W^\mu$. It is common in the literature, however, to also use W^μ representing the entire vector. This is done, for example, in the notation $\nabla_\nu W^\rho$ for the covariant derivative of $\boldsymbol{W}$ in Eq. (B.112). The covariant derivative of a function would just be its partial derivative, but $\nabla_\nu W^\rho$ includes the correction term for the covariant derivative of a vector. How do you know when terms like W^ρ are assumed to represent merely the component functions or the entire tensor? Usually this should be clear from the context, but a good rule of thump is that they represent the components if the basis vectors are explicitly present in the equation, but otherwise denote the tensor; cf. Eq. (B.110) with Eq. (B.112).

In order to define the covariant derivative of a covector field, we recall that a covector is defined through its action on vectors; cf. Eq. (B.16).

Def.: Let $\boldsymbol{\eta}$ be a covector field and $\boldsymbol{V}$, $\boldsymbol{W}$ be two vector fields. The covariant derivative of $\boldsymbol{\eta}$ is defined as the map

$$\nabla\boldsymbol{\eta} : \mathcal{T}_p(\mathcal{M}) \to \mathcal{T}_p^*(\mathcal{M}), \qquad \boldsymbol{V} \mapsto \nabla_{\boldsymbol{V}}\boldsymbol{\eta}, \quad \text{with}$$

$$(\nabla_{\boldsymbol{V}}\boldsymbol{\eta})(\boldsymbol{W}) \coloneqq \nabla_{\boldsymbol{V}}\big(\boldsymbol{\eta}(\boldsymbol{W})\big) - \boldsymbol{\eta}(\nabla_{\boldsymbol{V}}\boldsymbol{W}). \tag{B.119}$$

Note that $\boldsymbol{\eta}(\boldsymbol{W})$ is a function and $\nabla_{\boldsymbol{V}}\boldsymbol{W}$ is a vector, so that all terms on the right-hand side of Eq. (B.119) are already well defined. Equation (B.119) furthermore exhibits product rule explicitly for differentiating $\boldsymbol{\eta}(\boldsymbol{V})$ if we move the last term to the left-hand side.

$\nabla\boldsymbol{\eta}$ is a tensor of rank $\binom{0}{2}$ which can be seen as follows.

$$\begin{aligned}
(\nabla\boldsymbol{\eta})(\boldsymbol{V},\boldsymbol{W}) &:= \big(\nabla_{\boldsymbol{V}}\boldsymbol{\eta}\big)(\boldsymbol{W}) = \nabla_{\boldsymbol{V}}(\eta_\mu W^\mu) - \eta_\mu(\nabla_{\boldsymbol{V}}\boldsymbol{W})^\mu \\
&= V^\rho\partial_\rho(\eta_\mu W^\mu) - \eta_\mu\big(V^\rho\partial_\rho W^\mu + V^\rho\Gamma^\mu_{\nu\rho}W^\nu\big) \\
&= V^\rho W^\mu\partial_\rho\eta_\mu - \Gamma^\mu_{\nu\rho}\eta_\mu V^\rho W^\nu \\
&= \big(\partial_\rho\eta_\mu - \Gamma^\nu_{\mu\rho}\eta_\nu\big)V^\rho W^\mu.
\end{aligned} \tag{B.120}$$

So $\nabla\boldsymbol{\eta}$ is indeed a linear map taking two vectors as input and returning a number. Equation (B.120) further gives us the components of the covariant derivative of a one-form $\boldsymbol{\eta}$,

$$\boxed{\eta_{\mu;\rho} := \nabla_\rho\eta_\mu := (\nabla\boldsymbol{\eta})_{\rho\mu} = \partial_\rho\eta_\mu - \Gamma^\nu_{\mu\rho}\eta_\nu}. \tag{B.121}$$

We likewise define the covariant derivative of a tensor $\boldsymbol{T}$ of rank $\binom{r}{s}$ by filling all its slots with r one-forms and s vectors. The result is a number and we require Leibnitz rule to hold on the entire product. A straightforward calculation analogous to (B.120) shows that the result $\nabla \boldsymbol{T}$ is a tensor of rank $\binom{r}{s+1}$ and has the components

$$\boxed{\begin{aligned}\nabla_\rho T^{\mu_1\ldots\mu_r}{}_{\nu_1\ldots\nu_s} = \partial_\rho T^{\mu_1\ldots\mu_r}{}_{\nu_1\ldots\nu_s} &+ \Gamma^{\mu_1}_{\sigma\rho} T^{\sigma\mu_2\ldots\mu_r}{}_{\nu_1\ldots\nu_s} + \ldots + \Gamma^{\mu_r}_{\sigma\rho} T^{\mu_1\ldots\mu_{s-1}\sigma}{}_{\nu_1\ldots\nu_s} \\ &- \Gamma^{\sigma}_{\nu_1\rho} T^{\mu_1\ldots\mu_r}{}_{\sigma\nu_2\ldots\nu_s} - \ldots - \Gamma^{\sigma}_{\nu_s\rho} T^{\mu_1\ldots\mu_r}{}_{\nu_1\ldots\nu_{s-1}\sigma}\,.\end{aligned}} \tag{B.122}$$

This expression is simpler than it looks at first glance. First, we get a partial derivative and then for each tensor index one correction term constructed as follows. For each upstairs (downstairs) index of the tensor $\boldsymbol{T}$, we add (subtract) a term "ΓT". The derivative index (ρ in our case) is always the second downstairs index of the Γ whose other indices are combined with those of $\boldsymbol{T}$ in the only manner possible to make the free indices' positions agree with the left-hand side.

B.5 The Levi-Civita connection

Finally, it is time to answer the question about how to determine the connection coefficients in practice. The general answer is that you can choose any coefficients $\Gamma^\sigma_{\mu\nu}$ that obey the transformation rule (B.113) under a change of coordinates. In general manifolds, there is no more fundamental structure on the manifold that determines the connection, so it is part of defining the geometry to equip it with a connection of your choice. Note that it is not even necessary to have a metric on the manifold. Nothing of what we said in the previous section on covariant derivatives relied on having a metric available.

If we have a metric, however, there exists one special connection. This is a consequence of the **fundamental theorem of Riemannian geometry**.

Theorem: On a manifold $\mathcal{M}$ with metric $\boldsymbol{g}$ there exists a unique torsion free connection that is metric compatible, i.e. satisfies

$$\nabla \boldsymbol{g} = 0\,. \tag{B.123}$$

This connection is called the Levi-Civita connection and its components are given by the Christoffel symbols.

Proof: "$\Leftarrow$": Let

$$\Gamma^\alpha_{\beta\gamma} = \left\{ {\alpha \atop \beta\ \gamma} \right\} = \frac{1}{2} g^{\alpha\mu}(\partial_\beta g_{\gamma\mu} + \partial_\gamma g_{\mu\beta} - \partial_\mu g_{\beta\gamma})\,. \tag{B.124}$$

This connection is evidently symmetric in β and γ and therefore torsion free. Furthermore,

$$\begin{aligned}
\nabla_\alpha g_{\beta\gamma} &= \partial_\alpha g_{\beta\gamma} - \Gamma^\rho_{\beta\alpha} g_{\rho\gamma} - \Gamma^\rho_{\gamma\alpha} g_{\beta\rho} \\
&= \partial_\alpha g_{\beta\gamma} - \frac{1}{2} g^{\rho\sigma} \left[(\partial_\beta g_{\alpha\sigma} + \partial_\alpha g_{\sigma\beta} - \partial_\sigma g_{\beta\alpha})\, g_{\rho\gamma} \ + \ (\beta \leftrightarrow \gamma) \right] \\
&= \partial_\alpha g_{\beta\gamma} - \frac{1}{2} \left[\partial_\beta g_{\alpha\gamma} + \partial_\alpha g_{\gamma\beta} - \partial_\gamma g_{\beta\alpha} \ + \ (\beta \leftrightarrow \gamma) \right] \\
&= \partial_\alpha g_{\beta\gamma} - \frac{1}{2} (\partial_\alpha g_{\gamma\beta} + \partial_\alpha g_{\beta\gamma}) = 0\,,
\end{aligned} \tag{B.125}$$

since the metric is symmetric.

"$\Rightarrow$": Let $\Gamma^\alpha_{\beta\gamma}$ be a metric compatible, symmetric connection. Then

$$\begin{aligned}
& \nabla_\alpha g_{\beta\gamma} = 0 \\
\Rightarrow \quad & \partial_\alpha g_{\beta\gamma} = \Gamma^\rho_{\beta\alpha} g_{\rho\gamma} + \Gamma^\rho_{\gamma\alpha} g_{\beta\rho}\,.
\end{aligned} \tag{B.126}$$

By definition the Christoffel symbols are

$$\begin{aligned}
\left\{ {\mu \atop \beta\ \gamma} \right\} &= \frac{1}{2} g^{\mu\nu} (\partial_\beta g_{\gamma\nu} + \partial_\gamma g_{\nu\beta} - \partial_\nu g_{\beta\gamma}) \\
&= \frac{1}{2} g^{\mu\nu} \left(\Gamma^\rho_{\gamma\beta} g_{\rho\nu} + \underline{\Gamma^\rho_{\nu\beta} g_{\gamma\rho}} + \Gamma^\rho_{\nu\gamma} g_{\rho\beta} + \Gamma^\rho_{\beta\gamma} g_{\nu\rho} - \underline{\Gamma^\rho_{\beta\nu} g_{\rho\gamma}} - \Gamma^\rho_{\gamma\nu} g_{\beta\rho} \right) \\
&= \Gamma^\mu_{\beta\gamma}\,.
\end{aligned} \tag{B.127}$$

In general relativity we use the Levi-Civita connection and we shall henceforth assume the connection $\Gamma^\alpha_{\beta\gamma}$ to be the Levi-Civita one unless stated otherwise.

B.6 Parallel transport

The covariant derivative enables us to compare tensors at different points on the manifold. In particular, we can define when a tensor does not change along a curve.

Def.: Let $\boldsymbol{V}$ be a vector field and $\mathcal{C}$ an integral curve of $\boldsymbol{V}$. A tensor $\boldsymbol{T}$ is parallel transported along $\mathcal{C}$ if $\nabla_{\boldsymbol{V}} \boldsymbol{T} = 0$ at every point of the curve.

<u>**Example:**</u> Recall Eq. (B.93) for an affinely parametrized geodesic which we now write as

$$\frac{d^2x^\alpha}{d\lambda^2} + \Gamma^\alpha_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = 0\,. \tag{B.128}$$

The tangent vector of that curve is

$$U^\alpha = \frac{dx^\alpha}{d\lambda}\,, \tag{B.129}$$

which becomes the four velocity (B.70) for the case of timelike geodesics parametrized with proper time. Equation (B.128) then becomes

$$\begin{aligned} 0 &= \frac{d}{d\lambda}U^\alpha + \Gamma^\alpha_{\mu\nu}U^\mu U^\nu = \frac{dx^\beta}{d\lambda}\partial_\beta U^\alpha + \Gamma^\alpha_{\mu\nu}U^\mu U^\nu \\ &= U^\beta\partial_\beta U^\alpha + \Gamma^\alpha_{\mu\nu}U^\mu u^\nu = U^\beta\nabla_\beta U^\alpha = (\nabla_{\boldsymbol{U}}\boldsymbol{U})^\alpha\,. \end{aligned} \tag{B.130}$$

So the tangent vector of an affinely parameterized geodesic is parallel propagated along itself.
If κ is another parameter along the geodesic with $d\lambda/d\kappa > 0$, both vectors

$$\boldsymbol{U} = \frac{d}{d\lambda}\,, \qquad \text{and} \qquad \boldsymbol{V} = \frac{d}{d\kappa} = \frac{d\lambda}{d\kappa}\boldsymbol{U}\,, \tag{B.131}$$

are tangent to the geodesic. Defining $h := d\lambda/d\kappa$, we have

$$\nabla_{\boldsymbol{V}}\boldsymbol{V} = \nabla_{h\boldsymbol{U}}(h\boldsymbol{U}) = h\nabla_{\boldsymbol{U}}(h\boldsymbol{U}) = h^2\underbrace{\nabla_{\boldsymbol{U}}\boldsymbol{U}}_{=0} + \boldsymbol{U}(h)\,h\boldsymbol{U} = \frac{dh}{d\lambda}\boldsymbol{V}\,, \tag{B.132}$$

and

$$\nabla_{\boldsymbol{V}}\boldsymbol{V} = \frac{dh}{d\lambda}\boldsymbol{V}\,, \tag{B.133}$$

describes the same geodesic. κ is also affine if $dh/d\lambda = 0 \quad \Leftrightarrow \quad h = \text{const} \quad \Leftrightarrow \quad \kappa = c_1\lambda + c_2$ in agreement what we found in Eq. (B.98).

Parallel transport defines a tensor uniquely along the curve of transport. Let, for instance, $\boldsymbol{T}$ be a tensor of rank $\binom{1}{1}$ and let the curve be described by $x^\alpha(\lambda)$ with tangent vector $\boldsymbol{V}$. The parallel transport of $\boldsymbol{T}$ along the curve is then defined by the equation

$$\begin{aligned} & 0 = (\nabla_{\boldsymbol{V}}\boldsymbol{T})^\mu{}_\nu = V^\sigma\nabla_\sigma T^\mu{}_\nu = V^\sigma\partial_\sigma T^\mu{}_\nu + \Gamma^\mu_{\rho\sigma}T^\rho{}_\nu V^\sigma - \Gamma^\rho_{\nu\sigma}T^\mu{}_\rho V^\sigma \\ \Rightarrow \quad & \frac{d}{d\lambda}T^\mu{}_\nu + \Gamma^\mu_{\rho\sigma}T^\rho{}_\nu V^\sigma - \Gamma^\rho_{\nu\sigma}T^\mu{}_\rho V^\sigma = 0\,. \end{aligned} \tag{B.134}$$

The theory of ordinary differential equations ensures that a unique solution exists if initial conditions are provided for $T^\mu{}_\nu$ at some point on the curve. In the literature, you will sometimes find the notation

$$\boxed{\frac{D}{D\lambda} := \frac{dx^\rho}{d\lambda}\nabla_\rho}\,, \tag{B.135}$$

so that parallel transport of our $\binom{1}{1}$ tensor $\boldsymbol{T}$ along a curve is defined by $DT^\mu{}_\nu/D\lambda = 0$ and likewise for other tensors.
Note that parallel transport along $\boldsymbol{V}$ preserves the length of a vector $\boldsymbol{W}$

$$\frac{d}{d\lambda}(W_\alpha W^\alpha) = V^\mu \nabla_\mu (W_\alpha W^\alpha) = 2W_\alpha \underbrace{V^\mu \nabla_\mu W^\alpha}_{=0}, \tag{B.136}$$

and a similar calculation shows that the angle between two spatial vectors also remains unchanged under parallel transport. An important consequence of Eq. (B.136) is that the timelike, spacelike or null character of the tangent vector along a geodesic is constant along the geodesic. Unlike a normal curve, a geodesic that is timelike (spacelike, null) at some point is timelike (spacelike, null) everywhere. We have already seen this for the specific case of the four velocity of a timelike geodesic: $u_\alpha u^\alpha = -1$. In the context of timelike curves, one can also define the **acceleration**.

Def.: Let u^α be the tangent vector to a timelike curve parametrized by proper time τ. The acceleration is

$$a^\mu := \frac{Du^\mu}{D\lambda} = u^\rho \nabla_\rho u^\mu\,. \tag{B.137}$$

The curve is a geodesic if $a^\mu = 0$. Geodesics are therefore the analogs of the paths of freely moving particles in Newtonian dynamics. Note that a non-affinely parametrized geodesic satisfies $a^\alpha = f\,u^\alpha$ where f is a function.

It is instructive to contrast parallel transport in general relativity with that in special relativity. In the Minkowski spacetime with Cartesian coordinates, $\Gamma^\rho_{\mu\nu} = 0$ and Eq. (B.134) becomes

$$\frac{d}{d\lambda}T^\mu{}_\nu = 0\,, \tag{B.138}$$

so that in Cartesian coordinates parallel transport leaves tensor components unchanged and this result is independent of the curve we choose between points p to q. This is a key difference between special and general relativity. As we shall see in Sec. B.8.4 below, parallel transport of a tensor from p to q is dependent on which curve we choose.

B.7 Normal coordinates

In Sec. B.2.2 we stated that at a point $p \in \mathcal{M}$ we can construct coordinates such that the metric is Minkowskian at that point. We will now show how to construct these coordinates.

Def.: Let $\mathcal{M}$ be a manifold with connection Γ and let $p \in \mathcal{M}$. The exponential map is defined as

$$e : \mathcal{T}_p(\mathcal{M}) \to \mathcal{M}\,, \qquad \boldsymbol{X}_p \mapsto q\,, \tag{B.139}$$

where q is the point a unit affine parameter distance along the geodesic through p with tangent vector $\boldsymbol{X}_p$.

We make the following remarks.

(1) In a local neighborhood of p, the map e can be shown to be one-to-one and onto.

(2) The vector $\boldsymbol{X}_p$ fixes the parametrization of the geodesic: A straightforward calculation shows that e maps the vector $\lambda\boldsymbol{X}_p$, $0 \le \lambda \le 1$ to the point an affine parameter distance λ along the geodesic of $\boldsymbol{X}_p$.

The exponential map enables us to construct a special class of coordinates.

Def.: Let $(\mathbf{e}_\mu)$ be a basis of $\mathcal{T}_p(\mathcal{M})$. Normal coordinates in a neighborhood of $p \in \mathcal{M}$ are defined as the coordinate chart that assigns to $q = e(\boldsymbol{X}_p) \in \mathcal{M}$ the coordinates of the vector X^μ.

Note that this definition does not completely specify the coordinates. We still have the freedom to choose a basis for $\mathcal{T}_p(\mathcal{M})$.
Next, we will investigate how normal coordinates can be used to control the metric components at p.

Lemma: In normal coordinates constructed around the point p, the connection at p satisfies $\Gamma^\mu_{(\nu\rho)} = 0$. If the connection is torsion free, we furthermore have $\Gamma^\mu_{\nu\rho} = 0$.

Proof: In item (2) of the above set of comments we saw that the exponential map (B.139) maps the vector $\lambda\boldsymbol{X}_p$ to the point an affine parameter distance λ along the geodesic through $\boldsymbol{X}_p$. In the neighborhood of p, the affinely parametrized geodesic is therefore given by

$$\mathcal{C} : [0,1] \to \mathcal{M}\,, \qquad \lambda \mapsto x^\mu(\lambda) = \lambda X^\mu_p\,. \tag{B.140}$$

The geodesic equation for the affine parameter λ becomes

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\lambda}\frac{dx^\rho}{d\lambda} = \Gamma^\mu_{\nu\rho}X^\nu_p X^\rho_p = 0 \ \text{ at } p \in \mathcal{M} \ \ \forall \boldsymbol{X}_p \in \mathcal{T}_p(\mathcal{M})$$

$$\Rightarrow \quad \Gamma^\mu_{(\nu\rho)} = 0\,. \tag{B.141}$$

If the connection is torsion free, we also have $\Gamma^\mu_{[\nu\rho]} = 0$ and, hence, $\Gamma^\mu_{\nu\rho} = 0$.

Having chosen coordinates that lead to $\Gamma^\mu_{\nu\rho} = 0$ at $p \in \mathcal{M}$, we will in general not find the connection to also vanish at other points $q \ne p$. It is an interesting exercise to check which piece of the above proof breaks down at $q \ne p$. We will comment on this question in the actual lectures.

Lemma: If we have a manifold with metric $\boldsymbol{g}$ and choose the Levi-Civita connection, then in normal coordinates

$$\partial_\rho g_{\nu\sigma} = 0\,. \tag{B.142}$$

Proof: The Levi-Civita connection is torsion free, so that by the previous lemma

$$\Gamma^\rho_{\mu\nu} = 0$$

$$\Rightarrow \quad 2g_{\sigma\rho}\Gamma^\rho_{\mu\nu} = \partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu} = 0\,. \tag{B.143}$$

Next, we symmetrize the left-hand side over σ and μ and add the result to obtain $2\partial_\nu g_{\sigma\mu} = 0$.

Note again that this result holds at p but that in general we cannot make $\partial_\nu g_{\sigma\mu}$ vanish at other points $q \neq p$. It now remains to select the normal coordinates such that the metric components acquire the Minkowskian values.

Lemma: Let $\mathcal{M}$ be a manifold with a metric $\boldsymbol{g}$ of signature 2 and Γ the Levi-Civita connection. Then we can choose normal coordinates such that at p

$$\partial_\rho g_{\mu\nu} = 0\,, \qquad g_{\mu\nu} = \eta_{\mu\nu} = \mathrm{diag}(-1,\,+1,\,+1,\,+1)\,. \tag{B.144}$$

Proof: We already proved the first part. For the second part, let x^α be normal coordinates. By Eq. (B.140), the point an affine parameter distance λ along a geodesic through p with tangent $\boldsymbol{X}_p$ then has coordinates $\lambda\, X^\mu_p$. Now choose an orthonormal basis $(\mathbf{e}_\mu)$ of the tangent space $\mathcal{T}_p(\mathcal{M})$ (this can always be achieved, for example by Gram-Schmidt orthonormalisation) and consider the special case where $\boldsymbol{X}_p = \mathbf{e}_0$. The point an affine parameter distance λ along the geodesic through p with tangent $\mathbf{e}_0$ then has coordinates

$$\lambda X^\mu_p = \lambda\,(\mathbf{e}_0)^\mu = (\lambda,\,0,\,0,\,0)\,.$$

So the geodesic curve has coordinates $x^\mu(\lambda) = (\lambda,\,0,\,0,\,0)$. But in any coordinate system, the tangent vector to the curve $(\lambda,\,0,\,0,\,0)$ is $\partial_0 = \partial/\partial x^0$, so that $\partial_0 = \mathbf{e}_0$. We likewise show $\partial_\mu = \mathbf{e}_\mu$. It follows that the $\{\partial_\mu\}$ form an orthonormal basis and, hence,

$$g_{\mu\nu} = \boldsymbol{g}(\partial_\mu, \partial_\nu) = \eta_{\mu\nu}\,. \tag{B.145}$$

In summary, we can choose coordinates such that at $p \in \mathcal{M}$ the metric is Minkowskian and the connection coefficients vanish.

Def.: We call a coordinate frame with these properties a local inertial frame.

In a local inertial frame, we therefore recover the laws of special relativity. According to the equivalence principle, this frame represents freely falling observers.

B.8 The Riemann tensor

The fact that with all our efforts, we can at best recover the laws of special relativity locally at a point in the manifold is a consequence of spacetime curvature. The Riemann tensor is the mathematical object that encapsulates the curvature of our manifold. It is time to study this tensor in detail now.

B.8.1 The commutator

In Eq. (B.115) we saw that the partial derivative of a vector field, $\partial_\alpha W^\beta$ does not transform as a tensor. In consequence, $V^\alpha \partial_\alpha W^\beta$ is not a tensor either for any vector field $\boldsymbol{V}$. We can, however, define a tensor based on this partial derivative as follows.

Def.: The commutator $[\boldsymbol{V}, \boldsymbol{W}]$ of two vector fields $\boldsymbol{V}$, $\boldsymbol{W}$ is defined by

$$[\boldsymbol{V}, \boldsymbol{W}]^\alpha := V^\mu \partial_\mu W^\alpha - W^\mu \partial_\mu V^\alpha \,, \tag{B.146}$$

and is a vector field.

Proof: Using Eqs. (B.14), (B.115) for the transformation of a vector and its partial derivative under a change of coordinates $(x^\alpha) \to (\tilde{x}^\mu)$, we find for the commutator in the new coordinate system

$$\begin{aligned}
\tilde{V}^\nu \frac{\partial \tilde{W}^\mu}{\partial \tilde{x}^\nu} - \tilde{W}^\nu \frac{\partial \tilde{V}^\mu}{\partial \tilde{x}^\nu} &= \frac{\partial \tilde{x}^\nu}{\partial x^\beta} V^\beta \left[\frac{\partial \tilde{x}^\mu}{\partial x^\gamma} \frac{\partial x^\delta}{\partial \tilde{x}^\nu} \partial_\delta W^\gamma + \frac{\partial x^\alpha}{\partial \tilde{x}^\nu} W^\gamma \partial_\alpha \left(\frac{\partial \tilde{x}^\mu}{\partial x^\gamma} \right) \right] \\
&\quad - \frac{\partial \tilde{x}^\nu}{\partial x^\beta} W^\beta \left[\frac{\partial \tilde{x}^\mu}{\partial x^\gamma} \frac{\partial x^\delta}{\partial \tilde{x}^\nu} \partial_\delta V^\gamma + \frac{\partial x^\alpha}{\partial \tilde{x}^\nu} V^\gamma \partial_\alpha \left(\frac{\partial \tilde{x}^\mu}{\partial x^\gamma} \right) \right] \\
&= \frac{\partial \tilde{x}^\mu}{\partial x^\gamma} V^\beta \partial_\beta W^\gamma + V^\beta W^\gamma \frac{\partial^2 \tilde{x}^\mu}{\partial x^\beta \partial x^\gamma} \\
&\quad - \frac{\partial \tilde{x}^\mu}{\partial x^\gamma} W^\beta \partial_\beta V^\gamma - W^\beta V^\gamma \frac{\partial^2 \tilde{x}^\mu}{\partial x^\beta \partial x^\gamma} \\
&= \frac{\partial \tilde{x}^\mu}{\partial x^\gamma} \left(V^\beta \frac{\partial W^\gamma}{\partial x^\beta} - W^\beta \frac{\partial V^\gamma}{\partial x^\beta} \right) .
\end{aligned} \tag{B.147}$$

Note that we departed here from our usual path of defining tensors as linear maps and then deducing it's transformation properties. Instead, we define the commutator through its components and show that this definition satisfies the transformation rule of a vector under coordinate transformations.

One straightforwardly shows that with vector fields $\boldsymbol{U}$, $\boldsymbol{V}$, $\boldsymbol{W}$ and a function f the commutator satisfies

$$[\boldsymbol{V},\boldsymbol{W}] = -[\boldsymbol{W},\boldsymbol{V}]\,, \tag{B.148}$$

$$[\boldsymbol{V},\boldsymbol{W}+\boldsymbol{U}] = [\boldsymbol{V},\boldsymbol{W}]+[\boldsymbol{V},\boldsymbol{U}]\,, \tag{B.149}$$

$$[\boldsymbol{V},f\boldsymbol{W}] = f[\boldsymbol{V},\boldsymbol{W}]+\boldsymbol{V}(f)\,\boldsymbol{W}\,, \tag{B.150}$$

$$\big[\boldsymbol{U},[\boldsymbol{V},\boldsymbol{W}]\big]+\big[\boldsymbol{V},[\boldsymbol{W},\boldsymbol{U}]\big]+\big[\boldsymbol{W},[\boldsymbol{U},\boldsymbol{V}]\big] = 0 \qquad \text{“Jacobi Identity”}\,. \tag{B.151}$$

For a coordinate basis $\{\partial_\mu\}$, we obtain

$$\left[\frac{\partial}{\partial x^\mu},\frac{\partial}{\partial x^\nu}\right] = 0\,, \tag{B.152}$$

because the components of these vectors are constant by construction. We state without proof the following theorem about the inverse implication.

Theorem: If $\boldsymbol{V}_0,\,\ldots,\,\boldsymbol{V}_{m-1}\,,\quad m\le\dim(\mathcal{M})$ are vector fields that are linearly independent at every $p\in\mathcal{M}$ and whose commutators all vanish, then we can construct coordinates x^μ in a neighborhood of any $p\in\mathcal{M}$ such that

$$\boldsymbol{V}_i = \frac{\partial}{\partial x^i}\,,\quad i=0,\,\ldots,\,m-1\,. \tag{B.153}$$

B.8.2 Second derivatives and the Riemann tensor

From calculus in n dimensions we know that partial derivatives of functions commute, $\partial_\nu\partial_\mu f = \partial_\mu\partial_\nu f$. Let us see whether this holds for covariant derivatives of functions. We have

$$\nabla_\nu\nabla_\mu f = \partial_\nu(\nabla_\mu f) - \Gamma^\rho_{\mu\nu}\nabla_\rho f = \partial_\nu\partial_\mu f - \Gamma^\rho_{\mu\nu}\partial_\rho f = \nabla_\mu\nabla_\nu f - 2\Gamma^\rho_{[\mu\nu]}\partial_\rho f\,. \tag{B.154}$$

With a torsion free connection, such as Levi-Civita, we therefore find that second covariant derivatives of functions also commute. Note that in Eq. (B.154) we first took the outer covariant derivative. This avoids ending up with covariant derivatives of connection coefficients which are not well defined quantities.
Next, we consider second covariant derivatives of vectors. We find with the Levi-Civita connection

$$\begin{aligned}
&\nabla_\alpha\nabla_\beta V^\gamma = \partial_\alpha(\nabla_\beta V^\gamma) - \Gamma^\rho_{\beta\alpha}\nabla_\rho V^\gamma + \Gamma^\gamma_{\rho\alpha}\nabla_\beta V^\rho \\
\Rightarrow\quad &\nabla_\alpha\nabla_\beta V^\gamma = \partial_\alpha(\partial_\beta V^\gamma + \Gamma^\gamma_{\rho\beta}V^\rho) - \Gamma^\rho_{\beta\alpha}(\partial_\rho V^\gamma + \Gamma^\gamma_{\sigma\rho}V^\sigma) + \Gamma^\gamma_{\rho\alpha}(\partial_\beta V^\rho + \Gamma^\rho_{\sigma\beta}V^\sigma) \\
\Rightarrow\quad &\nabla_\alpha\nabla_\beta V^\gamma - \nabla_\beta\nabla_\alpha V^\gamma = \partial_\alpha\Gamma^\gamma_{\rho\beta}\,V^\rho + \Gamma^\gamma_{\rho\beta}\partial_\alpha V^\rho + \Gamma^\gamma_{\rho\alpha}\partial_\beta V^\rho + \Gamma^\gamma_{\rho\alpha}\Gamma^\rho_{\sigma\beta}V^\sigma \\
&\qquad\qquad\qquad - (\,\alpha\,\leftrightarrow\,\beta\,)\,,
\end{aligned} \tag{B.155}$$

where $(\alpha \leftrightarrow \beta)$ denotes the right-hand side of the preceding lines with α and β swapped.

Def.: The Riemann tensor is

$$\boxed{R^{\gamma}{}_{\rho\alpha\beta} := \partial_\alpha \Gamma^{\gamma}_{\rho\beta} - \partial_\beta \Gamma^{\gamma}_{\rho\alpha} + \Gamma^{\mu}_{\rho\beta}\Gamma^{\gamma}_{\mu\alpha} - \Gamma^{\mu}_{\rho\alpha}\Gamma^{\gamma}_{\mu\beta}}\,. \tag{B.156}$$

With Eq. (B.156), the second covariant derivative (B.155) becomes the so-called "Ricci Identity"

$$\nabla_\alpha \nabla_\beta V^\gamma - \nabla_\beta \nabla_\alpha V^\gamma = R^{\gamma}{}_{\rho\alpha\beta} V^\rho\,. \tag{B.157}$$

We conclude that covariant derivatives of vectors fail to commute and that the Riemann tensor (by definition) measures this failure.

An equivalent definition of the Riemann tensor is given as follows.

Def.: Let $\boldsymbol{U}$, $\boldsymbol{V}$, $\boldsymbol{W}$ be three vector fields. The Riemann tensor is the rank $\binom{1}{3}$ tensor $\boldsymbol{R}$ with

$$\big(\boldsymbol{R}(\boldsymbol{U},\boldsymbol{V})\big)(\boldsymbol{W}) = \nabla_{\boldsymbol{U}}\nabla_{\boldsymbol{V}}\boldsymbol{W} - \nabla_{\boldsymbol{V}}\nabla_{\boldsymbol{U}}\boldsymbol{W} - \nabla_{[\boldsymbol{U},\boldsymbol{V}]}\boldsymbol{W}\,. \tag{B.158}$$

Proof: Let f be a function. A straightforward calculation shows that

$$\boldsymbol{R}(f\boldsymbol{U},\boldsymbol{V})\boldsymbol{W} = f\boldsymbol{R}(\boldsymbol{U},\boldsymbol{V})\boldsymbol{W}\,,$$

$$\boldsymbol{R}(\boldsymbol{U},f\boldsymbol{V})\boldsymbol{W} = f\boldsymbol{R}(\boldsymbol{U},\boldsymbol{V})\boldsymbol{W}\,,$$

$$\boldsymbol{R}(\boldsymbol{U},\boldsymbol{V})f\boldsymbol{W} = f\boldsymbol{R}(\boldsymbol{U},\boldsymbol{V})\boldsymbol{W}\,. \tag{B.159}$$

So $\boldsymbol{R}$ is linear in its three vector arguments. Furthermore, the right-hand side of Eq. (B.158) is manifestly of vector type, so that contraction with a one-form is by construction a linear operation. Therefore, $\boldsymbol{R}$ is a tensor. In order to calculate the components, we fill the three vector slots of $\boldsymbol{R}$ with the basis vectors, i.e. substitute in (B.158) $\boldsymbol{U} = \mathbf{e}_\alpha$, $\boldsymbol{V} = \mathbf{e}_\beta$ and $\boldsymbol{W} = \mathbf{e}_\rho$. We use a coordinate basis $\mathbf{e}_\alpha = \partial_\alpha$ so that $[\mathbf{e}_\alpha, \mathbf{e}_\beta] = 0$ by Eq. (B.152),

$$\boldsymbol{R}(\mathbf{e}_\alpha,\mathbf{e}_\beta)\mathbf{e}_\rho = \nabla_\alpha\nabla_\beta \mathbf{e}_\rho - \nabla_\beta\nabla_\alpha \mathbf{e}_\rho\,. \tag{B.160}$$

Recalling Eq. (B.109), we find $\nabla_\alpha \mathbf{e}_\beta = \Gamma^{\mu}_{\beta\alpha}\mathbf{e}_\mu$ and therefore

$$\begin{aligned}
\boldsymbol{R}(\mathbf{e}_\alpha,\mathbf{e}_\beta)\mathbf{e}_\rho &= \nabla_\alpha(\Gamma^{\mu}_{\rho\beta}\mathbf{e}_\mu) - \nabla_\beta(\Gamma^{\mu}_{\rho\alpha}\mathbf{e}_\mu) \\
&= (\partial_\alpha\Gamma^{\mu}_{\rho\beta})\mathbf{e}_\mu + \Gamma^{\mu}_{\rho\beta}\nabla_\alpha\mathbf{e}_\mu - (\partial_\beta\Gamma^{\mu}_{\rho\alpha})\mathbf{e}_\mu - \Gamma^{\mu}_{\rho\alpha}\nabla_\beta\mathbf{e}_\mu \\
&= (\partial_\alpha\Gamma^{\nu}_{\rho\beta})\mathbf{e}_\nu + \Gamma^{\mu}_{\rho\beta}\Gamma^{\nu}_{\mu\alpha}\mathbf{e}_\nu - (\partial_\beta\Gamma^{\nu}_{\rho\alpha})\mathbf{e}_\nu - \Gamma^{\mu}_{\rho\alpha}\Gamma^{\nu}_{\mu\beta}\mathbf{e}_\nu \\
&= \big[\underbrace{\partial_\alpha\Gamma^{\nu}_{\rho\beta} - \partial_\beta\Gamma^{\nu}_{\rho\alpha} + \Gamma^{\mu}_{\rho\beta}\Gamma^{\nu}_{\mu\alpha} - \Gamma^{\mu}_{\rho\alpha}\Gamma^{\nu}_{\mu\beta}}_{=R^{\nu}{}_{\rho\alpha\beta}\text{ with the definition (B.156)}}\big]\mathbf{e}_\nu\,.
\end{aligned} \tag{B.161}$$

Equations (B.156) and (B.158) indeed define the same tensor. We have covered both definitions because from case to case, either one or the other may be more convenient in practical calculations.

B.8.3 Symmetries of the Riemann tensor

The Riemann tensor obeys a number of symmetries which we discuss and derive in this section.

(1) By definition, the Riemann tensor is antisymmetric in the last two indices

$$\boxed{R^{\alpha}{}_{\beta\gamma\delta} = -R^{\alpha}{}_{\beta\delta\gamma}} \qquad \Leftrightarrow \qquad \boxed{R^{\alpha}{}_{\beta(\gamma\delta)} = 0}\,. \tag{B.162}$$

(2) Let Γ be a torsion free connection, $p \in \mathcal{M}$ and x^{α} be normal coordinates at p. At the point p we then have

$$\Gamma^{\mu}_{\nu\rho} = 0 \qquad \Rightarrow \qquad R^{\mu}{}_{\nu\rho\sigma} = \partial_{\rho}\Gamma^{\mu}_{\nu\sigma} - \partial_{\sigma}\Gamma^{\mu}_{\nu\rho}\,. \tag{B.163}$$

Antisymmetrizing this equation over ρ, ν, σ yields

$$\underline{\partial_{\rho}\Gamma^{\mu}_{\nu\sigma}} - \underline{\underline{\partial_{\sigma}\Gamma^{\mu}_{\nu\rho}}} + \partial_{\nu}\Gamma^{\mu}_{\sigma\rho} - \underline{\partial_{\rho}\Gamma^{\mu}_{\sigma\nu}} + \underline{\underline{\partial_{\sigma}\Gamma^{\mu}_{\rho\nu}}} - \partial_{\nu}\Gamma^{\mu}_{\rho\sigma}$$

$$-\underline{\underline{\partial_{\sigma}\Gamma^{\mu}_{\nu\rho}}} + \underline{\partial_{\rho}\Gamma^{\mu}_{\nu\sigma}} - \partial_{\nu}\Gamma^{\mu}_{\rho\sigma} + \underline{\underline{\partial_{\sigma}\Gamma^{\mu}_{\rho\nu}}} - \underline{\partial_{\rho}\Gamma^{\mu}_{\sigma\nu}} + \partial_{\nu}\Gamma^{\mu}_{\sigma\rho} = 0$$

$$\Rightarrow \quad \boxed{R^{\mu}{}_{[\nu\rho\sigma]} = 0} \tag{B.164}$$

This is a tensorial equation and is therefore valid in any coordinate system. Furthermore, the point p was arbitrary, so that Eq. (B.164) holds at all points.

(3) Again we use normal coordinates at $p \in \mathcal{M}$ and a torsion free connection, so that at p we have $\Gamma^{\mu}_{\nu\rho} = 0$. Next, we take the partial derivative of the Riemann tensor as given in Eq. (B.156). Because the connection vanishes, the only terms surviving in this equation can be symbolically written as

$$\text{“}\partial R = \partial\partial\Gamma - \Gamma\partial\Gamma = \partial\partial\Gamma\text{''}\,. \tag{B.165}$$

Furthermore the vanishing connection at p implies that covariant and partial derivatives are the same in that point, so that

$$\nabla_\lambda R^{\mu}{}_{\nu\rho\sigma} = \partial_\lambda R^{\mu}{}_{\nu\rho\sigma} = \partial_\lambda\partial_\rho\Gamma^{\mu}_{\nu\sigma} - \partial_\lambda\partial_\sigma\Gamma^{\mu}_{\nu\rho}\,. \tag{B.166}$$

After a small calculation, antisymmetrization over ρ, σ, λ leads to the *Bianchi identities*

$$\boxed{\nabla_{[\lambda|}R^{\mu}{}_{\nu|\rho\sigma]} = R^{\mu}{}_{\nu[\rho\sigma;\lambda]} = 0}\,. \tag{B.167}$$

Again, this is a tensorial equation and the point p was arbitrary, so that the equality holds in general. Note the striking similarity with the Newtonian integrability condition (A.64).

(4) For this symmetry, we assume that the manifold is equipped with a metric and that the connection is the Levi-Civita one. At an arbitrary point $p \in \mathcal{M}$, using normal coordinates, we then have $\partial_\mu g_{\nu\rho} = 0$. This implies

$$0 = \partial_\mu\delta^{\nu}{}_{\rho} = \partial_\mu(g^{\nu\sigma}g_{\sigma\rho}) = g_{\sigma\rho}\partial_\mu g^{\nu\sigma} \qquad \Big|\ \cdot g^{\rho\lambda}$$

$$\Rightarrow\quad \partial_\mu g^{\nu\lambda} = 0$$

$$\Rightarrow\quad \partial_\rho\Gamma^{\lambda}_{\nu\sigma} = \frac{1}{2}g^{\lambda\mu}(\partial_\rho\partial_\nu g_{\sigma\mu} + \partial_\rho\partial_\sigma g_{\mu\nu} - \partial_\rho\partial_\mu g_{\nu\sigma})$$

$$\Rightarrow\quad R_{\mu\nu\rho\sigma} = \frac{1}{2}\big(\partial_\rho\partial_\nu g_{\sigma\mu} + \partial_\sigma\partial_\mu g_{\nu\rho} - \partial_\sigma\partial_\nu g_{\rho\mu} - \partial_\rho\partial_\mu g_{\nu\sigma}\big) + \underbrace{\text{“}\Gamma\Gamma - \Gamma\Gamma\text{''}}_{=0}$$

$$\Rightarrow\quad \boxed{R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu}}\,, \tag{B.168}$$

because $g_{\alpha\beta}$ is symmetric and $\partial_\alpha\partial_\beta$ commute. We can therefore swap the first with the second pair of indices. Together with Eq. (B.162), we directly obtain

$$\boxed{R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\gamma\delta}} \qquad \Leftrightarrow \qquad \boxed{R_{(\alpha\beta)\gamma\delta} = 0}\,. \tag{B.169}$$

Note that the first of our four symmetries always holds, the second and third hold if we have a torsion free connection, and the fourth holds if we have a metric and use the Levi-Civita

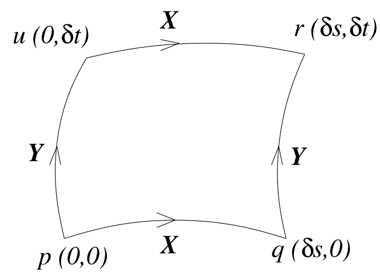


Figure 11: Integral curves and points along a closed loop along which a vector $\boldsymbol{Z}$ is parallel transported.

connection. In particular, all symmetries hold in general relativity.
The Riemann tensor describes the curvature of a manifold. We will next demonstrate the two main properties of a manifold with non-zero curvature and how these are mathematically related to the Riemann tensor. The first effect is the path dependency of parallel transporting a vector from one point to another and the second effect is geodesic deviation.

B.8.4 Parallel transport and curvature

Let $\mathcal{M}$ be a manifold with a torsion free connection Γ and $\boldsymbol{X}$, $\boldsymbol{Y}$ be two vector fields with the following properties: (i) they are linearly independent and (ii) their commutator vanishes, $[\boldsymbol{X}, \boldsymbol{Y}] = 0$. By the theorem of Eq. (B.153), we can then choose coordinates $x^\alpha = (s,\, t,\, \ldots)$ such that

$$\boldsymbol{X} = \frac{\partial}{\partial s}\,, \qquad \boldsymbol{Y} = \frac{\partial}{\partial t}\,. \tag{B.170}$$

Let us now consider the points p, q, r and s along integral curves of $\boldsymbol{X}$ and $\boldsymbol{Y}$ with coordinates $p = (0,\, \ldots,\, 0)\,,\;\; q = (\delta s,\, 0,\, \ldots,\,)\,,\;\; r = (\delta s,\, \delta t,\, 0,\, \ldots,\, 0)\,,\;\; u = (0,\, \delta t,\, 0,\, \ldots,\, 0)$ as illustrated in Fig. 11. Now we take a vector $\boldsymbol{Z}_p \in \mathcal{T}_p(\mathcal{M})$ at point p and parallel transport it along the closed loop $pqrup$ which gives us $\boldsymbol{Z}'_p \in \mathcal{T}_p(\mathcal{M})$. The difference between $\boldsymbol{Z}'_p$ and $\boldsymbol{Z}_p$ is related to the Riemann tensor by

$$\lim_{\delta s, \delta t \to 0} \frac{(\boldsymbol{Z}'_p - \boldsymbol{Z}_p)^\alpha}{\delta s\, \delta t} = \left(R^\alpha{}_{\beta\mu\nu} Z^\beta Y^\mu X^\nu\right)\big|_p\,. \tag{B.171}$$

<u>**Proof:**</u>

Let $\boldsymbol{Z}_p \in \mathcal{T}_p(\mathcal{M})$ and (x^μ) be normal coords. at p. Because of Eq. (B.170) the integral curves of $\boldsymbol{X}$ and $\boldsymbol{Y}$ are given by $(s,\, 0,\, \ldots,\, 0)$ and $(0,\, t,\, 0,\, \ldots,\, 0)$, respectively. We assume that δs and δt are small and related by $\delta t = a\delta s$ for $a = \text{const}$. We divide the closed path from p back to p into four parts.

(1) $p \to q$: We transport $\boldsymbol{Z}_p$ along the curve with tangent $\boldsymbol{X}$ and parameter s i.e. we have $\nabla_{\boldsymbol{X}} \boldsymbol{Z} = 0$, so that

$$X^\sigma \nabla_\sigma Z^\mu = X^\sigma \frac{\partial}{\partial x^\sigma} Z^\mu + \Gamma^\mu_{\rho\sigma} Z^\rho X^\sigma = \frac{dZ^\mu}{ds} + \Gamma^\mu_{\rho\sigma} Z^\rho X^\sigma = 0$$

$$\Rightarrow \quad \frac{dZ^\mu}{ds} = -\Gamma^\mu_{\rho\sigma} Z^\rho X^\sigma$$

$$\Rightarrow \quad \frac{d^2 Z^\mu}{ds^2} = -X^\lambda \partial_\lambda (\Gamma^\mu_{\rho\sigma} Z^\rho X^\sigma) \qquad \Big| \; \boldsymbol{X} = X^\mu \frac{\partial}{\partial x^\mu} = \frac{d}{ds}\,. \tag{B.172}$$

Next we Taylor expand Z^μ around p and use that $\Gamma^\mu_{\rho\sigma} = 0$ at p in our normal coordinate system,

$$\begin{aligned} Z^\mu_q - Z^\mu_p &= \left(\frac{dZ^\mu}{ds}\right)_p \delta s + \frac{1}{2}\left(\frac{d^2 Z^\mu}{ds^2}\right)_p \delta s^2 + \mathcal{O}(\delta s^3) \\ &= \underline{-\frac{1}{2}\big(X^\lambda Z^\rho X^\sigma \partial_\lambda \Gamma^\mu_{\rho\sigma}\big)_p \delta s^2} + \mathcal{O}(\delta s^3) \end{aligned} \tag{B.173}$$

(2) $q \to r$: We use again Taylor expansion, but this time around the point q and need to bear in mind that the connection coefficients do not vanish at q. We obtain

$$\begin{aligned} Z^\mu_r - Z^\mu_q &= \left(\frac{dZ^\mu}{dt}\right)_q \delta t + \frac{1}{2}\left(\frac{d^2 Z^\mu}{dt^2}\right)_q \delta t^2 + \mathcal{O}(\delta t^3) \\ &= -\big(\Gamma^\mu_{\rho\sigma} Z^\rho Y^\sigma\big)_q \delta t - \frac{1}{2}\big[Y^\lambda \partial_\lambda (\Gamma^\mu_{\rho\sigma} Z^\rho Y^\sigma)\big]_q \delta t^2 + \mathcal{O}(\delta t^3)\,. \end{aligned} \tag{B.174}$$

Using

$$\begin{aligned} \big(\Gamma^\mu_{\rho\sigma} Z^\rho Y^\sigma\big)_q \delta t &= \Big[(\Gamma^\mu_{\rho\sigma} Z^\rho Y^\sigma)_p + \big(X^\lambda \partial_\lambda (\Gamma^\mu_{\rho\sigma} Z^\rho Y^\sigma)\big)_p \delta s + \mathcal{O}(\delta s^2)\Big]\,\delta t \\ &= \big[0 + (Z^\rho Y^\sigma X^\lambda \partial_\lambda \Gamma^\mu_{\rho\sigma})_p \delta s + \mathcal{O}(\delta s^2)\big]\delta t\,, \end{aligned} \tag{B.175}$$

we find

$$\begin{aligned} \Rightarrow Z^\mu_r - Z^\mu_q &= -\big[(Z^\rho Y^\sigma X^\lambda \partial_\lambda \Gamma^\mu_{\rho\sigma})_p \delta s + \mathcal{O}(\delta s^2)\big]\delta t \\ &\quad -\frac{1}{2}\big[\big(Y^\lambda \partial_\lambda (\Gamma^\mu_{\rho\sigma} Z^\rho Y^\sigma)\big)_p + \mathcal{O}(\delta s)\big]\delta t^2 + \mathcal{O}(\delta t^3) \qquad (\text{B.176}) \\ &= \underline{-(Z^\rho Y^\sigma X^\lambda \partial_\lambda \Gamma^\mu_{\rho\sigma})_p \delta s \delta t - \frac{1}{2}(Z^\rho Y^\sigma Y^\lambda \partial_\lambda \Gamma^\mu_{\rho\sigma})_p \delta t^2} + \mathcal{O}(\delta s^3) \end{aligned}$$

For the first part of our loop we thus find

$$\left(Z^\mu_r - Z^\mu_p\right)_{pqr} = -\frac{1}{2}(\partial_\lambda \Gamma^\mu_{\rho\sigma})\left[Z^\rho\left(X^\sigma X^\lambda \delta s^2 + Y^\sigma Y^\lambda \delta t^2 + 2Y^\sigma X^\lambda \delta s\, \delta t\right)\right]_p + \mathcal{O}(\delta s^3)\,. \quad \text{(B.177)}$$

(3), (4): The change of $\boldsymbol{Z}_p$ under parallel transport along the alternative route $p \to u \to r$, follows from Eq. (B.177) by simply interchanging $\boldsymbol{X} \leftrightarrow \boldsymbol{Y}$, $s \leftrightarrow t$,

$$\left(Z^\mu_r - Z^\mu_p\right)_{pur} = -\frac{1}{2}(\partial_\lambda \Gamma^\mu_{\rho\sigma})\left[Z^\rho\left(Y^\sigma Y^\lambda \delta t^2 + X^\sigma X^\lambda \delta s^2 + 2X^\sigma Y^\lambda \delta t\, \delta s\right)\right]_p + \mathcal{O}(\delta s^3)\,. \quad \text{(B.178)}$$

The change of $\boldsymbol{Z}$ along the inverse path from rup is simply minus the result (B.178), so that the change of $\boldsymbol{Z}_p$ under parallel transport along the closed loop $pqrup$ is

$$\begin{aligned} Z'^\mu_p - Z^\mu_p &= (Z^\mu_r - Z^\mu_p)_{pqr} - (Z^\mu_r - Z^\mu_p)_{pur} = -\left[(Y^\sigma X^\lambda - X^\sigma Y^\lambda)(\partial_\lambda \Gamma^\mu_{\rho\sigma})\right]_p Z^\rho\, \delta s\, \delta t + \mathcal{O}(\delta s^3) \\ &= X^\sigma Y^\lambda Z^\rho \underbrace{(\partial_\lambda \Gamma^\mu_{\rho\sigma} - \partial_\sigma \Gamma^\mu_{\rho\lambda})}_{\overset{*}{=} R^\mu{}_{\rho\lambda\sigma}} \delta t\, \delta s + \mathcal{O}(\delta s^3) \\ &= \left(R^\mu{}_{\rho\lambda\sigma} Z^\rho Y^\lambda X^\sigma\right)_p \delta t\, \delta s + \mathcal{O}(\delta s^3)\,, \end{aligned} \quad \text{(B.179)}$$

where the symbol $\overset{*}{=}$ denotes equality in normal coordinates at p where $(\Gamma^\alpha_{\beta\gamma})_p = 0$. Taking the limit δs, $\delta t \to 0$, we recover Eq. (B.171).

We conclude that curvature measures the change of vectors under parallel transport along a closed curve or, equivalently, the path dependence of parallel transport.

B.8.5 Geodesic deviation

In flat Euclidean geometry, geodesics are straight lines and they are either parallel or cross in exactly one point which means that their separation either remains constant or changes linearly as we move along the geodesics. In curved spacetimes, geodesics undergo relative acceleration. As an example, we illustrate in Fig. 12 two great circles which are geodesics on the surface of a two-sphere. If two observers starting on these curves at different points on the equator measure their relative separation as a function of the distance from the equator, they would measure this function to have a negative second derivative. In this section, we will quantify this effect on arbitrary manifolds.

Def.: Let $(\mathcal{M}, \Gamma)$ be a manifold with connection. A "1-parameter family of geodesics" is a map

$$\gamma : I \times I' \to \mathcal{M} \;\; \text{with} \;\; I,\, I' \subset \mathbb{R}, \;\; \text{openand} \quad \text{(B.180)}$$

(i) for fixed s, $\gamma(s,t)$ is a geodesic with affine parameter t,

(ii) locally, $(s,t) \mapsto \gamma(s,t)$ is smooth, one-to-one and has a smooth inverse.

The family of geodesics then forms a 2-dim. surface $\Sigma \subset \mathcal{M}$.

Figure 12: Relative geodesic acceleration illustrated for great circles on planet Earth (red curves). Two such curves starting at the equator initially point perpendicular to the equator but converge at the north pole. Two observers, one moving along each great circle would find the second derivative of their separation with respect to their distance to the equator to be negative.

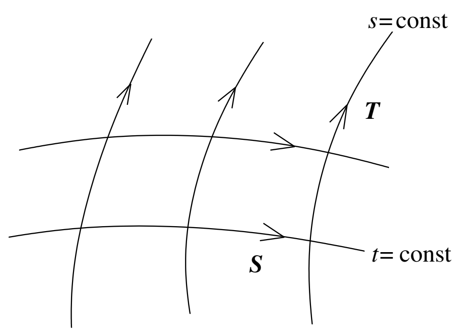


Figure 13: A one-parameter family of geodesics. Curves $s = \text{const}$ are geodesics and $T^\mu = dx^\mu/dt$ is their tangent vector. $\boldsymbol{S} = dx^\mu/ds$ is the vector pointing from one geodesic in the direction of neighboring geodesics.

Let $\boldsymbol{T}$ be the tangent vector to the geodesics $\gamma(s = \text{const}, t)$ and $\boldsymbol{S}$ the tangent vector to the curves $\gamma(s, t = \text{const})$. In coordinates (x^μ) we can write the vectors as

$$T^\mu = \frac{dx^\mu}{dt}\,, \qquad S^\mu = \frac{dx^\mu}{ds}\,. \tag{B.181}$$

We now consider two neighboring geodesics specified by parameters s_0 and $s_0 + \delta s$. These geodesics are given by $x^\mu(s_0, t)$ and $x^\mu(s_0+\delta s, t)$ and we Taylor expand their coordinate distance according to

$$x^\mu(s_0 + \delta s, t) = x^\mu(s_0, t) + \delta s\, S^\mu(s_0, t) + \mathcal{O}(\delta s^2)\,. \tag{B.182}$$

This equation motivates the following definitions.

Def.: $\delta s\, \boldsymbol{S}$ is the "geodesic deviation vector" that points from one geodesic with s_0 to a nearby one with parameter $s_0 + \delta s$.

The "relative velocity" of nearby geodesics is $\nabla_{\boldsymbol{T}}(\delta s\, \boldsymbol{S}) = \delta s\, \nabla_{\boldsymbol{T}} \boldsymbol{S}$

The "relative acceleration" of nearby geodesics is $\delta s\, \nabla_{\boldsymbol{T}} \nabla_{\boldsymbol{T}} \boldsymbol{S}$

Theorem: The geodesic deviation is given by

$$\boxed{\nabla_{\boldsymbol{T}} \nabla_{\boldsymbol{T}} \boldsymbol{S} = \boldsymbol{R}(\boldsymbol{T}, \boldsymbol{S}) \boldsymbol{T}} \tag{B.183}$$

$$\Leftrightarrow \quad T^\nu \nabla_\nu (T^\mu \nabla_\mu S^\alpha) = R^\alpha{}_{\lambda\mu\nu} T^\lambda T^\mu S^\nu \,. \tag{B.184}$$

Proof: We use coordinates (s, t) on the two-dimensional surface Σ spanned by the geodesics and extend the coordinates to $(s, t, \ldots)$ in a neighborhood of Σ. In this coordinate system, the vectors $\boldsymbol{S}$ and $\boldsymbol{T}$ have the particularly simple form

$$\boldsymbol{S} = \frac{\partial}{\partial s}\,, \qquad \boldsymbol{T} = \frac{\partial}{\partial t} \qquad \Rightarrow \quad [\boldsymbol{S}, \boldsymbol{T}] = 0\,, \tag{B.185}$$

because the commutator vanishes for basis vectors. For a torsion free connection, we further have for arbitrary vector fields $\boldsymbol{V}$, $\boldsymbol{W}$

$$\begin{aligned} V^\mu \nabla_\mu W^\alpha - W^\mu \nabla_\mu V^\alpha &= V^\mu \partial_\mu W^\alpha + V^\mu \Gamma^\alpha_{\rho\mu} W^\rho - W^\mu \partial_\mu V^\alpha - W^\mu \Gamma^\alpha_{\rho\mu} V^\rho \\ &= V^\mu \partial_\mu W^\alpha - W^\mu \partial_\mu V^\alpha = [\boldsymbol{V}, \boldsymbol{W}]^\alpha\,, \end{aligned} \tag{B.186}$$

by Eq. (B.146). For the vectors $\boldsymbol{T}$ and $\boldsymbol{S}$ this implies

$$\nabla_{\boldsymbol{T}} \boldsymbol{S} - \nabla_{\boldsymbol{S}} \boldsymbol{T} = [\boldsymbol{T}, \boldsymbol{S}] = 0\,, \tag{B.187}$$

and hence

$$\nabla_{\boldsymbol{T}} \nabla_{\boldsymbol{T}} \boldsymbol{S} = \nabla_{\boldsymbol{T}} \nabla_{\boldsymbol{S}} \boldsymbol{T} = \nabla_{\boldsymbol{S}} \underbrace{\nabla_{\boldsymbol{T}} \boldsymbol{T}}_{=0} + \boldsymbol{R}(\boldsymbol{T}, \boldsymbol{S}) \boldsymbol{T}\,, \tag{B.188}$$

where we have used the definition (B.158) of the Riemann tensor and the geodesic equation $\nabla_{\boldsymbol{T}} \boldsymbol{T} = 0$.

By Eq. (B.183), geodesic deviation is a manifestation of a non-vanishing Riemann tensor. The relative acceleration of geodesics is zero for all families of geodesics if and only if $R^\alpha{}_{\lambda\mu\nu} = 0$. Tidal forces are a physical consequence of geodesic deviation; recall Fig. 2 where two particles are accelerated towards each other when freely falling in the gravitational field of the Earth.

B.8.6 The Ricci tensor

We conclude our review of differential geometry with several tensors derived from the Riemann tensor which play a crucial role in Einstein's theory of general relativity.

Def.: The "Ricci tensor" is $\boxed{R_{\alpha\beta} := R^{\mu}{}_{\alpha\mu\beta}}$.

The "Ricci scalar" is $\boxed{R := g^{\mu\nu} R_{\mu\nu}}$.

The "Einstein tensor" is $\boxed{G_{\alpha\beta} := R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R}$.

A very important relation is obtained from the Bianchi identity (B.167),

$$
\begin{aligned}
& R_{\alpha\beta[\gamma\delta;\mu]} = 0 \qquad \Big| \cdot g^{\alpha\gamma} g^{\beta\delta} \\
\Rightarrow\ & \frac{1}{6} g^{\alpha\gamma} g^{\beta\delta} \big[R_{\alpha\beta\gamma\delta;\mu} + R_{\alpha\beta\delta\mu;\gamma} + R_{\alpha\beta\mu\gamma;\delta} - \underbrace{R_{\alpha\beta\delta\gamma;\mu}}_{=-R_{\alpha\beta\gamma\delta;\mu}} - R_{\alpha\beta\gamma\mu;\delta} - R_{\alpha\beta\mu\delta;\gamma} \big] = 0 \\
\Rightarrow\ & \frac{1}{3} g^{\alpha\gamma} g^{\beta\delta} \big[R_{\alpha\beta\gamma\delta;\mu} + R_{\alpha\beta\delta\mu;\gamma} + R_{\alpha\beta\mu\gamma;\delta} \big] = 0 \\
\Rightarrow\ & R_{;\mu} - g^{\alpha\gamma} R_{\alpha\mu;\gamma} - g^{\beta\delta} R_{\beta\mu;\delta} = 0 \\
\Rightarrow\ & \nabla_\mu R - 2 \nabla_\gamma R^{\gamma}{}_{\mu} = -2 \nabla^{\gamma} \left(R_{\gamma\mu} - \frac{1}{2} g_{\gamma\mu} R \right) \\
\Rightarrow\ & \boxed{\nabla^{\mu} G_{\mu\alpha} = 0}\,.
\end{aligned}
\tag{B.189}
$$

This relation is called the "contracted Bianchi identity" and bears a striking similarity to the Newtonian integrability condition (A.64).

C Physical laws in curved spacetimes

In the previous chapter we have constructed the mathematical framework for the formulation of Einstein's general relativity. We have seen that a connection or a metric add structure to a manifold and provide a measure for the manifold's curvature in the form of the Riemann tensor. At this point, however, we have no guidelines how to determine the metric corresponding to a specified physical system. Establishing the rules for this determination is the topic of this chapter. For this purpose, we will first motivate a recipe for converting physical laws of special relativity to the general case including gravity. We then explore how matter and energy are modelled in the form of the energy-momentum tensor which acts as the source of spacetime curvature and, in turn, obeys rules of motion dictated by the spacetime geometry. As suggested by this qualitative description, this interaction between matter and geometry is manifestly non-linear and it is determined by the Einstein field equations. In contrast to the theorems and equations we derived in Sec. B, most of the laws presented in this chapter cannot be derived from first principles. Instead they form a conjectured model describing physical phenomena involving gravitational interaction. Their correctness can only be tested by comparison with experiment and observation. In simple words, the previous chapter was predominantly of mathematical character; now we are entering the realm of physics. We assume from now on that we have a metric, so that the position, upstairs or downstairs, of tensor indices can be adjusted with the metric as convenient.

C.1 The covariance principle

Let us briefly recall some of the key observations we have made in our discussion so far.

- ***Equivalence principle:*** The physical laws governing non-gravitational experiments are the same in a (sufficiently small) freely falling frame as in an inertial frame in special relativity.
- ***Normal coordinates:*** There exist coordinates such that locally the spacetime metric is equal to the Minkowski metric, $g_{\alpha\beta} = \eta_{\alpha\beta}$, and the (Levi-Civita) connection vanishes, $\Gamma^{\alpha}_{\mu\nu} = 0$.
- The laws of special relativity are invariant under Lorentz transformations that relate different inertial frames.

These observations motivate the **covariance principle**.

Proposal: In general relativity, the laws of physics are stated in terms of tensorial equations and, thus, are invariant under coordinate transformations. The laws are obtained from those in special relativity by making the following substitutions,

(1) The Minkowski metric is replaced by the spacetime metric: $\eta_{\mu\nu} \quad \to \quad g_{\mu\nu}\,.$

(2) Partial derivatives are replaced by covariant derivatives: $\partial \quad \to \quad \nabla\,.$

Example: The Maxwell equations are conveniently formulated in terms of the antisymmetric Maxwell tensor $F_{\mu\nu} = F_{[\mu\nu]}$ related to the components E_i, B_i of the electric and magnetic field by

$$F_{0i} = -E_i\,, \qquad F_{ij} = \epsilon_{ijk} B_k\,, \tag{C.1}$$

where $i,\,j,\,\ldots = 1,\,2,\,3$ and ϵ_{ijk} is the completely antisymmetric symbol. The vacuum Maxwell equations in special relativity are

$$\eta^{\mu\nu}\partial_\mu F_{\nu\rho} = 0\,, \qquad \partial_{[\mu}F_{\nu\rho]} = 0\,. \tag{C.2}$$

The covariance principle predicts that the Maxwell equations in curved spacetimes are given by

$$g^{\mu\nu}\nabla_\mu F_{\nu\rho} = 0\,, \qquad \nabla_{[\mu}F_{\nu\rho]} = 0\,. \tag{C.3}$$

Note, however, that this covariance recipe is not unique. The Riemann and Ricci tensors vanish in special relativity, so that we could add terms involving them to the general relativistic equations without changing the corresponding special relativistic limit.

C.2 The energy momentum tensor

Postulate: In general relativity the mass, energy, momentum and strain of continuous matter distributions is described in the form of the **energy momentum tensor** (also sometimes called **stress-energy tensor**) $T_{\alpha\beta}$. This tensor is symmetric and conserved,

$$T_{\alpha\beta} = T_{\beta\alpha}\,, \qquad \nabla^\mu T_{\mu\nu} = 0\,. \tag{C.4}$$

We define the energy momentum in general terms as follows. Let x^α be a coordinate system. Then

$$T^{\alpha\beta} := \text{flux of } \alpha \text{ momentum across a surface of constant } x^\beta\,. \tag{C.5}$$

Recall that the tensor components are defined by filling the tensor slots with the basis one-forms, $T^{\alpha\beta} = \boldsymbol{T}(\mathbf{d}x^\alpha \mathbf{d}x^\beta)$. The components can be interpreted in a more intuitive manner by assuming that $x^0 = t$ is a timelike coordinates and $x^i\,,\; i = 1,\,2,\,3$ are spatial coordinates, so that

$$\begin{aligned}
T^{00} &= \text{flux of 0-momentum, i.e. energy, across surfaces } t = \text{const} \\
&= \text{energy density}\,, \\[1ex]
T^{0i} &= \text{energy flux across surface } x^i = \text{const}\,, \\[1ex]
T^{i0} &= \text{flux of momentum in the } x^i \text{ direction across surfaces } t = \text{const} \\
&= x^i\text{-momentum density}\,, \\[1ex]
T^{ij} &= \text{flux of } x^i \text{ momentum across surfaces } x^j = \text{const}\,,
\end{aligned} \tag{C.6}$$

where all fluxes are measured by an observer momentarily at rest in a local inertial frame comoving with the matter element at point p.

The construction of the energy momentum tensor often follows the covariance principle. We start with normal coordinates, find the energy momentum tensor from the local laws in special relativity and then generalize the tensor to arbitrary coordinates using the coordinate invariance of tensors. We will discuss below some of the most important types of matter used in applications of general relativity.

C.2.1 Particles

We begin this discussion with the special case of point particles which are not fully consistent forms of matter in general relativity because a finite amount of mass-energy contained inside an infinitesimally small volume will be a black hole. Nevertheless, point particles are a very useful concept and provide a good description of small objects that barely backreact on the spacetime geometry. They are exceptional in this discussion because they are not of continuous nature and are therefore conveniently described in terms of the four-momentum rather than the energy momentum tensor. Using an energy momentum tensor with δ distributions representing the particles would ultimately lead to the same relations that we develop here.

In special relativity (cf. Sec. A.3.5) we saw that the four momentum of a point particle of rest mass m in some given frame can be written as

$$p^\mu = m u^\mu = (E,\, p^i)\,, \tag{C.7}$$

where u^μ is the particle's four-velocity in this frame and E and p^i are the particle's energy and linear momentum in this frame. An observer at rest in this frame has four velocity is $w^\mu = (1,\, 0,\, 0,\, 0)$ and measures the particle's energy as

$$E = -\eta_{\mu\nu} w^\mu p^\nu\,. \tag{C.8}$$

The right-hand side is Lorentz invariant, but note that the E is the particle's energy *in the observer's rest frame*. The particle's rest mass can be expressed as

$$\eta_{\mu\nu} p^\mu p^\nu = -E^2 + \vec{p}^{\,2} = -m^2\,. \tag{C.9}$$

By the covariance principle, these equations only change by substituting the metric $g_{\mu\nu}$ for the Minkowski metric, so that

$$m^2 = -g_{\alpha\beta} p^\alpha p^\beta\,, \tag{C.10}$$
$$E = -g_{\alpha\beta} w^\alpha p^\beta\,. \tag{C.11}$$

A more important difference is that in general relativity Eq. (C.11) is only well defined if the vectors w^α and p^β are at the same point of the manifold; we have no recipe for multiplying vectors at different points and, unlike in special relativity, parallel transport is path dependent. An observer can therefore only measure the energy of the particle by being at the same location in the spacetime.

C.2.2 The electromagnetic field

In pre-relativistic formulation using Cartesian coordinates, the energy and momentum density and the stress tensor of the electromagnetic field are given by (note that we sum over repeated indices $i,\ j,\ \ldots = 1,\ 2,\ 3$)

$$\epsilon = \frac{1}{8\pi}(E_i E_i + B_i B_i)\,, \tag{C.12}$$

$$j_i = \frac{1}{4\pi}\epsilon_{ijk}E_j B_k\,. \tag{C.13}$$

$$S_{ij} = \frac{1}{4\pi}\left[\frac{1}{2}(E_k E_k + B_k B_k)\delta_{ij} - E_i E_j - B_i B_j\right]\,. \tag{C.14}$$

Here j^i is the so-called *Poynting vector* and also describes the energy flux. The conservation laws for energy and momentum density follow from the Maxwell equations and are

$$\frac{\partial \epsilon}{\partial t} + \partial_i j_i = 0\,, \qquad \frac{\partial j_i}{\partial t} + \partial_j S_{ij} = 0\,. \tag{C.15}$$

In special relativity, these equations are conveniently formulated in terms of the energy momentum tensor given in an inertial frame by (recall the example in Sec. C.1 for the Maxwell tensor $F_{\mu\nu}$)

$$T_{\mu\nu} = \frac{1}{4\pi}\left(F_{\mu\rho}F_\nu{}^\rho - \frac{1}{4}F^{\rho\sigma}F_{\rho\sigma}\eta_{\mu\nu}\right) = T_{\nu\mu}\,. \tag{C.16}$$

With the identification

$$T_{00} = \epsilon\,, \qquad T_{0i} = -j_i\,, \qquad T_{ij} = S_{ij}\,, \tag{C.17}$$

the conservation equations (C.15) can be shown to be equivalent to

$$\partial^\mu T_{\mu\nu} = \eta^{\mu\lambda}\partial_\lambda T_{\mu\nu} = 0\,. \tag{C.18}$$

The general relativistic analog follows straightforwardly from the covariance principle. The energy momentum tensor and its conservation are given by

$$T_{\alpha\beta} = \frac{1}{4\pi}\left(F_{\alpha\gamma}F_\beta{}^\gamma - \frac{1}{4}F^{\gamma\delta}F_{\gamma\delta}g_{\alpha\beta}\right) = 0\,, \tag{C.19}$$

$$\nabla^\alpha T_{\alpha\beta} = 0\,. \tag{C.20}$$

Let us now consider an observer $\mathcal{O}$ with four-velocity U^α and a local inertial frame at point $p \in \mathcal{M}$ where $\mathcal{O}$ is at rest. We can then construct an orthonormal basis starting with the timelike basis vector $\mathbf{e}_0 := \boldsymbol{U}$ and the choosing three spatial vectors $\mathbf{e}_i$ that are orthogonal to $\boldsymbol{U}$ and to each other and have unit length. By the equivalence principle, we can use the laws of special relativity in this frame and, using Eq. (C.17) obtain

$$\epsilon = T_{00} = T_{\alpha\beta}U^\alpha U^\beta\,, \tag{C.21}$$

which is the energy density at p measured by the observer $\mathcal{O}$. We likewise find

$$j_i = -T_{0i} = \text{momentum density}\,, \tag{C.22}$$

$$p^\alpha := -T^\alpha{}_\rho U^\rho = (\epsilon\,, j_i) \text{ in this basis} = \text{energy momentum flux}\,, \tag{C.23}$$

$$S_{ij} = T_{ij} = \text{stress tensor as measured by } \mathcal{O}\,, \tag{C.24}$$

in agreement with our general definition (C.6).

C.2.3 Dust

The simplest type of continuous matter is the so-called **dust**, defined as the continuum limit of a collection of non-interacting particles of rest mass m with a number density in the rest frame denoted by n. It is often convenient to define a **fluid element** or, in this case, a dust element as an infinitesimally small volume of particles with rest-frame density n.

The dust evolves purely under gravitational interaction, so that an observer comoving with a dust element is, by definition, freely falling. In a locally comoving inertial frame both, the particles and the observer are moving with four-velocity $u^\mu = (1,\, 0,\, 0,\, 0)$, the metric is locally Minkowskian and the energy density is $\rho = mn$. Since the particles are not moving in this frame, the momentum density is zero, $T^{i0} = 0$. Furthermore, the particles are not interacting, so no energy-momentum can be transferred in spatial directions, i.e. $T^{ij} = 0$, $T^{0j} = 0$. In this frame, the energy momentum tensor for dust is therefore given by

$$\boxed{T^{\alpha\beta} = \rho u^\alpha u^\beta = mnu^\alpha u^\beta}\,. \tag{C.25}$$

Here, m is merely a constant number and n, defined as the number density in the particles' rest frame is a scalar, so that the equation is tensorial and therefore valid in every coordinate system.

C.2.4 Perfect fluids

Probably the most important type of matter in applications of general relativity is the **perfect fluid** which is often used for the modeling of astrophysical systems such as neutron stars or accretion disks.

<u>**Def.:**</u> A perfect fluid is a continuous matter distribution that has no viscosity and no heat conduction in the locally comoving frame.

The form of the energy momentum tensor for this type of matter follows from looking more closely at the meaning of "no viscosity" and "no heat conduction".

No heat conduction: If the total energy m of a particle contains some internal energy, we require that this internal energy is not transferred to another particle. Energy can therefore only flow if the particles themselves flow.

No viscosity: Viscosity is defined as a force component exerted by one particle on another that is perpendicular to the line of sight between the two particles. In the absence of such a component, the force between two particles only changes the momentum in the direction along their line of sight. Without loss of generality we can rotate the coordinate system such that this direction coincides with the x^i direction for some fixed i. The only momentum that can flow in this direction is then the p^i component. By our general definition (C.6) of the components of the energy momentum tensor, this implies that $T^{ij} \neq 0$ only if $i = j$. Furthermore, our choice of the coordinate direction i was arbitrary, so that $T^{11} = T^{22} = T^{33}$. Let us call this quantity P, so that in special relativity in the locally comoving frame

$$T^{\alpha\beta} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix} \stackrel{!}{=} (\rho + P)u^\alpha u^\beta + P\eta^{\alpha\beta}\,. \tag{C.26}$$

Here, the last equality follows from the fact that in the comoving frame, $u^\alpha = (1, 0, 0, 0)$ in special relativity. The general relativistic expression follows from replacing $\eta^{\alpha\beta}$ with $g^{\alpha\beta}$ according to the covariance principle, so that

$$\boxed{T^{\alpha\beta} = (\rho + P)u^\alpha u^\beta + Pg^{\alpha\beta}}\,. \tag{C.27}$$

It is instructive to consider the implications of the energy conservation law $\nabla_\alpha T^{\alpha\beta} = 0$ for the perfect fluid (C.27). We find

$$\nabla_\alpha T^{\alpha\beta} = (\partial_\alpha\rho + \partial_\alpha P)u^\alpha u^\beta + (\rho + P)\left[u^\beta\nabla_\alpha u^\alpha + u^\alpha\nabla_\alpha u^\beta\right] + (\partial_\alpha P)g^{\alpha\beta} \stackrel{!}{=} 0\,. \tag{C.28}$$

First, we multiply this equation with u_β which gives

$$-u^\alpha(\partial_\alpha\rho + \partial_\alpha P) + (\rho + P)[-\nabla_\alpha u^\alpha + u^\alpha \underbrace{u_\beta\nabla_\alpha u^\beta}_{=0}] + u^\alpha\partial_\alpha P = 0$$

$$\Rightarrow \quad \boxed{u^\alpha\nabla_\alpha\rho + (\rho + P)\nabla_\alpha u^\alpha = 0}\,. \tag{C.29}$$

We can use this result to substitute for the first "$(\rho + P)$" term in Eq. (C.28), so that

$$(\partial_\alpha\rho + \partial_\alpha P)u^\alpha u^\beta + u^\beta(-u^\alpha\nabla_\alpha\rho) + (\rho + P)u^\alpha\nabla_\alpha u^\beta + \nabla^\beta P = 0$$

$$\Rightarrow \quad \boxed{(\rho + P)u^\alpha\nabla_\alpha u^\beta = -(g^{\alpha\beta} + u^\alpha u^\beta)\nabla_\alpha P}\,. \tag{C.30}$$

By taking the Newtonian limit, one can indeed show that Eqs. (C.29) and (C.30) become the law of mass conservation and the Euler equation of fluid dynamics. In order to model perfect fluid sources, one needs one additional ingredient that is not provided by general relativity: an equation of state relating pressure P and energy density ρ. This equation of state describes the form of matter and is a non-gravitational phenomenon. Practical applications often assume a power law dependency $P \propto \rho^\Gamma$ for some "polytropic" exponent Γ.

For the case of dust, i.e. $P = 0$, we see that Eq. (C.30) merely implies $u^\alpha \nabla_\alpha u^\beta = 0$, so that the dust particles move on geodesics. This is expected since they are non-interacting and, hence, freely falling.

It may have been noticed that all cases discussed here resulted in a symmetric energy momentum tensor, $T_{\alpha\beta} = T_{\beta\alpha}$. This is not trivially obvious but can be shown to hold in general for the energy momentum tensor. For example, energy flux in the x^i direction is by construction *energy density* $\times$ the *velocity* with which it flows in the x^i direction. This product, however, can be rewritten as *mass-energy* $\times$ *velocity / volume*, i.e. momentum density, and we have recovered $T^{0i} = T^{i0}$. The symmetry of T^{ij} can also be shown to hold generally. You may have come across the Newtonian limit of this symmetry: the stress tensor t_{ij} in Newtonian dynamics is symmetric. Readers interested in more details about the energy momentum tensor are referred to Chapter 4 of Schutz' book [24].

C.2.5 The Einstein equations

If you had the stamina to read up to this point, the reward is finally coming in the form of the postulates of general relativity. The very core of Einstein's theory is summarized as follows.

The Postulates of General Relativity

(1) Spacetime is a four-dimensional manifold with a metric of signature $-+++$ (or $+---$ if you use the opposite sign convention).

(2) Free test particles move on timelike or null geodesics.

(3) Energy, momentum and stress of continuous matter distributions are described by a symmetric tensor $T_{\alpha\beta}$, that is conserved according to $\nabla^\alpha T_{\alpha\beta} = 0$.

(4) Curvature is related to mass-energy by Einstein's equations

$$\boxed{G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} = \frac{8\pi G}{c^4} T_{\alpha\beta}}\,, \tag{C.31}$$

where we have restored the speed of light c and Newton's gravitational constant G.

Comments: (i) The proportionality factor $8\pi G/c^4$ is obtained from taking the Newtonian limit of the Einstein equations. We will return to this point in Sec. G.3 below.

(ii) Einstein's first guess at the field equations was $R_{\alpha\beta} = \kappa T_{\alpha\beta}$ with $\kappa = \text{const}$. The contracted Bianchi identities (B.189), however imply $\nabla^\alpha G_{\alpha\beta} = 0$, so that

$$\nabla^\alpha R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} \nabla^\alpha R = \kappa \underbrace{\nabla^\alpha T_{\alpha\beta}}_{=0} - \frac{1}{2} g_{\alpha\beta} \nabla^\alpha R = 0$$

$$\Rightarrow \quad \nabla^\alpha R = 0 \qquad \Rightarrow \qquad \nabla^\alpha T = 0\,. \tag{C.32}$$

This result is not satisfactory, however, since $T := T^\alpha{}_\alpha$ is non-zero inside a star but vanishes outside. The Bianchi identities instead suggest $G_{\alpha\beta} \propto T_{\alpha\beta}$.

(iii) In vacuum $T_{\alpha\beta} = 0$, so that

$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = 0 \qquad \Big| \; \cdot g^{\alpha\beta}$$

$$\Rightarrow \quad R = 0 \qquad \Rightarrow \qquad \boxed{R_{\alpha\beta} = 0}\,. \tag{C.33}$$

(iv) Finally, we emphasize that the Einstein equations represent 10 second-order, non-linear partial differential equations. Solving them is a very difficult task and, barring high degrees of symmetry, only possible using numerical methods or analytic approximations such as linearization.

We conclude this discussion with **Lovelock's theorem** and an important modification to the Einstein equations, long regarded as Einstein's biggest mistake, but by now rejuvenated to the status of critical importance for some of relativity's most important applications.

Theorem: Let $H_{\alpha\beta}$ be a symmetric tensor with

(1) In any coordinates and at every $p \in \mathcal{M}$, $H_{\alpha\beta}$ is a function only of the metric, its first and its second partial derivatives.

(2) $\nabla^\alpha H_{\alpha\beta} = 0$.

(3) $H_{\alpha\beta}$ is linear in the second partial derivatives of the metric $\partial_\sigma \partial_\rho g_{\mu\nu}$.

Then there exist constants a, $b \in \mathbb{R}$ so that

$$H_{\alpha\beta} = a G_{\alpha\beta} + b g_{\alpha\beta}\,. \tag{C.34}$$

We can thus modify Einstein's equation to

$$\boxed{G_{\alpha\beta} + \Lambda g_{\alpha\beta} = \frac{8\pi G}{c^4} T_{\alpha\beta}}\,, \tag{C.35}$$

where Λ is the cosmological constant presently estimated from observations to be about $\Lambda^{-1/2} \approx 10^9$ lightyears. As can be seen from Eq. (C.27) for the energy momentum tensor of a perfect fluid, the cosmological constant term in the Einstein equation is equivalent to a matter source of perfect fluid type with an equation of state

$$\rho = -P = \frac{\Lambda c^4}{8\pi G}\,. \tag{C.36}$$

This form of matter is called dark energy and trying to understand its nature is subject of considerable contemporary research. Note, however, that the interpretation as matter or as a cosmological constant term is mathematically indistinguishable.

D The Schwarzschild solution and classic tests of GR

When Einstein found his field equation, he was not very optimistic that physically meaningful solutions would be found anytime soon. Solving 10 second-order non-linear partial differential equations just looked too daunting a task. Of course, it is easy to construct solutions; just take some arbitrary metric, plug it into the definition of the Einstein tensor and call the resulting right-hand side of (C.35) your matter distribution. The problem with that approach is that matter distributions thus obtained will in general not describe any physical systems out there in the universe. Instead, we need to proceed the other way round, specify $T_{\alpha\beta}$ and solve (C.35) for the metric.

Notwithstanding Einstein's pessimism a physical solution of crucial importance was found in 1915 [25] by Karl Schwarzschild, shortly after Einstein published his theory. Tragically, Schwarzschild died in 1916 after contracting a disease in World War I. His solutions played a critical role in mathematical studies of general relativity ever since and, starting in the 1960s, acquired a similar importance as describing black holes in astrophysics. In this chapter, we will derive the Schwarzschild solution, study in detail the geodesics in this spacetime and resulting predictions by GR for the so-called classical tests of the theory, and then return to the Schwarzschild metric with a more-in-depth discussion of the causal structure of the spacetime.

D.1 Schwarzschild's solution

We are looking for spherically symmetric solutions to the Einstein equation in vacuum (C.33). Note that we do not require the spacetime to depend (or not depend) on time in any specific way.

D.1.1 Symmetric spacetimes

In order to make progress, we first need to translate the notion of spacetime symmetries into mathematical terms. This can be done in a mathematically elegant way using so-called Killing vector fields, but this approach is beyond the scope of this course (though you will encounter it in Part III General Relativity). Here, we will describe the symmetry properties of a spacetime in terms of conditions on the metric tensor.

Def.: A spacetime $(\mathcal{M}, \boldsymbol{g})$ is "symmetric in a variable s" if there exist coordinates x^α such that one of the $x^\alpha = s$ and the metric components are independent of s in this coordinate system.

Def.: A spacetime $(\mathcal{M}, \boldsymbol{g})$ is "stationary" if there exist coordinates x^α such that x^0 is a timelike coordinate and the metric components $g_{\alpha\beta}$ do not depend on x^0.

Def.: A spacetime $(\mathcal{M}, \boldsymbol{g})$ is "static" if it is stationary and in that coordinate system $g_{0i} = 0$ for $i = 1, 2, 3$.

In order to better understand the difference between stationary and static spacetimes, let us write the line element as

$$ds^2 = g_{00}dt^2 + 2g_{0i}dt\,dx^i + g_{ij}dx^i\,dx^j \,. \tag{D.1}$$

Under reversal of the time direction, $t \to -t$, the line element changes to

$$ds^2 = g_{00}dt^2 - 2g_{0i}dt\,dx^i + g_{ij}dx^i\,dx^j \,, \tag{D.2}$$

i.e. ds^2 is invariant under time reversal for static spacetimes with $g_{0i} = 0$ but not for stationary spacetimes with $g_{0i} \neq 0$.
Think of a pipe through which a fluid is flowing. If the fluid has the same constant density and velocity at every point, the flow is stationary; the system looks the same tomorrow as today. Under time reversal, however, the flow would change direction. The system is not static unless the flow velocity is zero.

D.1.2 Spherically symmetric spacetimes

Spherical symmetry means that there exists a special point, the origin O, such that the spacetime is invariant under rotations about this point. Let us fix the time for now and consider two points p and q infinitesimally close to each other and both with the same proper distance from O. As we rotate either point around O, it traces out a 2-sphere that can be parametrized by standard angular coordinates θ, ϕ,

$$0 \leq \theta \leq \pi \,, \qquad -\pi < \phi \leq \pi \,. \tag{D.3}$$

Spherical symmetry of the spacetime implies that the proper distance between these two points does not change under rotations. It can be shown that this condition implies that the angular part of the line element is given by the metric on a 2-sphere: $d\theta^2 + \sin^2\theta\, d\phi^2$.
Furthermore, we demand that the line element does not change under reflection of the angular coordinates $\theta \to \pi - \theta$, $\phi \to -\phi$. This implies that all metric cross terms involving the θ or ϕ component vanish. There must then exist a coordinate system $x^\alpha = (\tilde{t}, \tilde{r}, \theta, \phi)$ such that the spacetime metric is

$$ds^2 = -\tilde{A}d\tilde{t}^2 + 2\tilde{B}d\tilde{t}\,d\tilde{r} + \tilde{C}d\tilde{r}^2 + \tilde{D}(d\theta^2 + \sin^2\theta\, d\phi^2) \,, \tag{D.4}$$

where $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, $\tilde{D}$ are functions of $(\tilde{t}, \tilde{r})$ and $\tilde{D} > 0$.

We next define a new radial coordinate by $r := \sqrt{\tilde{D}}$, so that

$$ds^2 = -\hat{A}(\tilde{t}, r)d\tilde{t}^2 + 2\hat{B}(\tilde{t}, r)d\tilde{t}\,dr + \hat{C}(\tilde{t}, r)dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \,. \tag{D.5}$$

Now consider the term

$$-\hat{A}(\tilde{t}, r)d\tilde{t} + \hat{B}(\tilde{t}, r)dr \,. \tag{D.6}$$

The theory of ordinary differential equations tells us that there exists an integrating factor $I(\tilde{t}, r)$ such that we can rewrite the expression (D.6) as a total differential

$$\begin{aligned}
& d\hat{t} = I(\tilde{t}, r)\big[- \hat{A}(\tilde{t}, r)\, d\tilde{t} + \hat{B}(\tilde{t}, r)\, dr\big] \\
\Rightarrow\quad & d\hat{t}^2 = I^2\big(\hat{A}^2 d\tilde{t}^2 - 2\hat{A}\hat{B}\, d\tilde{t}\, dr + \hat{B}^2\, dr^2\big) \\
\Rightarrow\quad & -\hat{A}\, d\tilde{t}^2 + 2\hat{B}\, d\tilde{t}\, dr = -\frac{1}{\hat{A}I^2} d\hat{t}^2 + \frac{\hat{B}^2}{\hat{A}} dr^2 \\
\Rightarrow\quad & ds^2 = -\frac{d\hat{t}^2}{\hat{A}I^2} + \left(\hat{C} + \frac{\hat{B}^2}{\hat{A}}\right) dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \\
\Rightarrow\quad & ds^2 = -j(\hat{t}, r) d\hat{t}^2 + k(\hat{t}, r) dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\,, \qquad \text{(D.7)}
\end{aligned}$$

where in the last step we merely renamed the free functions in a more convenient manner. Note that up to this point, we have only used the coordinate freedom to adapt the line element to the spherical symmetry. In order to make further progress, we need to use the Einstein equations. A straightforward calculation leads to the non-vanishing components of the Ricci tensor

$$\begin{aligned}
R^{\hat{t}}{}_{\hat{t}} &= \frac{r^2\partial_r k + k^2 - k}{k^2 r^2} \overset{!}{=} 0\,, \qquad & R^{\hat{t}}{}_r &= \frac{\partial_{\hat{t}} k}{k^2 r} = 0\,, \\
R^r{}_{\hat{t}} &= -\frac{\partial_{\hat{t}} k}{jkr} = 0\,, \qquad & R^r{}_r &= \frac{-r\partial_r j + jk - j}{-jkr^2} = 0\,. \qquad \text{(D.8)}
\end{aligned}$$

The equations for $R^r{}_{\hat{t}}$ and $R^{\hat{t}}{}_r$ show that k only depends on r. Next, we solve $R^{\hat{t}}{}_{\hat{t}} = 0$ for the function k. Making the Ansatz $r/(r - 2M)$, $M = \text{const}$ turns out to give a solution. Plugging this result for k into the component $R^r{}_r$ gives us

$$\begin{aligned}
& -r\partial_r j + jk - j = 0 \\
\Rightarrow\quad & r\partial_r j - j\frac{r}{r - 2M} + j = 0 \\
\Rightarrow\quad & r(r - 2M)\partial_r j - 2Mj = 0\,. \qquad \text{(D.9)}
\end{aligned}$$

Again, knowing the solution simplifies our task, so we make the Ansatz $j = (r - 2M)f(\hat{t})/r$ which turns out to solve Eq. (D.9). Requiring a metric with Lorentzian signature implies that the otherwise arbitrary $f(\hat{t}) > 0$. Finally, we rescale the time coordinate through $dt = \sqrt{f(\hat{t})}\, d\hat{t}$ and obtain the Schwarzschild metric

$$\boxed{ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)}\,. \qquad \text{(D.10)}$$

We note some important points about this result.

- The Schwarzschild solution (D.10) is the unique solution of the vacuum Einstein equation in spherical symmetry.

- For large values of the radius r, the Schwarzschild metric approaches the Minkowski metric. This property is called *asymptotic flatness*.
- Even though we did not require any specific time dependence of the solution it turns out to be static.

This result is known as **Birkhoff's theorem**.

Theorem: Any spherically symmetric solution of the vacuum Einstein equations is given by the Schwarzschild metric and is therefore necessarily static and asymptotically flat.

The parameter M can be shown to denote the total mass-energy of the spacetime, the so-called Arnowitt-Deser-Misner or ADM mass [4] that coincides with the black-hole mass as defined through the apparent horizon. These concepts are beyond the scope of our course but more details may be found in [13, 30].
Note that the Schwarzschild metric (D.10) also describes the exterior of spherically symmetric stars; in its derivation we required the spacetime to be spherically symmetric but of vacuum nature only at those points where we calculated the solution. The metric *inside* a spherically symmetric matter distribution will differ from the Schwarzschild metric, but in the exterior vacuum, Eq. (D.10) is the solution.
We have a good deal more to say about the Schwarzschild metric but we leave that to a later section and first explore the geodesics in this spacetime.

D.2 Geodesics in the Schwarzschild spacetime

D.2.1 The geodesic equations and constants of motion

We derive the geodesics by varying the action (B.90) which we referred to as "version 2" above. Recall that with that version of the Lagrangian we require the parameter λ of the geodesic to be affine. The Lagrangian for the Schwarzschild metric is

$$\hat{\mathcal{L}} = -\left(1-\frac{2M}{r}\right)\dot{t}^2 + \left(1-\frac{2M}{r}\right)^{-1}\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2\,, \tag{D.11}$$

where the dot denotes $d/d\lambda$. First we consider the θ component of the Euler-Lagrange equation

$$\begin{aligned} &\frac{d}{d\lambda}\left(\frac{\partial\hat{\mathcal{L}}}{\partial\dot{\theta}}\right) - \frac{\partial\hat{\mathcal{L}}}{\partial\theta} = 2r^2\ddot{\theta} + 4r\dot{r}\dot{\theta} - 2r^2\sin\theta\,\cos\theta\,\dot{\phi}^2 = 0 \\ \Rightarrow\quad &\ddot{\theta} + 2\frac{\dot{r}\dot{\theta}}{r} - \sin\theta\,\cos\theta\,\dot{\phi}^2 = 0\,. \end{aligned} \tag{D.12}$$

We can always rotate our coordinate system such that the geodesic starts at $\theta = \pi/2$ with $\dot{\theta} = 0$. From Eq. (D.12) we then find $\theta = \pi/2$ along the entire geodesic. We can therefore set $\theta = \pi/2$ without loss of generality for all geodesics and shall do so in the remainder of this section.

The calculation of geodesic curves is further simplified by recalling Noether's theorem from Sec. B.3.2 and employing the resulting constants of motion. We have three such constants,

$$\text{(i)} \quad \frac{\partial \hat{\mathcal{L}}}{\partial t} = 0 \quad \Rightarrow \quad C_1 = \frac{\partial \hat{\mathcal{L}}}{\partial \dot{t}} = -2\left(1 - \frac{2M}{r}\right)\dot{t} =: -2E\,, \tag{D.13}$$

$$\text{(ii)} \quad \frac{\partial \hat{\mathcal{L}}}{\partial \phi} = 0 \quad \Rightarrow \quad C_2 = \frac{\partial \hat{\mathcal{L}}}{\partial \dot{\phi}} = 2r^2 \sin^2\theta\, \dot{\phi} = 2r^2\dot{\phi} =: 2L\,, \tag{D.14}$$

$$\text{(iii)} \quad \frac{\partial \hat{\mathcal{L}}}{\partial \lambda} = 0 \quad \Rightarrow \quad C_3 = -\left(1 - \frac{2M}{r}\right)\dot{t}^2 + \left(1 - \frac{2M}{r}\right)^{-1}\dot{r}^2 + r^2\dot{\phi}^2 =: Q\,. \tag{D.15}$$

Recall that the third constant of motion $Q = \hat{\mathcal{L}} = g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta$, so that $Q = -1$ if the geodesic is timelike and we choose proper time for the parametrization, $\lambda = \tau$. Likewise, $Q = 1$ if we have a spatial geodesic and parametrize it with proper distance $\lambda = s$, and $Q = 0$ if the geodesic is null. Recall that geodesics do not change their timelike, spacelike or null character. To summarize, we have the following constants of motion

$$E = \left(1 - \frac{2M}{r}\right)\dot{t}\,, \tag{D.16}$$

$$L = r^2\dot{\phi}\,, \tag{D.17}$$

$$Q = -\left(1 - \frac{2M}{r}\right)\dot{t}^2 + \left(1 - \frac{2M}{r}\right)^{-1}\dot{r}^2 + r^2\dot{\phi}^2 = \begin{cases} -1 & \text{timelike} \\ 0 & \text{null} \\ 1 & \text{spacelike} \end{cases}\,. \tag{D.18}$$

In order to identify the physical significance of the constants E and L for timelike geodesics, we consider the weak-field limit $r \gg M$. The Schwarzschild metric approaches the Minkowski limit in this case and we are in the regime of special relativity. In this limit,

$$E = \dot{t} = \frac{dt}{d\tau}\,, \tag{D.19}$$

where t is the time measured by an observer at rest and τ the proper time along the particles world line. In special relativity the two are related by Eq. (A.106), i.e. $dt/d\tau = \gamma$, so that

$$L = r^2\dot{\phi} = r^2\gamma\frac{d\phi}{dt}\,. \tag{D.20}$$

If we denote the particle mass by m, we can write this as

$$E\,m = m\gamma \qquad = \text{relativistic mass energy}\,, \tag{D.21}$$

$$L\,m = m\gamma r^2\frac{d\phi}{dt} \qquad = \text{relativistic angular momentum}\,, \tag{D.22}$$

so that E and L denote the energy and angular momentum per unit mass, respectively, of the particle.

Now we insert Eqs. (D.16), (D.17) into the equation (D.18) for Q and obtain

$$-E^2 + \dot{r}^2 + \frac{1}{r^2}\left(1 - \frac{2M}{r}\right) L^2 = \left(1 - \frac{2M}{r}\right) Q$$

$$\Rightarrow \quad \boxed{\frac{1}{2}\dot{r}^2 + V(r) = \frac{1}{2}E^2\,, \qquad V(r) = \frac{1}{2}\left(1 - \frac{2M}{r}\right)\left(\frac{L^2}{r^2} - Q\right)}\,. \tag{D.23}$$

D.2.2 Comparison with the Newtonian equations

It is instructive to compare the relativistic equation for timelike geodesics with the Newtonian equations of motion for a particle in a spherically symmetric gravitational field. For this purpose, we temporarily restore factors of G and c in Eq. (D.23). This also serves as an example of how this is done in practice. First, we multiply (D.23) with the particle mass m which gives

$$\frac{1}{2}m\dot{r}^2 + V(r)m = \frac{1}{2}E^2 m\,. \tag{D.24}$$

The term $m\dot{r}^2/2$ clearly represents kinetic energy, i.e. has units Nm = kg m^2/s^2 without requiring any factors of G or c. The factor $(1 - 2M/r)$ in the potential $V(r)$ is dimensionless, but M/r has SI units kg/m. In Sec. A.1 we saw that G/c^2 has units m/kg, so that $GM/(c^2 r)$ is dimensionless in SI units. The second factor in the potential is also dimensionless, since $Q = -1$. The constant of motion L, however, has units m^2/s according to Eq. (D.17) and, consequently, L^2/r^2 has units m^2/s^2. It turns out convenient to keep these SI units and instead apply a factor c^2 to the Q, so that the potential becomes

$$V(r) = \frac{1}{2}\left(1 - \frac{2GM}{c^2 r^2}\right)\left(\frac{L^2}{r^2} - Qc^2\right)\,,$$

and has SI units of (m/s)^2. After multiplication with the particle mass m, this gives Nm in agreement with the kinetic energy term $m\dot{r}^2/2$. There remains the term mE which we already identified as the relativistic mass. By Einstein's famous $\mathscr{E} = mc^2$, this term acquires the dimension of energy after multiplication with c^2. Equation (D.23) written in SI units therefore becomes

$$\frac{1}{2}m\dot{r}^2 + \frac{m}{2}\left(1 - \frac{2GM}{c^2 r}\right)\left(\frac{L^2}{r^2} - Qc^2\right) = \frac{1}{2}E^2 mc^2\,. \tag{D.25}$$

Of course, there is some freedom in absorbing factors of c in the constants of motion by redefining, for example, $\tilde{E} := cE$ or similar. Any such redefinition is, of course, equivalent to (D.25).

Now let us derive the Newtonian counterpart of this equation. It is obtained from energy conservation. The Newtonian kinetic energy has two contributions, a radial and an angular one,

$$E_{\text{kin}} = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\phi}^2 = \frac{1}{2}m\dot{r}^2 + \frac{m}{2}\frac{L^2}{r^2}\,, \tag{D.26}$$

where we defined the Newtonian angular momentum per unit mass $L := r^2\dot{\phi}$. Note that the dot denotes d/dt here, since we do not distinguish between proper time and coordinate time in Newtonian dynamics. The potential energy of a particle in a spherically symmetric field is

$$E_{\text{pot}} = -G\frac{Mm}{r}\,. \tag{D.27}$$

If we denote by $\mathcal{E}$ the total energy per unit mass, conservation of energy $E_{\text{kin}} + E_{\text{pot}}$ gives us

$$\frac{1}{2}m\dot{r}^2 + \frac{1}{2}m\frac{L^2}{r^2} - G\frac{Mm}{r} = m\mathcal{E} = \text{const}\,, \tag{D.28}$$

which we contrast with the relativistic Eq. (D.25) slightly rearranged as

$$\frac{1}{2}m\dot{r}^2 + \frac{1}{2}m\frac{L^2}{r^2} + QG\frac{Mm}{r} - \frac{G}{c^2}\frac{mML^2}{r^3} = \frac{1}{2}(E^2+Q)mc^2 = \text{const}\,. \tag{D.29}$$

In the weak-field regime, we had $E = \gamma$, so that for small v and setting $Q = -1$

$$E^2 - 1 = \frac{1}{1-v^2/c^2} - 1 \approx \left(1 + \frac{v^2}{c^2}\right) - 1 = \frac{v^2}{c^2} \quad \Rightarrow \quad \frac{1}{2}m\dot{r}^2 + \frac{1}{2}m\frac{L^2}{r^2} = \frac{1}{2}mv^2\,, \tag{D.30}$$

so in the limit of negligible gravitational field and low velocities, the relativistic equation merely reduces to the Newtonian kinetic energy balance. It just happens that the term E which we interpret as the relativistic energy in the absence of gravity enters the full blown geodesic equation of general relativity in the form $(E^2+Q)/2$.

Comparing the Newtonian and the relativistic equations (D.28) and (D.29), we see that they merely differ by the extra term $-GML^2/(c^2r^3)$ in the relativistic equation. For the following discussion it is convenient to write the two equations as follows

$$\frac{1}{2}\dot{r}^2 + V_{\text{N/GR}}(r) = \text{const}\,. \qquad V_{\text{N}}(r) = \frac{1}{2}\frac{L^2}{r^2} - \frac{GM}{r}\,,$$
$$V_{\text{GR}}(r) = \frac{1}{2}\frac{L^2}{r^2} + Q\frac{GM}{r} - \frac{G}{c^2}\frac{ML^2}{r^3}\,, \tag{D.31}$$

with $Q = -1$ for timelike and $Q = 0$ for null geodesics. The shape of the potential determines the possible trajectories, so let us explore the potential for the three cases in more detail. In doing so, we shall revert to natural units and set $G = c = 1$.

Newtonian: We immediately see that for $r \to 0$, the potential $V_{\text{N}} \to +\infty$ while for $r \to \infty$ the potential vanishes. A straightforward calculation shows

$$V'_{\text{N}}(r) = -\frac{L^2}{r^3} + \frac{M}{r^2} = 0 \qquad \Rightarrow \qquad r = \frac{L^2}{M}\,,$$
$$V''_{\text{N}}(r) = \frac{3L^2}{r^4} - \frac{2M}{r^3} \qquad \Rightarrow \qquad V''_{\text{N}}(L^2/M) = M^4/L^6 > 0\,. \tag{D.32}$$

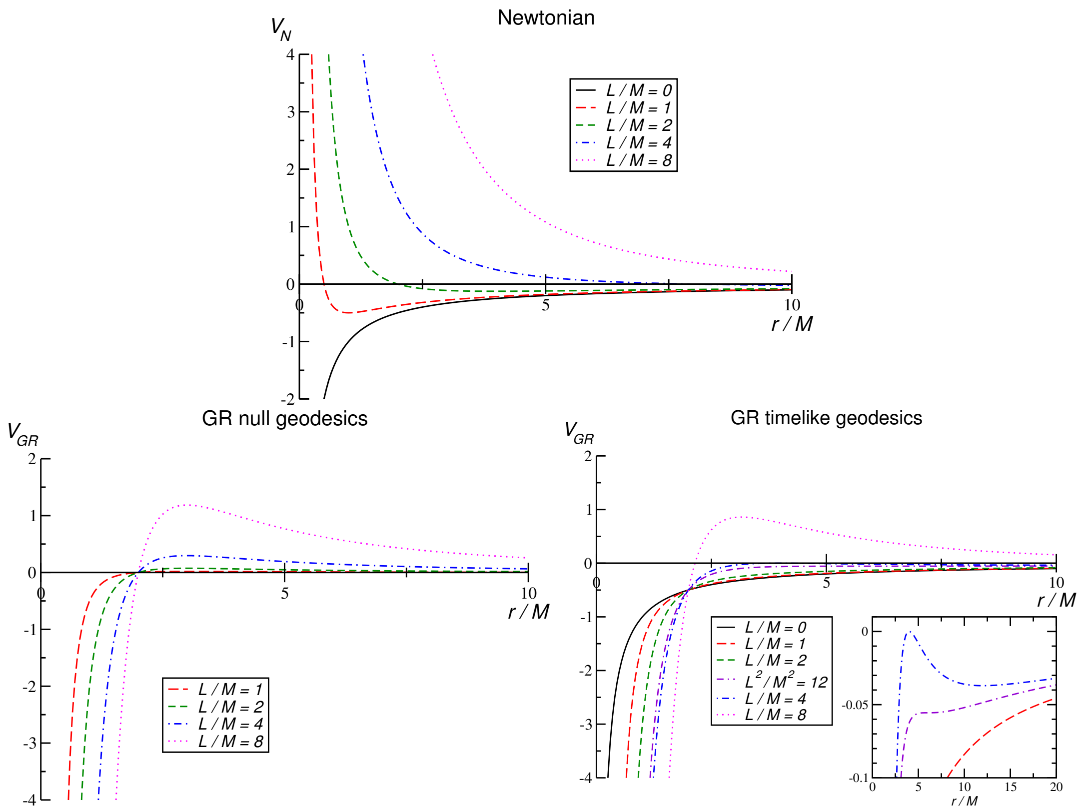


Figure 14: The Newtonian potential $V_{\rm N}$ (upper panel) and the relativistic potential $V_{\rm GR}$ for timelike (bottom left) and null geodesics (bottom right panel), all for selected values of the angular momentum parameter L/M.

The Newtonian potential has exactly one extremum and it is a minimum at $r = L^2/M$ except for the special case $L = 0$ which has no extremum. This behaviour is graphically illustrated in Fig. 14. For $L > 0$ the Newtonian potential always admits a stable circular orbit ($\dot{r} = 0$) which is located at $r = L^2/M$. Furthermore, a particle with non-zero angular momentum can never reach the origin, since the centrifugal repulsion dominates over the gravitational attraction; cf. top panel in Fig. 14.

GR null geodesics: The relativistic potential also approaches zero as $r \to \infty$, but in the limit $r \to 0$ we have $V_{\rm GR} \to -\infty$. A calculation of the extrema is quite easy for $Q = 0$ and leads to

$$V'_{\rm GR}(r) = -\frac{L^2}{r^3} + \frac{3ML^2}{r^4} = 0 \qquad \Rightarrow \qquad r = 3M\,,$$

$$V''_{\rm GR}(r) = \frac{3L^2}{r^4} - \frac{12ML^2}{r^5} \qquad \Rightarrow \qquad V''_{\rm GR}(3M) = -\frac{L^2}{81M^4} < 0\,. \tag{D.33}$$

For $L > 0$ there always exists an unstable circular orbit at $r = 3\ M$ which is often referred to as the *light ring*. The relativistic correction term $\propto r^{-3}$ furthermore implies an infinitely deep potential well at $r = 0$ which drags in all particles with insufficient energy; cf. bottom left panel in Fig. 14.

GR timelike geodesics: The equations are a little more complicated but after some crunching one finds

$$V'_{\rm GR}(r) = -\frac{L^2}{r^3} + \frac{M}{r^2} + \frac{3ML^2}{r^4} = 0 \quad \Rightarrow \quad r = r_\pm = \frac{L^2}{2M} \pm \sqrt{\frac{L^4}{4M^2} - 3L^2}\,,$$

$$V''_{\rm GR}(r) = \frac{3L^2}{r^4} - \frac{2M}{r^3} - \frac{12ML^2}{r^5}$$

$$\Rightarrow \qquad V''_{\rm GR}(r_+) = 16M^4 \frac{L^2 + L\sqrt{L^2-12M^2} - 12M^2}{L^3(L+\sqrt{L^2-12M^2})^5} > 0 \;\;\text{for}\;\; L^2 > 12\;M^2\,,$$

$$\wedge \qquad V''_{\rm GR}(r_-) = 16M^4 \frac{L^2 - L\sqrt{L^2-12M^2} - 12M^2}{L^3(L-\sqrt{L^2-12M^2})^5} < 0 \;\;\text{for}\;\; L^2 > 12\;M^2\,. \qquad \text{(D.34)}$$

The potential is shown for various values of L in the bottom right panel in Fig. 14 which also includes an inset zooming in on three curves to demonstrate the presence or absence of extrema. We see that extrema only exist for $L^2 > 12M^2$ and in that case we find a minimum, i.e. a stable circular orbit, at $r = r_+$ and a maximum, i.e. an unstable circular orbit, at $r = r_-$. One can furthermore show that r_+ (r_-) is monotonically increasing (decreasing) with L at fixed M and in the limit $L^2 \searrow 12M^2$, the two coincide: $r_+ = r_- = 6M$. Finally, in the limit of very large angular momentum parameter $L/M \to \infty$, the unstable circular orbit asymptotes towards the light ring limit $r_- = 3M$. In summary, stable circular orbits exist in the range $r > 6M$ and unstable circular orbits at $3M < r < 6M$. Note the contrast to the Newtonian case where stable circular orbits can be found for any r.

D.3 The classic tests of general relativity

In this section we will apply the geodesic framework developed above to contrast the general relativistic with the Newtonian predictions for three classic tests of Einstein's theory, (i) the perihelion precession of Mercury, (ii) light bending in a central gravitational field and (iii) the Shapiro time delay.

D.3.1 Mercury's perihelion precession

For this calculation, we model Mercury as a point mass orbiting in the gravitational field of the sun and ignore effects due to the other planets.

Newtonian calculation: Starting point for our Newtonian calculation is Eq. (D.28). It turns out convenient for this calculation to switch to an inverse radial coordinate

$$y = \frac{1}{r}\,, \qquad \text{(D.35)}$$

and parametrize the geodesic with the orbital angle ϕ rather than time t. We can do this because by definition of the angular momentum parameter

$$\dot{\phi} = \frac{L}{r^2}\,, \tag{D.36}$$

so that t and ϕ are monotonic functions of each other. Denoting time derivatives with a dot as before and ϕ derivatives with a prime, we obtain

$$\begin{aligned} \frac{d}{dt} &= \frac{d\phi}{dt}\frac{d}{d\phi} = \frac{L}{r^2}\frac{d}{d\phi} = Ly^2\frac{d}{d\phi} \\ \Rightarrow\quad \dot{r} &= Ly^2 r' = Ly^2\left(\frac{-1}{y^2}\right)y' = -Ly'\,. \end{aligned} \tag{D.37}$$

Equation (D.28) transformed into these variables becomes

$$L^2(y')^2 + L^2y^2 - 2My = 2\mathcal{E}\,. \tag{D.38}$$

Differentiating this equation with respect to ϕ gives

$$\begin{aligned} & 2L^2y'\,y'' + 2L^2yy' - 2My' = 0 \\ \Rightarrow\quad & y' = 0 \qquad \vee \qquad y'' + y = \frac{M}{L^2} \\ \Rightarrow\quad & y = \frac{M}{L^2}(1+\epsilon\cos\phi)\,, \end{aligned} \tag{D.39}$$

as is straightforwardly verified by inserting the solution. The resulting curve is a hyperbola for $\epsilon > 1$, a parabola for $\epsilon = 1$ or an ellipse (see Fig. 15) for $\epsilon < 1$. In the circular limit, $\epsilon = 0$, we find a constant radius $r = 1/y = L^2/M$. Most importantly for our calculation, the orbit is closed: y returns to the same value after every passage of $\Delta\phi = 2\pi$. Newtonian gravity predicts no perihelion precession for Mercury (barring for perturbations due to other planets that we ignore here).

General relativistic calculation: Here, the motion is governed by the geodesic equation (D.29) and we again change to the coordinate $y = 1/r$ and use the angle ϕ to parametrize the curve. This transformation proceeds in complete analogy to the Newtonian case above with proper time τ taking the place of the Newtonian t and leads to

$$\begin{aligned} & L^2(y')^2 + L^2y^2 + 2MQy - 2ML^2y^3 = E^2 + Q \\ \Rightarrow\quad & (y')^2 = \frac{E^2}{L^2} - (1-2My)\left(\frac{-Q}{L^2} + y^2\right)\,. \end{aligned} \tag{D.40}$$

Setting $Q = -1$ for a timelike geodesic and rearranging terms, we obtain

$$(y')^2 + y^2 = \frac{E^2-1}{L^2} + \frac{2M}{L^2}y + 2My^3\,. \tag{D.41}$$

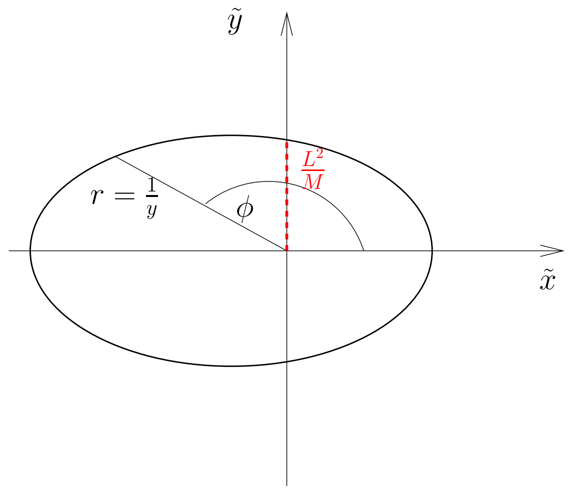


Figure 15: The solution (D.39) for the case $\epsilon < 1$ is an ellipse. Do not confuse the Cartesian coordinate $\tilde{y} = r \sin\phi$ with the inverse radius $y = 1/r$.

Differentiating with respect to ϕ leads to

$$\begin{aligned} 2y'y'' + 2yy' &= \frac{2M}{L^2}y' + 6My^2y' \\ \Rightarrow \quad y'' + y &= M/L^2 + 3My^2 \,, \end{aligned} \tag{D.42}$$

where we ignored the case $y' = 0$ which corresponds to a circular orbit that does not exhibit perihelion precession by construction. Note the similarity of our equation to the Newtonian case in the second line of Eq. (D.39): The only new feature is the extra term $2My^3$. This term, however, makes the solution significantly harder, so that we resort to perturbation theory. For this purpose we introduce the small parameter $\alpha \coloneqq 3M^2/L^2$ which is of the order of 10^{-7} for Mercury. Equation (D.42) than becomes the Newtonian case plus a perturbation $\propto \alpha$,

$$y'' + y = \frac{M}{L^2} + \alpha \frac{L^2}{M} y^2 \,, \tag{D.43}$$

and we likewise expand the solution in α as

$$y = y_0 + \alpha y_1 + \mathcal{O}(\alpha^2) \,. \tag{D.44}$$

Plugging this expansion into (D.43) and sorting terms according to the power of α leads to

$$\begin{aligned} y_0'' + \alpha y_1'' + y_0 + \alpha y_1 &= \frac{M}{L^2} + \alpha \frac{L^2}{M} y_0^2 \\ \Rightarrow \quad y_0'' + y_0 - \frac{M}{L^2} + \alpha \left(y_1'' + y_1 - \frac{L^2}{M} y_0^2 \right) &= 0 \,. \end{aligned} \tag{D.45}$$

In perturbation theory, equations of this type are solved order by order and we start with the terms $\propto \alpha^0 = 1$. At this order, we actually recover the Newtonian case (D.39), so that

$$y_0'' + y_0 - \frac{M}{L^2} = 0 \qquad \Rightarrow \qquad y_0 = \frac{M}{L^2}(1 + \epsilon \cos\phi)\,. \tag{D.46}$$

This expression for y_0 can now be used in those terms of the differential equation $\propto \alpha$ which become

$$\begin{aligned} y_1'' + y_1 &= \frac{L^2}{M} y_0^2 = \frac{M}{L^2}(1 + 2\epsilon\cos\phi + \epsilon^2\cos^2\phi) \\ &= \frac{M}{L^2}\left(1 + \frac{\epsilon^2}{2}\right) + \frac{2M}{L^2}\epsilon\cos\phi + \frac{M}{2L^2}\epsilon^2\cos 2\phi\,, \end{aligned} \tag{D.47}$$

where we used the idenity $\cos^2\phi = (1 + \cos 2\phi)/2$. As a solution, we make the Ansatz

$$\begin{aligned} & y_1 = A + B\phi\sin\phi + C\cos 2\phi \\ \Rightarrow\quad & y_1' = B\sin\phi + B\phi\cos\phi - 2C\sin 2\phi \\ \Rightarrow\quad & y_1'' = 2B\cos\phi - B\phi\sin\phi - 4C\cos 2\phi \\ \Rightarrow\quad & y_1'' + y_1 = A + 2B\cos\phi - 3C\cos 2\phi\,. \end{aligned} \tag{D.48}$$

Comparison with (D.47) gives us the coefficients A, B and C as

$$A = \frac{M}{L^2}\left(1 + \frac{\epsilon^2}{2}\right), \qquad B = \frac{M\epsilon}{L^2}, \qquad C = -\frac{M\epsilon^2}{6L^2}\,. \tag{D.49}$$

Putting together the results for y_0 and y_1, we obtain the solution to first perturbative order in α as

$$y = y_0 + \alpha y_1 = \frac{M}{L^2}(1 + \epsilon\cos\phi) + \alpha\frac{M}{L^2}\left[1 + \epsilon\phi\sin\phi + \epsilon^2\left(\frac{1}{2} - \frac{1}{6}\cos 2\phi\right)\right] \tag{D.50}$$

The last term in brackets is $\propto \epsilon$ and therefore very small for a nearly circular orbit such as Mercury's around the sun. To high accuracy we can therefore write

$$y \approx \frac{M}{L^2}(1 + \alpha + \epsilon\cos\phi + \alpha\epsilon\phi\sin\phi)\,. \tag{D.51}$$

The first two constant terms in parentheses merely give us the average radius of Mercury's orbit and play no role in the perihelion precession. The latter two terms can be approximated for small $\alpha \ll 1$ using the relation

$$\cos(\phi - \alpha\phi) = \cos\phi\,\cos\alpha\phi + \sin\phi\,\sin\alpha\phi \approx \cos\phi + \alpha\phi\sin\phi\,, \tag{D.52}$$

so that

$$y \approx \frac{M}{L^2}\left\{1+\alpha+\epsilon\cos[\phi(1-\alpha)]\right\} . \tag{D.53}$$

The key point is that the (inverse) radius returns to the same value as ϕ increases from ϕ_n to ϕ_{n+1} where

$$(1-\alpha)(\phi_{n+1}-\phi_n) = 2\pi \tag{D.54}$$

$$\Rightarrow \quad \phi_{n+1}-\phi_n = \frac{2\pi}{1-\alpha} \approx 2\pi(1+\alpha) \tag{D.55}$$

The angle traversed from one perihelion to the next therefore exceeds the Newtonian value 2π by the perihelion precession angle

$$\Delta\phi = 2\alpha\pi = 6\pi\frac{M^2}{L^2} . \tag{D.56}$$

For a nearly circular orbit, we can express the orbital angular momentum through the expression for r_+ in the first line of Eq. (D.34), which gives

$$L^2 = \frac{Mr}{1-3M/r} \approx Mr \qquad \Rightarrow \qquad \Delta\phi \approx 6\pi\frac{M}{r} . \tag{D.57}$$

The numbers for Mercury's orbit around the sun are

$$r = 5.55\times 10^7\ \text{km}\,, \qquad T = 0.24\ \text{yr}\,, \qquad M = 1.47\ \text{km}$$

$$\Rightarrow \quad \Delta\phi = 4.99\times 10^{-7}\frac{\text{rad}}{\text{orbit}} = \frac{43''}{\text{century}} . \tag{D.58}$$

D.3.2 Light bending

We now consider light passing close to the surface of a "strongly" gravitating body as for example the sun. Again, we contrast Newtonian with relativistic predictions.

Newtonian calculation: We start with the Newtonian equation of motion (D.38). We already know the solution (D.39), but it will be convenient here to shift the phase by $\pi/2$ so that

$$y = \frac{M}{L^2}(1+\epsilon\sin\phi) . \tag{D.59}$$

It is instructive to first consider the motion in the absence of a gravitational field. Equation (D.59) then simplifies to $y''+y=0$ and the solution can be written as

$$y = \frac{1}{b}\sin\phi . \tag{D.60}$$

A light ray in the absence of a gravitational field should travel on a straight line and, as illustrated in the upper panel of Fig. 16, Eq. (D.60) indeed describes a straight line, albeit in

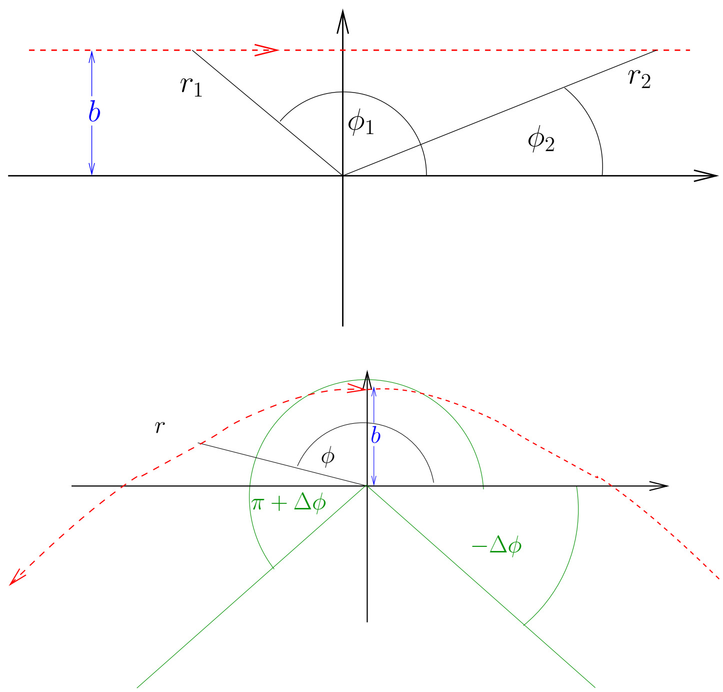


Figure 16: Upper panel: Illustration how Eq. (D.60) represents a straight line with impact parameter b. The deflection angle is zero in this case. Lower panel: In the presence of gravity, the light ray asymptotes to $\phi \to \pi + \Delta\phi$ to the left and $\phi \to -\Delta\phi$ to the right. In the figure, the magnitude of $\Delta\phi$ is vastly exaggerated.

slightly cryptic fashion. The parameter b represents the closest distance of the line to the origin and is often called the *impact parameter*. The light ray, assumed here to come from infinity from the left $\phi = \pi,\ y = 0$ and propagates to the right towards infinity at $\phi = 0,\ y = 0$.
Let us now return to the case with gravitational field described by Eq. (D.59). We are interested in small deflections of light rays that come in from infinity and, after the small deflection, move on towards infinity. At infinity, we are looking for solutions of

$$\frac{M}{L^2}(1 + \epsilon \sin\phi) = 0 \qquad \Rightarrow \qquad \sin\phi = -\frac{1}{\epsilon}\,. \tag{D.61}$$

Small deflection angles correspond to small corrections to the non-gravitational case where infinity corresponded to $\phi = \pi$ or $\phi = 0$, i.e. $\sin\phi \approx 0$. We therefore expect the small-deflection limit to be given by $1/\epsilon \ll 1$. Equation (D.61) will then be solved by $\phi = -\Delta\phi$ and $\phi = \pi + \Delta\phi$ with $\Delta\phi \ll 1$,

$$\sin(-\Delta\phi) \approx -\Delta\phi = -\frac{1}{\epsilon}\,, \qquad \sin(\pi + \Delta\phi) \approx -\Delta\phi = -\frac{1}{\epsilon}\,. \tag{D.62}$$

There remains the task to express ϵ in terms of the parameters L, M and b. As before, we define the impact parameter as the closest distance between the light ray and the origin. This is realized at $\phi = \pi/2$ where

$$\frac{1}{b} = y(\pi/2) = \frac{M}{L^2}(1 + \epsilon) \approx \frac{M}{L^2}\epsilon\,. \tag{D.63}$$

Furthermore, we can write the (conserved!) Newtonian angular momentum mass in terms of the particle's mass m and velocity c as

$$mL = |\vec{r} \times \vec{p}| = bmc = bm \qquad \Rightarrow \qquad L = b\,. \tag{D.64}$$

Using the last two equations we find the deflection angle as (see lower panel of Fig. 16 for an illustration with exaggerated magnitude of $\Delta\phi$)

$$2\Delta\phi = \frac{2}{\epsilon} = \frac{2Mb}{L^2} = \frac{2M}{b}\,. \tag{D.65}$$

General relativistic calculation: The starting point is again the geodesic equation (D.40) expressed in terms of the inverse radius y. We are considering null geodesics now and therefore set $Q = 0$ and obtain

$$L^2(y')^2 + L^2y^2 - 2ML^2y^3 = E^2\,. \tag{D.66}$$

We differentiate this equation with respect to ϕ and divide by $2L^2y'$ which gives

$$y'' + y = 3My^2\,. \tag{D.67}$$

In the absence of a gravitational field we have $M = 0$ and recover the Newtonian case with the solution (D.60). With gravitational field, we again assume the deflection angle to be small and make the Ansatz that the curve is perturbatively close to the straight line $y_0 = (\sin\phi)/b$,

$$y = y_0 + \frac{M}{b}\Delta y + \mathcal{O}\big((M/b)^2\big)\,. \tag{D.68}$$

Here $M/b \ll 1$ is our expansion parameter. Plugging this Ansatz into (D.67) and using that the background solution satisfies

$$y_0 = \frac{1}{b}\sin\phi \qquad \Rightarrow \qquad y_0'' + y_0 = 0\,, \tag{D.69}$$

we find to linear order in M/b for the perturbation Δy

$$\begin{aligned}
&\frac{M}{b}\Delta y'' + \frac{M}{b}\Delta y = 3M\left(\frac{1}{b}\sin\phi + \frac{M}{b}\Delta y\right)^2 \approx \frac{3M}{b^2}\sin^2\phi \\
\Rightarrow \quad &\Delta y'' + \Delta y = \frac{3}{b}\sin^2\phi \qquad \Big|\quad \cos 2\phi = \cos^2\phi - \sin^2\phi = 1 - 2\sin^2\phi \\
\Rightarrow \quad &\Delta y'' + \Delta y = \frac{3}{b}\,\frac{1-\cos 2\phi}{2}\,.
\end{aligned} \tag{D.70}$$

We solve this differential equation by first considering the homogeneous part $\Delta y'' + \Delta y = 0$ which is solved by

$$\Delta\tilde{y} = \frac{A}{b}\cos\phi + \frac{B}{b}\sin\phi\,, \tag{D.71}$$

where A and B are dimensionless constants that also satisfy $|A|,\ |B| \ll b/M$ in order to ensure our perturbative expansion in Eq. (D.68) remains valid. A particular solution for the inhomogeneous equation is

$$\Delta\hat{y} = \frac{1}{2b}(3 + \cos 2\phi)\,, \tag{D.72}$$

as is straightforwardly checked by inserting it into (D.70). We now choose $A = 2$ in the homogeneous part, so that gathering all terms together gives

$$y = y_0 + \frac{M}{b}\Delta y = \frac{1}{b}\sin\phi + \frac{M}{2b^2}(3 + \cos 2\phi) + \frac{2M}{b^2}\cos\phi + \frac{M}{b^2}B\sin\phi\,. \tag{D.73}$$

With this particular choice for A we have ensured that for $\phi \to \pi$ we have $y = 0$, i.e. the photon falls in directly from the left. This corresponds to a rotation of the bottom panel in Fig. 16 by $\Delta\phi$ but has no impact on the result for the deflection angle. As the photon travels to the right, it is deflected before escaping again to infinity $y = 0$ which now happens at an angle $\phi = \delta\phi$ determined by (D.73) to linear order as as

$$\begin{aligned} 0 &\approx \frac{\delta\phi}{b}\left(1 + \frac{M}{b}B\right) + \frac{M}{2b^2}(3+1) + \frac{2M}{b^2} \approx \frac{\delta\phi}{b} + \frac{M}{2b^2}(3+1) + \frac{2M}{b^2} \\ \Rightarrow \quad \delta\phi &\approx -\frac{4M}{b}\,. \end{aligned} \tag{D.74}$$

Note that in the Newtonian calculation we defined $\Delta\phi$ such that the total deflection angle was $2\Delta\phi$ whereas here $\delta\phi$ is the deflection angle. The relativistic result is twice as large as the Newtonian value (D.65).

For the sun with $M = 1.5$ km, $b \approx R_\odot \approx 7 \times 10^5$ km, we find

$$|\delta\phi| = 4 \times \frac{1.5\ \text{km}}{7 \times 10^5\ \text{km}}\frac{360}{2\pi} \times 60 \times 60'' \approx 1.77''\,. \tag{D.75}$$

This effect was famously tested in 1919 through observations by two expeditions to Sobral (Brazil) and to the Island of São Tomé e Principe off the west coast of Africa [10], both located in the path of totality of the solar eclipse on May 29, 1919. Both expeditions, run by Arthur Eddington and collaborators, measured the positions of stars near the sun (then located in the Taurus constellation) and generated results compatible with Einstein's theory of relativity. The confirmation of his theory catapulted Einstein to a global-star status that has lost nothing in the nearly one hundred years since.

D.3.3 Shapiro time delay

The experiment considered here consists in sending a radar signal to Venus and measure the time when the signal reflected off Venus' surface gets back to the Earth. Shapiro [26] predicted in 1964 that the effect of the solar gravitational field should be measurable if the Earth, the Sun and Venus are nearly aligned such that the radar signal passes through the gravitational well of the Sun near its surface. The effect is calculated by comparing the prediction of special relativity ignoring the Sun's gravitational field with that of general relativity where the field in

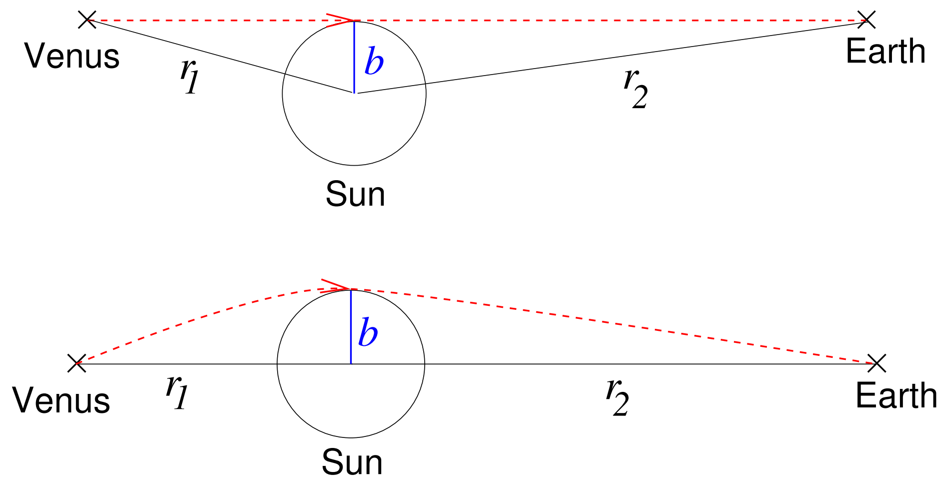


Figure 17: Illustration of the path of a radar signal from Earth to Venus and back in Minkowski spacetime (upper panel) where the gravitational field of the sun is ignored and in general relativity (lower panel) where the Sun's gravity bends the light path.

the solar exterior is modelled by the Schwarzschild metric. The two scenarios are illustrated in Fig. 17.

Without gravitational field: This scenario is shown in the upper panel of Fig. 17. We denote by r_1 and r_2 the distance of Venus and Earth from the sun, respectively. The impact parameter b is the solar radius. The time a radar signal needs to propagate to Venus and back then follows from the rules of flat geometry,

$$T = 2\left(\sqrt{r_1^2 - b^2} + \sqrt{r_2^2 - b^2}\right). \tag{D.76}$$

With gravitational field: We recall the geodesic equations (D.16) and (D.23) in the Schwarzschild spacetime and set $Q = 0$ for null geodesics,

$$\begin{aligned}
&\dot{r}^2 + \left(1 - \frac{2M}{r}\right)\frac{L^2}{r^2} = E^2\,, \qquad \dot{t} = \left(1 - \frac{2M}{r}\right)^{-1} E \\
\Rightarrow\quad & \left(\frac{dr}{dt}\right)^2 = \frac{\dot{r}^2}{\dot{t}^2} = \left(1 - \frac{2M}{r}\right)^2 \frac{1}{E^2}\left[E^2 - \left(1 - \frac{2M}{r}\right)\frac{L^2}{r^2}\right] \\
\Rightarrow\quad & \left(\frac{dr}{dt}\right)^2 = \left(1 - \frac{2M}{r}\right)^2 \left[1 - \left(1 - \frac{2M}{r}\right)\frac{L^2}{r^2 E^2}\right] \\
\Rightarrow\quad & \frac{dr}{dt} = \pm\left(1 - \frac{2M}{r}\right)\sqrt{1 - \left(1 - \frac{2M}{r}\right)\frac{L^2}{r^2 E^2}}\,.
\end{aligned} \tag{D.77}$$

At the point of closest approach to the sun, $r = b$ and $dr/dt = 0$, so that

$$\left(1 - \frac{2M}{b}\right)\frac{L^2}{b^2E^2} = 1 \qquad \Rightarrow \qquad \frac{L^2}{E^2} = \frac{b^2}{1 - \frac{2M}{b}}\,. \tag{D.78}$$

This enables us to replace E and L in Eq. (D.77) in terms of b,

$$\frac{dr}{dt} = \pm\left(1 - \frac{2M}{r}\right)\sqrt{1 - \frac{b^2}{r^2}\frac{1 - 2M/r}{1 - 2M/b}}\,. \tag{D.79}$$

Proper time on Earth is to very high precision identical with coordinate time of the Schwarzschild metric, so that the time of passage of the radar signal is

$$T = 2\int_b^{r_1}\frac{dr}{f(r)} + 2\int_b^{r_2}\frac{dr}{f(r)}\,, \qquad f(r) = \left(1 - \frac{2M}{r}\right)\sqrt{1 - \frac{b^2}{r^2}\frac{1 - 2M/r}{1 - 2M/b}}\,. \tag{D.80}$$

We approximate this integral by Taylor expanding $f(r)$ in M/r, $M/b \ll 1$,

$$\begin{aligned}
f(r) &\approx \left(1 - \frac{2M}{r}\right)\sqrt{1 - \frac{b^2}{r^2}\left(1 - \frac{2M}{r}\right)\left(1 + \frac{2M}{b}\right)} \\
&\approx \left(1 - \frac{2M}{r}\right)\sqrt{1 - \frac{b^2}{r^2}\left(1 - \frac{2M}{r} + \frac{2M}{b}\right)} \\
&= \left(1 - \frac{2M}{r}\right)\sqrt{\frac{r^2 - b^2}{r^2} - \frac{2Mb}{r^3}(r - b)} \\
&= \left(1 - \frac{2M}{r}\right)\sqrt{\frac{r^2 - b^2}{r^2}}\sqrt{1 - \frac{2Mb}{r(r+b)}} \approx \left[1 - \frac{2M}{r}\right]\sqrt{\frac{r^2 - b^2}{r^2}}\left[1 - \frac{Mb}{r(r+b)}\right].
\end{aligned} \tag{D.81}$$

For our integrand $1/f(r)$ we thus obtain

$$\begin{aligned}
\frac{1}{f(r)} &\approx \left(1 + \frac{2M}{r}\right)\sqrt{\frac{r^2}{r^2 - b^2}}\left[1 + \frac{Mb}{r(r+b)}\right] \\
&\approx \sqrt{\frac{r^2}{r^2 - b^2}}\left(\underset{①}{1} + \underset{②}{\frac{2M}{r}} + \underset{③}{\frac{M}{r}\frac{b}{r+b}}\right).
\end{aligned} \tag{D.82}$$

Let us handle the labeled terms one by one.

① $$\int\sqrt{\frac{r^2}{r^2 - b^2}}dr = \int\frac{r}{\sqrt{r^2 - b^2}}dr = \sqrt{r^2 - b^2}$$

$$\Rightarrow \int_b^{r_1} ①\; dr = \sqrt{r_1^2 - b^2}\,.$$

$$② \qquad \int \frac{r}{\sqrt{r^2-b^2}} \frac{2M}{r} dr = \int \frac{2M}{\sqrt{r^2-b^2}} dr = 2M \ln(r + \sqrt{r^2-b^2})$$

$$\Rightarrow \int_b^{r_1} ② \; dr = 2M \ln \frac{r_1 + \sqrt{r_1^2-b^2}}{b} .$$

$$③ \qquad \int \frac{r}{\sqrt{r^2-b^2}} \frac{M}{r} \frac{b}{r+b} dr = \int \frac{Mb}{(r+b)\sqrt{r^2-b^2}} dr = \ldots = M \frac{r-b}{\sqrt{r^2-b^2}} = M \sqrt{\frac{r-b}{r+b}}$$

$$\Rightarrow \int_b^{r_1} ③ \; dr = M \sqrt{\frac{r_1-b}{r_1+b}} .$$

The second integral from b to r_2 in Eq. (D.80) is obtained by merely substituting $r_1 \to r_2$ in the expressions we have just calculated. Gathering all terms and applying a factor of 2 for the return trip, we find

$$\begin{aligned} T \;=\; & \underbrace{2\left(\sqrt{r_1^2-b^2} + \sqrt{r_2^2-b^2}\right)}_{=:T_{\text{Mink}}} + 4M \left(\ln \frac{r_1 + \sqrt{r_1^2-b^2}}{b} + \ln \frac{r_2 + \sqrt{r_2^2-b^2}}{b} \right) \\ & + 2M \left(\sqrt{\frac{r_1-b}{r_1+b}} + \sqrt{\frac{r_2-b}{r_2+b}} \right) . \end{aligned} \tag{D.83}$$

The first term is just the result (D.76) we obtained in the absence of gravity using the Minkowski metric. The second and third term describe the time delay ΔT relative to the Minkowski result. Using

$$\begin{aligned} M &= M_\odot = 1.47 \text{ km} \\ r_1 &= r_♀ = 1.08 \times 10^8 \text{ km} \\ r_2 &= r_♁ = 1.496 \times 10^8 \text{ km} \\ b &= R_\odot = 6.96 \times 10^5 \text{ km} , \end{aligned} \tag{D.84}$$

(the astronomical symbols for Venus and Earth are ♀ and ♁) we obtain $\Delta T \approx 77$ km $= 257\ \mu$s. In practice, the radar signal passes a bit away from the solar surface which decreases the delay to about 200 μs. The effect was first measured with the Massachusetts Institute of Technology's Haystack antenna a few years after Shapiro's prediction and has been reinvestigated with increasing accuracy in numerous experiments since, all compatible with the general relativistic result. A chronology of experimental and observational tests of Einstein's theory is given in Sec. 15.9 of d'Inverno's book [9]. We should add to this list the Nobel Prize winning observations of the Hulse-Taylor pulsar [16, 29, 32] and the ground breaking first detection of gravitational waves from the black-hole binary system `GW150914` [2] that kicked US presidential hopefuls off the news headlines on February 11, 2016.

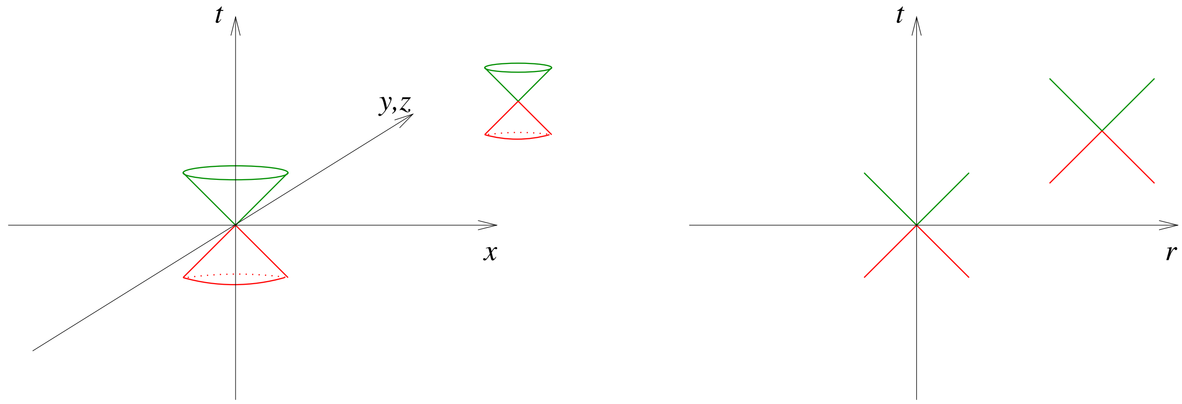


Figure 18: Left panel: Light cones in the Minkowski spacetime in Cartesian coordinates. One spatial direction is suppressed and time points upwards. The future pointing light cone is shown in green, the past one in red color. Right panel: Often we are interested in the limiting curves of outgoing and ingoing radial geodesics. We then use spherical coordinates (t, r) with the angular dependency suppressed and show the light cones by the out and ingoing curves.

D.4 The causal structure of the Schwarzschild spacetime

In the previous two sections we have derived the Schwarzschild metric and studied in detail the motion of test particles in that spacetime with a particular focus on the differences to the predictions by Newtonian gravity. Yet, there remain several open questions, as for example what happens at $= 0$ and $r = 2M$ where the metric (D.10) becomes irregular. In this section, we address these questions and also see that a more in-depth study of the Schwarzschild spacetime has a few surprises in stall for us.

D.4.1 Light cones in the Schwarzschild metric

Light cones are a very convenient tool to explore and understand the causal structure of spacetimes. They represent the possible trajectories of null curves and the boundary of timelike curves which must be inside the light cones. We illustrate this for the case of Minkowski spacetime in Cartesian coordinates in the left panel of Fig. 18 where time points upwards and we suppress one of the spatial directions (we represent y and z by one axis). Most of the time, we will focus on the future light cones which we color in green. Sometimes, we also show the past light cones and do so in red color for distinction. The most important curves for displaying the causal structure are the radial in and outgoing null geodesics which are most conveniently displayed by switching to spherical coordinates and suppressing the angular directions. An example is shown in the right panel of Fig. 18 which shows the resulting light cones in Minkowski spacetime in the time-radius diagram.

Let us now apply this technique to the Schwarzschild metric (D.10)

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\,.$$

We study the geodesic equation using an affine parameter λ and set $d\theta = d\phi = 0$, i.e. we consider radial geodesics. We therefore need the t and r component of the geodesic equation. The former we have already obtained in Eq. (D.16),

$$\left(1 - \frac{2M}{r}\right)\dot{t} = E = \text{const}\,,$$

but the r component we still have to work out. The Euler-Lagrange equation applied to the Schwarzschild metric gives us

$$\begin{aligned}
& \frac{d}{d\lambda}\frac{\partial\mathcal{L}}{\partial\dot{r}} = \frac{\partial\mathcal{L}}{\partial r} \\
\Rightarrow\quad & \frac{d}{d\lambda}\left[2\left(1 - \frac{2M}{r}\right)^{-1}\dot{r}\right] = -\left(1 - \frac{2M}{r}\right)^{-2}\frac{2M}{r^2}\dot{r}^2 - \frac{2M}{r^2}\dot{t}^2 \\
\Rightarrow\quad & -2\left(1 - \frac{2M}{r}\right)^{-2}\frac{2M}{r^2}\dot{r}^2 + 2\left(1 - \frac{2M}{r}\right)^{-1}\ddot{r} = -\left(1 - \frac{2M}{r}\right)^{-2}\frac{2M}{r^2}\dot{r}^2 - \frac{2M}{r^2}\dot{t}^2 \\
\Rightarrow\quad & -2\dot{r}^2 + \left(1 - \frac{2M}{r}\right)\frac{r^2}{M}\ddot{r} = -\dot{r}^2 - \left(1 - \frac{2M}{r}\right)^2\dot{t}^2 = -\dot{r}^2 - E^2 \\
\Rightarrow\quad & \left(1 - \frac{2M}{r}\right)\frac{r^2}{M}\ddot{r} = \dot{r}^2 - E^2\,,
\end{aligned} \tag{D.85}$$

where we plugged in the above equation for $\dot{t}$. This equation is clearly solved by $\dot{r} = \pm E$. It follows that $r = \pm E\lambda + r_0$ is also an affine parameter. We use that observation to reparametrize the geodesic by r,

$$\begin{aligned}
& \frac{dt}{dr} = \frac{\dot{t}}{\dot{r}} = \pm\frac{r}{r - 2M} \\
\Rightarrow \ldots \Rightarrow\quad & \boxed{t(r) = \pm(r + 2M\,\ln|r - 2M|) + k}\,, \qquad k = \text{const}\,.
\end{aligned} \tag{D.86}$$

In Fig. 19 we plot several curves given by Eq. (D.86) and also show some corresponding light cones. Clearly, $r = 2M$ separates two regions which we discuss in turn.

$r > 2M$: The $+$ sign in (D.86) gives us outgoing and the $-$ sign ingoing geodesics. At any given point in the spacetime, a time like curve must be inside the light cones constructed from the radial geodesics. For example, curves $r = \text{const}$ are clearly timelike and located inside the light cones.

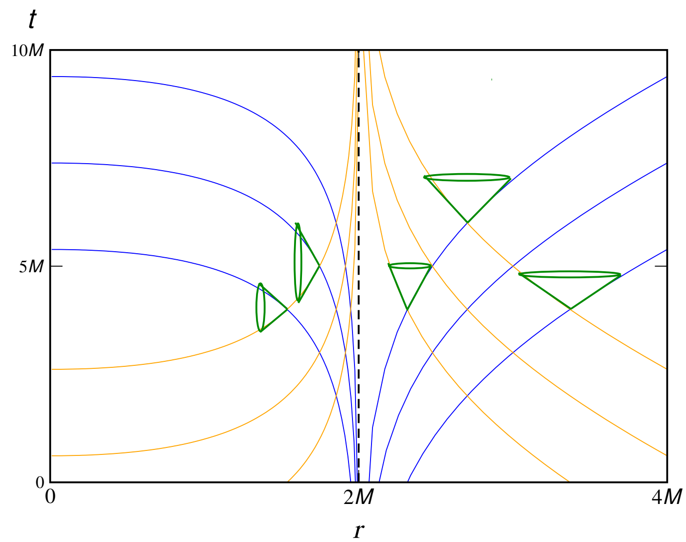


Figure 19: Geodesic curves in the Schwarzschild spacetime according to Eq. (D.86). Curves corresponding to the + sign are shown in in blue, those with the − sign in orange. A few light cones are shown in green. The dotted black line marks the location $r = 2M$ where the Schwarzschild metric (D.10) becomes singular.

$r < 2M$: This case is more complicated. First, we note that the line element (with $d\theta = d\phi = 0$) can now be written in the form

$$ds^2 = -\left(\frac{2M}{r} - 1\right)^{-1} dr^2 + \left(\frac{2M}{r} - 1\right) dt^2\,,$$

so that now $g_{rr} < 0$ and $g_{tt} > 0$ and, hence, r is the timelike coordinate. Curves $t =$ const are now timelike. In our diagram this means that horizontal lines must be inside the light cones which are, accordingly, tilted horizontally. There remains the question whether the future light cones point to the left or right in our diagram. Based on physical arguments, we expect them to point towards $r = 0$, since we expect the gravitational field to pull objects towards the center. We already note at this point, however, that we do not have a mathematical proof for this. For example, we cannot use continuity of the light cones from the exterior across $r = 2M$ because there the metric (D.10) is singular and does not allow for a calculation of light cones.

D.4.2 An infalling observer

It is instructive to calculate the trajectory of an observer freely falling from a large distance in the Schwarzschild metric. This amounts to solving the timelike geodesic equation. Again, we consider radial geodesics with $\dot{\phi} = 0$. For this purpose we need Eq. (D.16) and Eq. (D.18) for the case $Q = -1$,

$$\left(1 - \frac{2M}{r}\right)\dot{t} = E\,, \qquad -\left(1 - \frac{2M}{r}\right)\dot{t}^2 + \left(1 - \frac{2M}{r}\right)^{-1}\dot{r}^2 = -1\,.$$

$$\Rightarrow \quad -E^2 + \dot{r}^2 = -1 + \frac{2M}{r}\,. \tag{D.87}$$

We set $E = 1$ which by Eq. (D.21) implies that the observer's energy corresponds to being at rest at infinity. Furthermore we use proper time τ as the affine parameter, so that our equation becomes

$$\begin{aligned}
&\dot{r}^2 = \frac{2M}{r} \qquad \Rightarrow \qquad \left(\frac{d\tau}{dr}\right)^2 = \frac{r}{2M} \\
\Rightarrow\quad &\frac{d\tau}{dr} = -\sqrt{\frac{r}{2M}} < 0 \qquad \text{for an infalling observer} \\
\Rightarrow\quad &\int \sqrt{2M}\,d\tau = -\int \sqrt{\tilde{r}}\,d\tilde{r} \\
\Rightarrow\quad &\tau - \tau_0 = \frac{2}{3\sqrt{2M}}\left(r_0^{3/2} - r^{3/2}\right).
\end{aligned} \tag{D.88}$$

The constants of integration merely imply that the observer's clock shows time τ_0 at some fixed initial position r_0. Even without solving this expression for $r(\tau)$, we make two important observations: (i) The observer's trajectory passes through $r = 2M$ at finite time τ and (ii) the radius r decreases monotonically as τ increases. The observer is falling to ever decreasing radii which is our physical motivation for having future light cones pointing towards $r = 0$ in Fig. 19.

For comparison, we now describe the same timelike geodesic in terms of Schwarzschild time t instead of proper time τ. Note that t is equal to the proper time of an observer staying fixed at very large radius r. We obtained the expressions

$$\dot{t} = \left(1 - \frac{2M}{r}\right)^{-1} E\,, \qquad \dot{r}^2 = \frac{2M}{r}\,,$$

in the preceding calculation and thus find

$$\frac{dt}{dr} = \frac{\dot{t}}{\dot{r}} = -\sqrt{\frac{r}{2M}}\left(1 - \frac{2M}{r}\right)^{-1}. \tag{D.89}$$

After some crunching, this equation can be integrated to give us

$$t - t_0 = -\frac{2}{3\sqrt{2M}}\left[r^{3/2} - r_0^{3/2} + 6M(\sqrt{r} - \sqrt{r_0})\right] + 2M\,\ln\frac{\left(\sqrt{r} + \sqrt{2M}\right)\left(\sqrt{r_0} - \sqrt{2M}\right)}{\left(\sqrt{r_0} + \sqrt{2M}\right)\left(\sqrt{r} - \sqrt{2M}\right)}\,. \tag{D.90}$$

In Fig. 20 we compare $\tau(r)$ from Eq. (D.88) and $t(r)$ from Eq. (D.90) for an observer starting to fall from $r_0 = 20\ M$ at $t_0 = \tau_0 = 0$. The coordinate time t diverges as the observer approaches

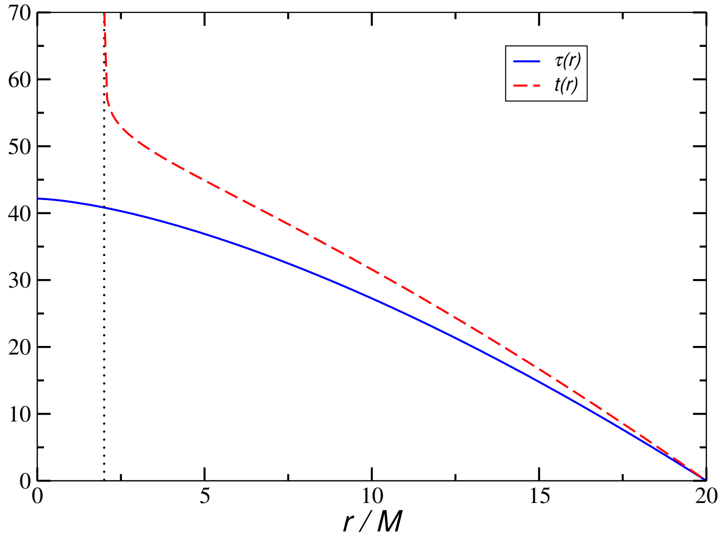


Figure 20: The trajectory of a falling observer in the Schwarzschild spacetime measured in terms of the observer's proper time τ (D.88) and coordinate time t (D.90) which corresponds to the proper time of an observer staying behind at large radius. Both trajectories start from $r_0 = 20\ M$ at $t_0 = \tau_0 = 0$.

$r = 2\ M$. A second observer remaining behind at fixed r_0 will therefore never see his sibling cross the threshold $r = 2\ M$ as that would only happen at $t \to \infty$. On the other hand, we have already seen that the falling observer has quite another experience, crossing $r = 2M$ after finite proper time without anything special happening (besides gradually being spaghettified due to the effect of tidal forces, but that's another story).
We could imagine a scenario where the falling observer emits light signals outwards at regular intervals of proper time. These are picked up by the less adventurous friend who will not detect them at regular intervals in time t but instead sees them arrive with ever increasing delays (and redshift).

D.4.3 Ingoing Eddington Finkelstein coordinates

Our calculations performed so far in the Schwarzschild metric revealed important insights, but encountered considerable difficulties at the point $r = 2M$. The key tool to make further progress is to switch to a new coordinate system. For this purpose, we recall that radial ingoing null geodesics in the Schwarzschild metric are given by Eq. (D.86) using the minus sign therein, i.e.

$$t + 2M\ \ln|r - 2M| = -r + \text{const}\,. \tag{D.91}$$

This motivates the definition of a new time coordinate

$$\bar{t} = t + 2M\ln|r - 2M| \tag{D.92}$$

$$\Rightarrow\quad d\bar{t} = dt + \frac{2M}{r-2M}dr\,, \qquad \text{valid for } r > 2M \text{ or } r < 2M \tag{D.93}$$

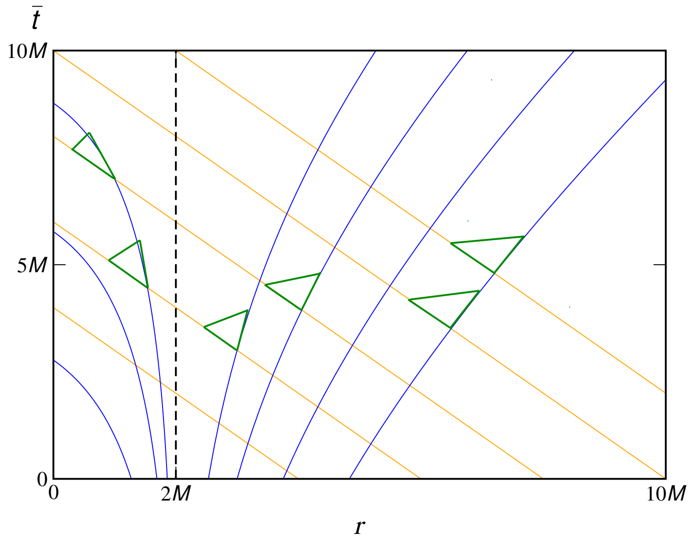


Figure 21: Geodesic curves in the Schwarzschild spacetime in ingoing Eddington Finkelstein coordinates according to Eqs. (D.95), (D.96). The former are shown in orange, the latter in blue. A few light cones are shown in green. The dotted black line marks the location $r = 2M$ where the Schwarzschild metric (D.10) becomes singular.

The Schwarzschild line element (D.10) becomes in this new coordinate system

$$ds^2 = -\left(1 - \frac{2M}{r}\right)\left(d\bar{t} - \frac{2M}{r-2M}dr\right)^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi)$$

$$\Rightarrow \boxed{ds^2 = -\left(1 - \frac{2M}{r}\right) d\bar{t}^2 + \frac{4M}{r} d\bar{t}dr + \left(1 + \frac{2M}{r}\right) dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi)}\,. \quad \text{(D.94)}$$

Ingoing and outgoing radial null geodesics are given in terms of $\bar{t}$ and r by

$$\bar{t} = -r + \text{const}\,, \quad \text{(D.95)}$$

$$\bar{t} = r + 4M\, \ln|r - 2M| + \text{const}\,. \quad \text{(D.96)}$$

An illustration of these geodesics together with the resulting light cones is shown in Fig. 21. We note the following observations.

(1) The light cones now smoothly vary across $r = 2M$. They tilt over in the inward direction such that at $r < 2M$ even outgoing null geodesics are directed towards decreasing r.

(2) At large distances, the light cones approach their Minkowskian structure with 45° inclination.

The location $r = 2M$ marks a semi-transparent membrane in the sense that light rays can move towards $r < 2M$ from the outside, but not the other way round. Even outgoing light rays are drawn in by the gravitational field. Since time like curves are bounded by the light cones, all timelike observers inside $r < 2M$ also inevitably fall towards smaller r. This motivates the following definition.

Def.: The outermost boundary of a region of spacetime from which no null geodesics and, hence, no timelike curves can escape to infinity, is called an event horizon.

This horizon motivated, of course, the term *black hole* coined by John Wheeler in the 1960s. Without proof, we state *Israel's theorem* on the uniqueness of static spacetimes containing a horizon.

Theorem: If a spacetime is static, asymptotically flat and contains a regular horizon then it is a Schwarzschild spacetime.

A simplification of the line element (D.94) is obtained by transforming to the null coordinate

$$\begin{aligned}
v &= \bar{t} + r \qquad \Rightarrow \qquad d\bar{t} = dv - dr \\
\Rightarrow ds^2 &= -\left(1 - \frac{2M}{r}\right)(dv^2 - 2dr dv + dr^2) + \frac{4M}{r}(dv\,dr - dr^2) + \left(1 + \frac{2M}{r}\right)dr^2 + r^2 d\Omega^2 \\
&= -\left(1 - \frac{2M}{r}\right)dv^2 + 2dr\,dv + dr^2\left[-\left(1 - \frac{2M}{r}\right) - \frac{4M}{r} + 1 + \frac{2M}{r}\right] + r^2 d\Omega^2 \\
\Rightarrow ds^2 &= -\left(1 - \frac{2M}{r}\right)dv^2 + 2dr\,dv + r^2 d\Omega^2\,, \qquad \text{(D.97)}
\end{aligned}$$

where we introduced the notation $d\Omega^2 := d\theta^2 + \sin^2\theta\, d\phi^2$. In this line element, the null character of our ingoing radial null geodesics is manifest: the tangent vector to the curves $v = \text{const}$ is ∂_r and clearly $\boldsymbol{g}(\partial_r, \partial_r) = 0$.

You may wonder whether the coordinate transformation (D.92) is really a legitimate way to transform from Schwarzschild to Eddington Finkelstein coordinates; after all, (D.92) is singular at $r = 2M$. This viewpoint, however, looks at the situation the wrong way round. The Eddington Finkelstein version (D.94) of the Schwarzschild metric is a perfectly legitimate solution of the Einstein equations (C.35). It is regular at $r = 2M$ and has a clean structure of light cones. Transforming to the Schwarzschild metric through (D.92) introduces a coordinate singularity at $r = 2M$ which is not surprising given that the transformation itself is singular there.

D.4.4 Outgoing Eddington Finkelstein coordinates

Our transformation (D.92) adapted the time coordinate to the ingoing null geodesics. Nothing stops us from playing the same game with the outgoing null geodesics given by

$$t - 2M\,\ln|r - 2M| = r + \text{const}\,. \qquad \text{(D.98)}$$

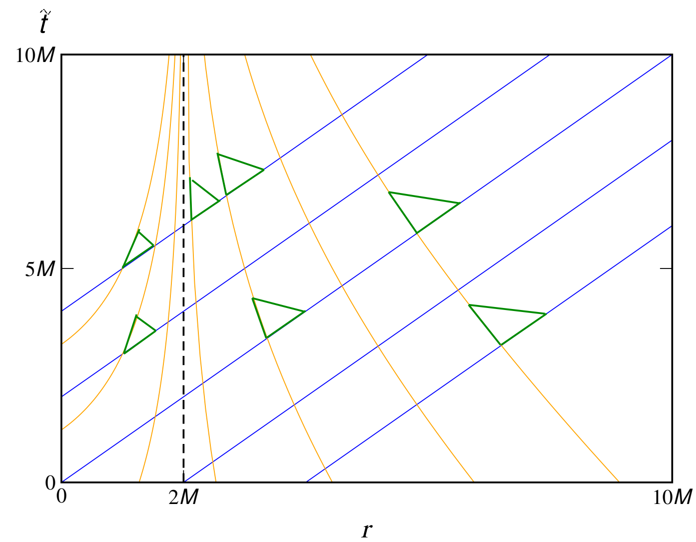


Figure 22: Geodesic curves in the Schwarzschild spacetime in outgoing Eddington Finkelstein coordinates according to Eqs. (D.101), (D.102). The former are shown in orange, the latter in blue. A few light cones are shown in green. The dotted black line marks the location $r = 2M$ where the Schwarzschild metric (D.10) becomes singular.

This equation motivates a new time coordinate given by

$$\tilde{t} = t - 2M \ln|r - 2M| \tag{D.99}$$

$$\Rightarrow \quad d\tilde{t} = dt - \frac{2M}{r - 2M} dr\,, \qquad \text{valid for } r > 2M \text{ or } r < 2M\,. \tag{D.100}$$

Ingoing and outgoing radial null geodesics are now given by

$$\tilde{t} = -r - 4M \ln|r - 2M| + \text{const}\,, \tag{D.101}$$

$$\tilde{t} = r + \text{const}\,. \tag{D.102}$$

Comparing these equations with (D.95) and (D.96), we see that the resulting curves are obtained from those in Fig. 21 by flipping the curves upside down and reversing the "ingoing" and "outgoing" label. The resulting curves are shown in Fig. 22. Clearly, outgoing light rays now always point outwards at 45° and inside $r < 2M$, even ingoing light rays now point towards increasing r. In the limit $r \to \infty$ we again recover the light cones of flat spacetime.

We should be a little puzzled now. With ingoing Eddington Finkelstein coordinates we have just shown that all future pointing light cones tilt over inwards inside $r < 2M$ and that therefore all null geodesics and timelike curves fall inwards. Here, we use outgoing Eddington Finkelstein coordinates and demonstrate the exact opposite; all future pointing light cones inside $r < 2M$ point completely outwards. What is going on and which of the results is correct? The answer is that both are correct. And at second glance the puzzle looks less paradoxical. By construction, the Schwarzschild spacetime is static. We should therefore expect symmetry under time reversal. In order to fully grasp how the puzzle is resolved, we need to go one coordinate transformation further: to *Kruskal-Szekeres coordinates*.

D.4.5 Kruskal-Szekeres coordinates and the maximal extension of Schwarzschild

This derivation requires a few steps, all straightforward, but a bit complex when all put together. Let us proceed step by step.

Step 1: We start by calculating the line element in outgoing Eddington-Finkelstein coordinates and transform to a null version analogous to Eq. (D.97). Using (D.100), the Schwarzschild metric becomes

$$ds^2 = -\left(1 - \frac{2M}{r}\right)\left(d\tilde{t} + \frac{2M}{r-2M}dr\right)^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

$$\Rightarrow \quad \boxed{ds^2 = -\left(1 - \frac{2M}{r}\right) d\tilde{t}^2 - \frac{4M}{r} d\tilde{t}\, dr + \left(1 + \frac{2M}{r}\right) dr^2 + r^2 d\Omega^2}\,. \tag{D.103}$$

The outgoing null coordinate is

$$u = \tilde{t} - r \qquad \Rightarrow \qquad d\tilde{t} = du + dr$$

$$\Rightarrow \quad ds^2 = -\left(1 - \frac{2M}{r}\right)(du + dr)^2 - \frac{4M}{r}(du + dr)dr + \left(1 + \frac{2M}{r}\right) dr^2 + r^2 d\Omega^2$$

$$\Rightarrow \quad ds^2 = -\left(1 - \frac{2M}{r}\right) du^2 - 2du\, dr + r^2 d\Omega^2\,. \tag{D.104}$$

Step 2: Now we collect both, the ingoing and outgoing, coordinate transformations

$$v = \bar{t} + r = t + r + 2M\ln(r - 2M) - 2M\ln r_* = t + r + 2M\ln\frac{r - 2M}{r_*}\,,$$

$$u = \tilde{t} - r = t - r - 2M\ln\frac{r - 2M}{r_*}\,, \tag{D.105}$$

where we wrote the integration constant in the geodesic equations (D.91), (D.99) in the form of a constant r_* that ensures the argument of the logarithm is dimensionless. Now we combine the in and outgoing Eddington Finkelstein coordinates into one coordinate transformation

$$\frac{1}{2}(v + u) = t\,, \qquad \frac{1}{2}(v - u) = r + 2M\ln\frac{r - 2M}{r_*}\,, \tag{D.106}$$

$$\Rightarrow \quad dt = \frac{1}{2}(dv + du)\,, \qquad dr = \frac{r - 2M}{2r}(dv - du)\,, \tag{D.107}$$

which transforms the Schwarzschild metric into

$$ds^2 = -\left(1 - \frac{2M}{r}\right)\frac{1}{4}(dv + du)^2 + \frac{1}{4}\left(1 - \frac{2M}{r}\right)(dv - du)^2 + r^2 d\Omega^2$$

$$\Rightarrow \quad ds^2 = -\left(1 - \frac{2M}{r}\right) du\, dv + r^2 d\Omega^2\,. \tag{D.108}$$

It will be noted here that we dropped the modulus in the logarithmic argument, i.e. use $\ln(r-2M)/r_*$ instead of $\ln|(r-2M)/r_*|$. In fact, all results we have obtained for the ingoing and outgoing Eddington Finkelstein coordinates remain the same, with or without modulus. So we can simply accept the transformation to involve complex intermediate expressions and see where it leads us. The end product will be real.

Step 3: Next we introduce an exponential version of u and v through

$$\tilde{v} = e^{\frac{v}{4M}}\,, \qquad \tilde{u} = -e^{-\frac{u}{4M}}$$

$$\Rightarrow d\tilde{v} = \frac{1}{4M}\tilde{v}dv\,, \qquad d\tilde{u} = -\frac{1}{4M}\tilde{u}du$$

$$\Rightarrow \quad ds^2 = \frac{16M^2}{\tilde{u}\tilde{v}}\left(1-\frac{2M}{r}\right)d\tilde{u}\,d\tilde{v} + r^2 d\Omega^2\,. \tag{D.109}$$

Step 4: The coordinates $\tilde{u}$, $\tilde{v}$ are null coordinates. Since we are more used to time and radius, we now switch back to this type of coordinates. First, we realize that

$$\tilde{u}\tilde{v} = -e^{\frac{v-u}{4M}} = -\exp\left[\frac{r}{2M} + \ln\frac{r-2M}{r_*}\right] = -\frac{r-2M}{r_*}\,e^{\frac{r}{2M}}$$

$$\Rightarrow \quad ds^2 = -\frac{16M^2 r_*}{r-2M}\left(1-\frac{2M}{r}\right)e^{-\frac{r}{2M}}\,d\tilde{u}\,d\tilde{v} + r^2 d\Omega^2$$

$$\Rightarrow \quad ds^2 = -\frac{16M^2}{r/r_*}e^{-\frac{r}{2M}}\,d\tilde{u}\,d\tilde{v} + r^2 d\Omega^2\,. \tag{D.110}$$

Our new time and radius are defined by

$$\hat{t} = \frac{1}{2}(\tilde{v}+\tilde{u})\,, \quad \hat{r} = \frac{1}{2}(\tilde{v}-\tilde{u}) \qquad \Leftrightarrow \qquad \tilde{v} = \hat{t}+\hat{r}\,, \quad \tilde{u} = \hat{t}-\hat{r}$$

$$\Rightarrow \quad d\tilde{v}\,d\tilde{u} = (d\hat{t}+d\hat{r})(d\hat{t}-d\hat{r}) = d\hat{t}^2 - d\hat{r}^2$$

$$\Rightarrow \quad \boxed{ds^2 = \frac{16M^2}{r/r_*}e^{-\frac{r}{2M}}(-d\hat{t}^2 + d\hat{r}^2) + r^2 d\Omega^2}\,, \qquad \hat{t}^2 - \hat{r}^2 = -\frac{r-2M}{r_*}e^{\frac{r}{2M}}\,. \tag{D.111}$$

This is the Schwarzschild metric in Kruskal-Szekeres coordinates. Note that the original radius r is implicitly defined through the last expression and still present in the metric components.

From now on we will set the integration constant $r_* = 1$ as is customary in the literature. This constant represents merely the unit in which we measure radius r and mass M. Note that we have gained a lot with the new form of the Schwarzschild metric:

(1) The metric (D.111) is manifestly regular at $r = 2M$.

(2) Radial null geodesics now have the pleasantly simple form

$$\hat{t} = \hat{r} + \text{const}\,, \qquad \hat{t} = -\hat{r} + \text{const}\,. \tag{D.112}$$

(3) The third and probably most dramatic benefit only becomes clear if we consider the allowed range of our new coordinates. This requires a little work.

- $r = 2M$ is now given by $\hat{t}^2 - \hat{r}^2 = 0 \quad \Rightarrow \quad \hat{t} = \pm\hat{r}\,.$
- $r = 0$ now corresponds to $\hat{t}^2 - \hat{r}^2 = 2M \quad \Rightarrow \quad \hat{t} = \pm\sqrt{\hat{r}^2 + 2M}$.
  Furthermore, $\hat{t}^2 - \hat{r}^2 = -e^{r/(2M)}(r - 2M)$ is a monotonically decreasing function of r, so that for any $r > 0$ we have $\hat{t}^2 - \hat{r}^2 < 2M$.
- There are no other restrictions on our coordinates, so that the allowed range is

$$\boxed{\hat{r} \in (-\infty, \infty)\,, \quad \hat{t}^2 \le \hat{r}^2 + 2M}\,. \tag{D.113}$$

It seems that we have somehow extended our spacetime. Unlike the Schwarzschild radius r, our new radial coordinate $\hat{r}$ can take on negative values. Furthermore, we have two different expressions of $\hat{t}$ and $\hat{r}$ for each of the locations $r = 2M$ and $r = 0$.

In order to understand these issues better, we draw the Kruskal diagram. For this purpose, we consider the following curves.

(i) Curves $r = r_0 = \text{const}$ are hyperbolic curves

$$\hat{t}^2 - \hat{r}^2 = -e^{\frac{r_0}{2M}}(r_0 - 2M) =: C$$

$$\Rightarrow \quad \boxed{\hat{t} = \pm\sqrt{\hat{r}^2 + C}} \quad \vee \quad \boxed{\hat{r} = \pm\sqrt{\hat{t}^2 - C}}\,. \tag{D.114}$$

(ii) Curves $t = t_0 = \text{const}$ are obtained as follows. Equation (D.105) gives us u, v as functions of t, r. This implies

$$\tilde{v} = e^{\frac{v}{4M}} = e^{\frac{t+r}{4M}}\sqrt{r - 2M}\,, \qquad \tilde{u} = -e^{-\frac{u}{4M}} = -e^{\frac{r-t}{4M}}\sqrt{r - 2M}$$

$$\Rightarrow \quad \hat{t} = \frac{1}{2}(\tilde{v} + \tilde{u}) = \sqrt{r - 2M}e^{\frac{r}{4M}}\sinh\frac{t}{4M}$$

$$\wedge \quad \hat{r} = \frac{1}{2}(\tilde{v} - \tilde{u}) = \sqrt{r - 2M}e^{\frac{r}{4M}}\cosh\frac{t}{4M}$$

$$\Rightarrow \quad \boxed{\tanh\frac{t}{4M} = \frac{\hat{t}}{\hat{r}}}\,. \tag{D.115}$$

Curves $t = \text{const}$ therefore correspond to $\hat{t} = C\hat{r}$, $C = \text{const}$.

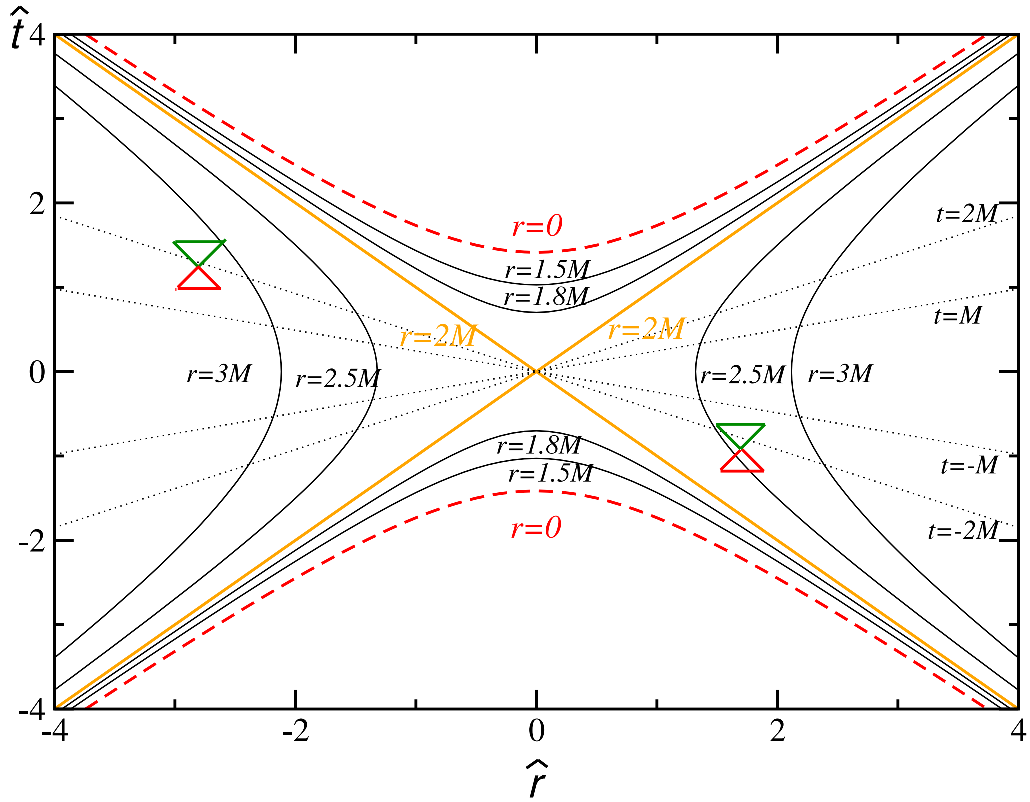


Figure 23: Kruskal diagram of the Schwarzschild spacetime with curves $r = \text{const}$ and $t = \text{const}$ as labeled. For each value $r = \text{const}$ there exist two curves in the spacetime.

Several examples of these curves are plotted in Fig. 23. Note that each value $r = \text{const}$ corresponds to two curves. In particular, there are two singularities $r = 0$ and two horizons $r = 2M$. We now also understand the apparent paradox of the outgoing Eddington-Finkelstein coordinates. The singularity in the future is a *black hole*, everything passing inside $r = 2M$ is doomed to fall ever inwards until it hits $r = 0$. The past singularity $r = 0$, however, is a *white hole* from which all light and timelike curves move outwards. We also have two asymptotically flat regions, one at $\hat{r} \rightarrow \infty$ and one at $\hat{r} \rightarrow -\infty$. These two regions, however, are causally disconnected. Since all light cones have the shape $\hat{t} = \pm\hat{r} + \text{const}$, they open up at 45° and no information can pass from the left to the right region or vice versa. Finally, we note that the horizon $r = 2M$ is a null surface ($\hat{t} = \pm\hat{r}$) and the singularity at $r = 0$ is spacelike.

D.5 Hawking radiation

General relativity is a classical theory and does not take into account quantum effects. We do not yet have a theory of quantum gravity and the search for it remains an active field of research. Quantum effects can be estimated in an approximate manner, however, through semi-classical calculations which model quantum fields on a classical curved background spacetime. This is not the topic of our notes, but we quote here one key result that is of special relevance

for Schwarzschild black holes, the *Hawking radiation*.
The idea behind this effect is pair creation of virtual particles near the horizon. One of the virtual particles has a negative overall energy and therefore falls into the horizon while the other escapes to infinity. This type of quantum tunneling facilitates a mechanism for radiation from a black hole. A quantitative treatment of this process shows that the Hawking radiation is of black-body type with a characteristic temperature that depends on the black-hole mass M through [12]

$$T = \frac{\hbar c^3}{8\pi G M k_B}\,, \tag{D.116}$$

where k_B is the Boltzmann constant. Note that the temperature is inversely proportional to the total mass-energy of the black hole! This is different from standard thermodynamic systems we are used to where more energy implies hotter objects. This has a very important consequence: black holes are thermodynamically unstable objects. As they radiate energy through Hawking radiation, their mass-energy decreases, their temperature increases and they radiate even more. We can calculate the expected life time of a black hole from the Stefan-Boltzmann law that gives the energy flux per unit area from a body of temperature T as

$$-\frac{1}{A}\frac{dM}{dt} = \sigma T^4\,, \qquad \sigma = \frac{\pi^2 k_B^4}{60\hbar^3 c^2} = 5.67\times 10^{-8}\frac{\mathrm{J}}{\mathrm{m}^2\,\mathrm{s}\,\mathrm{K}^4}\,. \tag{D.117}$$

Plugging in Eq. (D.116) for the temperature and $A = 4\pi(2GM/c^2)^2$ for the surface area of a black hole gives us an ordinary differential equation for $M(t)$,

$$\frac{dM}{dt} = -\frac{\hbar c^6}{15360\pi\, G^2\, M^2} \qquad \Rightarrow \qquad t = 5120\frac{\pi G^2}{\hbar c^6}M^3\,. \tag{D.118}$$

For a black hole of one solar mass, $M_\odot = 2\times 10^{30}$ kg, the evaporation time is $\mathcal{O}(10^{60})$ yr. For macroscopic black holes, this is such an extreme value that we can treat them as effectively stable objects. Primordial black holes with masses $M \ll M_\odot$, however, have been conjectured to have formed in the very early universe's density fluctuations. They would have evaporation times much closer to the life time of our universe. If these objects exist, Hawking radiation provides a potentially testable observational signature. Note, however, that our calculations assume that no energy is added through accretion onto the holes. Accretion of some sort should happen, even if only from the 2.7 K cosmic microwave background radiation, modifying the expected evaporation times.

E Cosmology

Cosmology is the attempt to describe the entire Universe using simplifying assumptions that still enable us to capture the essential properties of the Universe. The central concepts are those of *homogeneity* and *isotropy*. These provide us with sufficient degrees of symmetry such that analytic solutions of the Einstein equations are available and predict non-trivial consequences that can be tested through astrophysical observations.

E.1 Homogeneity and Isotropy

Let us start with a collection of fundamental astrophysical observations that guide our construction of cosmological models.

- Telescopes roughly enable us to observe the Universe out to distances of the order of 10^{11} pc. Recall that one parsec is about 3.26 light years.
- Galaxies have a size of the order of 10^5 pc. Even allowing for considerable variation in the size of different types of galaxies, we can approximate them as point particles on the scale of 10^{11} pc.
- On length scales of about 10^9 pc, the universe looks very much the same, in an averaged sense, everywhere. For example, the density of ordinary, observable matter is of the order of $10^{-28}\ \mathrm{kg\,m^{-3}}$ everywhere when averaged over sufficiently large volumes.
- On such large scales, the universe looks the same in every direction.
- The universe appears to be expanding; far away galaxies are increasingly redshifted.

These observations suggest the following basic principles.

(1) On large scales, the universe is spatially homogeneous.

(2) The universe is isotropic around every point.

(3) We can model the matter of the universe as a fluid, i.e. the continuum limit of a large number of particles.

Note that homogeneity and isotropy do not generally imply each other. For example, a homogeneous universe with a magnetic field of constant magnitude pointing in the same direction everywhere is not isotropic. A universe that is isotropic around every point, however, is necessarily homogeneous. The fundamental ideas about our universe can be formalized by the following two postulates.

Cosmological principle: At a given moment in time, the universe is spatially homogeneous and isotropic when viewed on a large scale.

Weyl's postulate: The world lines of the fluid elements, that model the universe's matter content, are orthogonal to hypersurfaces of constant time, Σ_t, to which the cosmological principle applies.

Note that we have been a bit vague so far about defining a time coordinate in this context and, correspondingly, which spatial hypersurfaces are isotropic and homogeneous. Clearly, this is not the case for arbitrary choices of time. For example, if an observer $\mathcal{O}$ finds the universe to be isotropic, a second observer moving with constant velocity $v \neq 0$ relative to $\mathcal{O}$ will not see the universe as isotropic. Weyl's postulate fixes this ambiguity: The spatial hypersurfaces with isotropy and homogeneity are those defined by constant proper time as measured by an observer comoving with the cosmological fluid, i.e. with the galaxy distribution averaged over a large volume. You may wonder at this stage what that has to do with hypersurface orthogonality. We will shortly come to this.

First, though, we will define suitable coordinates and explore the structure of the metrics satisfying the cosmological principle. The galaxies are assumed, by construction, to have no peculiar motion relative to the averaged large-scale motion of the cosmological fluid elements and therefore remain at fixed positions $(x^1,\, x^2,\, x^3)$ in coordinates comoving with the fluid. Furthermore, we define time t to be the proper time measured along the world lines of the galaxies or fluid elements. Note that we assume the universe to be homogeneous in space but not necessarily in time. We therefore allow metric components to have arbitrary time dependency. The spatial part of the line element (i.e. setting $dt = 0$) at time t is

$$d\ell^2 = h_{ij}(t, x^k) dx^i dx^j \,. \tag{E.1}$$

Isotropy at every point implies that the time evolution is the same in every direction, so that none of the h_{ij} components can have a preferred time dependency. With all h_{ij} depending on time in the same way, we can factor out a time dependent term and write the spatial line element as

$$d\ell^2 = a(t)^2 h_{ij}(x^k) dx^i dx^j \,. \tag{E.2}$$

The spacetime metric with this spatial part and using a time coordinate given by the proper time of comoving observers is

$$ds^2 = -dt^2 + g_{0i} dt\, dx^i + a(t)^2 h_{ij}(x^k) dx^i dx^j \,. \tag{E.3}$$

Now we use the hypersurface orthogonality of Weyl's postulate. Let $\mathbf{e}_0 = \partial_t$ and $\mathbf{e}_i = \partial_i$ denote the coordinate basis vectors. Clearly, ∂_t is tangent to the world lines of observers comoving with the cosmological fluid elements, since these are curves $x^i = \text{const}$. By Weyl's postulate, these curves are orthogonal to the surface $t = \text{const}$. The spatial basis vectors $\mathbf{e}_i$ are tangent to this surface and we therefore have the condition

$$\begin{aligned} & g_{0i} = \boldsymbol{g}(\mathbf{e}_0, \mathbf{e}_i) = \mathbf{e}_0 \cdot \mathbf{e}_i = 0 \\ \Rightarrow\quad & ds^2 = -dt^2 + a(t)^2 h_{ij}(x^k) dx^i dx^j \,. \end{aligned} \tag{E.4}$$

Now we consider an observer moving with constant velocity relative to the fluid elements. The metric in the frame of such an observer would be obtained from (E.4) by a Lorentz transformation. This transformation would mix time and spatial coordinates and therefore lead to $g_{0i} \neq 0$; cf. Eq. (A.85). The world line of this observer would not be orthogonal to a surface of constant time in that frame and, as we already mentioned, such an observer would not see the universe as isotropic.

We can further constrain the line element by considering the symmetry requirements on the components h_{ij}. In Sec. D.1.2, we have seen that the spatial part of a spherically symmetric metric can be written in the form [cf. Eq. D.4]

$$d\ell^2 = C(t,r)dr^2 + D(t,r)(d\theta^2 + \sin^2\theta\, d\phi^2)\,. \tag{E.5}$$

Note that spherical symmetry means isotropy around one point. Our assumption of isotropy around every point amounts up to a so-called *maximally symmetric* spacetime which is a stronger symmetry condition that implies spherical symmetry and more besides. One of the "besides" that we have already identified is that the time dependency of $C(t,r)$ can be factored out as in Eq. (E.2). It turns out convenient to write this in the form $C(t,r) = a(t)^2 e^{2\beta(r)}$. As we have seen in Sec. D.1.2, we can also rescale the radius to simplify the function $D(t,r)$. Instead of rescaling to $D(t,r) = r^2$ as in the derivation of the Schwarzschild metric, we now use $D(t,r) = a^2(t)r^2$, so that our line element (E.4) becomes

$$ds^2 = -dt^2 + a(t)^2 d\ell^2\,, \qquad d\ell^2 = e^{2\beta(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\,. \tag{E.6}$$

For further simplification, we focus on the spatial line element $d\ell^2$. The framework of differential geometry we have developed in Sec. B applies to general manifolds and can therefore be used as well to describe the three-dimensional hypersurface $t =$ const. The only difference is that we use Latin indices $i,\, j,\, \ldots = 1,\, 2,\, 3$ in place of the Greek $\alpha,\, \beta,\, \ldots = 0,\, \ldots,\, 3$ and that the metric is now of signature $(+++)$ instead of $(-+++)$. The quantities of particular interest for our calculation are the three-dimensional Ricci tensor and scalar which we denote by $\mathcal{R}_{ij}$ and $\mathcal{R} = \mathcal{R}^i{}_i$. A straightforward calculation gives us

$$\mathcal{R} = \frac{2}{r^2}\left[1 - \partial_r\left(re^{-2\beta}\right)\right]\,. \tag{E.7}$$

This is a scalar quantity and therefore invariant under a coordinate transformation $(x^i) \to (\tilde{x}^m)$. Furthermore we demand spatial homogeneity so that this quantity must be the same at every point on the hypersurface $t =$ const,

$$\frac{2}{r^2}\left[1 - \partial_r\left(re^{-2\beta}\right)\right] = \tilde{k} = \text{const.} \tag{E.8}$$

This can be integrated to

$$e^{2\beta} = \frac{1}{1 - \frac{1}{6}\tilde{k}r^2 - \frac{A}{r}}\,, \qquad \text{with} \quad A = \text{const}\,. \tag{E.9}$$

The constant A is determined by requiring that there be no conical singularity at $r = 0$. The meaning of a conical singularity is best illustrated in two dimensions, so let us consider a metric in polar coordinates,

$$ds^2 = f(r)^2\left[dr^2 + g(r)^2 r^2 d\phi^2\right]\,. \tag{E.10}$$

Proper circumference and proper radius at r_0 are given by

$$c = \int_0^{2\pi} f(r_0)g(r_0)r_0 d\phi = 2\pi f(r_0)g(r_0)r_0\,, \tag{E.11}$$

$$\rho = \int_0^{r_0} f(r)dr\,. \tag{E.12}$$

In the limit of small radius, their ratio is

$$\lim_{r_0\to 0}\frac{c}{\rho} = 2\pi \lim_{r_0\to 0}\frac{f(r_0)g(r_0)r_0}{f(r_0)r_0} = 2\pi g(0)\,. \tag{E.13}$$

The result is 2π only if $g(0) = 1$; on a cone, for instance, one measures such a deviation from 2π and this *deficit angle* is precisely the angle you would cut out from a circular sheet of paper when manufacturing a cone. We require our metric (E.6) to not contain such a singularity and, hence, that in the limit $r \to 0$, $d\ell^2 \propto (dr^2 + r^2 d\Omega^2)$. This implies

$$\lim_{r\to 0} h_{rr} = \frac{1}{1-\frac{A}{r}} \overset{!}{=} 1 \qquad \Rightarrow \qquad A = 0\,. \tag{E.14}$$

Redefining $k := \tilde{k}/6$, we obtain the **Robertson-Walker metric**

$$\Rightarrow \quad \boxed{ds^2 = -dt^2 + a(t)^2\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right]}\,. \tag{E.15}$$

Note that we can trivially rescale r and a such that the constant k is always $+1$, 0 or -1. Say, for instance, $k = -3$. We then set $\tilde{r} = \sqrt{3}r$, $\tilde{a} = a/\sqrt{3}$ and obtain

$$ds^2 = -dt^2 = a(t)^2\left[\frac{dr^2}{1+3r^2} + r^2 d\Omega^2\right] = -dt^2 + \tilde{a}(t)^2\left[\frac{d\tilde{r}^2}{1+\tilde{r}^2} + \tilde{r}^2 d\Omega^2\right]\,. \tag{E.16}$$

It is not possible, however, to scale away in a similar manner the sign of k, so that we have three cases to consider.

1) $k = 0$ **:** In this case, we have

$$d\ell^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) = dx^2 + dy^2 + dz^2\,, \tag{E.17}$$

which is the flat metric on $\mathbb{R}^3$ but may also describe a topologically more complex space such as a cylinder. Models with $k = 0$ are often called *flat*.

2) $k = +1$**:** We introduce a new radial coordinate χ through

$$r = \sin\chi \qquad \Rightarrow \qquad \frac{dr^2}{1-r^2} = \frac{\cos^2\chi}{1-\sin^2\chi}d\chi^2 = d\chi^2\,, \tag{E.18}$$

$$\Rightarrow \quad d\ell^2 = d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta\, d\phi^2)\,. \tag{E.19}$$

This is the metric of a three-sphere, i.e. the surface $w^2 + x^2 + y^2 + z^2 = r^2$ in $\mathbb{R}^4$. Models with $k = +1$ are often called *closed*.

3) $k = -1$**:** We introduce a new radial coordinate ψ through

$$r = \sinh\psi \qquad \Rightarrow \qquad \frac{dr^2}{1+r^2} = \frac{\cosh^2\psi}{1+\sinh^2\psi}d\psi^2 = d\psi^2 \tag{E.20}$$

$$\Rightarrow \quad d\ell^2 = d\psi^2 + \sinh^2\psi(d\theta^2 + \sin^2\theta\, d\phi^2)\,. \tag{E.21}$$

This space can be viewed as the surface $w^2 - x^2 - y^2 - z^2 = \text{const}$ in the flat manifold with metric $-dw^2 + dx^2 + dy^2 + dz^2$. It is commonly viewed as a saddle. Models with $k = -1$ are often called *open*.

E.2 The Friedmann equations

E.2.1 Ricci tensor and Christoffel symbols

In the previous section we have substantially simplified the line element by exploiting the symmetries of the spacetimes under consideration. We thus arrived at the Robertson-Walker metric (E.15). In order to make further progress, however, we need to use the Einstein equations. A straightforward calculation gives the Ricci tensor and Christoffel symbols of (E.15) as

$$\begin{aligned}
R_{00} &= -3\frac{\ddot{a}}{a}\,, & \Gamma^0_{11} &= \frac{a\dot{a}}{1-kr^2}\,,\quad \Gamma^0_{22} = a\dot{a}r^2\,,\quad \Gamma^0_{33} = a\dot{a}r^2\sin^2\theta\,,\\
R_{11} &= \frac{a\ddot{a} + 2\dot{a}^2 + 2k}{1-kr^2}\,, & \Gamma^1_{10} &= \Gamma^1_{20} = \Gamma^1_{30} = \frac{\dot{a}}{a}\,,\\
R_{22} &= r^2(a\ddot{a} + 2\dot{a}^2 + 2k)\,, & \Gamma^1_{22} &= -r(1-kr^2)\,,\quad \Gamma^1_{33} = -r\sin^2\theta(1-kr^2)\,,\\
R_{33} &= \sin^2\theta\, R_{22}\,, & \Gamma^2_{21} &= \Gamma^3_{31} = \frac{1}{r}\,,\\
R &= \frac{6}{a^2}(a\ddot{a} + \dot{a}^2 + k)\,, & \Gamma^2_{33} &= -\sin\theta\,\cos\theta\,,\quad \Gamma^3_{32} = \cot\theta\,,
\end{aligned} \tag{E.22}$$

with all other non-vanishing components following by symmetry.

E.2.2 The cosmological matter fields

In order to solve the Einstein equations, we need the energy momentum tensor describing the cosmological matter distribution. For this purpose we recall from Sec. C.2.4 that perfect fluids are by definition isotropic in their rest frame. This corresponds exactly to the isotropy we require from the cosmological spacetime and we therefore set

$$T_{\mu\nu} = (\rho + P)u_\mu u_\nu + P g_{\mu\nu}\,. \tag{E.23}$$

In the comoving coordinate frame we have $u^\mu = (1,\,0,\,0,\,0)$ and $u_\nu = (-1,\,0,\,0,\,0)$ and, hence

$$T^\mu{}_\nu = (\rho + P)u^\mu u_\nu + P\delta^\mu{}_\nu = \text{diag}(-\rho,\,P,\,P,\,P) \qquad \Rightarrow \qquad T = T^\mu{}_\mu = -\rho + 3P\,. \tag{E.24}$$

Conservation of energy and momentum is given by

$$\nabla_\mu T^\mu{}_\nu = \partial_\mu T^\mu{}_\nu + \Gamma^\mu_{\rho\mu}T^\rho{}_\nu - \Gamma^\rho_{\nu\mu}T^\mu{}_\rho = 0\,. \tag{E.25}$$

Using the expressions (E.22), we obtain for the $\nu = 0$ component

$$\nabla_\mu T^\mu{}_0 = \partial_0 T^0{}_0 + \Gamma^\mu_{0\mu} T^0{}_0 - \Gamma^\rho_{0\mu} T^\mu{}_\rho = -\partial_0 \rho + 3\frac{\dot{a}}{a}(-\rho) - 3\frac{\dot{a}}{a}P = 0$$

$$\Rightarrow \quad \dot{\rho} = -3\frac{\dot{a}}{a}(\rho + P)\,. \tag{E.26}$$

Next we need an equation of state. The type of matter considered in most cosmological studies has an equation of state of the form

$$P = w\rho\,, \qquad w = \text{const}\,, \tag{E.27}$$

so that

$$\frac{\dot{\rho}}{\rho} = -3(1+w)\frac{\dot{a}}{a} \qquad \Rightarrow \qquad \rho \propto a^{-3(1+w)}\,. \tag{E.28}$$

The important cases are dust, radiation and dark energy.

(1) ***Dust:*** Here we have

$$w = 0 \qquad \Rightarrow \qquad \rho \propto a^{-3}\,. \tag{E.29}$$

Dust represents a matter dominated Universe. The pressure between the individual galaxies is negligible, so that this type of cosmological fluid is well approximated by dust.

(2) ***Radiation:*** In the Statistical Physics lecture you have learned/will learn that photons can be regarded as gas with equation of state $P = \rho/3$. This corresponds to

$$w = \frac{1}{3} \qquad \Rightarrow \qquad \rho \propto a^{-4}\,. \tag{E.30}$$

As we will see below in Sec. E.3, cosmological expansion leads to a redshift of the photons whose wavelength $\lambda \propto a$. The four powers of a in Eq. (E.30) are therefore composed of three factors for the density of photons and one factor for the energy per photon.

(3) ***Dark energy:*** The third type of matter is literally more obscure. Recall from Lovelock's theorem in Sec. C.2.5 that we could add a cosmological term to the Einstein equations without affecting the contracted Bianchi identities nor any of the fundamental properties of the "left-hand side" of the Einstein equations (C.35). The cosmological term can alternatively be interpreted as part of the energy momentum tensor,

$$\begin{aligned} & G_{\alpha\beta} + \Lambda g_{\alpha\beta} = 8\pi T_{\alpha\beta} \\ \Rightarrow \quad & G_{\alpha\beta} = 8\pi \left[T_{\alpha\beta} + T^{(\text{vac})}_{\alpha\beta}\right] \qquad \text{with} \qquad T^{(\text{vac})}_{\alpha\beta} = -\frac{\Lambda}{8\pi} g_{\alpha\beta}\,. \end{aligned} \tag{E.31}$$

This special form of the energy momentum tensor actually is a perfect fluid with the equation of state

$$\begin{aligned} & -P = \rho = \frac{\Lambda}{8\pi}\,, \\ \Rightarrow \quad & w = -1 \qquad \Rightarrow \qquad \rho \propto a^0 = 1\,. \end{aligned} \tag{E.32}$$

This type of matter is interpreted as the non-zero ground state energy of the vacuum and called *dark energy*. Do not confuse it with *dark matter* which is a separate dark-sector component of the Universe that falls into either the dust or radiation category in this discussion. As one might expect from a vacuum energy, its density is independent of the size of the universe.

To summarize, the energy density of the different types of matter considered is

$$\rho_{\text{rad}} \propto a^{-4}\,, \qquad \rho_{\text{mat}} \propto a^{-3}\,, \qquad \rho_{\text{vac}} \propto a^{0}\,. \tag{E.33}$$

In an ever expanding universe, dark energy will therefore dominate over the other forms of energy while the very early stages would be radiation dominated.

Before moving on with the Einstein equations and their solutions, we list here some parameters that are frequently used in the literature on cosmology.

Def.: $H := \dfrac{\dot{a}}{a}$ is the Hubble parameter.

$q := -\dfrac{a\ddot{a}}{\dot{a}^2}$ is the deceleration parameter.

$\rho_{\text{crit}} := \dfrac{3H^2}{8\pi}$ is the critical density; its significance will be revealed below.

$\Omega = \dfrac{\rho}{\rho_{\text{crit}}} = \dfrac{8\pi}{3H^2}\rho$ is the density parameter.

Note that these quantities are in general time dependent. They are often referred to as "parameters" because observations measure their present value which then is a number.

E.2.3 The Einstein equations in cosmology

We now plug the metric (E.15) and the energy momentum tensor (E.24) into the Einstein equations in the form $G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}$. Note that we can take account of the dark energy component in two ways, setting $\Lambda \neq 0$ or as a perfect fluid inside $T_{\mu\nu}$ with $w = -1$. We will usually use the cosmological constant for this purpose. In cases where $\Lambda = 0$ but we consider a dark energy fluid with $w = -1$, we will emphasize doing so. After some crunching, the Einstein field equations give us

$$\boxed{3\frac{\dot{a}^2+k}{a^2}-\Lambda=8\pi\rho} \qquad \text{(I)} \qquad \Rightarrow \frac{1}{2}\frac{\dot{a}^2+k}{a^2}-\frac{\Lambda}{6}=\frac{4\pi}{3}\rho\,,$$

$$\boxed{\frac{2a\ddot{a}+\dot{a}^2+k}{a^2}-\Lambda=-8\pi P} \qquad \text{(II)} \qquad \Rightarrow \frac{\ddot{a}}{a}+\frac{1}{2}\frac{\dot{a}^2+k}{a^2}-\frac{\Lambda}{2}=-4\pi P\,,$$

$$\boxed{\frac{\ddot{a}}{a}=-\frac{4\pi}{3}(\rho+3P)+\frac{\Lambda}{3}} \qquad \text{(III)}\,.$$

The first two equations (I) and (II) are the Friedmann equations and we have rewritten both in a slightly different way on the right side, since these are useful in some of the calculations we will do later on. The third equation (III) follow from the other two but will be frequently used in its specific form. Since we will use these equations quite often in the remainder of this section, we distinguish them by the special labels (I)-(III).

An interesting consequence is obtained by taking the derivative of Eq. (I) and multiplying Eq. (II) with $3\dot{a}/a$ which leads to

$$3\left[\frac{2\dot{a}\ddot{a}}{a^2}-2\frac{\dot{a}(\dot{a}^2+k)}{a^3}\right]=8\pi\dot{\rho}\,, \qquad 3\left[\frac{2\dot{a}\ddot{a}}{a^2}+\frac{\dot{a}^3}{a^3}+\frac{k\dot{a}}{a^3}\right]-3\frac{\dot{a}}{a}\Lambda=-24\pi\frac{\dot{a}}{a}P$$

$$\Rightarrow \quad 3\left(-3\frac{\dot{a}^3+\dot{a}k}{a^3}+\frac{\dot{a}}{a}\Lambda\right)=8\pi\left(\dot{\rho}+3\frac{\dot{a}}{a}P\right)\,. \tag{E.34}$$

Using Eq. (I) on the left-hand side gives

$$\begin{aligned} -24\pi\frac{\dot{a}}{a}\rho &= 8\pi\left(\dot{\rho}+3\frac{\dot{a}}{a}P\right) \\ \Rightarrow \quad \dot{\rho}+3\frac{\dot{a}}{a}(\rho+P) &= 0 \qquad \Big| \cdot a^3 \\ \Rightarrow \quad \frac{d}{dt}(a^3\rho)+P\frac{d}{dt}a^3 &= 0\,. \end{aligned} \tag{E.35}$$

The volume element of the metric (E.15) scales with $V \propto a^3$, so that our last equation can be written as $dE + PdV = 0$, i.e. in the form of the first law of thermodynamics. This equation can be shown to also follow from $\nabla_\mu T^\mu{}_\alpha = 0$. Here, instead, we obtained this equation by differentiating the Einstein field equations. This is a direct manifestation of the contracted Bianchi identities $\nabla_\mu(G^{\alpha\mu}+\Lambda g^{\alpha\mu})=0$.

E.3 Cosmological redshift

Before solving the Friedmann equations, we calculate the redshift of light in an evolving universe by studying radial null geodesics. Setting $d\theta = d\phi = 0$ in the Robertson-Walker metric (E.15),

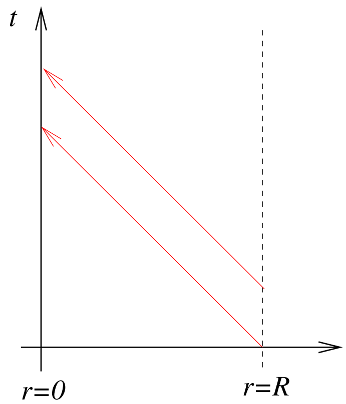


Figure 24: A galaxy at $r = R$ emits two signals to an observer at $r = 0$.

we obtain for null curves

$$ds^2 = -dt^2 + \frac{a^2}{1-kr^2}dr^2 \stackrel{!}{=} 0$$
$$\Rightarrow \quad \frac{dt}{a(t)} = \pm \frac{dr}{\sqrt{1-kr^2}} . \tag{E.36}$$

One can straightforwardly show that the curves obtained from this equation also solve the geodesic equation. Let the observer be located at $r = 0$ and a galaxy at $r = R$ from where it emits light towards the observer; cf. Fig. 24. A first signal is emitted at t_e and a second at $t_e + \Delta t_e$. These reach the observer at t_o and $t_o + \Delta t_o$, respectively. The signals travel on ingoing (towards $r = 0$) null geodesics, so we take the $-$ sign in (E.36). For the two signals we thus obtain

$$\int_{t_e}^{t_o} \frac{dt}{a} = -\int_R^0 \frac{dr}{\sqrt{1-kr^2}} , \qquad \int_{t_e+\Delta t_e}^{t_o+\Delta t_o} \frac{dt}{a} = -\int_R^0 \frac{dr}{\sqrt{1-kr^2}} . \tag{E.37}$$

The right-hand side is the same in both equations, so that

$$\int_{t_o}^{t_o+\Delta t_o} \frac{dt}{a} = \int_{t_e}^{t_e+\Delta t_e} \frac{dt}{a} . \tag{E.38}$$

Furthermore, we assume that Δt_e, $\Delta t_o \ll t_o - t_e$, as realized for example for two consecutive crests in a light wave. We can then regard a as nearly constant in the integrands. Finally the wavelength of a photon is $\lambda \propto \Delta t$, so that

$$\frac{\Delta t_o}{a(t_o)} = \frac{\Delta t_e}{a(t_e)} \qquad \Rightarrow \qquad \boxed{\frac{\lambda_o}{\lambda_e} = \frac{a(t_o)}{a(t_e)}} . \tag{E.39}$$

For relatively nearby galaxies, we can Taylor expand a around t_o,

$$a(t_e) \approx a(t_o) - (t_o - t_e)\dot{a}(t_o) , \qquad (t_o - t_e)\dot{a}(t_o) \ll a(t_o)$$
$$\Rightarrow \quad \frac{a(t_o)}{a(t_e)} \approx \frac{a(t_o)}{a(t_o) - (t_o - t_e)\dot{a}(t_o)} \approx \left[1 - (t_o - t_e)\frac{\dot{a}}{a}\right]^{-1} \approx 1 + (t_o - t_e)\frac{\dot{a}(t_o)}{a(t_o)} . \tag{E.40}$$

The cosmological redshift z for nearby galaxies therefore becomes

$$1 + z := \frac{\lambda_o}{\lambda_e} = \frac{a(t_o)}{a(t_e)} \approx 1 + (t_o - t_e)\frac{\dot{a}(t_o)}{a(t_o)} = 1 + (t_o - t_e)H(t_o)\,, \tag{E.41}$$

where we used the Hubble parameter defined in Sec. E.2.2. In natural units ($c = 1$), $t_o - t_e$ is identical to the distance of the galaxy and we have obtained Hubble's law.

A final comment concerns the notion of distance in cosmology. A radial coordinate frequently used in general relativity is the so-called *areal radius* $R_{\rm ar}$ defined such that a sphere of constant $R_{\rm ar}$ has a proper surface area $4\pi R_{\rm ar}^2$. On a surface of constant radius, the Robertson-Walker line element (E.15) becomes

$$ds^2 = a(t)^2 r^2 (d\theta^2 + \sin^2\theta\, d\phi^2)\,. \tag{E.42}$$

The area of a sphere of constant r is $4\pi a^2 r^2$, so r is not an areal radius, but ar is. Now consider the intensity of light collected at $r = 0$ from a source at $r = R$. The intensity is

$$I := \frac{\text{energy}}{\text{area}} = \frac{E}{4\pi a^2 R^2 (1+z)^2}\,. \tag{E.43}$$

The two factors of $1+z$ arise from (i) the redshift of each individual photon and (ii) the reduced rate at with which photons hit the observer relative to their emission rate. Astrophysicists often use the so-called *luminosity distance* defined by

$$D_L^2 := \frac{E}{4\pi I (1+z)^2}\,, \tag{E.44}$$

which incorporates the redshift factors and therefore is identical to the areal radius, $D_L = ar$.

E.4 Cosmological models

Now we will solve the Friedmann equations (I), (II) for different combinations of the matter sources, parameter k and values of the cosmological constant Λ.

E.4.1 General considerations

(1) From Eq. (I) we have in general

$$H^2 = \frac{\dot{a}^2}{a^2} = \frac{8\pi}{3}\rho + \frac{\Lambda}{3} - \frac{k}{a^2} \qquad \Rightarrow \qquad \frac{k}{\dot{a}^2} = \Omega - 1 + \frac{\Lambda}{3H^2}\,, \tag{E.45}$$

where $\Omega = \rho/\rho_{\rm crit}$, $\rho_{\rm crit} = 3H^2/(8\pi)$ is the density parameter from Sec. E.2.2. In the case of vanishing cosmological constant, $\Lambda = 0$, we therefore have

$$\begin{array}{lllllll} \rho > \rho_{\rm crit} & \Leftrightarrow & \Omega > 1 & \Leftrightarrow & k = +1 & \text{“closed”}\,, \\ \rho = \rho_{\rm crit} & \Leftrightarrow & \Omega = 1 & \Leftrightarrow & k = 0 & \text{“flat’}\,, \\ \rho < \rho_{\rm crit} & \Leftrightarrow & \Omega < 1 & \Leftrightarrow & k = -1 & \text{“open”}\,. \end{array}$$

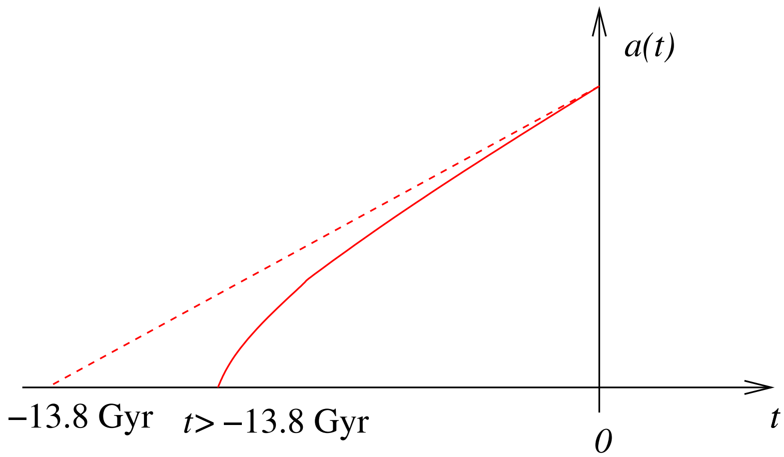


Figure 25: Illustration of the function $a(t)$ for $\ddot{a} = 0$ (dashed curve) and $\ddot{a} < 0$ (solid curve). The measured Hubble parameter $H_0 \approx 71$ km/s/Mpc corresponds to an age of the Universe of 13.8 Gyr. For $\ddot{a} < 0$, this value is an upper limit of the Universe's age.

(2) Let us again consider $\Lambda = 0$ and further assume that the energy density is positive and the pressure is non-negative, $\rho > 0$, $P \geq 0$. Then Eq. (III) tells us $\ddot{a} < 0$. From observations we furthermore know that $\dot{a} > 0$; the Universe is expanding. For vanishing $\ddot{a}$, the curve $a(t)$ would be a straight line reaching the singularity $a = 0$ at time $\Delta t = -a/\dot{a} = -1/H$, where H would then be genuinely constant. Astrophysical observations determine the present value of the Hubble constant $H_0 \approx 71$ km/(s Mpc) corresponding to $-\Delta t = 1/H_0 \approx 13.8$ Gyr. With $\ddot{a} < 0$, $\dot{a}$ must have been larger in the past and Δt is only an upper limit for the age of the Universe; cf. Fig. 25. The singularity $a = 0$ is called the *Big Bang*. Near this point, quantum effects will become important and general relativity is no longer expected to provide an accurate description.

(3) We again consider the case $\Lambda = 0$, $\rho > 0$, $P \geq 0$. From Eq. (I) we find

$$\dot{a}^2 = \frac{8\pi}{3} a^2 \rho - k \,. \tag{E.46}$$

For $k = 0$ or $k = -1$, the right-hand side is manifestly positive, so that $\dot{a}^2 > 0$ always and $\dot{a}$ never reaches zero. Since $\dot{a} > 0$ today, we have $\dot{a} > 0$ always for open and flat Universes. Next, we consider Eq. (E.35), which we write as

$$\frac{d}{dt}(a^3 \rho) = -P \frac{d}{dt} a^3 = -3a^2 P \dot{a} \,. \tag{E.47}$$

The right-hand side is non-positive, so that $d(a^3\rho)/dt \leq 0$. On the other hand, ρa^3 is by construction non-negative and must therefore approach a non-negative constant at late times. This implies

$$\lim_{t\to\infty} a^2 \rho = 0 \,. \tag{E.48}$$

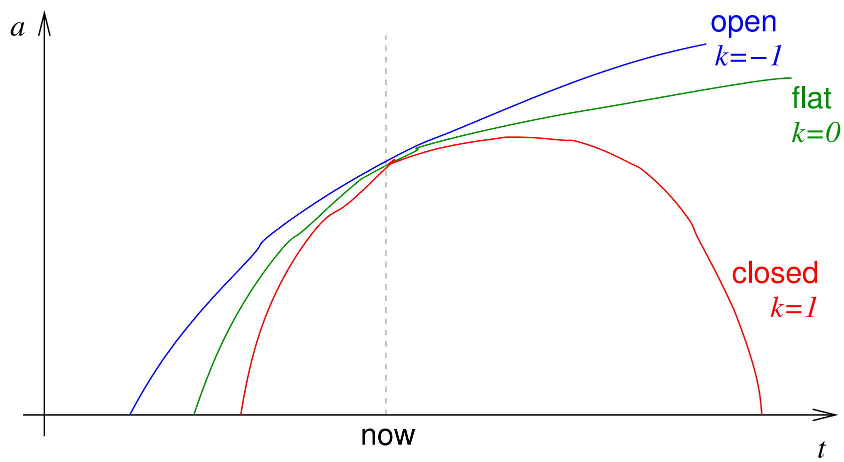


Figure 26: Illustration of the function $a(t)$ for open ($k = -1$), flat ($k = 0$) and closed ($k = +1$)

Using this behaviour in Eq. (E.45), we obtain

$$\dot{a}^2 = a^2 H^2 = \frac{8\pi}{3} a^2 \rho - k \to -k \qquad \Rightarrow \qquad \lim_{t\to\infty} \dot{a} = |k| \,. \tag{E.49}$$

In open Universes, $\dot{a} \to 1$ at late times, while in the flat case $\dot{a} \to 0$. In both cases, the expansion never stops; cf. Fig. 26.

The case $k = +1$ is different. Here, Eq. (I) gives

$$\dot{a}^2 = \frac{8\pi}{3} a^2 \rho - 1 \,. \tag{E.50}$$

As before, $\lim_{a\to\infty} a^2 \rho = 0$, but now $\dot{a}$ will reach zero at $a = a_{\text{max}} = \sqrt{3/(8\pi\rho)}$. Furthermore, we find from Eq. (III) that

$$\lim_{a\to a_{\text{max}}} \ddot{a} = -\frac{4\pi}{3} (\rho + P) a_{\text{max}} < 0 \,. \tag{E.51}$$

At a_{max}, $\dot{a}$ will therefore become negative and, since $\ddot{a}$ remains manifestly negative as long as $a > 0$, a will drop all the way back to zero. This is called the *Big Crunch*.

E.4.2 Selected solutions to the Friedmann equations

According to our investigations up to this point, the solutions to the Friedmann equation are characterized by the following main parameters: (i) the parameter k which separates open, flat and closed models, (ii) the cosmological constant Λ and (iii) the dominant form of matter which we quantify in terms of the equation-of-state parameter w. We will now solve the Friedmann equations (I), (II) for some of the most important combinations of these parameters.

(1) ***Flat, matter dominated models***: $k = 0$, $P = 0$.

We set $P = 0$ in Eq. (II) and multiply with $a^2\dot{a}$,

$$2a\dot{a}\ddot{a} + \dot{a}^3 + k\dot{a} - \Lambda a^2\dot{a} = 0 \qquad \Big| \; \frac{d}{dt}(a\dot{a}^2) = \dot{a}^3 + 2a\dot{a}\ddot{a}$$

$$\Rightarrow \quad a\dot{a}^2 + ka - \frac{1}{3}\Lambda a^3 = C = \text{const}$$

$$\Rightarrow \quad \dot{a}^2 = \frac{C}{a} + \frac{1}{3}\Lambda a^2 - k\,. \tag{E.52}$$

The constant C can be identified by writing Equation (I) as (recall that $a^3\rho = \text{const}$ in a matter dominated Universe)

$$8\pi a^3\rho = 3\left(a\dot{a}^2 + ka - \frac{1}{3}\Lambda a^3\right) \stackrel{!}{=} 3C \quad \Rightarrow \quad \boxed{C = \frac{8\pi}{3}a^3\rho}\,. \tag{E.53}$$

We first consider the case $\Lambda > 0$, set $k = 0$ in Eq. (E.52) and introduce a new variable

$$u = \frac{2\Lambda}{3C}a^3 \qquad \Rightarrow \qquad \dot{u} = \frac{2\Lambda}{C}a^2\dot{a}$$

$$\Rightarrow \quad \dot{u}^2 = \frac{4\Lambda^2}{C^2}a^4\left(\frac{C}{a} + \frac{\Lambda}{3}a^2\right) = \frac{4\Lambda^2}{C}a^3 + \frac{4\Lambda^3}{3C^2}a^6 = 6\Lambda u + 3\Lambda u^2$$

$$\Rightarrow \quad \dot{u}^2 = 3\Lambda(2u + u^2)$$

$$\Rightarrow \quad \dot{u} = \sqrt{3\Lambda}(2u + u^2)^{1/2}\,, \tag{E.54}$$

Assuming that the Universe starts with a big bang, we use initial conditions $a = u = 0$ at $t = 0$, so that

$$\int_0^u \frac{1}{\sqrt{2\tilde{u} + \tilde{u}^2}}d\tilde{u} = \sqrt{3\Lambda}\,t\,. \tag{E.55}$$

The integral on the left-hand side is solved with $u = -1 + \cosh w$,

$$\int_0^u \frac{d\tilde{u}}{\sqrt{\tilde{u}^2 + 2\tilde{u}}} = \int_0^u \frac{d\tilde{u}}{\sqrt{(\tilde{u}+1)^2 - 1}} = \int_0^w \frac{\sinh\tilde{w}\,d\tilde{w}}{\sqrt{\cosh^2\tilde{w} - 1}} = \int_0^w d\tilde{w} = w$$

$$\Rightarrow \quad u + 1 = \cosh w = \cosh(\sqrt{3\Lambda}t)$$

$$\Rightarrow \quad \frac{2\Lambda}{3C}a^3 = \cosh(\sqrt{3\Lambda}\,t) - 1 \qquad \Rightarrow \qquad \boxed{a^3 = \frac{3C}{2\Lambda}\left[\cosh(\sqrt{3\Lambda}\,t) - 1\right]}\,. \tag{E.56}$$

For $\Lambda < 0$, we perform a similar calculation introducing

$$u = -\frac{2\Lambda}{3C}a^3\,, \tag{E.57}$$

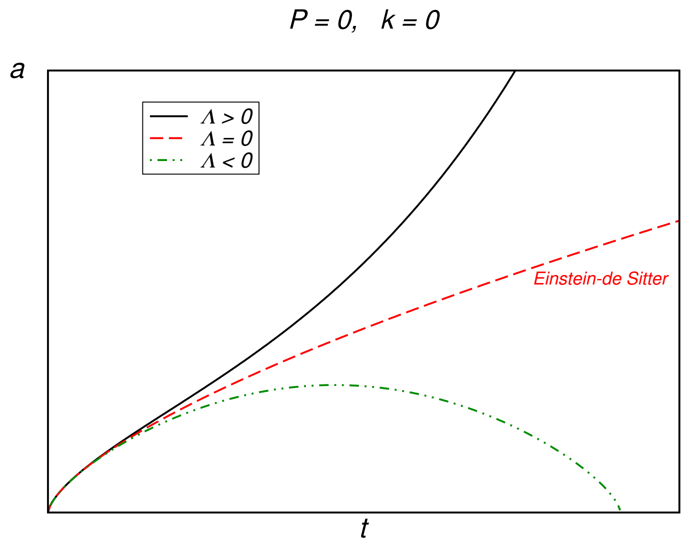


Figure 27: Flat, matter dominated cosmological models for $\Lambda > 0$, $\Lambda = 0$, $\Lambda < 0$.

which eventually leads to

$$\boxed{a^3 = \frac{3C}{2(-\Lambda)}\left[1 - \cos(\sqrt{-3\Lambda}\,t)\right]}\,. \tag{E.58}$$

The case $\Lambda = 0$ is obtained directly from Eq. (E.52) which, with $\Lambda = k = 0$, becomes

$$\dot{a}^2 = \frac{C}{a} \qquad \Rightarrow \qquad \int \sqrt{a}da = \int \sqrt{C}dt$$

$$\Rightarrow \quad \frac{2}{3}a^{3/2} = \sqrt{C}t \qquad \Rightarrow \qquad \boxed{a^3 = \frac{9C}{4}t^2}\,. \tag{E.59}$$

This model is known as the *Einstein-de Sitter model*. For this case, $k = \Lambda = 0$, we find

$$H = \frac{\dot{a}}{a} = \frac{2}{3t}\,,$$

$$q = -\frac{a\ddot{a}}{\dot{a}^2} = -\left(\frac{\dot{a}}{a}\right)^{-1}\left(\frac{\ddot{a}}{\dot{a}}\right) = \frac{1}{2}\,. \tag{E.60}$$

The three different types of models (E.56), (E.59) and (E.58) are graphically illustrated in Fig. 27.

(2) ***Matter dominated models with vanishing cosmological constant:*** $\Lambda = 0$, $P = 0$

Equation (E.52) now gives us

$$\dot{a}^2 = \frac{C}{a} - k\,. \tag{E.61}$$

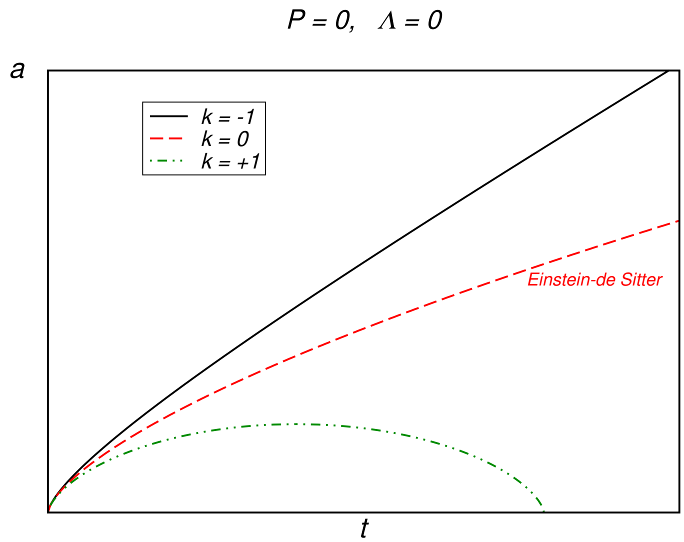


Figure 28: Matter dominated cosmological models with $\Lambda = 0$. Note that the model $k = 0$ is the Einstein-de Sitter model also shown in Fig. 27.

For $k = +1$, we change to the variable

$$u^2 = \frac{a}{C} \quad \Rightarrow \quad 2u\dot{u} = \frac{\dot{a}}{C} \quad \Rightarrow \quad 4u^2\dot{u}^2 = \frac{\dot{a}^2}{C^2}$$

$$\Rightarrow \quad \dot{u}^2 = \frac{\dot{a}^2}{4u^2C^2} = \frac{1}{4u^2C^2}\left(\frac{C}{a} - k\right) = \frac{1}{4u^2C^2}\left(\frac{1}{u^2} - 1\right) = \frac{1}{4u^4C^2}(1-u^2)$$

$$\Rightarrow \quad 2\int \frac{u^2 du}{\sqrt{1-u^2}} = \pm\frac{1}{C}\int dt = \pm\frac{t}{C} + b_\pm\,. \tag{E.62}$$

The constants $b_\pm$ are determined, for example, by requiring that $a = 0$ at $t = 0$ and continuity in $a(t)$ over both branches of the solution. We solve the integral on the left-hand side by setting $u = \sin\chi$ and find after some calculation

$$\arcsin u - u\sqrt{1-u^2} = \pm\frac{t}{C} + b_\pm \;\Rightarrow\; \boxed{C\left[\arcsin\sqrt{\frac{a}{C}} - \sqrt{\frac{a}{C}}\sqrt{1-\frac{a}{C}}\right] = \pm t + b_\pm}\,. \tag{E.63}$$

A similar calculation for $\Lambda = 0$, $k = -1$ gives

$$\boxed{C\left[\sqrt{\frac{a}{C}}\sqrt{1+\frac{a}{C}} - \operatorname{arsinh}\sqrt{\frac{a}{C}}\right] = \pm t + b_\pm}\,. \tag{E.64}$$

Without loss of generality we can set $b_\pm = 0$. Furthermore, we consider future oriented models, so that $t \geq 0$. The case $k = 0$ is the Einstein-de Sitter model which we have already calculated in Eq. (E.59). The three different types of models (E.63), (E.59) and (E.64) are graphically illustrated in Fig. 28.

(3) ***The static Einstein Universe:*** $\dot{a} = \ddot{a} = 0$**,** $P = 0$

Einstein's original motivation for introducing the cosmological constant came from his attempts to construct a static cosmological model which is not possible for $\Lambda = 0$. We are now so used to the fact that the Universe is expanding, that we spend little thought on static models. On philosophical grounds, however, we may be more than a bit puzzled that the cosmos exhibits homogeneity in space, but not in time. After all, the equal footing of space and time was a key foundation of relativity. Extending the principle of homogeneity to time was also the basis of the so-called *Steady-State Model* of Bondi, Gold and Hoyle [6, 14] where the Universe's properties are kept stationary by evoking continuous creation of new energy. The steady-state model is no longer regarded as viable since it is incompatible with the cosmic microwave background observations. But let us return to Einstein's static model. From Eqs. (I), (II) we have

$$\begin{aligned} &\frac{3k}{a^2} = \Lambda + 8\pi\rho\,, && \frac{k}{a^2} = \Lambda \\ \Rightarrow\ & 2\frac{k}{a^2} = 8\pi\rho && \Rightarrow\quad k = 4\pi a^2\rho\,, \end{aligned} \tag{E.65}$$

so that $k = +1$ is a necessary condition of this model. We therefore have

$$a^2 = \frac{1}{\Lambda}\,, \tag{E.66}$$

and using Eq. (E.53) for C, we obtain

$$\begin{aligned} & 3a = \Lambda a^3 + 8\pi\rho a^3 = a + 3C \\ \Rightarrow\ & \boxed{a = \frac{3C}{2} \qquad \wedge \qquad \Lambda = \frac{4}{9C^2}}\,. \end{aligned} \tag{E.67}$$

Unfortunately, this model is not stable. Let us use (E.67) as a background solution a_0 and perturb around this background using $a = a_0 + \epsilon\,,\quad \epsilon \ll a_0$. From Eq. (III) we obtain

$$\begin{aligned} & \frac{\ddot{a}}{a} = -\frac{4\pi}{3}\rho + \frac{\Lambda}{3} \\ \Rightarrow\quad & a^2\ddot{a} = -\frac{C}{2} + \frac{4}{27C^2}a^3 \\ \Rightarrow\quad & a_0^2\ddot{\epsilon} \approx \underbrace{-\frac{C}{2} + \frac{4}{27C^2}(a_0^3}_{=0} + 3a_0^2\epsilon + \ldots) = \frac{4}{9C^2}\,\frac{9}{4}C^2\epsilon = \epsilon \\ \Rightarrow\quad & \ddot{\epsilon} = \frac{4}{9C^2}\epsilon = \Lambda\epsilon\,. \end{aligned} \tag{E.68}$$

The solutions are exponential functions $\exp(\pm\sqrt{\Lambda}t)$. The negative exponent can be ruled out on physical grounds. Say, $\epsilon > 0$, then the Universe is less dense, the gravitational attraction is reduced which leads to further expansion.

(4) ***The de Sitter Universe:*** $\rho = P = 0$**,** $\Lambda > 0$

This model contains no matter other than the dark energy represented by the cosmological constant. Even though this may not appear as a particularly realistic model for our universe, is is of high historical and mathematical interest. Furthermore, it describes a Universe dominated by dark energy, quite possibly the future of our cosmos.

From Eq. (I) we find

$$3\frac{\dot{a}^2 + k}{a^2} = \Lambda\,. \tag{E.69}$$

We consider the three cases for k separately.

Case 1: For $k = -1$, we have

$$3\frac{\dot{a}^2 - 1}{a^2} = \Lambda \qquad \Rightarrow \qquad a(t) = \sqrt{\frac{3}{\Lambda}}\sinh\left(\sqrt{\frac{\Lambda}{3}}\,t\right) \tag{E.70}$$

Case 2: For $k = 0$, we have

$$3\frac{\dot{a}^2}{a^2} = \Lambda \qquad \Rightarrow \qquad a(t) \propto e^{\pm\sqrt{\Lambda/3}\,t} \tag{E.71}$$

Case 3: For $k = +1$, we have

$$3\frac{\dot{a}^2 + 1}{a^2} = \Lambda \qquad \Rightarrow \qquad a(t) = \sqrt{\frac{3}{\Lambda}}\cosh\left(\sqrt{\frac{\Lambda}{3}}\,t\right) \tag{E.72}$$

The result is a bit misleading since all the three solutions can be shown to represent the same spacetime, merely in different coordinates. Readers interested in more details are referred to Hawking & Ellis [13]. In Fig. 29 we display the scale factor $a(t)$ as given by Eq. (E.70) for $k = -1$.

We mention in passing that for $\Lambda < 0$, there also exists a solution known as the *Anti-de Sitter* spacetime. It has attracted less interest in a cosmological context, but plays a central role in a fairly new branch of gravitational research known as the *gauge-gravity duality*, sometimes also called the *AdS/CFT correspondence* (CFT stands for conformal field theory).

(5) ***Radiation dominated, vanishing cosmological constant:*** $P = \rho/3$**,** $\Lambda = 0$

We recall from Eq. (E.35) that in general (even if $\Lambda \neq 0$),

$$\frac{d}{dt}(a^3\rho) + P\frac{d}{dt}a^3 = 0 \qquad \Big| \text{ Now set } P = \frac{\rho}{3}$$

$$\Rightarrow \quad \frac{d}{dt}(a^3\rho) + \frac{1}{3}\rho\frac{d}{dt}a^3 = \frac{d}{dt}(a^3\rho) + \rho a^2\frac{da}{dt} = 0\,. \tag{E.73}$$

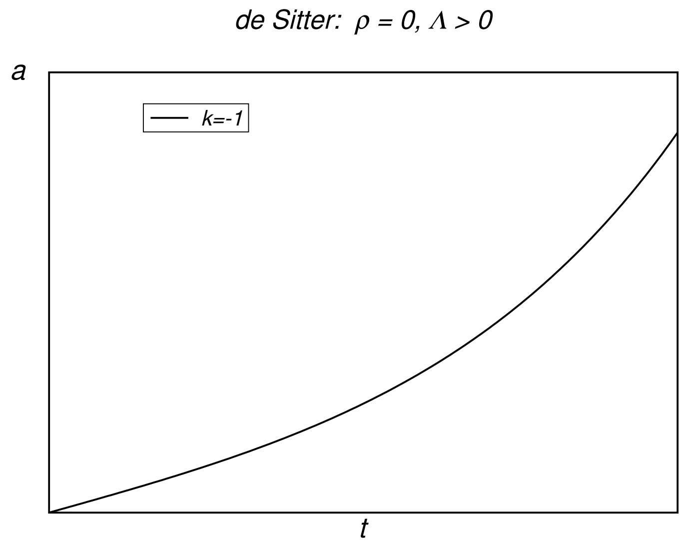


Figure 29: The de Sitter Universe contains no matter other than dark energy corresponding to $\Lambda > 0$. The solutions for $k = -1$, 0, +1 describe the same spacetime merely in different coordinates. The figure shows $a(t)$ for $k = -1$ as given in Eq. (E.70)

We also have in general

$$\frac{d}{dt}(a^4\rho) = \frac{d}{dt}(aa^3\rho) = a^3\rho\frac{da}{dt} + a\frac{d}{dt}(a^3\rho) = a\left[\frac{d}{dt}(a^3\rho) + \rho a^2\frac{da}{dt}\right] = 0\,, \tag{E.74}$$

so in radiation dominated universes $a^4\rho$ is constant which we define as

$$B := \frac{8\pi}{3}a^4\rho. \tag{E.75}$$

For $k = 0$, we have from Eq. (I)

$$\begin{aligned} & 3\frac{\dot{a}^2}{a^2} = 8\pi\rho && \Rightarrow \quad \dot{a}^2 a^2 = \frac{8\pi}{3}a^4\rho \stackrel{!}{=} B \\ \Rightarrow \quad & \int a\,da = \int \pm\sqrt{B}dt && \Rightarrow \quad \frac{1}{2}a^2 = \pm\sqrt{B}\,t\,. \end{aligned} \tag{E.76}$$

The scale factor a is real and non-negative, so that we take the positive square root on both occasions and obtain

$$\boxed{a = \sqrt{2}B^{1/4}\sqrt{t}}\,. \tag{E.77}$$

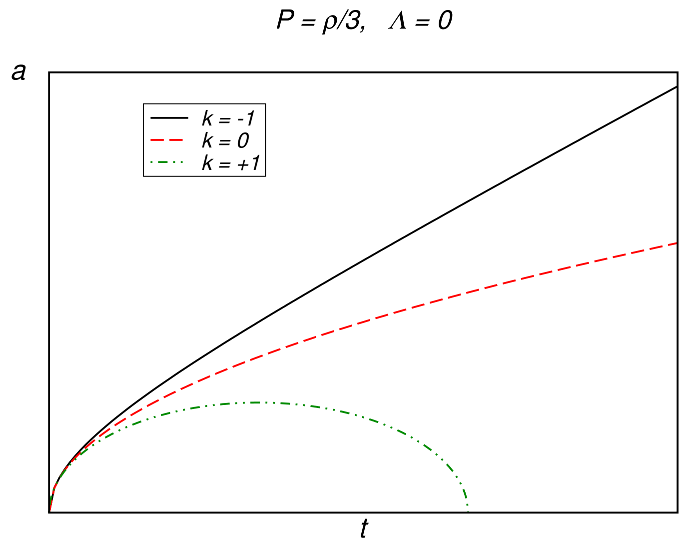


Figure 30: The scale factor for radiation-dominated universes with vanishing cosmological constant $\Lambda = 0$ and $k = -1$, $k = 0$ and $k = +1$. The behaviour is similar to the matter dominated counterparts in Fig. 28.

One can show that the solutions for $k = \pm 1$ are given by

$$k = +1 \qquad \Rightarrow \qquad \boxed{a = \sqrt{B}\sqrt{1 - \left(1 - \frac{t}{\sqrt{B}}\right)^2}}\,, \tag{E.78}$$

$$k = -1 \qquad \Rightarrow \qquad \boxed{a = \sqrt{B}\sqrt{\left(1 + \frac{t}{\sqrt{B}}\right)^2 - 1}}\,. \tag{E.79}$$

The three solutions (E.79), (E.77) and (E.78) are displayed in Fig. 30.

A brief summary of our observations is as follows. We have the following conservation laws for the energy density.

(1) Radiation: $\rho a^4 = \text{const}\,,$

(2) Matter: $\rho a^3 = \text{const}\,,$

(3) Vacuum energy: $\rho \propto \Lambda = \text{const}\,.$

Going back into the past when the Universe was smaller, we therefore find radiation to become the increasingly dominant form of energy. Likewise, as a increases to the future, dark energy will become more and more dominant. Only a stop of the expansion and an ensuing contraction phase would then put an end to the dominance of dark energy. Our observations indicate that at present, about 75 % of the Universe's energy are in the form of dark energy, about 25 % in the form of matter and only a negligible fraction as radiation. The 25 % of matter subdivide into about 4 % of visible matter (such as stars or gas) and about 21 % dark matter whose gravitational effect is apparent, for example in the rotation curve of galaxies, but whose nature is unknown. It is an open puzzle that our present era coincides with a time where neither of the forms of energy is completely dominating over the others. Bear in mind, however, that modifications of Einstein's theory cannot be ruled out and may change the picture we are drawing here. As we have seen in our discussion of the motion of planets in Sec. A.2.5, history has seen both, the revelation of previously unknown matter (Neptune) and a case where the theory of gravity needed to be modified (Mercury). So stay tuned...

F Singularities and geodesic incompleteness

In our study of the Schwarzschild solution and cosmological models we have encountered various instances where the metric components become singular [$r = 0$ and $r = 2M$ in the Schwarzschild metric (D.10) or $a = 0$ in the cosmological spacetimes]. We have also realized that theses singularities may in some cases be cured by switching to more benign coordinates. In this section we will discuss some techniques that enable us to obtain more information about the nature of such singular points in a systematic way.

F.1 Coordinate versus physical singularities

Let us first consider the Schwarzschild metric

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi)^2 . \tag{F.1}$$

Clearly something goes bad in this metric at $r = 2M$ where $g_{rr} \to \infty$. We have seen in Sec. D.4.5, however, that switching to Kruskal-Szekeres coordinates, we were able to cure this singularity. We saw that $r = 2M$ is still a special point, namely the location of the event horizon that marks Schwarzschild's solution as a black hole. But nothing really bad is happening at that point. Likewise, the metric components diverge at $r = 0$ and this is still the case in the Kruskal line element (D.111). This raises two questions. First, can we determine without a priori knowledge of better coordinates whether such coordinates exist? Second, could there be a further improvement over Kruskal coordinates that may even cure the singularity at $r = 0$? Both questions amount up to finding a criterion whether we have a coordinate singularity or a genuine physical singularity.

In order to answer that question, we turn to our rules of tensor calculus, where we saw that scalars are invariant under coordinate transformations. Finding a suitable *curvature scalar* should then tell us more about the nature of a singularity no matter which coordinates we happen to be using. One might first turn towards the Ricci scalar (B.8.6), but this is not too helpful: any vacuum spacetime satisfies the vacuum version of Einstein's field equations $R_{\alpha\beta} = 0$, so that the Ricci scalar also vanishes in such spacetimes by construction. A more powerful variable is the *Kretschmann scalar* constructed out of the Riemann tensor

$$\boxed{\kappa := R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma}} . \tag{F.2}$$

While the Ricci tensor vanishes for vacuum spacetimes such as Schwarzschild, the Riemann tensor only vanishes for the Minkowski metric. After a straightforward but tedious calculation (preferably performed with computation packages such as Mathematica [33] or GRTensor in Maple [34, 35]), one finds for the Schwarzschild metric that

$$\kappa = \frac{48M^2}{r^6} . \tag{F.3}$$

This tells us that the curvature diverges at $r = 0$ which therefore represents a genuinely singular point in the spacetime whereas the curvature at $r = 2M$ is regular. Likewise, we find for the Einstein-de Sitter Universe (E.59) that at $t = 0$

$$\kappa = \frac{80}{27t^4}\,, \tag{F.4}$$

which therefore is also a physical singularity.

F.2 Geodesic incompleteness

The concept of geodesic incompleteness is best introduced in a concrete example. Consider for this purpose the so-called Kasner V spacetime given by [see e.g. [18]]

$$ds^2 = -\frac{1}{z}dt^2 + z^2(dx^2 + dy^2) + z dz^2\,, \tag{F.5}$$

with $t,\ x,\ y,\ z \in \mathbb{R},\ \ z > 0$. From our discussion of Noether's theorem in Sec. B.3.2, we have the following constants of motion

$$c_0 = \frac{\dot{t}}{z}\,, \quad c_1 = z^2\dot{x}\,, \quad c_2 = z^2\dot{y}\,, \tag{F.6}$$

where the dot denoted differentiation with respect to an affine parameter λ. Furthermore, the Lagrangian does not explicitly depend on λ, so that

$$g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = -\frac{1}{z}\dot{t}^2 + z^2(\dot{x}^2 + \dot{y}^2) + z\dot{z}^2 = \epsilon\,, \tag{F.7}$$

where $\epsilon = +1$ $(-1,\ 0)$ for spacelike (timelike, null) geodesics with suitable affine parameter. A straightforward calculation shows that the geodesic equation is solved by

$$\dot{z}^2 + \frac{c_1^2 + c_2^2}{z^3} - \frac{\epsilon}{z} = c_0^2\,, \tag{F.8}$$

with $\dot{t}$, $\dot{x}$ and $\dot{y}$ directly following from the constants of motion (F.6).

Let us now consider the special case of null geodesics with initial conditions

$$\dot{x} = \dot{y} = 0\,, \quad \dot{z} < 0\,, \quad z = z_0 \quad \text{at} \ \ t = 0\,. \tag{F.9}$$

Without loss of generality, we assume time to be increasing towards the future, i.e. $\dot{t} = zc_0 > 0 \ \ \Rightarrow \ \ c_0 > 0$. Clearly, $\dot{x} = 0$, $\dot{y} = 0$ remain valid along the entire geodesic, so that all we need is to solve

$$\dot{z}^2 = c_0^2 \qquad \Rightarrow \qquad \dot{z} = \pm c_0\,. \tag{F.10}$$

For our initial condition $\dot{z} < 0$ we use the minus sign in the square root and our solution is

$$z = -c_0\lambda + z_0\,. \tag{F.11}$$

From this equation we conclude that the geodesic hits the point $z = 0$ at finite affine parameter λ. Is $z = 0$ a physical singularity? "Yes" screams the Kretschmann scalar

$$\kappa = \frac{12}{z^6} \,. \tag{F.12}$$

We see here an example of geodesic incompleteness.

Def.: A geodesic is defined to be incomplete if it "cannot be extended to arbitrarily large values of its parameter, either to the future or the past. The termination point is then a singularity." (quoted from Ryder [21]).

Does the same happen at coordinate singularities? To answer this question, we consider a second example, the *Rindler spacetime* (see e.g. [17]). Its metric is given by

$$ds^2 = -z^2 dt^2 + dx^2 + dy^2 + dz^2 \,, \tag{F.13}$$

with t, x, y, $z \in \mathbb{R}$, $z > 0$. We use Noether's theorem again on the Lagrangian for geodesics with affine parameter λ which does not depend on t, x or y

$$-\frac{\partial \mathcal{L}}{\partial \dot{t}} = 2z^2 \dot{t} = 2c_0 \;\Rightarrow\; \dot{t} = \frac{c_0}{z^2} \,, \qquad \dot{x} = c_1 \,, \qquad \dot{y} = c_2 \,, \tag{F.14}$$

Furthermore, $\mathcal{L}$ does not depend on λ so that

$$-z^2 \dot{t}^2 + \dot{x}^2 + \dot{y}^2 + \dot{z}^2 = \epsilon \,, \tag{F.15}$$

where $\epsilon = +1$ $(-1,\ 0)$ for spacelike (timelike, null) geodesics with suitable affine parameter. We consider geodesics with initial conditions

$$\dot{x} = \dot{y} = 0 \,, \quad \dot{z} < 0 \,, \quad z = z_0 \quad \text{at} \quad t = 0 \,. \tag{F.16}$$

We assume again future pointing time, so that $\dot{t} = c_0/z^2 > 0$, so that Eq. (F.15) becomes

$$\begin{aligned} \mathcal{L} &= -z^2 \frac{c_0^2}{z^4} + \underbrace{c_1^2 + c_2^2}_{=0} + \dot{z}^2 = \epsilon \\ \Rightarrow \quad \dot{z}^2 &= \epsilon + \frac{c_0^2}{z^2} \,. \end{aligned} \tag{F.17}$$

The solution for timelike geodesics ($\epsilon = -1$ which implies $\lambda = \tau$) is given by

$$z(\tau) = \sqrt{z_0^2 - \tau^2} \,, \qquad\qquad t(\tau) = \operatorname{artanh} \frac{\tau}{z_0} \,. \tag{F.18}$$

We see that we cannot extend the geodesic beyond the affine parameter $|\tau| = z_0$.

In order to see what is happening here, we transform the Rindler metric (F.13) to new coordinates T, X, Y, Z defined by

$$x = X \,, \quad y = Y \,, \quad z = \sqrt{Z^2 - T^2} \,, \quad t = \operatorname{artanh} \frac{T}{Z} \,. \tag{F.19}$$

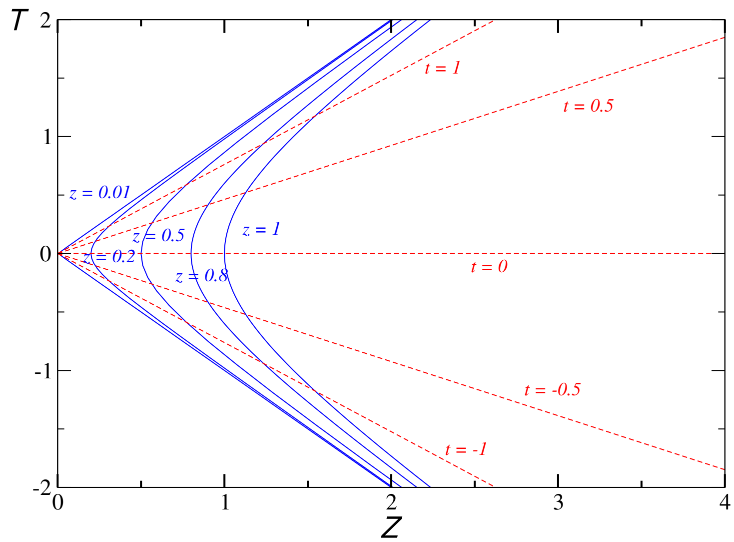


Figure 31: The Rindler wedge. Curves of constant t and z in the T-Z plane of the Minkowski spacetime. Note that curves $t \to \pm\infty$ coincide with the curve $z = 0$.

We obtain [note that $(\operatorname{artanh} x)' = 1/(1-x^2)$]

$$\begin{aligned} ds^2 &= -(Z^2-T^2)\left[\left(\frac{Z}{Z^2-T^2}\right)^2 dT^2 + \left(\frac{-T}{Z^2-T^2}\right)^2 dZ^2\right] \\ &\quad + \left(\frac{Z}{\sqrt{Z^2-T^2}}dZ\right)^2 + \left(\frac{-T}{\sqrt{Z^2-T^2}}dT\right)^2 + dX^2 + dY^2 \\ &= dT^2\left(-\frac{Z^2}{Z^2-T^2} + \frac{T^2}{Z^2-T^2}\right) + dZ^2\left(\frac{-T^2}{Z^2-T^2} + \frac{Z^2}{Z^2-T^2}\right) + dX^2 + dY^2 \\ &= -dT^2 + dX^2 + dY^2 + dZ^2 \,. \end{aligned} \tag{F.20}$$

This is simply the Minkowski spacetime which contains no singularity. The geodesic incompleteness in the Rindler spacetime signifies a coordinate singularity. For illustration of the so-called *Rindler wedge* we invert the coordinate transformation,

$$T = \frac{z \tanh t}{\sqrt{1-\tanh^2 t}} \,, \qquad Z = \frac{z}{\sqrt{1-\tanh^2 t}} \,, \tag{F.21}$$

and show in Fig. 31 curves of constant t and z in the Minkowski spacetime spanned by T and Z.

Of course, we benefited greatly in this case from "knowing" the correct coordinate transformation. Can we systematically identify the "correct" coordinates for extending a spacetime in this way once we have identified geodesic incompleteness and convinced ourselves that the singularity is not physical? In general, there is no recipe. But in two dimensions (which includes, for example, four-dimensional spacetimes with spherical symmetry), there exists a systematic procedure based on using affine parameters of ingoing and outgoing null geodesics. For more details, we refer to Sec. 6.4 in Wald [30].

G Linearized theory and gravitational waves

All analytic solutions of Einstein's equations, those we have discussed and many more we do not cover in these notes, rely on high symmetry assumptions that simplify Einstein's equations and make an analytic treatment possible. Many physical systems of interest, however do not obey these symmetries and one needs to find other ways to model them in the framework of general relativity. One way is to resort to numerical methods and solve Einstein's equations on super computers. Alternatively, one can apply perturbative techniques provided that the physical system is fairly close to an analytically known configuration. The formalism to do this is called *perturbation theory* and has found rich applications in many fields including black hole or neutron star physics and cosmology, where it supports an entire industry. General perturbation theory is beyond the scope of these notes, but we will introduce the basic methods for the case of a flat Minkowski background. These methods apply with little modifications to arbitrary background spacetimes. We start this discussion with a slight departure into plane wave solutions in general relativity. After introducing the perturbative formalism, we will discuss one of the most important applications of the weak-field theory, gravitational waves. We will also close the grand circle we have taken in these notes and see how Newtonian gravity is recovered in the limit of weak gravitational fields and slow velocities.

G.1 Plane waves and *pp* metrics

Plane waves are a very general phenomenon in physics. In electromagnetism, plane electromagnetic waves represent a propagating pattern of electric and magnetic fields described by

$$\vec{E},\ \vec{B} \propto e^{i(\vec{k}\cdot\vec{x}-\omega t)}\,, \tag{G.1}$$

where $\vec{k}$ is the wave propagation vector. This is most easily seen by rotating the coordinate system such that $\vec{k}$ points in the direction of one coordinate, say z. Then

$$\vec{k} = (0,\, 0,\, k) \qquad \Rightarrow \qquad \vec{E},\ \vec{B} \propto e^{i(kz-\omega t)} = e^{ik(z-vt)}\,, \tag{G.2}$$

where $v = \omega/k$ is the phase velocity. Plane electromagnetic waves solve the wave equation

$$\Box f = -\partial_t^2 f + \vec{\nabla}^2 f = 0\,, \tag{G.3}$$

where f stands for any of the field components. Plugging (G.1) into the wave equation we obtain the condition

$$\omega^2 - \vec{k}^2 = 0\,. \tag{G.4}$$

For a plane wave traveling in the z direction, this implies a phase velocity $v = \omega/k = \pm 1$, i.e. the wave propagates at unit speed. In relativistic notation, we write solutions to the wave equation (G.3) as

$$f \propto e^{ik_\alpha x^\alpha}\,, \qquad k_\alpha = (-\omega,\, \vec{k}) \quad \text{with} \quad k_\alpha k^\alpha = 0\,. \tag{G.5}$$

For a plane wave traveling in the z direction, $k_\alpha = (-\omega,\, 0,\, 0,\, k)$ and $\omega = |k|$.

Plane waves also exist in general relativity, either in the perturbative regime or in the fully non-linear theory. We briefly consider the latter case before focusing on the linearized case in the remainder of this section.

Def.: In general relativity, spacetimes admitting planar wave solutions are called pp wave spacetimes and defined in more mathematical terms as spacetimes that admit a covariantly constant vector field $\boldsymbol{V}$.

A class of spacetimes which satisfies this property is given by the so-called *Brinkmann metrics*

$$ds^2 = H(u,x,y)du^2 + 2du\,dv + dx^2 + dy^2\,. \tag{G.6}$$

It satisfies the above definition, since $\boldsymbol{V} := \partial_v$ is a null vector field with

$$\nabla_\alpha V^\beta = \partial_\alpha V^\beta + \Gamma^\beta_{\mu\alpha}V^\mu = 0 + \Gamma^\beta_{\mu\alpha}\delta^\mu{}_v = \Gamma^\beta_{v\alpha} = 0\,, \tag{G.7}$$

since a straightforward calculation shows that all Christoffel symbols $\Gamma^\alpha_{\mu\nu}$ with $\mu = v$ or $\nu = v$ vanish.

The vacuum Einstein equations $R_{\alpha\beta} = 0$ for the metric (G.6) has only one non-trivial component

$$R_{uu} = 0 \qquad \Rightarrow \qquad \partial_x^2 H + \partial_y^2 H = 0\,. \tag{G.8}$$

A plane wave propagating in the z direction

$$H = H_0\, e^{ik_\alpha x^\alpha}\,, \qquad H_0 = \text{const}\,, \qquad k_\alpha = (-\omega,\, 0,\, 0,\, \omega)\,, \tag{G.9}$$

therefore solves the Einstein equations as well as the wave equation (G.3). We have only introduced the Brinkmann metrics here to illustrate how plane waves can arise in general relativity and how they are represented mathematically. The concept of Brinkmann metrics and covariantly constant vectors, however, has more far-reaching consequences for the construction of analytic solutions to the Einstein equations. For example, one can allow for more general wave solutions with axisymmetry; the wave amplitude is no longer constant in the plane. One application of this technique leads to the Aichelburg-Sexl metric [3] that describes a Schwarzschild black hole moving at the speed of light. Analytic solutions of this type play important roles in contemporary research.

G.2 Linearized theory

We now consider weak gravitational fields. Gravitational waves are an example of this type. Astrophysically relevant gravitational waves are generated in the strong-field regime near strongly gravitating sources such as black-hole binaries, but when they propagate far away from their sources in the so-called *wave zone*, they represent weak perturbations on a Minkowski background and are well modelled by the weak-field formalism. Another example is the Newtonian limit that describes with good accuracy many phenomena we are used to from daily experience. Let us consider therefore spacetimes that only differ mildly from the Minkowski metric

$$\eta_{\mu\nu} = \text{diag}(-1,\, 1,\, 1,\, 1)\,. \tag{G.10}$$

A metric that is close to Minkowski is conveniently described in terms of its deviation from $\eta_{\mu\nu}$,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}\,, \qquad h_{\mu\nu} = \mathcal{O}(\epsilon) \ll 1\,, \tag{G.11}$$

where $\epsilon \ll 1$ is an expansion parameter. We regard $h_{\mu\nu}$ as a tensor field on the Minkowski background manifold. We therefore have two metrics, the background metric $\eta_{\mu\nu}$ and the physical metric $g_{\mu\nu}$. Next, we look at the inverse metric defined by $g^{\mu\nu}g_{\nu\lambda} = \delta^{\mu}{}_{\lambda}$. We expect $g^{\mu\nu}$ to be close to $\eta^{\mu\nu}$, but make no further assumption about its form, so that

$$\begin{aligned} & g^{\mu\nu} = \eta^{\mu\nu} + k^{\mu\nu}\,, \qquad k^{\mu\nu} = \mathcal{O}(\epsilon) \ll 1 \\ \Rightarrow \quad & g^{\mu\nu}g_{\nu\rho} = \delta^{\mu}{}_{\rho} + k^{\mu\nu}\eta_{\nu\rho} + \eta^{\mu\nu}h_{\nu\rho} + \underbrace{k^{\mu\nu}h_{\nu\rho}}_{=\mathcal{O}(\epsilon^2)} \overset{!}{=} \delta^{\mu}{}_{\rho}\,. \end{aligned} \tag{G.12}$$

In linearized theory we drop all terms beyond linear order $\mathcal{O}(\epsilon)$. Here lies the key simplification achieved with the perturbative technique. For the inverse metric perturbation we thus obtain

$$\begin{aligned} & k^{\mu\nu}\eta_{\nu\rho} + \eta^{\mu\nu}h_{\nu\rho} = 0 \qquad\qquad \Big|\ \cdot\,\eta^{\rho\sigma} \\ \Rightarrow \quad & k^{\mu\sigma} = -\eta^{\mu\nu}\eta^{\rho\sigma}h_{\nu\rho} =: -h^{\mu\sigma} = \mathcal{O}(\epsilon)\,. \end{aligned} \tag{G.13}$$

Here we have raised the indices of $h_{\mu\nu}$ with the Minkowski metric $\eta^{\alpha\beta}$. Note, however, that at linear order, raising the indices instead with $g^{\mu\nu}$ would have led to the same result. Nevertheless, we need to be watchful in raising and lowering indices and bear in mind which metric is used. Unless specified otherwise, we shall from now on use the physical metric $\boldsymbol{g}$ to raise and lower indices. Note also that $k^{\mu\nu} \neq h^{\mu\nu}$. This is a general result: the perturbation of a tensor with upstairs indices is not obtained by raising (either with $g^{\mu\nu}$ or $\eta^{\mu\nu}$) those of the downstairs tensor perturbations.

Let us next calculate the perturbations of the Christoffel symbols. To linear order in ϵ,

$$\Gamma^{\mu}_{\nu\rho} = \frac{1}{2}\eta^{\mu\sigma}(\partial_{\nu}h_{\rho\sigma} + \partial_{\rho}h_{\sigma\nu} - \partial_{\sigma}h_{\nu\rho}) + \mathcal{O}(\epsilon^2)\,. \tag{G.14}$$

For the Riemann tensor we obtain

$$R_{\mu\nu\rho\sigma} = \eta_{\mu\tau}\big(\partial_{\rho}\Gamma^{\tau}_{\nu\sigma} - \partial_{\sigma}\Gamma^{\tau}_{\nu\rho}\big) \qquad \Big|\ \Gamma\cdot\Gamma = \mathcal{O}(\epsilon^2)$$

$$\Rightarrow \quad R_{\mu\nu\rho\sigma} = \frac{1}{2}\big(\partial_{\rho}\partial_{\nu}h_{\mu\sigma} + \partial_{\sigma}\partial_{\mu}h_{\nu\rho} - \partial_{\rho}\partial_{\mu}h_{\nu\sigma} - \partial_{\sigma}\partial_{\nu}h_{\mu\rho}\big) \tag{G.15}$$

$$\Rightarrow \quad R_{\mu\nu} = \partial^{\rho}\partial_{(\mu}h_{\nu)\rho} - \frac{1}{2}\partial^{\rho}\partial_{\rho}h_{\mu\nu} - \frac{1}{2}\partial_{\mu}\partial_{\nu}h \qquad \Big|\ h := h^{\mu}{}_{\mu}\,, \quad \partial^{\mu} := g^{\mu\rho}\partial_{\rho} \tag{G.16}$$

$$\Rightarrow \quad G_{\mu\nu} = \partial^{\rho}\partial_{(\mu}h_{\nu)\rho} - \frac{1}{2}\partial^{\rho}\partial_{\rho}h_{\mu\nu} - \frac{1}{2}\partial_{\mu}\partial_{\nu}h - \frac{1}{2}\eta_{\mu\nu}(\partial^{\rho}\partial^{\sigma}h_{\rho\sigma} - \partial^{\rho}\partial_{\rho}h) \overset{!}{=} 8\pi T_{\mu\nu}\,. \tag{G.17}$$

Note that the Einstein tensor $G_{\mu\nu} = \mathcal{O}(\epsilon)$ and, hence, the energy momentum tensor is also of perturbative order $T_{\mu\nu} = \mathcal{O}(\epsilon)$. Equation (G.17) gives us the Einstein equations at first order in ϵ. It turns out that these equations are more conveniently expressed in terms of the trace reversed metric perturbation.

Def.: The trace reversed metric perturbation is

$$\bar{h}_{\mu\nu} := h_{\mu\nu} - \frac{1}{2} h\, \eta_{\mu\nu} \qquad \Leftrightarrow \qquad h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2}\bar{h}\, \eta_{\mu\nu}\,, \tag{G.18}$$

where $\bar{h} = \bar{h}^{\mu}{}_{\mu} = -h$.

Plugging this definition into Eq. (G.17), we obtain after a little calculation

$$G_{\mu\nu} = -\frac{1}{2}\partial^{\rho}\partial_{\rho}\bar{h}_{\mu\nu} + \partial^{\rho}\partial_{(\mu}\bar{h}_{\nu)\rho} - \frac{1}{2}\eta_{\mu\nu}\partial^{\rho}\partial^{\sigma}\bar{h}_{\rho\sigma} = 8\pi T_{\mu\nu}\,. \tag{G.19}$$

Further simplification of the linearized Einstein equations is achieved by using the coordinate freedom. Note that we have specified the background coordinates, Cartesian coordinates in an inertial frame of the Minkowski spacetime. But we can still change the coordinates at order $\mathcal{O}(\epsilon)$. We denote this change by a difference $\xi^{\alpha} = \mathcal{O}(\epsilon)$,

$$\begin{aligned} \tilde{x}^{\alpha} &= x^{\alpha} - \xi^{\alpha} & &\Leftrightarrow & x^{\alpha} &= \tilde{x}^{\alpha} + \xi^{\alpha} \\ \frac{\partial \tilde{x}^{\alpha}}{\partial x^{\mu}} &= \delta^{\alpha}{}_{\mu} - \partial_{\mu}\xi^{\alpha} & &\Leftrightarrow & \frac{\partial x^{\nu}}{\partial \tilde{x}^{\beta}} &= \delta^{\nu}{}_{\beta} + \tilde{\partial}_{\beta}\xi^{\nu}\,. \end{aligned} \tag{G.20}$$

The physical metric transforms according to the tensor transformation law (B.34), so that

$$\tilde{g}_{\mu\nu} = \eta_{\mu\nu} + \tilde{h}_{\mu\nu} = (\delta^{\alpha}{}_{\mu} + \partial_{\mu}\xi^{\alpha})(\delta^{\beta}{}_{\nu} + \partial_{\nu}\xi^{\beta})(\eta_{\alpha\beta} + h_{\alpha\beta}) = \eta_{\mu\nu} + \partial_{\mu}\xi_{\nu} + \partial_{\nu}\xi_{\mu} + \mathcal{O}(\epsilon^2)$$

$$\Rightarrow \quad \boxed{\tilde{h}_{\mu\nu} = h_{\mu\nu} + \partial_{\mu}\xi_{\nu} + \partial_{\nu}\xi_{\mu}}\,. \tag{G.21}$$

We have four free functions and can use these to satisfy four relations. A particularly convenient transformation is to choose the ξ_{μ} such that

$$\partial^{\nu}\partial_{\nu}\xi_{\mu} = -\partial^{\nu}\bar{h}_{\mu\nu} \tag{G.22}$$

$$\Rightarrow \quad \bar{\tilde{h}}_{\mu\nu} = \tilde{h}_{\mu\nu} - \frac{1}{2}\eta^{\rho\sigma}\tilde{h}_{\rho\sigma}\eta_{\mu\nu} = h_{\mu\nu} + \partial_{\mu}\xi_{\nu} + \partial_{\nu}\xi_{\mu} - \frac{1}{2}\eta^{\rho\sigma}(h_{\rho\sigma} + \partial_{\rho}\xi_{\sigma} + \partial_{\sigma}\xi_{\rho})\eta_{\mu\nu}$$

$$\Rightarrow \quad \bar{\tilde{h}}_{\mu\nu} = \bar{h}_{\mu\nu} + \partial_{\mu}\xi_{\nu} + \partial_{\nu}\xi_{\mu} - \eta_{\mu\nu}\partial^{\sigma}\xi_{\sigma} \tag{G.23}$$

$$\Rightarrow \quad \partial^{\nu}\bar{\tilde{h}}_{\mu\nu} = \partial^{\nu}\bar{h}_{\mu\nu} + \partial^{\nu}\partial_{\mu}\xi_{\nu} + \partial^{\nu}\partial_{\nu}\xi_{\mu} - \partial^{\sigma}\partial_{\mu}\xi_{\sigma} = \partial^{\nu}\bar{h}_{\mu\nu} + \partial^{\nu}\partial_{\nu}\xi_{\mu} = 0\,. \tag{G.24}$$

Note that the expression (G.19) for the Einstein tensor is valid in unchanged form if we replace $h_{\mu\nu}$ with $\tilde{h}_{\mu\nu}$, since we could have started the entire derivation with either $\boldsymbol{h}$ or $\tilde{\boldsymbol{h}}$. With the gauge condition (G.24), however, Eq. (G.19) simplifies to

$$\boxed{\partial^{\rho}\partial_{\rho}\bar{\tilde{h}}_{\mu\nu} = -16\pi T_{\mu\nu}}\,. \tag{G.25}$$

This is a quite remarkable simplification: we merely have to solve the flat-space wave equation for the metric components. Because the tilde is not a convenient notation, especially in combination with the bar for the trace reverse metric perturbation, we will drop the tilde now and write $h_{\mu\nu}$ which we implicitly assume to satisfy the so-called "Lorentz gauge" condition (G.22).

G.3 The Newtonian limit

Newtonian gravity is described by the Poisson equation

$$\vec{\nabla}\Phi = 4\pi\rho\,, \tag{G.26}$$

where Φ is the gravitational potential. In Eqs. (A.9) and (A.10), we have seen that the Newtonian potential $\Phi \propto v^2$ where v is the velocity of objects moving in this field due to gravitational attraction. This is indeed a generic feature of Newtonian gravity and we therefore define the expansion parameter ϵ of the previous section as

$$\epsilon = \frac{v^2}{c^2} = v^2 \propto \frac{M}{R}\,, \tag{G.27}$$

where M is the characteristic mass of the gravitational source and R the distance of moving particles from this source. For non-relativistic motion we have $\epsilon \ll 1$ as required for a perturbative treatment. From our discussion of the energy-momentum tensor in Sec. C.2, we furthermore know that the component T_{00} represents mass-energy density ρ, the T_{0i} components represent momentum density $\propto \rho v^i$ and the T_{ij} components denote the flux of this momentum in spatial directions, i.e. $T_{ij} \propto \rho v^i v^j$. For Newtonian sources of gravitational waves, we already know from the discussion following Eq. (G.17) that the energy density is $\rho = \mathcal{O}(\epsilon)$, so that

$$\begin{aligned} T_{00} &= \rho = \mathcal{O}(\epsilon)\,,\\ T_{0i} &\sim \rho v^i \sim \mathcal{O}(\epsilon^{3/2})\,,\\ T_{ij} &\sim \rho v^i v^j \sim \mathcal{O}(\epsilon^2)\,. \end{aligned} \tag{G.28}$$

Consider, for example, solar interior modelled as a perfect fluid

$$T_{\mu\nu} = (\rho + P)u_\mu u_\nu + P g_{\mu\nu}\,, \qquad P \sim \rho v^2 \approx 10^{-5}\rho \;\text{ in the sun}\,. \tag{G.29}$$

In Newtonian gravity, temporal changes in the field Φ are caused by the motion of the matter sources. Again, we use the fact that these velocities v are small, so that

$$\begin{aligned} &\frac{\partial}{\partial t} \sim v\frac{\partial}{\partial x^i} = \mathcal{O}(\epsilon^{1/2})\frac{\partial}{\partial x^i}\\ \Rightarrow\quad &\Box \bar{h}_{\mu\nu} = \partial^\rho\partial_\rho \bar{h}_{\mu\nu} = \partial^i\partial_i \bar{h}_{\mu\nu} = \vec{\nabla}^2 \bar{h}_{\mu\nu} = -16\pi T_{\mu\nu}\\ \Rightarrow\quad &\vec{\nabla}^2 \bar{h}_{00} = -16\pi T_{00} = -16\pi\rho + \mathcal{O}(\epsilon^{3/2})\,, \qquad \bar{h}_{0i} = \mathcal{O}(\epsilon^{3/2})\,, \qquad \bar{h}_{ij} = \mathcal{O}(\epsilon^2)\,. \end{aligned} \tag{G.30}$$

This is Newton's law (G.26) with the identification $\bar{h}_{00} = -4\Phi$. Now we merely need to reverse-engineer the metric perturbations from $\bar{h}_{00}$. We have

$$\bar{h} = \eta^{\mu\nu}\bar{h}_{\mu\nu} = 4\Phi + \mathcal{O}(\epsilon^{3/2}) = -h$$
$$\Rightarrow \quad h_{00} = \bar{h}_{00} - \frac{1}{2}\eta_{00}\bar{h} = -2\Phi\,, \qquad h_{ij} = \bar{h}_{ij} - \frac{1}{2}\eta_{ij}\bar{h} = -2\Phi\delta_{ij}\,, \tag{G.31}$$

which gives us the metric in the Newtonian limit as

$$\boxed{ds^2 = -(1+2\Phi)dt^2 + (1-2\Phi)(dx^2+dy^2+dz^2)}\,, \tag{G.32}$$

which is the line element we have used in the redshift calculation in Eq. (A.42).

Let us next calculate particle motion in the Newtonian limit by studying the geodesics of (A.42). Using proper time and time like geodesics, we obtain [note that $\dot{x}^i \sim v^i = \mathcal{O}(\epsilon^{1/2})$]

$$\mathcal{L} = (1+2\Phi)\dot{t}^2 - \delta_{ij}(1-2\Phi)\dot{x}^i\dot{x}^j \overset{!}{=} 1$$
$$\Rightarrow \quad \dot{t}^2 = (1+2\Phi)^{-1}[1+\delta_{ij}\dot{x}^i\dot{x}^j + \mathcal{O}(\epsilon^2)]$$
$$\Rightarrow \quad \dot{t} = 1 - \Phi + \frac{1}{2}\delta_{ij}\dot{x}^i\dot{x}^j + \mathcal{O}(\epsilon^2)\,. \tag{G.33}$$

The Euler-Lagrange equation for the x^k component is given by

$$\frac{d}{d\tau}\frac{\partial\mathcal{L}}{\partial\dot{x}^k} = \frac{d}{d\tau}[-2\delta_{jk}(1-2\Phi)\dot{x}^j] = \frac{\partial\mathcal{L}}{\partial x^k} = 2\partial_k\Phi\,\underbrace{(\dot{t}^2 + \delta_{ij}\dot{x}^i\dot{x}^j)}_{=1+\mathcal{O}(\epsilon^2)}$$
$$\Rightarrow \quad -2\delta_{jk}\ddot{x}^j + \mathcal{O}(\epsilon) = 2\partial_k\Phi$$
$$\Rightarrow \quad \frac{d^2x_k}{dt^2} = \frac{d^2x_k}{d\tau^2} + \mathcal{O}(\epsilon^2) = -\partial_k\Phi\,. \tag{G.34}$$

This is exactly the equation of motion for a test particle in Newtonian gravity. Note that this calculation also confirms that the factor 8π in the Einstein equations $\boldsymbol{G} = 8\pi\boldsymbol{T}$ is the correct number to reproduce the Newtonian limit.

G.4 Gravitational waves

Gravitational waves are modulations in the spacetime fabric that propagate at the speed of light and that induce, as we shall see, variations in the length of objects they pass through. For their modeling in perturbation theory, we consider vacuum spacetimes but allow for relativistic velocities. The linearized Einstein equations then become

$$\Box\bar{h}_{\mu\nu} = (\partial_t^2 - \vec{\nabla}^2)\bar{h}_{\mu\nu} = 0\,. \tag{G.35}$$

This is exactly the wave equation (G.3) we discussed in the context of plane waves in Sec. G.1. Plane wave solutions to this equation are given by

$$\bar{h}_{\mu\nu} = H_{\mu\nu} e^{ik_\rho x^\rho}\,, \qquad H_{\mu\nu} = \text{const}\,. \tag{G.36}$$

This solution has the following properties.

(1) Plugging (G.36) into (G.35), we find $k_\mu k^\mu = 0$, i.e. the wave propagates at the speed of light.

(2) The Lorentz gauge condition $\partial^\nu \bar{h}_{\mu\nu} = 0$ implies $k^\mu H_{\mu\nu} = 0$, which means that the waves are transverse to the direction of propagation. For a plane wave traveling in the z direction, for example, we have $k^\mu = \omega\,(1,\,0,\,0,\,1)$ and, hence, $H_{\mu 0} + H_{\mu 3} = 0$.

We still have some remaining gauge freedom to exploit. Taking

$$\xi_\mu = X_\mu e^{ik_\rho x^\rho} \qquad \Rightarrow \qquad \partial^\nu \partial_\nu \xi_\mu = 0\,, \tag{G.37}$$

leaves the Lorentz gauge condition (G.22) unaffected. A short calculation shows that the transformation (G.37) changes the plane wave (G.36) according to

$$H_{\mu\nu} \quad \rightarrow \quad H_{\mu\nu} + i(k_\mu X_\nu + k_\nu X_\mu - \eta_{\mu\nu} k^\rho X_\rho)\,. \tag{G.38}$$

It can be shown that there exists a choice X_μ such that (G.38) leads to

$$H_{0\mu} = 0\,, \qquad H^\mu{}_\mu = 0\,. \tag{G.39}$$

This is the "traceless" condition and combined with the transverse condition above, it is often referred to as the *transverse-traceless gauge*. In this gauge, the gravitational wave solution has two important properties.

(1) $h = 0 \quad \Rightarrow \quad h_{\mu\nu} = \bar{h}_{\mu\nu}$, so that we need not distinguish between the trace-reversed and the original metric perturbation.

(2) For a plane wave propagating in the z direction, we find $H_{0\mu} = H_{3\mu} = H^\mu{}_\mu = 0$, so that $H_{\mu\nu}$ can be written as

$$H_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & H_+ & H_\times & 0 \\ 0 & H_\times & -H_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{G.40}$$

So what happens if such a gravitational wave passes through some arrangement of test particles? To answer this question, we study the geodesic equation for the metric $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with $h_{\mu\nu}$ given by Eqs. (G.36), (G.40). Let us consider a test particle initially at rest in a background inertial frame, i.e. the four-velocity of this particle is initially $u^\alpha = (1,\,0,\,0,\,0)$. The geodesic equation at the initial time is given by

$$\frac{d}{d\tau} u^\alpha + \Gamma^\alpha_{\mu\nu} u^\mu u^\nu = \dot{u}^\alpha + \Gamma^\alpha_{00} = 0\,. \tag{G.41}$$

From the metric perturbation, we obtain

$$\Gamma^{\alpha}_{00} = \frac{1}{2}\eta^{\alpha\beta}(\partial_0 h_{\beta 0} + \partial_0 h_{0\beta} - \partial_\beta h_{00}) = 0 \quad \text{since } H_{0\mu} = 0\,. \tag{G.42}$$

The particle therefore never acquires velocity components in the x^i directions and remains at fixed position x^μ in this gauge as the gravitational wave passes through. Physical experiments, however, measure the proper distance that is obtained from

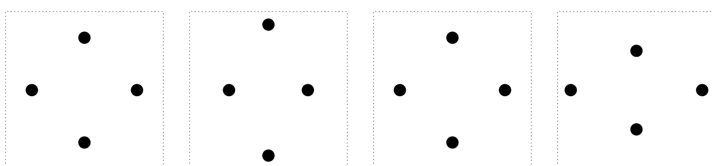
$$ds^2 = -dt^2 + (1 + h_+)dx^2 + (1 - h_+)dy^2 + 2h_\times dx\,dy + dz^2\,, \tag{G.43}$$

where $h_{+,\times} = H_{+,\times}e^{ik_\rho x^\rho}$. We consider two cases.

Case 1: $H_\times = 0\,,\quad H_+ \neq 0$, so that h_+ oscillates. The proper distance between specific particles can be summarized as follows.

2 particles at $(-\delta,\,0,\,0)$, $(\delta,\,0,\,0)$ have $ds^2 = (1 + h_+)4\delta^2$.

2 particles at $(0,\,-\delta,\,0)$, $(0,\,\delta,\,0)$ have $ds^2 = (1 - h_+)4\delta^2$.

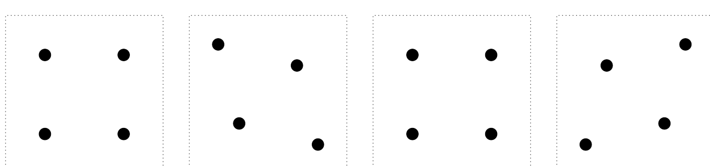


The figure illustrates the motion of the four test particles as the gravitational wave generates the oscillating perturbation. This pattern motivates the index "+" in h_+.

Case 2: $H_+ = 0\,,\quad H_\times \neq 0$, so that $h_\times$ oscillates. The proper distance between specific particles can be summarized as follows.

2 particles at $(-\delta,\,-\delta,\,0)/\sqrt{2}$, $(\delta,\,\delta,\,0)/\sqrt{2}$ have $ds^2 = (1{+}h_\times)4\delta^2$.

2 particles at $(\delta,\,-\delta,\,0)/\sqrt{2}$, $(-\delta,\,\delta,\,0)/\sqrt{2}$ have $ds^2 = (1{-}h_\times)4\delta^2$.



The figure illustrates the motion of the four test particles as the gravitational wave generates the oscillating perturbation. This pattern motivates the index "×" in $h_\times$.

Gravitational waves have been conjectured to exist soon after Einstein published his theory, but their nature remained under constant debate for about 40 years, including Einstein himself who vacillated on the issue. It was only in the late 1950s, that results by Bondi, Pirani, Sachs and others demonstrated convincingly that gravitational waves are not merely a gauge effect but carry physical energy; for an overview of of the history on these debates, see for instance [22]. By now, there remains no doubt that gravitational waves carry energy and the leading order term can be calculated analytically for a wide variety of sources. This is contained in the famous *quadrupole formula* which we merely quote here; for a derivation of this formula see for example [36]. Consider for this purpose a distribution of energy density $\rho(t,\vec{y})$ contained inside a domain of compact support. The quadrupole tensor is defined as

$$I_{ij} := \int \rho(t,\vec{y})\, y^i\, y^j\, d^3y\,. \tag{G.44}$$

The quadrupole formula predicts the energy flux at a distance r from the source averaged over times that are large compared with the period of the gravitational wave signal. This flux is

$$\boxed{\langle p\rangle_t = \frac{G}{5c^5}\langle \dddot{Q}_{ij}\dddot{Q}_{ij}\rangle_{t-r}\,; \quad Q_{ij} := I_{ij} - \frac{1}{3}I_{kk}\,\delta_{ij}}\,, \tag{G.45}$$

where Q_{ij} is the *reduced quadrupole tensor* and the indices t and $t-r$ means that a gravitational wave observed at time t is sourced by time variations of the sources at retarded time $t-r$. The dots denote time derivatives and the symbols $\langle\ .\ \rangle$ the averaging over sufficiently long times.

Let us consider as an example a system of two equal point masses in circular orbit according to Newtonian gravity. The energy density is

$$\rho(\vec{x}) = m\delta(\vec{x}-\vec{x}_1)+m\delta(\vec{x}-\vec{x}_2)\,, \qquad x_1^i = r\,(\cos\phi,\, \sin\phi,\, 0)\,, \quad x_2^i = -r\,(\cos\phi,\, \sin\phi,\, 0)\,. \tag{G.46}$$

The motion of two such bodies is governed by the Newtonian gravitational and centrifugal forces

$$G\frac{m^2}{(2r)^2} \overset{!}{=} \frac{mv^2}{r} \quad\Rightarrow\quad G\frac{m}{4r^2} = \frac{v^2}{r} \quad\Rightarrow\quad \omega = \frac{v}{r} = \sqrt{G\frac{m}{4r^3}} \tag{G.47}$$

The quadrupole tensor is

$$\begin{aligned} I_{xx} &= 2mr^2\cos^2\omega t = mr^2(1+\cos 2\omega t) \\ I_{yy} &= 2mr^2\sin^2\omega t = 2mr^2(1-\cos^2\omega t) = mr^2(1-\cos 2\omega t) \\ I_{xy} &= I_{yx} = 2mr^2\cos\omega t\,\sin\omega t = mr^2\sin 2\omega t\,. \end{aligned} \tag{G.48}$$

Note that we traded the quadratic cos and sin functions for linear ones to simplify taking derivatives. The traceless quadrupole tensor is $Q_{ij} = I_{ij} - 2mr^2/3$ and thus only differs from I_{ij} by a constant. The time derivatives of the two are therefore equal,

$$\begin{aligned} \dddot{Q}_{xx} &= 8\omega^3 mr^2\sin 2\omega t\,, \\ \dddot{Q}_{yy} &= -8\omega^3 mr^2\sin 2\omega t\,, \\ \dddot{Q}_{xy} = \dddot{Q}_{yx} &= -8\omega^3 mr^2\cos 2\omega t\,. \end{aligned} \tag{G.49}$$

Adding all up gives

$$\langle p\rangle_t = \frac{2}{5}\frac{G^4}{c^5}\frac{m^5}{r^5}\,. \tag{G.50}$$

This loss of energy was famously identified in observations of the Hulse-Taylor pulsar starting in the 1970s [16]. The observations were compared with higher-order predictions going beyond the quadrupole formula and revealed excellent agreement with the predictions of general relativity leading to the 1993 Nobel Prize. Finally, in September 2015, the LIGO gravitational wave detectors in Hanford and Livingston, US, made the first direct detection of a gravitational wave signal [1] using an instrumental setup that is reminiscent of the Michelson-Morley interferometer but uses a wealth of highly advanced technology. Even though gravitational waves carry a

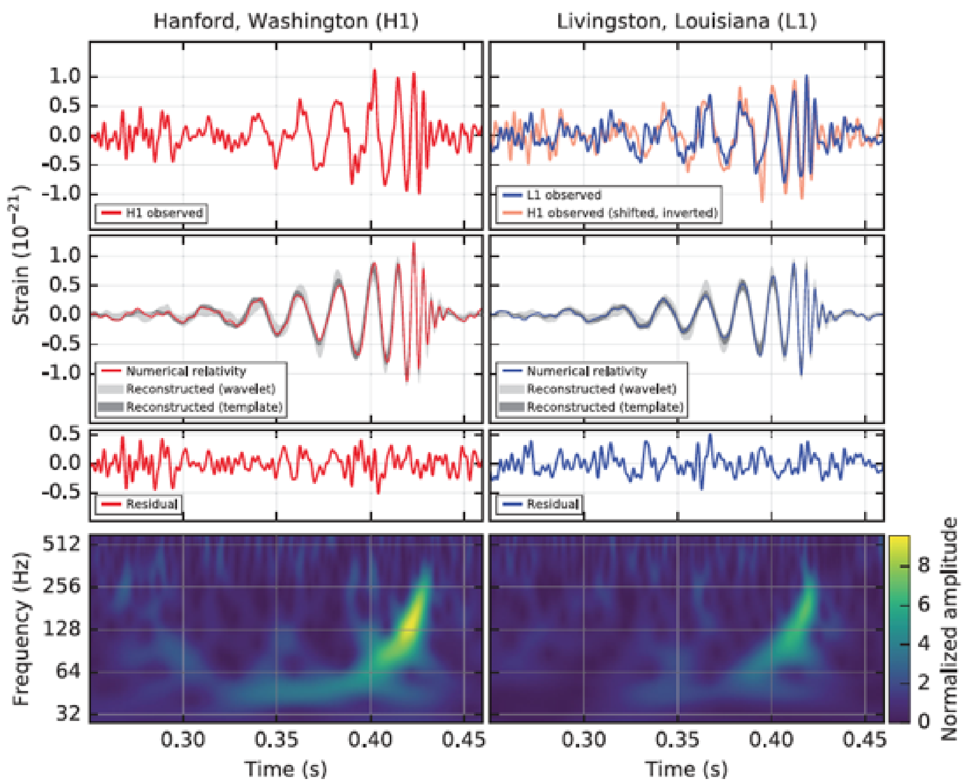


Figure 32: Observed signal of the black-hole binary signal GW150914 as measured with the LIGO detectors at Hanford and Livingston (upper panels), numerical relativity predictions for a black-hole binary using the most likely mass parameters (upper middle panels), the difference between signal and prediction (lower middle panels) and the power spectrum in the time-frequency domain (bottom panels). Taken from [1].

tremendous amount of energy, they interact very weakly with matter including the detectors. The variation in length we have displayed for the arrangements of test particles above has been vastly exaggerated. For realistic sources the change in length $\Delta l/l = \mathcal{O}(10^{-21})$ which corresponds to about the width of a hair in the distance to the next star, Proxima Centauri. The detected signal together with the theoretical predictions and power spectra is shown in Fig. 32. A second event has by now been detected [2], demonstrating that the first detection was not merely a fluke. The LIGO detectors are being upgraded to higher sensitivity and other detectors, Virgo, LIGO India and Japan's KAGRA will join the network over the coming years. Throughout these notes, we have encountered a number of questions that remain open to this day (dark energy, dark matter, possible modifications of the theory of relativity). It is not unlikely that the new field of gravitational wave astronomy will revolutionize our understanding of the Universe. But that is a story to be told on some other future occasion...

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